MAT321: APPLYING BURNSIDE'S LEMMA TO CALCULATE NUMBER OF UNIQUE BRACELETS

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1. Introduction

Imagine you are cleaning out your room and find a box of string and colorful beads. You decide to make bracelets that have five beads, and challenge yourself to make as many unique bracelets as you can using the purple, blue, and orange beads that you have. It seems like a reasonable prediction that you can make 243 bracelets, as there are five beads chosen for each bracelet and each bead is selected from a set of three colors. Thus, it seems like you could create $3^5 = 243$ different bracelets.

However, as you are making your bracelets you run into a problem. You made two bracelets that you thought were unique, but you realized that when you rotated one bracelet clockwise, it ended up being the exact same bracelet as the other!

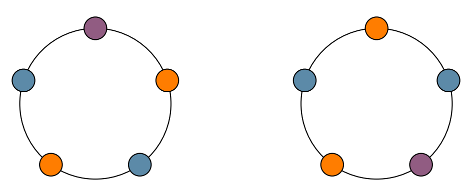

Figure 1. The same 5-bead bracelet when rotated by two beads clockwise.

You alter your bracelet-making approach to account for these rotations, but encounter another problem! If you flip one of the bracelets you made over the vertical axis crossing the bead at the top, it is the same as yet another bracelet you made.

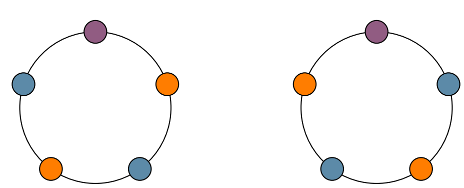

Figure 2. The same 5-bead bracelet when reflected over the vertical axis.

You realize that, due to these rotations and reflections, you cannot create 243 unique bracelets with your given craft supplies. So how many unique bracelets can actually be created? Is there an equation that accurately calculates the number of unique bracelets, accounting for these rotations and reflections? This idea will be explored in subsequent sections.

2. Background information

To answer the questions posed in the introduction, we can apply the following equation, called Burnside's Lemma. To understand this theorem, we must gain familiarity with concepts such as orbits and group actions.

Theorem 2.1 (**Burnside's Lemma / Burnside's Counting Theorem**)**.** [1, Prop 5.2.2] *Let a finite group G act on a finite set X. Then the number of orbits of the action is*

$$\frac{1}{|G|}\sum_{g\in G}|Fix(g)|.$$

Proof. The proof of Burnside's Lemma can be found in [3, Section 14]. □

In words, this equation states that the number of **orbits** is equal to the average number of **fixed points** in the **group action**, taken over all the group elements. In terms of our bracelet example, this equation means that the number of objects, or number of unique bracelets, is equal to the sum of the symmetrical bracelets divided by the number of symmetries. Let's break down this complex theorem into its individual components and define the terms. First, we will define our first new term, **orbit**.

Definition 2.2. [2, Def 7.2.1] Let G be a group and X be a set. If G acts on X, then the **orbit** of an element $x \in X$ is defined to be

$$\text{orb}(x) = \{g \cdot x | g \in G\}.$$

But what does it mean for a group G to **act on** a set X?

Definition 2.3. [1, Def 5.1.1] An **action** of a group G on a set X is a homomorphism from G into $\text{Sym}(X)$.

From [2, List of Common Groups], $\text{Sym}(X)$ is defined to be the set of all bijections $X \to X$ under the operation of composition. A more comprehensive definition of group action is as follows.

Definition 2.4 ((Left) Group Action)**.** [2, Def 7.1.2] Let X be a set and G be a group. An **action of** G **on** X is a *set* function $\cdot : G \times X \to X$, where we use $g \cdot x$ to denote the function $\cdot$ evaluated on $(g, x) \in G \times X$, that satisfies the following:

(i) $e \cdot x = x$ for all $x \in X$, and
(ii) $g_2 \cdot (g_1 \cdot x) = (g_2 g_1) \cdot x$ for all $g_1, g_2 \in G$ and $x \in X$.

Finally, we must define $\mathbf{Fix}(g)$ for g in group G. This is found in [2, Def 7.2.2], which states that it is the set of $g \in G$ such that $g \cdot x = x$. Let's explore an example of a group action to gain familiarity with this new concept.

Example 2.5. Let $X = \{a, b, c\}$ and $G = \mathbb{Z}_3 = \{0, 1, 2\}$. Consider the group action $\cdot : G \times X \to X$, where $(g, x) \in G \times X$ maps to $g \cdot x \in X$. In this case, the group G acts on the set X by determining where x ends up in the set X. Consider $(0, a) \in G \times X$. This element maps to $0 \cdot a = a \in X$. Similarly, we have $0 \cdot b = b$ and $0 \cdot c = c$. So, the element $0 \in G$ fixes an $x \in X$. For $1 \in G$, we get $1 \cdot a = b$,

$1 \cdot b = c$, and $1 \cdot c = a$. So, the element $1 \in G$ shifts an $x \in X$ by one position. Lastly, with $2 \in G$, observe that $2 \cdot a = c$, $2 \cdot b = a$, and $2 \cdot c = b$. So, the element $2 \in G$ shifts an $x \in X$ by two positions.

Now, let's consider an example of group actions on a bracelet.

Example 2.6. Let X be a set with one bracelet that has four beads (one pink bead, one green bead, and two yellow beads) and let $G = \{e, r, r^2, r^3, s, sr, sr^2, sr^3\}$, where r is a rotation by one bead clockwise and s is a reflection over the vertical axis. The group G acts on X by rotating and reflecting it.

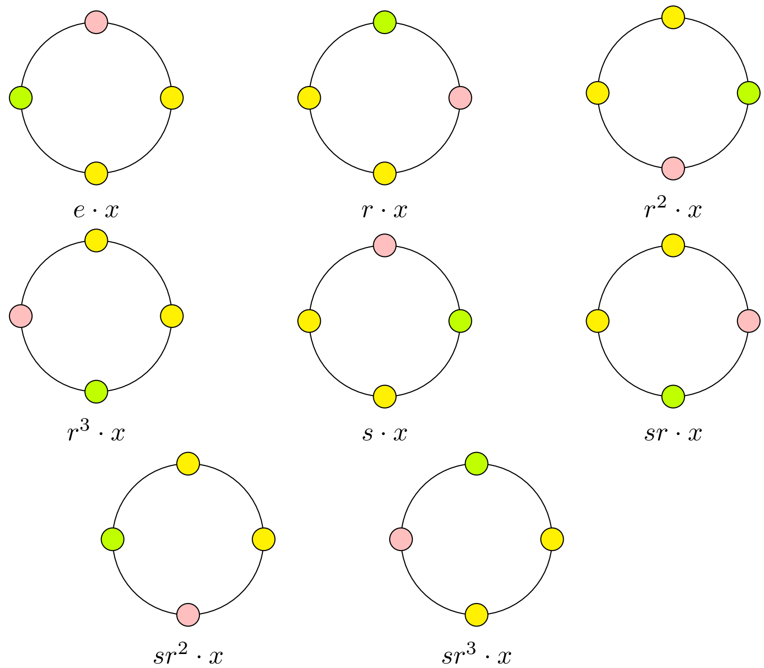

FIGURE 3. The elements in orb(x).

Notice that the bracelet in this example has has 8 different rotations/reflections that result in an exhaustive set of unique configurations of the bracelet. If you are not convinced that the clockwise rotations and reflections over the vertical axis result in a complete set of unique configurations, consider the counter-clockwise rotation of the original bracelet by two beads, followed by a reflection over the horizontal axis. Notice that we are left with the same bracelet as configuration as $s \cdot x$ (a reflection over the vertical axis). This is just one example of an action on the bracelet, and it can be shown that all unique bracelet configurations are contained in the orbit $\{g \cdot x | g \in G\}$ through a similar process.

We must generalize this example of $G = D_4$ by defining the group of symmetries of an n-bead bracelet.

Lemma 2.7. *The group of symmetries of an n-bead bracelet is $G = D_n = \{e, r, ..., r^{n-1}, s, rs, ..., r^{n-1}s\}$ where r is a rotation by one bead clockwise and s is a reflection over the vertical axis through the top bead.*

In order to apply Burnside's Lemma, we must also explore the idea of **conjugacy classes**.

Definition 2.8. [1, Def 2.6.16] Let G be a group and fix $a, b \in G$. We say b is **conjugate** to a if there exists a $g \in G$ such that $b = gag^{-1}$.

Now that we have defined what it means for one element to be conjugate to another, we can explore this idea further.

Definition 2.9. [1, Def 2.6.17] The equivalence classes for conjugacy are called **conjugacy class**.

Recall that equivalence classes partition a set into pairwise disjoint subsets [5, Theorem 9.1]. We will apply this idea of conjugacy classes on our group G in order to determine which symmetric operations belong to the same conjugacy class.

Proposition 2.10. *The elements in a given conjugacy class have the same order.*

Proof. Let G be a group. Fix $x, y, g \in G$ and let $y = gxg^{-1}$. We will prove that $o(x) = o(y)$. First, let the order of x be n. That is, $n \in \mathbb{Z}^+$ is the smallest positive integer such that $x^n = e$. Consider the equation $(gxg^{-1})^n = (gxg^{-1})(gxg^{-1})(gxg^{-1})...(gxg^{-1})$ (repeated n times). This simplifies to gx^ng^{-1}, because all the inner $g^{-1}g$ terms cancel, due to associativity. Since the order of x is n, this implies $geg^{-1} = gg^{-1} = e$. Thus the order of $y = gxg^{-1} \leq o(x)$. Now, start with the order of $y = gxg^{-1}$ to be n. This gives us $y^n = (gxg^{-1})(gxg^{-1})(gxg^{-1})...(gxg^{-1})$ (repeated n times) $= e$, which simplifies, as above, to $gx^ng^{-1} = e$, which is only possible if $x^n = e$. Thus the order of x is less than or equal to the order of y. So, we have $o(x) \leq o(y)$ and $o(y) \leq o(x)$, so $o(x) = o(y)$. □

It turns out that in D_5 and D_9, the converse of this proposition is also true. That is, if elements in the group of symmetries are of the same order, then they are in the same conjugacy class. We will not prove this particular fact, but one could check this result. Based on this result, we have the following lemma.

Lemma 2.11. *The conjugacy classes for D_5 are $\{e\}$, $\{r, r^2, r^3, r^4\}$, and $\{s, rs, r^2s, r^3s, r^4s\}$. The conjugacy classes for D_9 are $\{e, \}$, $\{r, r^2, r^4, r^5, r^7, r^8\}$, $\{s, rs, r^2s, r^3s, ..., r^8s\}$, and $\{r^3, r^6\}$.*

Proof. This lemma can be proven using an exhaustive approach. □

We need one final proposition in order to to apply Burnside's Lemma.

Proposition 2.12. *If g and h are in the same conjugacy class, then*

$$|Fix(g)| = |Fix(h)|.$$

Proof. Let g and h be elements of G in the same conjugacy class. We will prove that $|\text{Fix}(g)| = |\text{Fix}(h)|$ by proving that there exists a bijection between $\text{Fix}(g)$ and $\text{Fix}(h)$.

Define $\phi : \text{Fix}(g) \to \text{Fix}(h)$ such that for $x \in \text{Fix}(g)$, we have $\phi(x) = k \cdot x \in \text{Fix}(h)$. Note that because g and h are in the same conjugacy class, there exists $k \in G$ such that $h = kgk^{-1}$. We will first prove that $k \cdot x \in \text{Fix}(h)$. We start with $h \cdot (k \cdot x) = (kgk^{-1})(k \cdot x) = k \cdot g \cdot x$. Since $x, g \in \text{Fix}(g)$, we have $g \cdot x = x$, so $k \cdot g \cdot x = k \cdot x$. This implies that $k \cdot x$ is fixed by h.

Now, we must prove that ϕ is a bijection by proving it is one-to-one and onto. We will first prove that ϕ is one-to-one. Let $\phi(x) = \phi(a)$. We will prove $x = a$. By definition of ϕ, we have $\phi(x) = k \cdot x$ and $\phi(a) = k \cdot a$, which implies that $x = a$. Now, we will prove that ϕ is one-to-one. Let $y \in \mathrm{Fix}(h)$. We will prove there exists $b \in \mathrm{Fix}(g)$ such that $\phi(b) = x$. Let $b = k^{-1} \cdot y$. We have $\phi(b) = \phi(k^{-1} \cdot y) = k \cdot (k^{-1} \cdot y) = y$.

Thus there exists a bijection ϕ between $\mathrm{Fix}(g)$ and $\mathrm{Fix}(h)$, so $|\mathrm{Fix}(g)| = |\mathrm{Fix}(h)|$. □

As shown in Example 2.6, any one bracelet can be put in different orientations based on rotations and reflections, but it is still the same bracelet. When calculating the possible number of unique bracelets that contain certain colors, we must be careful and take rotations and reflections into account so as to not count the same bracelet twice, mistakenly thinking it is two distinct bracelets. The next section will provide concrete examples on how to apply Burnside's lemma, and we will work to answer the question posed in the introduction.

3. Applications

Let's apply our new knowledge to prove some claims about different bracelets. First, consider the example from [1, Example 5.2.4].

Claim 3.1. *There can be 1219 bracelets made with 9 beads of three different colors, if any number of beads of each color can be used.*

Proof. I found this stack exchange post particularly helpful when approaching this example: [6].
When we apply Burnside's Lemma, we get the equation

$$\frac{1}{|G|}\sum_{g \in G} |\mathrm{Fix}(g)| = \frac{1}{18}(3^9 + 2 \cdot 3^3 + 6 \cdot 3 + 9 \cdot 3^5) = 1219.$$

Let's break this down and explain how it relates to rotations and reflections. Since the bracelet has 9 beads, the group of unique rotations/reflections is

$$G = \{e, r, r^2, r^3, ..., r^8, s, rs, r^2s, r^3s, ...r^8s\}.$$

Notice that $|G| = 18$, since there are 18 symmetric operations in G.
Our main task is to determine which bracelets are identical under the symmetric operations of rotations and reflections.

First, why do we use powers of 3 in our calculation? When we apply the identity e on the bracelet (that is, we do not rotate or reflect it) we have 3^9 unique bracelets because all 9 beads can be one of 3 different colors. Now consider rotations of the bracelet. The bracelet only remains identical for all rotations $r, r^2, r^3, ..., r^8$ if all the beads are the same color, so we have 3 options for bracelets, since we are working with 3 different colors.

First consider the element e. We have proven above that e applied to the bracelet allows for 3^9 different bracelets. Since $e \in Z(G)$, the conjugacy class for e is just e

[1, Sec 2.6].

Next, consider the rotation r^3. For this operation, we are able to select the color of 3 beads, and because of the rotation, the colors of the other beads must depend on the color choices of the first 3. So for this case we have 3^3 options. Notice that r^3 and r^6 both have order 3, so we have that $\{r^3, r^6\}$ forms a conjugacy class, so we can add $2 \cdot 3^3$ to our equation.

Now, think about the element r. The bracelet is only identical for all rotations $r, r^2, ..., r^8$ if all beads are the same color. This leaves us with 3 options to create a bracelet of this form, because we have 3 colors to work with. The element r has order 9, and so do the elements r^2, r^4, r^5, r^7, r^8, so we have the conjugacy class $\{r, r^2, r^4, r^5, r^7, r^8\}$, which has 6 elements. Thus we can add $6 \cdot 3$ to our equation.

Finally, consider the element s (a reflection of the bracelet over the vertical axis through the bead at the top). The top bead is fixed by this reflection, and the other 8 beads get moved to a different position. In order for the bracelet to be the same before and after the reflection, the 8 beads that switch positions must be the same color as the bead they are swapped with. Taking into account the fixed bead at the top and 4 beads that get swapped on each side, we have a total of 3^5 different bracelets. The conjugacy class with s is $\{s, rs, r^2s, r^3s, ..., r^8s\}$, as all the elements in this conjugacy class have the same order. Therefore we must add $9 \cdot 3^5$ to our equation.

Overall, this gives us the $3^9 + 2 \cdot 3^3 + 6 \cdot 3 + 9 \cdot 3^5$ written above.

□

Now, consider the question posed in the introduction. We will still consider bracelets with three color options and no constraints on the number of beads of each color. However this time, we will consider bracelets that have 5 beads to demonstrate how our calculations differ from the 9-bead bracelet example above.

Claim 3.2. *There can be 39 5-bead bracelets made with purple, blue, and orange beads, if any number of beads of each color can be used.*

Proof. When we apply Burnside's Lemma, we get the equation

$$\frac{1}{|G|} \sum_{g \in G} |\mathrm{Fix}(g)| = \frac{1}{10}(3^5 + 4 \cdot 3 + 5 \cdot 3^3) = 39.$$

Let's break this down and explain how it relates to rotations and reflections.

First, note that since we are working with 5-bead bracelets, our group G of unique rotations and reflections is $G = \{e, r, r^2, r^3, r^4, s, rs, r^2s, r^3s, r^4s\}$, containing 10 elements.

The identity element e, as we explored in the proof of Claim 3.1, is in a conjugacy class that just contains itself: $\{e\}$. For this bracelet, when we apply e there are 3^5 unique bracelets created, as all 5 beads can be any of the 3 colors.

Next, consider $r \in G$, which has order 5. Its conjugacy class $\{r, r^2, r^3, r^4\}$ contains all elements of G that have order 5. The bracelet is only identical for all rotations $r, r_2, ..., r_4$ if all beads are the same color. This leaves us with 3 options to create a bracelet of this form, because we have 3 colors to work with. Since there are 4 elements in the conjugacy class we have $4 \cdot 3$.

The final conjugacy class is $\{s, rs, r^2s, r^3s, r^4s\}$, which contains elements of order 2. Consider the reflection s over the vertical axis through the top bead. The top bead remains unchanged by this reflection, so we can choose any of the three colors for this bead. We are also able to choose the color of two of the beads on one side of the bracelet, but the bead opposite each bead on the other side must have that same color, in order for the bracelet to remain the same after the reflection. Thus the color of the two other beads depend on the color of the first two. So we get to choose the color (out of three colors) for three beads in the bracelet and have a total of 3^3 unique bracelets for s. Since there are 5 elements in the conjugacy class of s, we have $5 \cdot 3^3$ overall.

This gives us the $3^5 + 4 \cdot 3 + 5 \cdot 3^3$ in our equation. Notice that our result of 39 is far fewer than our initial estimate of 243 in the introduction.

□

What happens if we change our constraints for the colors in the bracelet. What if, instead of allowing any bead to be any color, we require a certain number of beads for each color in a bracelet. Consider the claim for the example from [1, Example 5.2.3] .

Claim 3.3. *There can be 76 bracelets made with four pink beads, three purple beads, and two green beads.*

Proof. The bracelets we are working with has 9 beads, which act as the vertices of a nonagon. Let $G = \{e, r, r^2, r^3, ..., r^8, s, sr, sr^2, sr^3, ..., sr^8\}$. Apply the formula given in [1, Example 5.1.20] and let X be the set of $\frac{9!}{4!3!2!} = 1260$ bead arrangements.

Consider $e \in G$. Each of the 1260 bead arrangements are fixed by e.

Next, consider an element $x \in \langle r \rangle$. As stated in the previous two proofs, a rotation can only fix the a bracelet if that bracelet has all beads of the same color. Since we now require a bracelet with beads of multiple colors, this is not possible.

Next consider the reflection $s \in G$ through a vertex of the nonagon bracelet and through the center of the opposite edge. The vertex must be colored purple, as there are an odd number of purple-colored beads, and beads of the same color must swap potisions in order for a reflection to fix the bracelet. For the remaining orbits, one must be colored purple, two must be colored pink, and one must be colored green in order for the corrosponding beads to be the same color after the reflection. We have $\frac{4!}{2!1!1!} = 12$ possible ways to achieve this result, so s has 12 fixed points in X. Notice that there are 9 elements of G for which this is the case.

In total, there are

$$\frac{1}{|G|}\sum_{g\in G}|\mathrm{Fix}(g)| = \frac{1}{18}(1260 + 9\cdot 12) = 76$$

possible unique necklaces that can be created. $\square$

4. Conclusion

Burnside's Lemma has applications other than bracelets. For example, we can apply Burnside's Lemma to count the number of ways to color the edges of a cube. We could also seek to answer more general questions about bracelets such as the number of unique bracelets that can be created with four beads, having k colors to choose from. These applications are beyond the scope of this paper, but the ideas explored in this paper can be utilized to answer them.

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