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# Final Exam

**Instructions:**

- Do not open this quiz booklet until directed. Read all the instructions on this page.
- Write your name below, then write your name at the top of **every** subsequent page.
- Write your solutions in the space provided. Stay inside the frame, to ensure nothing is lost during scanning.
- If you need more space, continue on the Extra Pages at the back of the test. Be sure to leave pointers in both locations (e.g., "Continued on Extra Page 1" and "Continuation of Problem 12(a)") so we can find all of your solutions.
- If the Extra Pages are still not enough, you may ask for additional paper.
- **You are allowed three single-sided, letter-sized sheets with your own notes**. No calculators or programmable devices are permitted. No cell phones or other communication devices are permitted.

**Advice:**

- You have 180 minutes to earn a maximum of 120 points. **Do not spend too much time on any single problem**: we recommend roughly $n$ minutes on an $n$ point problem. Read all the problems first, and attack them in the order that allows you to make the most progress. **This is especially true for multiple choice problems.**
- Please show your work **on long-answer problems (not multiple choice)**. Partial credit cannot be given for a wrong answer if your work isn't shown.
- Be neat and write legibly and clearly. You will be graded not only on the correctness of your answers, but also on the clarity with which you express them.
- Do not waste time re-deriving results presented in class, pests, or recitations. You may simply state and cite them.

**Your Name**: ______________________________

**MIT Email or MIT ID Number**: ______________________________

# Multiple Choice Section

Each multiple choice problem below will specify whether you should **choose one answer** or **choose all that apply**. These will be automatically graded by gradescope, so when making your selections, please **clearly fill in** the appropriate box or boxes. We recommend using **pencil** so you can erase answers later if needed; you may borrow one from us. Do not spend time providing explanations or justifying your computations. We recommend spending only a few minutes for each problem; if stuck, fill in a guess and come back later if you can.

## Problem 1. [4 points]

Let $P$ and $Q$ be arbitrary propositions. Which of the following are equivalent to the **negation** of $P \to Q$? Select all that apply.

- ☐ $Q \to (\neg P)$
- ☐ $P \to (\neg Q)$
- ☐ $Q \wedge (\neg P)$
- ■ $P \wedge (\neg Q)$
- ☐ None

**Solution.** $P \wedge (\neg Q)$ ■

## Problem 2. [4 points]

Which of the following are true for the function $f(n) := n^3 \log n$? Select all that apply.

- ☐ $f(n) \in o(n^3)$
- ☐ $f(n) \in O(n^3)$
- ☐ $f(n) \in \Theta(n^3)$
- ■ $f(n) \in \Omega(n^3)$
- ■ $f(n) \in \omega(n^3)$
- ☐ $f(n) \sim n^3$

## Problem 3. [4 points]

Which of the following are correct applications of Fermat's Little Theorem? Select all that apply.

- ☐ $\mathrm{rem}(a^{11}, 12) = 1$, where $a = 42^{56}$
- ■ $\mathrm{rem}(a^{12}, 13) = 1$, where $a = 42^{56}$
- ☐ $\mathrm{rem}(a^{12}, 13) = 1$, where $a = 26^{56}$

## Problem 4. [4 points]

True or false? If $k \geq 3$ and $G$ is an undirected graph such that $\chi(G) = k$, then $G$ must contain a $k$-node complete graph as a subgraph.

- ○ True
- ● False

**Solution.** False (e.g. $k = 3$, and an odd-length cycle larger than a triangle). ■

## Problem 5. [4 points]

Suppose we have a parallel schedule on a DAG, where node $a$ is completed at the 4th step, while node $b$ is completed earlier, at the 2nd step. Suppose we modify our graph by adding a directed edge from $a$ to $b$. True or false?: the modified graph always has a directed cycle.

- ○ True
- ● False

## Problem 6. [4 points]

Consider the "greater than" relation ($>$) defined on the real numbers. Which of the following properties does this relation have? Select all that apply.

- ☐ Reflexive
- ☐ Symmetric
- ■ Antisymmetric
- ■ Transitive
- ☐ Equivalence Relation
- ☐ Weak Partial Order

## Problem 7. [4 points]

75 students attended an event with unlimited free ice cream, where there were three different flavors being served: chocolate, vanilla, and strawberry. 60 students at the event ate vanilla ice cream, 45 ate chocolate ice cream, and 25 ate strawberry ice cream. 20 students ate all three flavors of ice cream. Every student ate at least one flavor of ice cream. How many students ate exactly two distinct flavors?

- ● 15
- ○ 20
- ○ 25

○ 30
○ Not enough information

**Problem 8.** [4 points]

In a standard deck of cards, a ***triple*** is a set of three cards that all have the same rank. What is the smallest number of cards you would need to blindly select from the deck (without replacement) to be sure that you have selected at least two disjoint ***triples***?

○ 14
○ 15
○ 16
○ 27
○ 28
● 29
○ 40
○ 41
○ 42

**Problem 9.** [4 points]

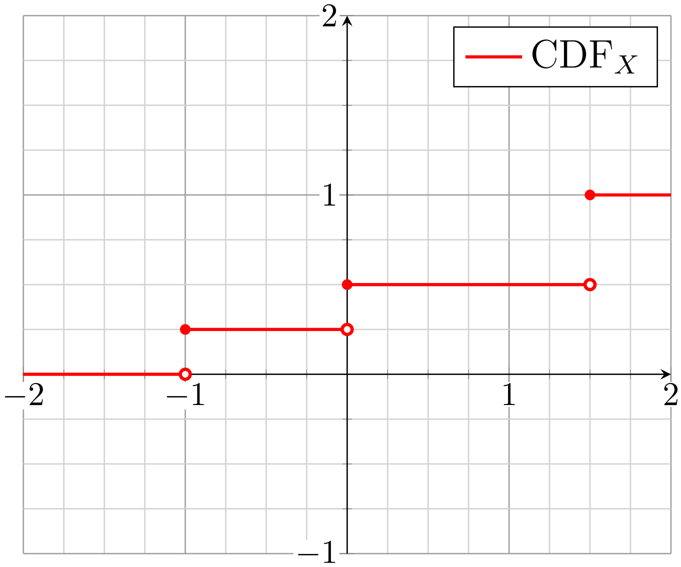
True or false? Suppose $X$ and $Y$ are **indicator** random variables such that $\text{Ex}[XY] = \text{Ex}[X]\,\text{Ex}[Y]$. Then $X$ and $Y$ must be independent.

● True
○ False

**Solution.** True ■

**Problem 10.** [4 points]

Suppose $X$ is a random variable whose CDF is given by the following plot:




Note: graph is to scale. All relevant coordinates are multiples of $\frac{1}{4}$.

What is Var($X$)?

- ○ 0
- ○ 1/2
- ○ 7/8
- ○ 1
- ● 9/8
- ○ 11/8
- ○ 3/2

**Solution.** By inspection of the graph, the PDF is given by

$$\mathrm{PDF}_X(x) = \begin{cases} \frac{1}{4} & x = -1 \\ \frac{1}{4} & x = 0 \\ \frac{1}{2} & x = \frac{3}{2} \\ 0 & \text{otherwise} \end{cases}$$

Thus the expectation is

$$\begin{aligned}
\mathrm{Ex}[X] &= \sum_x x \cdot \mathrm{PDF}_X(x) \\
&= -1 \cdot \frac{1}{4} + 0 \cdot \frac{1}{4} + \frac{3}{2} \cdot \frac{1}{2} \\
&= \frac{1}{2}
\end{aligned}$$

Next,

$$\begin{aligned}
\mathrm{Ex}[X^2] &= \sum_x x^2 \cdot \mathrm{PDF}_X(x) \\
&= (-1)^2 \cdot \frac{1}{4} + 0^2 \cdot \frac{1}{4} + \left(\frac{3}{2}\right)^2 \cdot \frac{1}{2} \\
&= \frac{11}{8} \\
\mathrm{Var}(X) &= \mathrm{Ex}[X^2] - \mathrm{Ex}[X]^2 \\
&= \frac{11}{8} - \left(\frac{1}{2}\right)^2 \\
&= \boxed{\frac{9}{8}}
\end{aligned}$$

$\blacksquare$

# Long Answer Section

Please justify answers and show your work on the problems below, as in earlier quizzes.

## Problem 11. Computing GCDs [20 points]

**(a)** [6 pts] Define $\gcd(a, b, c)$ as the largest integer that divides all of $a$, $b$, and $c$. Prove that $\gcd(a-b, b, c) = \gcd(a, b, c)$ for all triples of positive integers $(a, b, c)$.

**Solution.** Define $g = \gcd(a, b, c)$ and $h = \gcd(a-b, b, c)$. Since $g$ is a factor of $a$, $b$, and $c$, it is also a factor of $a-b$ and $b$ and $c$, and thus $g$ is less than or equal to the *largest* number with this property, namely $h$. Similarly, $h$ divides $a-b$, $b$, and $c$, and thus also divides $(a-b)+b = a$, so by similar reasoning as before, $h \le g$. This guarantees that $g = h$, as needed. ■

**(b)** [2 pts] Define a state machine whose states are triples of **positive** integers $(a, b, c)$. A state $(a, b, c)$ may transition to a new state by subtracting a smaller coordinate from a larger coordinate. Specifically, state $(a, b, c)$ may transition to any of these six new states, as long as each coordinate remains strictly positive after the transition:

$$\begin{array}{lll} (a, b-a, c) \text{ if } a < b & (a-b, b, c) \text{ if } b < a & (a-c, b, c) \text{ if } c < a \\ (a, b, c-a) \text{ if } a < c & (a, b, c-b) \text{ if } b < c & (a, b-c, c) \text{ if } c < b. \end{array}$$

Call the start state $(x_0, y_0, z_0)$. In this problem, we'll see how this state machine is able to compute $\gcd(x_0, y_0, z_0)$, using only subtractions.

To begin, **briefly explain** (in just a sentence or two) why the predicate $P(a, b, c) :=$ "$\gcd(a, b, c) = \gcd(x_0, y_0, z_0)$" is preserved by this state machine. You may cite part (a) even if you haven't solved it.

**Solution.** If $(a, b, c)$ transitions to $(a', b', c')$, part (a) (similarly applied to the other 5 cases) guarantees that $\gcd(a, b, c) = \gcd(a', b', c')$. So if we assume $P(a, b, c)$ (i.e., $\gcd(a, b, c) = \gcd(x_0, y_0, z_0)$), it follows that $\gcd(a', b', c') = \gcd(a, b, c) = \gcd(x_0, y_0, z_0)$, which proves $P(a', b', c')$. ■

**(c)** [4 pts] Demonstrate a strictly decreasing derived variable whose value is always a positive integer, and **explain** why it is strictly decreasing. (Note: This proves that the algorithm terminates, because the derived variable can't decrease forever.)

**Solution.** $a + b + c$. After a transition, the value $a + b + c$ changes to one of $a + b$, $b + c$, or $a + c$, which is strictly smaller because all three coordinates are positive. ■

**(d)** [3 pts] How can you tell whether a state $(a, b, c)$ is a final state (i.e., has no outgoing transitions)? Briefly explain why.

**Solution.** $a = b = c$. Assuming $(a, b, c)$ has no outgoing transitions, we must have $a = b$, because otherwise we could use one of the first two types of transitions. We similarly require $b = c$, so in fact the state is $(a, a, a)$. And indeed, every such state is final, because no subtractions are possible without introducing zeroes. So the final states are precisely those with three equal coordinates. ■

**(e)** [5 pts] Once the state machine terminates, how can we obtain our desired answer of $\gcd(x_0, y_0, z_0)$ from the final state? Use the Invariant Principle, and the fact that $P(a, b, c)$ is preserved, to **carefully prove** why the result really does have value $\gcd(x_0, y_0, z_0)$.

**Solution.** Suppose we terminate at final state $(w, w, w)$ (all three coordinates are equal by the previous part). Since $P$ is true at the start state and is preserved, it is true at all reachable states, so it is true at $(w, w, w)$. In other words, $\gcd(w, w, w) = \gcd(x_0, y_0, z_0)$. But $\gcd(w, w, w) = w$ since this is certainly a common divisor and nothing larger can divide $w$, so this value $w$ is our desired answer. ■

## Problem 12. One Hundred Sides [12 points]

Suppose we roll a fair 100-sided die 60 times, producing a list of numbers $\ell$. (The faces of the die are labelled $1, 2, \ldots, 100$, and each is equally likely on each roll, and the rolls are mutually independent.) Define the following three events:

$$\begin{aligned} W &= [\text{the elements of } \ell \text{ are in weakly increasing order}] \\ E &= [\text{every element of } \ell \text{ is even}] \\ D &= [\text{the elements of } \ell \text{ are all distinct}] \end{aligned}$$

Compute the following quantities and justify your answers.

**(a)** [4 pts] Compute $\Pr[W \mid E]$.

**Solution.** Let $S = \{x \in \mathbb{Z} : 1 \le x \le 100 \text{ and } x \text{ is even}\}$. The number of ways to choose 60 numbers from $S$ in weakly increasing order is the same as the number of ways to assign 60 donuts into 50 flavors, which is $\binom{60+50-1}{50-1} = \binom{109}{49}$. Since the conditional distribution of $\ell$ is uniform (that is, all $50^{60}$ sequences of 60 numbers in $S$ have the same conditional probability of occurring), the answer is $\boxed{\dfrac{\binom{109}{49}}{50^{60}}}$. ■

**(b)** [4 pts] Compute $\Pr[D \mid E]$.

**Solution.** There are only 50 elements in $S$, so by the Pigeonhole Principle it is impossible to pick 60 distinct elements from them. Thus the conditional probability is 0. ■

**(c)** [4 pts] Compute $\Pr[W \mid D]$.

**Solution.** Considering only the relative order of elements in $\ell$, all $60!$ possible orderings have equal conditional probability. However, only one order is in increasing order. So the answer is $\boxed{\dfrac{1}{60!}}$. ■

## Problem 13. Independent Coins [10 points]

Flip a fair coin four times, mutually independently. Let $H_i$ (for $1 \leq i \leq 4$) be the indicator variable for the $i$th coin coming up Heads. Now let

$$Y = H_1 + H_2, \qquad Z = H_1 + H_3, \qquad W = H_1 + H_4.$$

**(a)** [5 pts] Are $Y, Z, W$ mutually independent? Justify.

**Solution.** $\Pr(Z = 2) = \frac{1}{2} * \frac{1}{2} = \frac{1}{4}$, but $\Pr(Z = 2 | Y = 2) = \frac{1}{2} \neq \frac{1}{4}$ ■

**(b)** [5 pts] Let $P_Y$ the event that $Y$ is even. Let $P_Z$ and $P_W$ be defined similarly. Are $P_Y, P_Z, P_W$ pairwise independent? Justify.

**Solution.** By symmetry, we only have to consider $P_Y$ and $P_Z$.

$\Pr(P_Y = 1) = \frac{1}{2}$, same with $P_Z$ and $P_W$.

$\Pr(P_Y \cap P_Z) = \Pr(\text{first three coin flips are all heads}) + \Pr(\text{first three coin flips are all tails}) = \frac{1}{2}^3 + \frac{1}{2}^3 = \frac{1}{4} = \Pr(P_Y) * \Pr(P_Z)$.

■

## Problem 14. Expected Point Total [9 points]

Consider the following game: You roll a fair 6-sided die 10 times. On the $i$'th roll (for each $2 \le i \le 10$), if you roll a number higher than the previous roll, then you score $i$ points. (No points can be scored on the first roll, i.e., when $i = 1$.) For example, if the rolls were $[1, \underline{3}, 1, 1, \underline{3}, 3, 3, 3, 3, 3]$ then you would score $2 + 5 = 7$ points in total.

Compute the expected number of total points scored. Be sure to **very precisely** justify your work.

**Solution.** Let $D_1, \ldots, D_{10}$ be the outcomes of the dice, and let $X_2, \ldots, X_{10}$ be the number of points scored on each roll. By definition we have $X_i = i \cdot 1_{D_i > D_{i-1}}$, and we can compute $\Pr[D_i > D_{i-1}] = \frac{\binom{6}{2}}{36} = \frac{5}{12}$.

Then by linearity of expectation:

$$\begin{aligned}
\operatorname{Ex}\left[\sum_{i=2}^{10} X_i\right] &= \sum_{i=2}^{10} \operatorname{Ex}[X_i] \\
&= \sum_{i=2}^{10} i \cdot \Pr[D_i > D_{i-1}] \\
&= \frac{5}{12} \sum_{i=2}^{10} i \\
&= \frac{5}{12} \cdot 54 \\
&= \boxed{\frac{45}{2}}
\end{aligned}$$

■

## Problem 15. Picking Marbles [14 points]

**(a)** [7 pts] A box contains red and green balls, and we draw balls uniformly at random, **with replacement**, until we draw a red ball. Let $0 < p \leq 1$ be the probability of drawing a red ball, and let $X$ be the random variable representing the number of balls drawn during this process, including the final red ball. For example, if you pick (green, green, red), then $X = 3$. Prove that the **variance** of $X$ equals $\frac{1-p}{p^2}$.

You may use the fact that $\sum_{k=1}^{\infty} k^2 y^{k-1} = \frac{1+y}{(1-y)^3}$ when $-1 < y < 1$.

**Solution.** $Var(X) = E[X^2] - E[X]^2$. By the "mean time to failure" formula (i.e., since $X$ is a geometric random variable with parameter $p$), $E[X] = \frac{1}{p}$.

We have

$$E[X^2] = \sum_{k=1}^{\infty} k^2 P[X = k] = \sum_{k=1}^{\infty} k^2 (1-p)^{k-1} p = \frac{(2-p)}{p^2}$$

$$Var(X) = E[X^2] - E[X]^2 = \frac{2-p}{p^2} - \left(\frac{1}{p}\right)^2 = \frac{1-p}{p^2}.$$

■

**(b)** [7 pts] In a different experiment, a box initially contains $n$ balls, all colored red. A ball is drawn from the box at random. If the drawn ball is red, then it is **replaced with a green ball**. If the drawn ball is green, the green ball is just **returned to the box**. This process is repeated until the box contains $n$ green balls. Let $D$ be the number of balls drawn until the process ends with the box full of green balls. Compute the **variance** of $D$, and carefully justify your work. You may express your answer in terms of $H_n := \sum_{k=1}^{n} \frac{1}{k}$ and $L_n := \sum_{k=1}^{n} \frac{1}{k^2}$.

**Solution.** Solution: Let $D_k$ be the number of draws until a red ball appears, in a bag that has $n - k$ green balls and $k$ red balls. We have $D = D_n + D_{n-1} + \ldots + D_1$. Since all the $D_k$s are independent, we can write

$$Var(D) = Var(D_n) + \ldots + Var(D_1)$$

Each $D_k$ is a geometric random variable with parameter $p = \frac{k}{n}$ and therefore variance $\frac{1-(k/n)}{(k/n)^2} = \frac{n(n-k)}{k^2}$. We have,

$$Var(D) = \sum_{k=1}^{n} \frac{n(n-k)}{k^2} = \sum_{k=1}^{n} \frac{n^2}{k^2} - \sum_{k=1}^{n} \frac{n}{k} = n^2 L_n - n H_n.$$

■

## Problem 16. Goblin Invasion [15 points]

Green math goblins invaded the universe's fabric of logic and won't let you sleep at night. Instead, they pull out a jar containing 11 strips of paper and propose a game to you. The greenest goblin tells you that every strip is labeled with a number in the range $[-50, 50]$ (not necessarily distinct) and that the mean of the numbers equals 5. If you pick a number $X$ that is *not* in the range $-20 < X < 30$, you'll win and the math goblins will disappear. (In other words, you win if you pick a number that is *far away* from the mean.) But you suspect the game might be rigged against you. Armed with Markov and Chebyshev, you begin to investigate.

Recall:

**Theorem** (Markov's Inequality)**.** *Let $R$ be a **non-negative** random variable. Then,*

$$\Pr[R \geq k] \leq \frac{\operatorname{Ex}[R]}{k}.$$

**Theorem** (Chebyshev's Inequality)**.** *For every $k > 0$ and for every random variable $R$,*

$$\Pr[|R - \operatorname{Ex}[R]| \geq k] \leq \frac{\operatorname{Var}[R]}{k^2} = \left(\frac{\sigma(R)}{k}\right)^2.$$

**(a)** [6 pts] Use Markov's inequality to find an upper bound on the probability that you pick a strip with value at least 30.

**Solution.** Let $X$ be the (random variable describing the) value of the chosen strip. Then $X + 50$ is always nonnegative, and using Markov's inequality on $X + 50$ gives

$$\Pr[X \geq 30] = \Pr[X + 50 \geq 30 + 50] \leq \frac{\operatorname{Ex}[X + 50]}{80} = \frac{55}{80} = \frac{11}{16}.$$

■

**(b)** [3 pts] Find a possible set of strips that have values in $[-50, 50]$ and mean 5 where it is *never* possible to win the game.

**Solution.** Perhaps they listed 5 on all the strips! ■

**(c)** [6 pts] The goblin with the pointiest ears tries to reassure you by telling you that the variance of the 11 numbers in the jar (i.e., $\operatorname{Var}[X]$) equals 300. Prove that the game is still not fair: use Chebyshev's inequality to show that your probability of winning is less than $1/2$.

**Solution.** By Chebyshev's inequality,

$$\Pr[X \leq -20 \text{ OR } X \geq 30] = \Pr[|X - \operatorname{Ex}[X]| \geq 25] \leq \frac{\operatorname{Var}[X]}{25^2} = \frac{300}{25^2} = \frac{12}{25} < \frac{1}{2}.$$

■