Predicate

Contrapositive (equivalent)

$(x \to y) \longleftrightarrow (\neg y \to \neg x)$

converse (not equivalent)

$(x \to y) \quad (y \to x)$

SETS $A = \{6, 1, 2, 3\}$ $\{1, 2\} \subseteq A$ IS ✓

$\{\}$ ϕ $S \subseteq T$ $\{1, 2\} \in A$ IS ✗ set

Set BUILDER

$\{n \in \mathbb{N} \mid \text{isPrime}(n)\} = \{2, 3, 5, 7, 11 \ldots\}$

$A \cap B$ AND $A \cup B$ OR $A - B = A \setminus B$ $x \in A$ AND $x \notin B$

NOTATION

"Student x is immediately to the right of student y."

No other student can be between x and y.

isnext$(x, y) := F(y, x)$ AND NOT$(\exists z.\, F(y, z)$ AND $F(z, x))$

(c) "There are at least two students after x."

$\exists y \exists z.$ NOT$(y = z)$ AND $(F(x, y)$ AND $F(x, z))$.

(d) "Student x is second in line."

$\exists y.$first(y) AND isnext(x, y)

"Student y is somewhere between x and z."

$(F(x, y)$ AND $F(y, z))$ OR $(F(z, y)$ AND $F(y, x))$.

Definition 3. *A* preserved predicate *is a predicate $P(\cdot)$ defined on states, such that if $P(s)$ is true for a state s, and there exists a transition $s \mapsto t$ from s to t, then $P(t)$ is true as well.*

Definition 4. *An* invariant *is a property that is true for all reachable states.*

Theorem 2 (The Invariant Principle). *Suppose $P(\cdot)$ is true for the initial state and is a preserved predicate. Then $P(\cdot)$ is an invariant.*

Definition 1. *A state machine is defined by a collection of* states, *a specified* initial state, *and for each state a (possibly empty) list of possible* transitions *to other states. If a state has multiple possible transitions, we say the state machine is* non-deterministic; *otherwise, it is* deterministic.

Corollary 3. *If $P(s)$ is false and P is an invariant, then s in unreachable.*

egral Bound (Increasing): If $f : [1, n] \to \mathbb{R}$ is weakly **increasing**, then

$$f(1) + \int_1^n f(x)\,\mathrm{d}x \le \sum_{k=1}^n f(k) \le f(n) + \int_1^n f(x)\,\mathrm{d}x.$$

egral Bound (Decreasing): If $f : [1, n] \to \mathbb{R}$ is weakly **decreasing**, then

$$f(n) + \int_1^n f(x)\,\mathrm{d}x \le \sum_{k=1}^n f(k) \le f(1) + \int_1^n f(x)\,\mathrm{d}x.$$

egral Bound (Improper): If $f : [1, \infty) \to \mathbb{R}$ is weakly decreasing, then the sum $\sum_{k=1}^\infty f(k$ verges iff the improper integral $\int_1^\infty f(x)\,\mathrm{d}x$ converges. If they converge, then

$$\int_1^\infty f(x)\,\mathrm{d}x \le \sum_{k=1}^\infty f(k) \le f(1) + \int_1^\infty f(x)\,\mathrm{d}x.$$

5 Asymptotics

5.1 Definitions

- **Big O**: $f(n) \in O(g(n))$ if there exists some $n_0 \in \mathbb{N}$ and $c > 0$ such that for all $n \ge n_0$, we have $|f(n)| \le c \cdot g(n)$. Intuition: all functions that are "at most" $g(n)$.
- **Big Omega**: $f(n) \in \Omega(g(n))$ if $g(n) \in O(f(n))$. (We can apply the Big O definition, but with f and g swapped.) Intuition: all functions that are "at least" $g(n)$.

- **Big Theta**: $f(n) \in \Theta(g(n))$ if $f \in O(g)$ and $g \in O(f)$. Intuition: f and g are about the same, up to constant factors.
- **Little O**: $f(n) \in o(g(n))$ precisely when $\lim_{n\to\infty} f(n)/g(n) = 0$. Intuition: f grows much slower than g.
- **Little Omega**: $f(n) \in \omega(g(n))$ precisely when $g(n) \in o(f(n))$, i.e., $\lim_{n\to\infty} f(n)/g(n) = \infty$. Intuition: f grows much faster than g.
- **Asymptotic Equivalence**: $f(n) \sim g(n)$ (f and g are asymptotically equivalent) precisely when $\lim_{n\to\infty} f(n)/g(n) = 1$.

$$n! \sim \sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n$$

Theorem 1 (Master Theorem). *Let $a \ge 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the nonnegative integers by the recurrence*

$$T(n) = aT(\lfloor \tfrac{n}{b} \rfloor) + f(n).$$

Then $T(n)$ has the following asymptotic bounds:

1. *If $f(n) \in O(n^{(\log_b a) - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) \in \Theta(n^{\log_b a})$.*
2. *If $f(n) \in \Theta(n^{\log_b a})$, then $T(n) \in \Theta(n^{\log_b a} \log n)$.*
3. *If $f(n) \in \Omega(n^{(\log_b a) + \epsilon})$ for some constant $\epsilon > 0$, and if $af(\lfloor n/b \rfloor) \le cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) \in \Theta(f(n))$.*

This theorem is also true if we replace $\lfloor n/b \rfloor$ with $\lceil n/b \rceil$ everywhere.

We can model this as a state machine whose states are pairs (n_a, n_b) where $n_a \ge 0$ equals the number of ten cent tokens, and $n_b \ge 0$ equals the number of seven cent tokens. The allowed transitions are

$$(n_a, n_b) \to (n_a - 1, n_b + 1) \quad \text{if } n_a \ge 1, \text{ and}$$
$$(n_a, n_b) \to (n_a + 3, n_b - 6) \quad \text{if } n_b \ge 6.$$

$$1 + r + r^2 + r^3 + \ldots + r^n + r^{n+1} = \frac{1 - r^{n+1}}{1 - r} + r^{n+1}$$
$$= \frac{1 - r^{n+1} + (1 - r) \cdot r^{n+1}}{1 - r}$$
$$= \frac{1 - r^{n+2}}{1 - r}$$

$$S = 1 + r + r^2 + r^3 + \ldots + r^n$$
$$rS = r + r^2 + r^3 + \ldots + r^{n+1}$$
$$S - rS = 1 - r^{n+1}$$
$$S = \frac{1 - r^{n+1}}{1 - r}$$

Problem 3. Contradiction

Prove by contradiction that if $a \cdot b = n$, then either a or b must be $\le \sqrt{n}$, where a, b, and n are nonnegative real numbers.

Solution. Suppose to the contrary that $a > \sqrt{n}$ and $b > \sqrt{n}$. Then

$$a \cdot b > \sqrt{n} \cdot \sqrt{n} = n,$$

contradicting the fact that $a \cdot b = n$.

Here we have used the rule that for nonnegative real numbers u, v, w, x,

$$[u > v \text{ AND } w > x] \text{ IMPLIES } u \cdot w > v \cdot x.$$

This rule applies since a, b, and $\sqrt{n}$ are nonnegative.

Note that we are using the fact that if n is nonnegative, then $\sqrt{n}$ denotes the nonnegative square root. If we had used the negative square root of n, the above rule would not hold, and the proof would not be correct. Indeed, if we used the negative square root of 2 with $a = b = 1$, the claim to be proved would be false.

(b) Suppose we start with 100 of each token. Prove carefully that the state (50,8) is unreachable. If you would like to use a fact from the previous part, you must prove it here.

Solution. Define the predicate $P(n_a, n_b) := \text{rem}(n_a + n_b, 3) = 2$; we'll prove that P is invariant.

At the start state (100, 100), the property $P(100, 100)$ holds because $100 + 100 = 200 \equiv 2 \mod 3$. Now assume we have any state (n_a, n_b) where $P(n_a, n_b)$ holds; we must show that P still holds after following any transition from (n_a, n_b). The first kind of transition takes us to $(n_a - 1, n_b + 1)$, and $(n_a - 1) + (n_b + 1) = n_a + n_b$, which has remainder 2 when divided by 3 (by assumption), so the property holds. For the second kind of transition, $(n_a + 3) + (n_b - 6) = n_a + n_b - 3$, which has the same remainder as $n_a + n_b$, namely, 2. Since $P(n_a, n_b)$ is true at the start state and is preserved across transitions, the Invariant Principle shows that it is true at all reachable states.

However (50,8) has $50 + 8$, which has remainder 1, thus it is unreachable. ■

$$S = \sum_{i=1}^{\infty} \frac{1}{(2i-1)^2}$$

$u \le S \le V$ and $|V - u| \le \frac{1}{100}$

$$u = f(1) + f(2) + f(3) + f(4) + f(5) + \int_6^\infty \frac{1}{(2i-1)^2}$$

$$V = f(1) + f(2) + f(3) + f(4) + f(5) + f(6) + \int_6^\infty \frac{1}{(2i-1)^2}$$

$$\begin{aligned}
T(n) &= 3T(\tfrac{n}{3}) + 2n - 3 \\
&= 3(3T(\tfrac{n}{9}) + \tfrac{2n}{3} - 3) + 2n - 3 \\
&= 9T(\tfrac{n}{9}) + (2n - 9) + (2n - 3) \\
&= \cdots \\
&= 3^k T(\tfrac{n}{3^k}) + 2kn - (3^k + 3^{k-1} + \cdots + 3) \\
&= 3^k T(\tfrac{n}{3^k}) + 2kn - \frac{3(3^k - 1)}{3 - 1} \\
&= 3^k T(\tfrac{n}{3^k}) + 2kn - \frac{3}{2}3^k + \frac{3}{2}
\end{aligned}$$

Now let $k = \log_3 n$. The formula then becomes

$$\begin{aligned}
T(n) &= 3^k T(\tfrac{n}{3^k}) + 2kn - \frac{3}{2}3^k + \frac{3}{2} \\
&= 3^{\log_3 n} T(\tfrac{n}{3^{\log_3 n}}) + 2n(\log_3 n) - \frac{3}{2}3^{\log_3 n} + \frac{3}{2} \\
&= nT(1) + 2n \log_3 n - \frac{3}{2}n + \frac{3}{2} \\
&= 2n \log_3 n - \frac{3}{2}n + \frac{3}{2}
\end{aligned}$$

5.4 Asymptotics Tips

(a) Something is Θ if and only if it is O and Ω.

(b) If something is o, then it is also O. The other way around is not for certain.

(c) If something is ω, then it is also Ω. The other way around is not for certain.

(d) Refer to the following order of largeness (biggest is highest up on the list, smallest is lowest on the list):

- $O(n!)$ (factorial)
- $O(2^n)$ or any constant raised to n (exponential)
- $O(n^c)$ or n raised to any constant (polynomial)
- $O(n \log n)$ (log linear)
- $O(n)$ (linear)
- $O(\log n)$ (log)

Problem 6. Recurrences

Use the Plug and Chug method to find an exact closed form formula for $a(n)$ as a function of n, where $a(n) = 2a(n-1) + 3^n$ for $n \ge 1$, and $a(0) = 5$.

Solution. We perform a few substitutions to look for a pattern:

$$\begin{aligned}
a(n) &= 3^n + 2a(n-1) \\
&= 3^n + 2(3^{n-1} + 2a(n-2)) \\
&= 3^n + 2 \cdot 3^{n-1} + 4a(n-2) \\
&= 3^n + 2 \cdot 3^{n-1} + 2^2 \cdot 3^{n-2} + 8 \cdot a(n-3) \\
&\vdots
\end{aligned}$$

After k expansions, we can guess that the formula takes the form

$$\begin{aligned}
a(n) &= 3^n + 2 \cdot 3^{n-1} + \cdots + 2^k 3^{n-k} + 2^{k+1} a(n-k-1) \\
&= \left(\sum_{i=0}^{k} 3^n \cdot (2/3)^i\right) + 2^{k+1} a(n-k-1).
\end{aligned}$$

Choosing $k = n - 1$, we find that

$$\begin{aligned}
&= \left(\sum_{i=0}^{n-1} 3^n \cdot (2/3)^i\right) + 2^n \cdot a(0) \\
&= 3^n \cdot \frac{1 - (2/3)^n}{1 - (2/3)} + 5 \cdot 2^n,
\end{aligned}$$

assume P true for all k <= n for some n > 1

Inductive Hypothesis: $F(n)$ is even IFF $F(n+3)$ is even

Base case: $F(0) = 0$ $F(3) = 2$ both even ✓

$F(1) = 1$ $F(4) = 3$ both not even ✓

Induction step:

- Assume $P(n)$ true for $0 \le n \le k$ WTS true for $n = k+1$

$n = k - 1$, k are true WTS true for $n = k+1$

- $F(n+1)$ is even IFF $F(n) + F(n-1)$ is even
- $F(n) + F(n-1)$ is even IFF $F(n)$ and $F(n-1)$ are both even or odd. [sums addition]

$F(n)$ even $\leftrightarrow$ $F(n+3)$ even $\quad$ $F(n-1)$ even $\longleftrightarrow$ $F(n+2)$ even

$\therefore F(n+4) = F(n+3) + F(n+2)$

even $\longleftrightarrow$ even

Even + Even = Even
Odd + odd = Even
Even + odd = Odd

Even and odd sums

$F(n+1)$ even IFF $F(n)$ & $F(n-1)$ are both even or both odd

Same even/odd status by assumption.

$F(n+4)$ even IFF $F(n+3)$ & $F(n+2)$ are both even or odd.

$F(n+1)$ is even IFF $F(n+4)$ is even.

Total value of annuity

- $V = \sum_{i=1}^{n} \frac{m}{(1+p)^{i-1}} = m \sum_{j=0}^{n-1} \left(\frac{1}{1+p}\right)^j \quad j = i - 1 \qquad V = m \cdot \frac{1 - x^n}{1 - x}$

$\sum^{\infty} = \frac{1}{1 - x} \qquad x = \frac{1}{(1+p)}$

- $\sum_{k=1}^{n} k^2 = 1^2 + 2^2 + 3^2 + \ldots + n^2 = \frac{n(n+1)(2n+1)}{6}$
- $\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots n = \frac{n(n+1)}{2}$
- $\sum_{i=0}^{n-1} ar^i = a + ar^2 + ar^3 + \ldots ar^{n-1} = a\frac{r^n - 1}{r - 1}$ $\Rightarrow$ a can be $= 1$
- $\sum_{i=1}^{n} r^i = r\left(\frac{r^n - 1}{r - 1}\right)$
- $\sum_{k=1}^{n} kx^k = \frac{(x + x^{n+1}(nx - n - 1))}{(1 - x)^2}$

(b) [5 pts] Use the perturbation method to find a closed-form expression for $S = \sum_{k=1}^{n} k^2 3^k$. *Hint:* Perturb once, then use the formula listed above.

Solution. We pertrube once to get

$$3S = 3\sum_{k=1}^{n} k^2 3^k = \sum_{k=1}^{n} k^2 3^{k+1}.$$

We now subtract $3S$ from S:

$$\begin{aligned}
S - 3S = {} & 1^2 \cdot 3^1 + 2^2 \cdot 3^2 + 3^2 \cdot 3^3 + \cdots + n^2 \cdot 3^n \\
& - 1^2 \cdot 3^2 - 2^2 \cdot 3^3 - \cdots - (n-1)^2 \cdot 3^n - n^2 \cdot 3^{n+1},
\end{aligned}$$

ich becomes

$$\begin{aligned}
S - 3S &= 1^2 \cdot 3^1 - n \cdot 3^{n+1} + \sum_{k=2}^{n} (k^2 - (k-1)^2) 3^k \\
&= 3 - n^2 \cdot 3^{n+1} + \sum_{k=2}^{n} (2k - 1) 3^k \\
&= 3 - n^2 \cdot 3^{n+1} + 2\sum_{k=2}^{n} k3^k - \sum_{k=2}^{n} 3^k \\
&= 3 - n^2 \cdot 3^{n+1} + 2\left(\frac{3(n \cdot 3^{n+1} - (n+1)3^n + 1)}{(3-1)^2} - 3\right) - \left(3\frac{3^n - 1}{3 - 1} - 3\right) \\
&= 3 - n^2 \cdot 3^{n+1} + \frac{3}{2}(2n \cdot 3^n - 3^n - 3) - \frac{3}{2}(3^n - 3) \\
&= 3 - n^2 \cdot 3^{n+1} + n \cdot 3^n - 3 \cdot 3^n
\end{aligned}$$

(d) Explain why the Master Theorem **cannot** be used to analyze $T(n) = 2T(\lfloor n/2 \rfloor) + n \log n$. (You don't need to find a Θ-bound yourself, but it could be a fun optional challenge!)

Solution. Since $a = b = 2$, the critical exponent is $\log_b a = 1$. The ratios $(n \log n)/n^1$ and $n^{1+\epsilon}/(n \log n)$ both limit to ∞ as $n \to \infty$, no matter how small $\epsilon > 0$ is chosen, so $f(n)$ grows too fast to belong to $\Theta(n^1)$ (so cases 1 and 2 don't apply), and at the same time, $f(n)$ grows and too slow for $\Omega(n^{1+\epsilon})$ (so case 3 doesn't apply).

This example shows that the **Master Theorem is not exhaustive**: there are cases like this one that can "fall through the cracks" by being simultaneously too big for case 2 but too small for case 3. It's possible to fit between cases 1 and 2 similarly; try to come up with such an example.

So how would we analyze this algorithm? Some options include a more detailed recursion tree analysis, or a careful induction argument. Try it yourself! ■

Example: Mergesort: $T(n) = 2T(\lceil n/2 \rceil) + n - 1$ has $a = b = 2$ and $f(n) = n - 1$. This is case 2, because $n^{\log_2 2} = n^1$. So $T(n) \in \Theta(n^1 \log n)$, as we hoped.

Example: Binary Search (saw in homework): $T(n) = T(\lceil n/2 \rceil) + 1$. This has $a = 1$, $b = 2$, $f(n) = 1$, $n^{\log_b a} = n^0 = 1$, so this is *also* case 2: $T(n) \in \Theta(n^0 \log n) = \Theta(\log n)$.

Example: Karatsuba, $T(n) = 3T(\lceil n/2 \rceil) + \Theta(n)$. This has $a = 3$, $b = 2$, $n^{\log_b a} = n^{\log_2 3}$. This is case 1 because $f(n) \in O(n^{\log_2 3 - \epsilon})$ because $\log_2 3 > 1$. So $T(n) \in \Theta(n^{\log_2 3})$.

Example: Hanoi: $T(n) = 2T(n-1) + 1$. Can't apply Master Theorem, because doesn't have correct form. Doesn't follow the "divide and conquer" paradigm. And to see a difference, consider the recursion tree: 1 node at n, 2 and $n - 1$, 4 at $n - 2$, etc. Since we subtract 1 instead of dividing by b, there are *linearly* many levels and *exponentially* many nodes in the tree, which is why the solution is exponential. By contrast, divide-and-conquer-style recurrences have log-depth and polynomial-size trees, so those solutions are typically much smaller.

Example: Master Theorem does have gaps! For example, $T(n) = 2T(n/2) + n \log n$ *cannot* be solved by master theorem. It is not case 2 because $f(n) = n \log n$ is strictly larger than $n^{\log_b a} = n$ asymptotically. But it is also not case 3, because it is only larger by a log factor, not by a polynomial factor n^ϵ. So we would need to resort to more careful recursion tree analysis, or some form of induction instead.