6.1200J/18.062J *Mathematics for Computer Science* Friday 17th May, 2024
Massachusetts Institute of Technology, Spring 2024
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Final Exam

Instructions:

- Do not open this quiz booklet until directed. Read all the instructions on this page.
- Write your name below, then write your name at the top of **every** subsequent page.
- Write your solutions in the space provided. Stay inside the frame, to ensure nothing is lost during scanning.
- If you need more space, continue on the Extra Pages at the back of the test. Be sure to leave pointers in both locations (e.g., "Continued on Extra Page 1" and "Continuation of Problem 12(a)") so we can find all of your solutions.
- If the Extra Pages are still not enough, you may ask for additional paper.
- **You are allowed <u>three</u> single-sided, letter-sized sheets with your own notes**. No calculators or programmable devices are permitted. No cell phones or other communication devices are permitted.

Advice:

- You have 180 minutes to earn a maximum of 120 points. **Do not spend too much time on any single problem**: we recommend roughly n minutes on an n point problem. Read all the problems first, and attack them in the order that allows you to make the most progress. **This is especially true for multiple choice problems.**
- Please show your work **on long-answer problems (not multiple choice)**. Partial credit cannot be given for a wrong answer if your work isn't shown.
- Be neat and write legibly and clearly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- Do not waste time re-deriving results presented in class, psets, or recitations. You may simply state and cite them.

Your Name: ______________________________

Recitation Instructor or Recitation Number: ____________________

Multiple Choice

Each multiple choice problem below will specify whether you should **choose one answer** or **choose all that apply**. These will be automatically graded by gradescope, so when making your selections, please **clearly fill in** the appropriate box or boxes. We recommend using **pencil** so you can erase answers later if needed; you may borrow one from us. Do not spend time providing explanations or justifying your computations. We recommend spending only a few minutes for each problem; if stuck, fill in a guess and come back later if you can.

Problem 1. [4 points]

Is the following statement true or false?

> *If q is rational and x is irrational, then $q \cdot x$ is always irrational.*

○ True
○ False

Problem 2. [4 points]

Which of the following are equivalent to the proposition $P \to (Q \to R)$? (Choose all that apply.)

☐ $P \to (Q \land R)$
☐ $P \to (Q \lor R)$
☐ $(P \land Q) \to R$
☐ $(P \lor Q) \to R$

Problem 3. [4 points]

Is the following statement true or false?

> *If P is an invariant of a state machine, and s is a state for which $P(s)$ is true, then s must be reachable.*

○ True
○ False

Problem 4. [4 points]

What values of n are counterexamples for the following **false** statement? (Choose all that apply.)

$$\forall n \in \mathbb{N}.\ 2 \mid n \text{ IMPLIES } 4 \mid n$$

☐ $n = 2$
☐ $n = 3$
☐ $n = 4$
☐ $n = 61200$

Problem 5. [4 points]

What is the remainder when 6^{1200} is divided by 15? (Choose one answer.)

○ 0
○ 1
○ 6
○ 14

Problem 6. [4 points]

How many ways are there to properly color the complete graph K_n with k colors, where $k \geq n$? (Choose one answer.)

○ $k!$
○ $k \cdot n!$
○ $k!/(k-n)!$
○ k^n
○ $n!/(n-k)!$

Problem 7. [4 points]

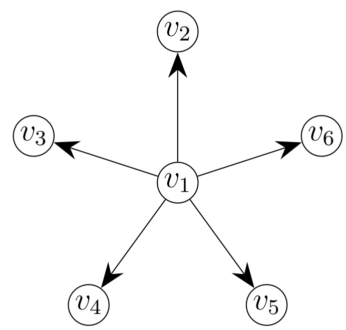
Let $G = (V, E)$ be an undirected graph with at least 3 vertices. Which of the following imply that G is a tree? (Choose all that apply.)

- ☐ G is connected and acyclic
- ☐ $|E| = |V| - 1$
- ☐ $\sum_{v \in V} \deg(v) = 2|E|$
- ☐ G is acyclic, but adding any edge would create a cycle
- ☐ G is connected, bipartite, and doesn't contain any even length cycles

Problem 8. [4 points]

Let G be the n-vertex DAG with n vertices $v_1, v_2, \ldots, v_n$ and $n-1$ edges (v_1, v_i) for each $2 \le i \le n$. An example for $n = 6$ is illustrated on the right. How many topological orderings does G have? (Choose one answer.)

- ○ $n!$
- ○ $(n-1)!$
- ○ n
- ○ 2^n
- ○ 2^{n-1}



Problem 9. [4 points]

Little Johnny flips two independent coins. The first coin comes up heads with probability 3/4 and the second coin comes up heads with probability 1/2. Let H_1 be the indicator random variable that is 1 if the first coin comes up heads, and 0 otherwise. Define H_2 similarly to be 1 if the second coin comes up heads, and 0 otherwise. What is $\text{Var}[H_1 + H_2]$? (Choose one answer.)

- ○ 9/64
- ○ 3/8
- ○ 7/16
- ○ 13/16
- ○ 5/4

Problem 10. [4 points]

Assume random variable X takes values in the range $[6, 120]$ and $\text{Ex}[X] = 10$. Which of the following is the strongest **upper bound** on $\Pr[X \geq 42]$ that can be derived from Markov's inequality (stated below for convenience)? (Choose one answer.)

- ○ 0
- ○ 1/10
- ○ 1/9
- ○ 1/8
- ○ 1/7
- ○ 1/6
- ○ 1/2
- ○ 1

Markov: If R is a nonnegative random variable and $a > 0$, then $\Pr[R \geq a] \leq \dfrac{\text{Ex}[R]}{a}$.

Long Answer Section

Please justify answers and show your work on the problems below, as in earlier quizzes.

Problem 11. Asymptotics [10 points]

Define $g(n) = n^n$ and $f(n) = g(n+1)$. Which of the following is the strongest (i.e. most specific) asymptotic relationship that holds between f and g, from among these options? **Carefully** justify your answer.

$$f \sim g, \qquad f \in \Theta(g), \qquad f \in O(g), \qquad f \in o(g), \qquad f \in \omega(g), \qquad f \in \Omega(g), \qquad \text{None.}$$

Problem 12. Combinatorial Identity [10 points]

Prove the following identity using **Double Counting**. In other words, choose a set S and prove separately that each side of the identity counts $|S|$.

$$\sum_{k=0}^{n} k \cdot \binom{n}{k} = n \cdot 2^{n-1}$$

Hint: Imagine that in our n-person class, we wish to choose a subset of students to form the Cookie Committee. One member of the committee must be chosen as the Cookie Captain. How many committees like this are possible?

Problem 13. Pirate Law [14 points]

Eleven pirates find a chest filled with gold coins. Pirate law requires them to divide up the gold coins in such a way that the difference in the number of coins given to any two pirates is not divisible by 10, and there are no coins left over.

(a) [6 pts] Prove that it is impossible for the 11 pirates to divide the coins in this way.

[Problem continues on the next page.]

(b) [8 pts] One of the pirates gives up and sails off into the sunset. The remaining ten pirates divide the coins according to pirate law. Prove that the total number of gold coins must be odd.

Hint: Let $g_1, g_2, \ldots, g_{10}$ be the number of gold coins given to the 10 pirates, and consider $\operatorname{rem}\left(\sum_{1 \le i \le 10} g_i,\ 10\right)$.

Problem 14. Counting Big Elements [20 points]

In this experiment, a subset S of m different integers is selected uniformly from the set $\{1, 2, \ldots, 2m\}$. Define the random variable B that counts the number of elements in S that are strictly greater than m. **Carefully** justify all of your answers.

(a) [6 pts] Let X_{2m} be the indicator random variable for the event $[2m \in S]$, i.e., X_{2m} is 1 when $2m \in S$, and 0 otherwise. What is $\Pr[X_{2m} = 1]$, in fully simplified form?

Hint: The answer is a number that does not depend on m.

(b) [6 pts] Are the events $[B = m]$ and $[2m \in S]$ independent?

[Problem continues on the next page.]

(c) [8 pts] What is $\text{Ex}[B]$? Find a closed-form formula in terms of m, and be sure to justify your answer.

Hint: Carefully describe how to express B as a sum of indicator random variables. Part (a) might be helpful.

Problem 15. Dicey Diancie [14 points]

Dicey Diancie rolls two fair tetrahedral dice (with four sides numbered 1 to 4). Let D be the ***absolute value*** of the difference between them. (E.g. if the two rolls are 1 and 3 in either order, then $D = 2$.)

(a) [4 pts] Fill in the blank: D is ______. (Choose all that apply, and do not justify.)

- ☐ a probability
- ☐ an outcome
- ☐ an event
- ☐ a random variable
- ☐ none of the above

(b) [5 pts] What is $\mathrm{PMF}_D(x)$? Specify it as precisely as you can.

[Problem continues on the next page.]

(c) [5 pts] What is $\text{Ex}[D]$?

Problem 16. Misty's Missing Psyduck [12 points]

Misty's Psyduck has wandered into one of two forests overnight. It is in Forest A with probability 0.4 and in Forest B with probability 0.6.

Assuming Psyduck is in Forest A and Misty spends the day searching for it in Forest A, the probability that she will find Psyduck is 0.25. Similarly, assuming Psyduck is in Forest B and Misty spends a day searching for it in Forest B, then she will find Psyduck with probability 0.15. (If she searches the wrong forest, she will not find Psyduck.) Being indecisive, Misty uses a fair coin to decide which forest to search.

(a) [6 pts] Draw a decision tree to describe this experiment. Be sure to clearly label what the outcomes *mean*, as well as the conditional probabilities that belong to each edge.

[Problem continues on the next page.]

(b) [6 pts] Assuming that Misty searches in Forest A and doesn't find Psyduck, what is the probability that Psyduck is in Forest A?

Use your decision tree to compute a numerical answer (which does *not* need to be simplified), and be sure to explain your work.

Hint: It may help to first describe the answer in terms of the following events:

$$P_A := \textbf{P}\text{syduck is in Forest } A.$$
$$S_A := \text{Misty spends the day } \textbf{S}\text{earching Forest } A.$$
$$F := \text{Misty } \textbf{F}\text{inds Psyduck!}$$

The End!

Extra Page 1: If you need extra space to write your solutions, you may continue here. Be sure to leave pointers in both locations (e.g., "Continued on Extra Page 1" and "Continuation of Problem 12") so we can find all of your solutions.

Extra Page 2: If you need extra space to write your solutions, you may continue here. Be sure to leave pointers in both locations (e.g., "Continued on Extra Page 2" and "Continuation of Problem 14") so we can find all of your solutions.

Extra Page 3: If you need extra space to write your solutions, you may continue here. Be sure to leave pointers in both locations (e.g., "Continued on Extra Page 3" and "Continuation of Problem 17") so we can find all of your solutions.