Dynamic modeling of a simple pendulum underwater

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The problem discussed in this short report is the dynamic modeling of a simple 2D pendulum underwater as shown in Fig.1.

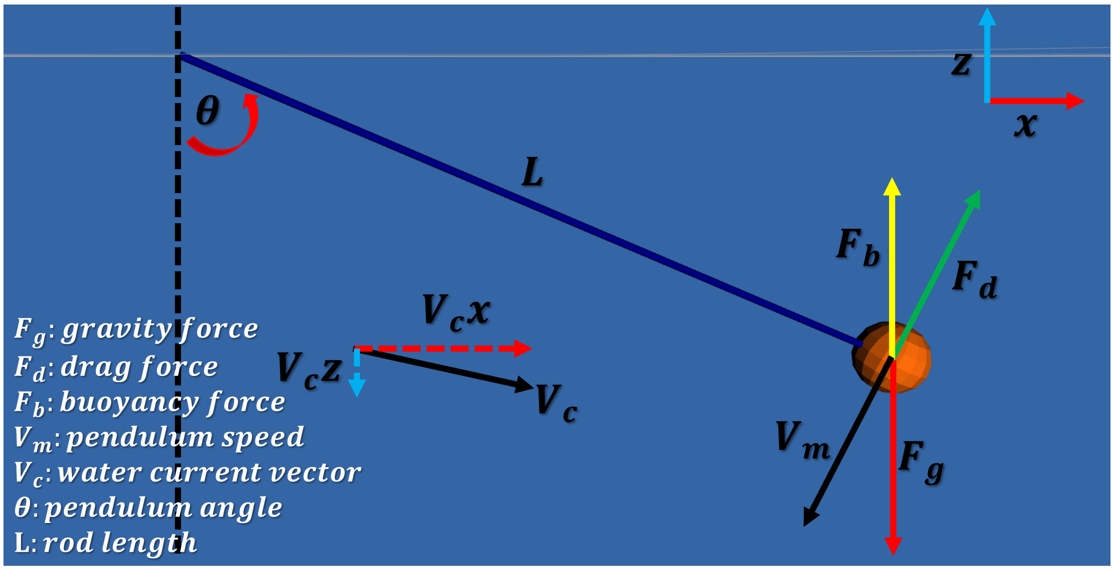

Figure 1: An illustrative figure of a simple pendulum

Assumptions:

1- The pendulum joint allows the pendulum to move in 2d only. So we will ignore any force components in the y-direction.
2- Water current is a 3D vector but we will ignore the y component for the previous reason.
3- The pendulum rod link is assumed to have no mass or drag effect.
4- The centrifugal forces and the friction forces are ignored in this simulation.
5- The pendulum angle θ is measured from the vertical axis as shown in the figure. So, the equilibrium angles of the pendulum are 0, π, and $-\pi$.
6- When we consider the pendulum speed V_m or the water current speed V_c, we must consider their components that are perpendicular to the position vector L.

One efficient approach to solve this problem is to apply Newton's second law of motion:

" The acceleration of an object as produced by a net force is directly proportional to the magnitude of the net force, in the same direction as the net force, and inversely proportional to the mass of the object. "

Or in our language:

$$F_{net} = ma \tag{1}$$

But since we are dealing with a rotational object, it is better to say [1]:

$$\tau_{net} = I\alpha \tag{2}$$

Where τ_{net} is the total torque forces applied on the object, I is the second moment of interia, and α is the angular acceleration. But since we are ignoring the mass of the pendulum rod, we can easily calculate I for the pendulum by:

$$I = mL^2 \tag{3}$$

Regarding the total torques applied on the pendulum mass, we have:
1- Torque due to gravity force

$$\tau_g = -mgLsin(\theta) \tag{4}$$

2- Torque due to water current and drag
We have the drag force formula as:

$$F_d = -0.5\rho_w \times A_p \times k_d \left|V_m - V_c\right| (V_m - V_c) \tag{5}$$

$$\tau_d = F_d L \tag{6}$$

where V_m, V_c are the instantaneous speeds of the pendulum mass and water current respectively. k_d is the drag coefficient, ρ_w is the water density, and A_p is the projected area which in our case will πr^2. Moreover:

$$V_m = L\dot{\theta} \tag{7}$$

$$V_c = V_c x \times cos(\theta) + V_c z \times sin(\theta) \tag{8}$$

3- Torque due to buoyancy Since we are simulating a pendulum underwater, it is important to take buoyancy into consideration. Buoyancy force mainly depends on the submerged volume and the fluid density. Therefore the torque due to buoyancy can be calculated as the following

$$\tau_b = \rho_w \times V \times gLsin(\theta) \tag{9}$$

Where V is the submerged volume of the pendulum which in our case will be a volume of a sphere ($\frac{4}{3}\pi r^3$)
So to sum up:

$$mL^2\ddot{\theta} = \tau_g + \tau_d + \tau_b \tag{10}$$

$$\ddot{\theta} = \frac{\tau_g + \tau_d + \tau_b}{mL^2} \tag{11}$$

Up to here, we may solve this differential equation by using several techniques and perhaps making some assumptions like the small angle approximation. But we can also utilize the computing power of the simulation and solve it numerically where:

$$\ddot{\theta} = \frac{-mgLsin(\theta) - 0.5\rho_w \times A_p \times k_d \left|V_m - V_c\right| (V_m - V_c)L + \rho_w \times V \times gLsin(\theta)}{mL^2} \tag{12}$$

And we can provide initial values for θ and $\dot{\theta}$ to calculate $\ddot{\theta}$ and then repeat the simulation loop where:

$$\dot{\theta} = \dot{\theta} + \ddot{\theta}\Delta t \tag{13}$$

$$\theta = \theta + \dot{\theta}\Delta t \tag{14}$$

The previous process in numerical integration and the smaller the Δt the better. In the simulation, Δt was set to be 0.01s.

References

[1] Varghese Mathai, Laura A. W. M. Loeffen, Timothy T. K. Chan, and Sander Wildeman. Dynamics of heavy and buoyant underwater pendulums. *Journal of Fluid Mechanics*, 862:348–363, 2019. doi:10.1017/jfm.2018.867.