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Global Energy Conservation in Gravitational Lensing

by

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ABSTRACT

Global properties of energy conservation in gravitational microlensing are discussed. We consider the influence of a gravitating body on the observations done at different locations on a sphere surrounding the source. We show that a mass located outside this sphere only redistributes the photons, so some observers register a larger energy flux, photon count rate *etc.* at the cost of other observers, but there is no net effect on the whole sphere.

A lensing mass located inside the observers' sphere reduces its surface area, and also redshifts photons on average. The redistribution of photons is accompanied by a net decrease of averaged energy flux because of the redshift. However, the total photon count from a bursting source is increased on average as the area of the observers' sphere is reduced.

Key words: *gravitational lensing*

1. Introduction

If a point mass M is located close to the line of sight joining the observer O and a point source S then the two images of the source have a combined magnification A given as

$$A = \frac{u^2+2}{u\sqrt{u^2+4}}, \tag{1}$$

(*cf.* Paczyński 1996, and references therein), where u is the dimension-less impact parameter expressed in units of Einstein ring radius r_E:

$$r_E = \sqrt{\frac{2r_g(D-R_S)R_S}{D}}; \quad D > R_S \tag{2}$$

where R_S is the distance of the mass from the source, D is the observer's distance from the source, and:

$$r_g \equiv \frac{2GM}{c^2} \tag{3}$$

is the Schwarzschild radius of the mass M. Therefore, the observers located close to the line joining the source S and the mass M (*i.e.*, close to the optical axis) observe an increase in the apparent source brightness by a factor $A > 1$. A question is sometimes asked: if there is an apparent excess of radiative flux at some locations then is there a corresponding reduction of the flux in some other directions. A similar question of the flux conservation in cosmology has been considered by Weinberg (1976), Ehlers and Schneider (1986), and in the monograph devoted to gravitational lensing by Schneider, Ehlers and Falco (1992). Recently the problem of luminosity changes due to the defocusing microlensing, as authors call it, has been addressed by Capozziello *et al.* (1996a,b).

Presumably the question is stimulated by a superficial similarity between a gravitational lens and an ordinary glass lens. However, there is a fundamental difference between the two types of lenses. A conceptual glass lens is inserted into a fixed flat Euclidean space while a gravitational lens modifies the space geometry. Therefore it is not obvious at all that there must be any region from which the excess of radiative flux is "stolen" and therefore in that region there must be a flux reduction. There is another possibility: the total area illuminated by the source may be reduced, so the flux may in fact increase everywhere, with the peak of flux increase given with the Eq. (1). The purpose of this paper is to clarify this issue, which turns out to be somewhat more complex than naively imagined, because the difference in the gravitational redshifts of the source and the observer is of the same order of magnitude as the geometrical effects.

The magnification of the source A tells us how many times the combined solid angle of all the source images exceeds the solid angle of its image uninfluenced by any lensing. The apparently larger angular size of a source gives an impression, that its distance has decreased. Similarly in the case of unresolved sources the magnification increases the measured flux of energy from the source, since the surface brightness is unchanged and the source has larger (albeit unmeasurable) apparent size (*cf.* Schneider *et al.* 1992). This also might be interpreted as a decrease in the source – observer distance. Thus the influence of lensing on the measured distance to the source seems to be a convenient way of quantifying the effect.

The measurements of the angular size or the apparent brightness, which we mention above, are not the only methods of estimating distances in astronomy. There are other which we describe in the next Section. The methods used in astronomy are based on various source characteristics which, at least in principle, can be measured. In the flat space all the methods give the same answer but in the curved space of gravitating bodies their results can be different. We are going to describe the influence of a mass, not necessarily close to the source – observer line, on the various measurements done by the observer and express the results as the changes in the related distances. For comparison we use the metric distance between the source and the observer. There is probably no practical way to measure it directly, but simultaneously it is in a sense absolute. We imagine a class of static observers which are at the same metric distance from the source, so they are, in a sense, on a "sphere". We shall see that the surface defined in this way has some interesting properties. The distances measured by different observers on the "sphere" depend on their location and the method of measurement.

We consider an idealized situation treating the source of radiation and the observers as test particles in the static gravitational field of a point mass. We also neglect the relative motions of the source, mass, and observers. All these simplifications can be relaxed, and we discuss possible effects they can have on the results.

In the next Section we define various measures of the distance. Section 3 describes the influence of a gravitating body on the measured distances after averaging over possible observer's positions. In Section 4 we compare the observer – averaged results with the situation, where the gravitating body is replaced by a spherical layer of the same total mass. In Section 5 we calculate the dependence of the measured distances on the observer's position in the first order approximation in r_g/R_S. We also show the results of numerical calculations based on following the rays in Schwarzschild spacetime. The discussion follows in the last Section.

2. Different Distance Definitions

The distance between the source and the observer can be defined in many ways. All our definitions are based on measurements along the paths of light rays and all are equivalent to usual definition of the distance in the limit of flat space.

The most natural, but not the easiest to measure is the metric distance between the source and the observer. We use the Schwarzschild metric in the isotropic coordinates (t, x, y, z):

$$\mathrm{d}s^2 = -X^2(R)c^2\,\mathrm{d}t^2 + A^2(R)(\mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2) \tag{4}$$

where c is the speed of light and $R \equiv (x^2 + y^2 + z^2)^{1/2}$. The metric functions X and A are given as (Misner, Thorne and Wheeler 1973):

$$X = \frac{1 - \frac{r_g}{4R}}{1 + \frac{r_g}{4R}}, \quad A = \left(1 + \frac{r_g}{4R}\right)^2 \tag{5}$$

where r_g is the Schwarzschild radius. The angles measured in rectangular coordinates (x, y, z) are the same as measured in space. The origin of coordinate system can be placed anywhere but R must be measured from the mass.

We consider the situation in Fig. 1. The source of light (S) is located at a coordinate distance R_S from the point mass serving as a "lens". The source is surrounded by observers on a "sphere". By assumption all observers are at the same metric distance D from the source. An observer at the position angle α is at the coordinate radius $r(\alpha)$ measured from the source, which can be obtained from the condition:

$$D = D^{\rm met}(\alpha) = \int\limits_{0}^{r(\alpha)} A(R(r))\,{\rm d}r \tag{6}$$

where R is the distance from the mass. In a calculation to the first order in r_g/R the integral is calculated along a straight line. The curvature of a light ray influences the results in the second order only. The details are shown in the Appendix. The result is:

$$r(\alpha) = D - \frac{r_g}{2}\int\limits_{0}^{D} \frac{{\rm dr}}{R(r)} \equiv D - \frac{1}{2} r_g \ln \frac{R_S(1+\cos\alpha)}{R_O(1+\cos\beta)} \tag{7}$$

where R_O is the coordinate distance of the observer from the mass and β is the angle between the ray and the observer-mass line (compare Fig. 1). Thus the observers' "sphere" is defined.

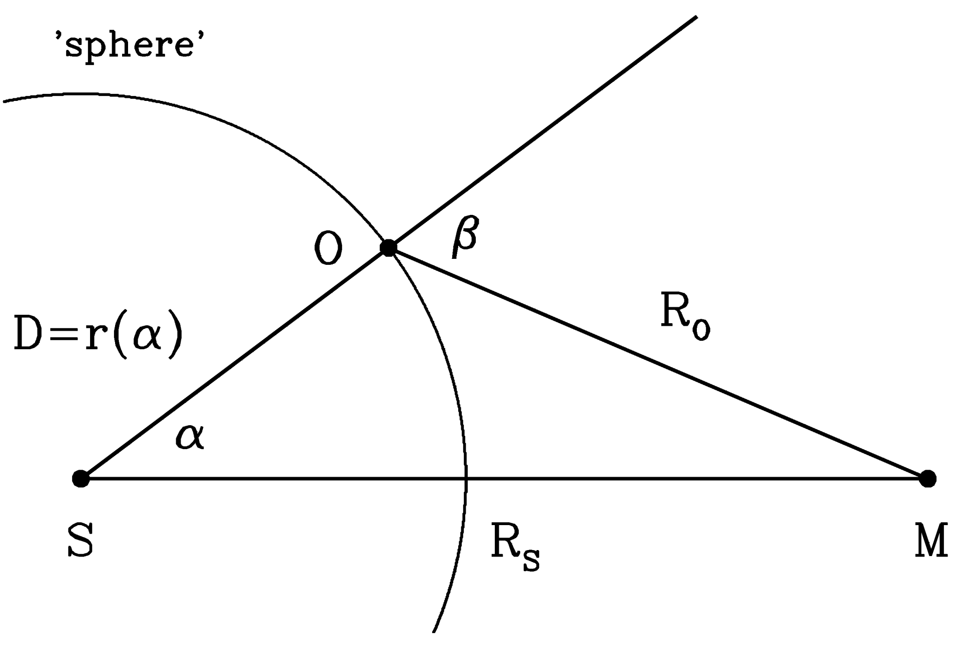

Fig. 1. The relative positions of the mass (M), the source (S) and an observer (O) are shown schematically for a case with mass outside the observers' "sphere". R_S and R_O are the distances of the source and an observer from the mass, measured in isotropic coordinates. The coordinate distance of an observer from the source $(r(\alpha))$ is close to, but not equal to the metric distance D between the source and an observer (compare Eq. (7)). The positions of other observers are shown schematically as a "sphere". α is the original direction of a ray measured at the source. β is the angle between the ray and the observer-mass (OM) line.

It is also shown in the Appendix, that in the case of a mass far away ($R_S > D$) the surface area of the observers' sphere is independent of the lens location. In the case of a mass inside the observers' sphere, its surface area decreases. Depending on the case we have:

$$S = \begin{cases} 4\pi D^2 & \text{if } D < R_S, \\ 4\pi D^2 \left(1 - \frac{r_g}{D} \ln \frac{D}{R_S}\right) & \text{if } D > R_S. \end{cases} \tag{8}$$

The metric distance, which is conceptually simple, is difficult to measure. In the Solar System one uses radar to measure the distances. The idea is to measure the time of electromagnetic waves propagation forth and back between two points. Multiplying the time lapse by the speed of light and dividing by two one gets the distance, which we shall call the radar distance, $D^{\rm rad}$. The time lapse actually measured by the observer in gravitational field of a distant mass depends on its position and it is related to the time measured at infinity by the gravitational redshift factor $\mathrm{d}t_O = X_O\,\mathrm{d}t_\infty$ where the subscript O stands for the quantities measured at the observer's position. We assume that the observers know their position relative to the massive body and can readjust their measurements expressing them in units of the time at infinity. This gives the following definition of the radar distance:

$$D^{\rm rad}(\alpha) = \int\limits_0^{r(\alpha)} \frac{A(R(r))}{X(R(r))}\,\mathrm{d}r\,. \tag{9}$$

Since we have $X(R) < 1$ at any position, the radar distance is always greater than the metric distance between any given points.

The main question of this paper is the influence of the gravitating body on the measurements of the energy flux from the source by different observers. The related questions are the influence of the mass on the measurements of the photon count rate, of the time integrated photon counts and of the observed angular diameter of the source. All these measurements lead to the related definitions of distances, which can in principle be used in astronomy, if some intrinsic properties of the source (*i.e.*, luminosity, photon emission rate, linear size) are known. In fact the luminosity distance and the angular diameter distance are widely used in cosmology (Weinberg 1972, Schneider *et al.* 1992).

Despite the fact that all the observers are at the same metric distance from the source, their observations may differ. Let us start from the integrated photon counts, which may be applied to a source emitting photons mostly through outbursts. Suppose the source emits isotropically a total number of $N_{\rm ph}$ photons during an outburst. The photons emitted into a solid angle $\delta\Omega_S = \sin\alpha\delta\alpha$ with the initial directions in the range $\alpha \div \alpha + \delta\alpha$ will eventually pass the "sphere" of observers at position angles $\theta \div \theta + \delta\theta$, encompassing a surface area of $\delta S = A^2(R) r^2(\theta) \sin\theta\delta\theta$. Due to the deflection of rays by the mass the angles are slightly different ($\theta < \alpha$), but the difference is small. An observer at the position angle θ will register

$N_{\rm ph}\delta\Omega/(4\pi\delta S)$ photons using a detector of unit surface area. This integrated count rate should be inversely proportional to the square of the distance. The appropriate definition of the distance based on integrated photon counts is given below. Schneider *et al.* (1992) call it the corrected luminosity distance:

$$D^N = \sqrt{\frac{\delta S}{\delta\Omega_S}} = A(R)r(\theta)\sqrt{\frac{\sin\theta\delta\theta}{\sin\alpha\delta\alpha}}\,. \qquad (10)$$

We may note in passing that the above definition involves angles measured at the source and sizes at the observer position. Thus it is close to the definition of the angular diameter distance measured from the source. Since the beams of light are distorted due to the deflection, the angular size of an object placed at the observer position and measured from the source position depends on its orientation in the sky. The geometrical mean of angular sizes of an object having two principal orientations leads to the angular diameter distance identical to the above definition based on the integrated photon counts, $D^{\rm ang}_{SO} = D^N$, where the first subscript defines the location where the measurement is done. The parallax is the change in the position of a source in the sky due to the motion of Earth around the Sun. This is also the angle between the two extremal Earth positions as observed from the source, or the angular size of the Earth orbit as seen from the source. Thus the orientation averaged parallax measurement would also give the distance D^N as a result (compare Weinberg 1972).

The total number of photons detected after an outburst is independent of the redshift/blueshift factors. We assume that the detector is active long enough to detect all photons sent in its direction. The rate at which photons arrive at the detector does depend, however, on the redshift factor. Let E_S be a photon energy measured at the source and E_O – its energy measured by an observer. The redshift factor is given as:

$$1+z = \frac{E_S}{E_O} \approx 1 + \frac{1}{2}\frac{r_g}{R_S} - \frac{1}{2}\frac{r_g}{R_O}\,. \qquad (11)$$

Note that z is negative for observers who are closer to the mass than the source is. Suppose the photons are emitted at the rate $\dot{N}_{\rm ph}$ by the source according to a clock placed there. Since $dt_O = (1+z)\,dt_S$, the detection rate differs and for the count rate at the observer location we get

$$C = \frac{\dot{N}_{\rm ph}}{4\pi(D^N)^2(1+z)} \quad \Rightarrow \quad D^C = D^N\sqrt{1+z} \qquad (12)$$

where D^C is the distance based on the measurement of photon count rate. The flux of energy is affected by an another $(1+z)$ factor, since each photon's energy changes by this factor. For a source of bolometric luminosity L, the bolometric flux of energy F can be expressed as:

$$F = \frac{L}{4\pi(D^N)^2(1+z)^2} \quad \Rightarrow \quad D^F = D^N(1+z) \qquad (13)$$

where D^F stands for the luminosity distance. Finally, one can use the reciprocity relation (Etherington 1933) to define the angular diameter distance measured at the observer's location:

$$D_{OS}^{\mathrm{ang}} = \frac{D_{SO}^{\mathrm{ang}}}{1+z} = \frac{D^N}{1+z}\,. \tag{14}$$

Again this definition assumes orientation-averaged measurements of the source's angular size. We also derive the above relation in the first order calculation of Section 5.

The distance to the source clearly depends on the method one uses to measure it. For a given observer's location the ordering of various distances depends mostly on the redshift factor and can easily be found. The ordering is not global, unless the sign of the redshift z is the same for all the observers. This happens when the mass is closer to the source than to any observer. If the mass is closer to some observers than to the source, the sign of z changes and ordering of distances is location dependent.

3. Observer-averaged Measurements

A good characteristics of the influence of gravitating mass are the observer-averaged effects of its presence. We make our calculations to the first order in r_g/R. In the case of mass inside the observers' "sphere", the linear approach is not valid for observers nearly aligned with the source and mass (true lensing situation) and we shall discuss the limitations of our approach later on.

The easiest to evaluate is the averaged integrated photon count. In the case of mass outside the "sphere" the situation is very simple: the number of photons per outburst is the same and the surface area remains unchanged (see the Appendix). Thus the observer-averaged integrated photon count number does not change under the influence of gravitating mass. That means that the averaged reversed square of the distance D^N is equal to D^{-2}. To the first order it is equivalent to $\langle D^N\rangle = D$. In the case of mass inside the sphere some of the photons are lost and never reach the observers. For an extended mass this effect depends on the size of gravitating body. To avoid extra parameters we consider here a point mass. Rays which pass closer than the photon orbit never return to infinity and are captured by the black hole (Bardeen 1973). The maximal encounter parameter for lost rays is equal to $3\sqrt{3}r_g/2$ (Bardeen 1973) and it corresponds to the solid angle of:

$$\delta\Omega_{\mathrm{lost}} = \frac{27}{4}\pi\frac{r_g^2}{R_S^2}\,, \tag{15}$$

which is a second order quantity. We neglect the influence of capturing the rays in our further analytical, first order treatment. We now use the result from the Appendix. Since the same number of photons crosses a smaller area, a given detector registers more of them and the observers have (on average) an impression of being closer to the source. Summarizing both cases, we have quantitatively:

$$\langle D^N \rangle = D\sqrt{\frac{S}{4\pi D^2}} = \begin{cases} D & \text{if } D < R_S, \\ D\left(1 - \frac{1}{2}\frac{r_g}{D}\ln\frac{D}{R_S}\right) & \text{if } D > R_S. \end{cases} \tag{16}$$

For other methods of distance measurements, based on the observed light from the source, the results differ due to the dependence on the redshift factor. We are going to include only the first order effects, so the corrections depend on the averaged redshift value only. The redshift dependence on the observer location (Eq. (11)) is such that we use the results of integration in the Appendix, obtaining:

$$\langle z \rangle = \begin{cases} 0 & \text{if } D < R_S, \\ \frac{1}{2}\left(\frac{r_g}{R_S} - \frac{r_g}{D}\right) > 0 & \text{if } D > R_S. \end{cases} \tag{17}$$

From the distance definitions it follows that:

$$\langle D^C \rangle = \langle D^N \rangle + \frac{1}{2}D\langle z \rangle \,, \tag{18}$$

$$\langle D^F \rangle = \langle D^N \rangle + D\langle z \rangle \,, \tag{19}$$

$$\langle D^{\rm ang}_{OS} \rangle = \langle D^N \rangle - D\langle z \rangle \tag{20}$$

which means that in the case of mass laying outside the observers' sphere ($D < R_S$) all measures of distance are equal to each other on average and equal to the metric distance D. If the mass is inside the sphere the situation is more complicated.

4. Spherically Symmetric Distribution of Mass

Suppose we now replace the point mass by a thin spherical layer of constant surface density and the same total mass surrounding the source. We still consider observers at the metric distance D from the source, and R_S is now the radius of the mass sphere. If mass is distributed farther away from the source than the observers ($R_S > D$), the source and the observers are in a flat space, and all their measurements of the distance agree, regardless of the method. In the opposite case ($R_S < D$) the space outside mass shell is curved. The observers are on the coordinate radius r, in isotropic coordinates. The equation for r reads:

$$D = A(R_S)R_S + \int_{R_S}^{r} A(r)\,{\rm d}r \,, \tag{21}$$

and to the first order in r_g/D gives:

$$r = D - \frac{1}{2}r_g - \frac{1}{2}r_g \ln\frac{D}{R_S} \,. \tag{22}$$

The surface area of the sphere is obtained as:

$$S = 4\pi A^2(r)r^2 = 4\pi D^2\left(1 - r_g \ln\frac{D}{R_S}\right). \quad (23)$$

One can also repeat the redshift calculation obtaining a value which is identical to the former, observer-averaged, result. As one can see, there is a strict correspondence between the averaged results of measurements in the case of point mass at a given distance from the source, and the situation where the point mass is replaced by a spherical, transparent shell of matter of the same radius and mass, surrounding the source.

5. Observer-dependent Measurements

We describe here the influence of gravitating mass on the measurements of individual observers. The location of an observer is defined by its metric distance from the source D and the polar angle θ measured from the optical axis. (This is not exactly the same as the angle α between the ray and optical axis due to the bending of rays.) In principle a first order, approximate treatment of the bending of rays is sufficient and one can obtain the relation between angles α and θ in this way.

The usual weak field calculation of the ray deflection involves the integration of the gravitational force perpendicular to the ray along the straight line from minus to plus infinity. Dyer (1977) has noticed, in different context, that the integration between infinite limits is not always appropriate. The deflection angle measures the change in the direction of the photon momentum. We are interested in the deflection on the way between the source and the observer. This can be calculated by an integration between these two points; similarly integration up to a given point gives the deflection up to this point. In Fig. 2 we show schematically the ray as being deflected from its original direction along the straight line SA. At the coordinate distance r from the source, the ray is displaced by $\delta r(r)$ from this line. The derivative of the displacement measures the angle between the actual and the original directions of the ray. Thus we have:

$$\frac{\mathrm{d}\delta r}{\mathrm{d}r} = \frac{r_g}{R_S \sin\alpha}(\cos\alpha - \cos\beta). \quad (24)$$

The deflection angle in the RHS is the result of integration, which we mention above, between finite limits. (When $\alpha = 0$ and $\beta = \pi$ we recover the standard result for the bending angle.) The integration over r is straightforward, especially after the change of the independent variable to $s = R_S \cos\alpha - r$. Finally one obtains:

$$\delta r(D) = \frac{r_g}{R_S \sin\alpha}(D\cos\alpha - R_S + R_O) \quad (25)$$

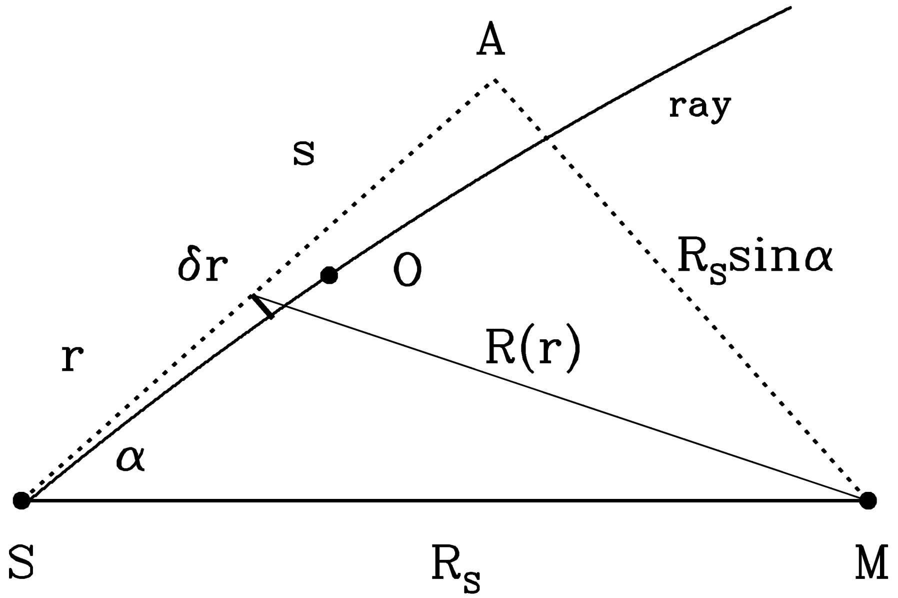

Fig. 2. The deflection of a ray. We schematically show (with exaggeration) how a ray is deflected from its original direction. All the distances shown are coordinate distances measured in the isotropic coordinate system we use, so the angles on the plot and the angles in curved space are the same. The dotted line starting from the source (S) is the tangent to the ray at this point. It is shown up to the point A where it crosses the perpendicular dotted line from the mass M. The displacement δr is shown at the distance r from S (or the distance s from A, where s is the variable used in the integrations). $R(r)$ is the coordinate distance from the mass to a point on the tangent to the ray. $R_S \sin\alpha$ is the encounter parameter for the ray.

(compare Fig. 1 for the meaning of symbols), which gives:

$$\theta = \alpha - \frac{\delta r(D)}{D} . \tag{26}$$

Using the expression for δr we get to the first order:

$$\frac{\sin\theta \, \mathrm{d}\theta}{\sin\alpha \, \mathrm{d}\alpha} = 1 + \frac{r_g}{R_S} - \frac{r_g}{R_O} . \tag{27}$$

It is now straightforward to obtain the distance related to the integrated photon counts using Eqs. (7,10) and the above. After some algebra one gets:

$$D^N = D + \frac{1}{2} r_g \left(\frac{D}{R_S} - \ln \frac{R_S(1+\cos\alpha)}{R_O(1+\cos\beta)} \right) . \tag{28}$$

Suppose, the positions of the observer and the source were mutually interchanged. The coordinate distance r between them would remain the same, variables R_S and R_O would also be interchanged in Eq. (27), and $A(R_S)$ would replace $A(R_O)$ in Eq. (10) defining the distance. As a result we would get (to the first order) the RHS of Eq. (28) divided by $(1 + r_g/2R_S)/(1 + r_g/2R_O) \approx (1 + z)$. But this is just, what we use in Eq. (14) as the definition of the angular diameter distance measured by the observer. Our derivation proves the reciprocity relation in the first order calculation.

The above expression remains finite for any α value between 0 and π. In the case of true lensing (observer behind the lens: $D > R_S$, $\alpha \to 0$, $\beta \to \pi$), the expressions based on the linear approximation are no longer valid because the bending angles may become large. The linear approach is limited in this case to the rays remaining outside the it Einstein radius of the lens:

$$\alpha > \alpha_E \equiv \sqrt{\frac{2r_g(D - R_S)}{DR_S}}; \quad D > R_S\,. \tag{29}$$

For $\alpha < \alpha_E$ some other approach is needed.

To handle the case of small angles one can use the usual lens theory. In particular the source magnified A times seems to be $\sqrt{A}$ times closer to the observer and the angular diameter distance to the source decreases by this factor. For the observer on the axis behind the lens, the magnification goes to infinity for point sources and the distances based on counting photons or measuring their energy flux become zero. This is clearly beyond the validity of a linear approximation.

We repeat our calculations following the null geodesics in Schwarzschild geometry. For a ray leaving the source at an angle α to the optical axis we find its crossing with the observers' "sphere" (which lays at metric distance D along the ray) and then we find the corresponding angle $\theta(\alpha)$. Further calculation follows the linear approach but this time the small values of the angle α are not excluded. In the lensing situation an observer on the optical axis detects the rays with initial direction $\alpha \approx \alpha_E$. Smaller shooting angles ($\alpha < \alpha_E$) lead to crossing the optical axis by the rays which corresponds to negative values of observer position angle θ. We call rays which do not cross the optical axis primary and these which do – secondary rays. We neglect higher order rays which reach the observer after crossing the optical axis more than once. Reversing the relation between angles and using the absolute value of the angle θ, we get two possible shooting angles for each position of the observer: $\alpha_{\rm prim}(\theta)$ and $\alpha_{\rm sec}(\theta)$, where $\alpha_{\rm prim} > \alpha_E$ while $\alpha_{\rm sec} < \alpha_E$. Taking into account both kinds of rays we generalize the distance definition of Eq. (7):

$$D^N = \sqrt{\frac{\delta S}{\delta\Omega_S}} = A(R)r(\theta)\sqrt{\frac{\sin\theta\delta\theta}{\sin\alpha_{\rm prim}\delta\alpha_{\rm prim} + \sin\alpha_{\rm sec}\delta\alpha_{\rm sec}}} \tag{30}$$

where we understand that $\delta\alpha_{\rm sec}$ is positive. Other distance measures, which differ from the above one by redshift factors in different powers, change similarly.

We show the results of our numerical calculations and the comparison with analytical results in Fig. 3 and Fig. 4. We plot the differences between the actual distance measure and the metric distance from the source to an observer as a function

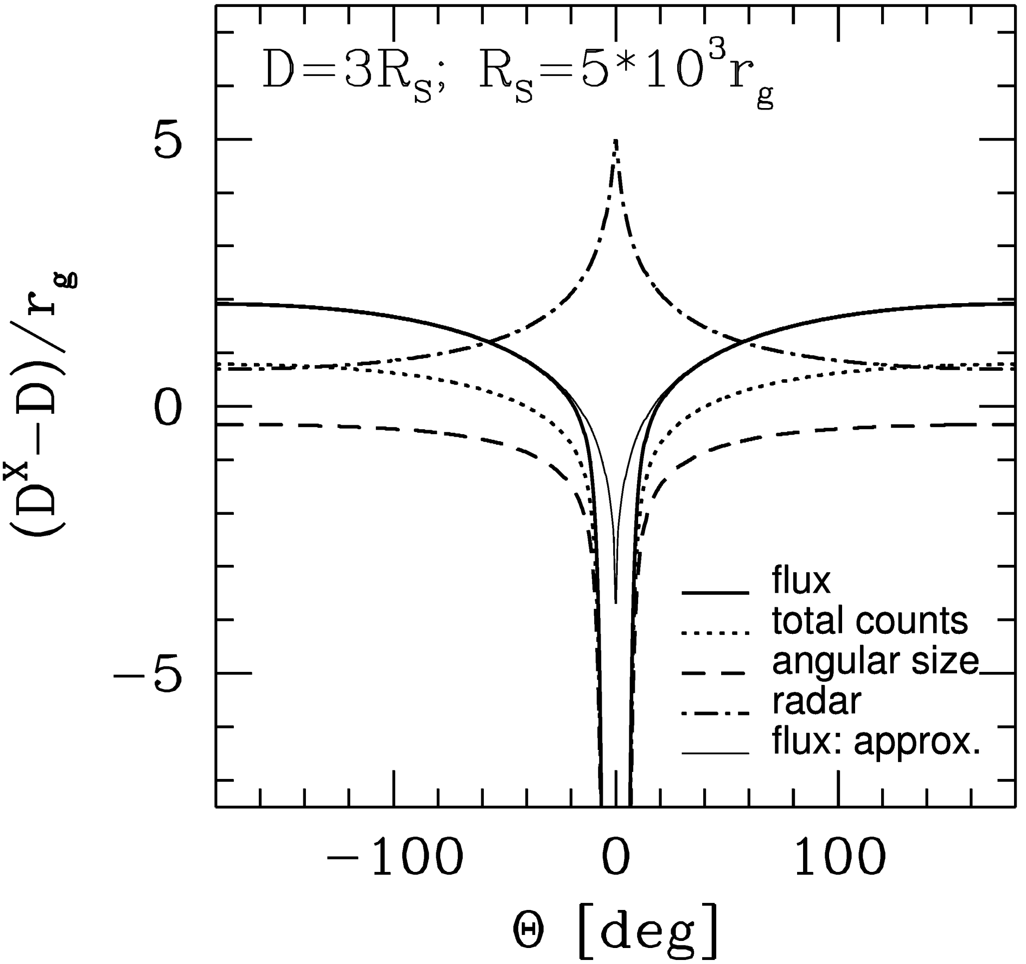

Fig. 3. The influence of a mass inside observers' "sphere" on the distances measured by different observers. The angle θ [degrees] defines an observer position on the "sphere" relative to the optical axis; $\theta = 0$ corresponds to the observer position behind the mass. The differences between various measures of the distance (D^X , which stands for distance estimated by measuring the flux of radiation, the total counts of photon, the angular size of the source or the time lapse of a radar signal sent to the source) and the metric distance (assumed to be D for all considered observers) are expressed in units of Schwarzschild radius r_g . The meaning of different lines is shown on the graph. The thin solid line shows the result of the approximate treatment for the distance related to the measurements of the energy flux. Near optical axis the true lensing is possible, and the observer measuring the flux, counting the photons or measuring the angular size of the source has an impression of being much closer to the source. The answer given by the first order approximation calculation becomes wrong close to the optical axis.

of the observer position relative to the optical axis, θ. Fig. 3 illustrates the case of the lens closed inside the observers' "sphere". The strong effects for observers behind the lens are clearly visible. One can also see that the approximate, first order approach gives excellent results apart from optical axis. In Fig. 4 we show the case of the mass outside the "sphere". The effects are small for all observers.

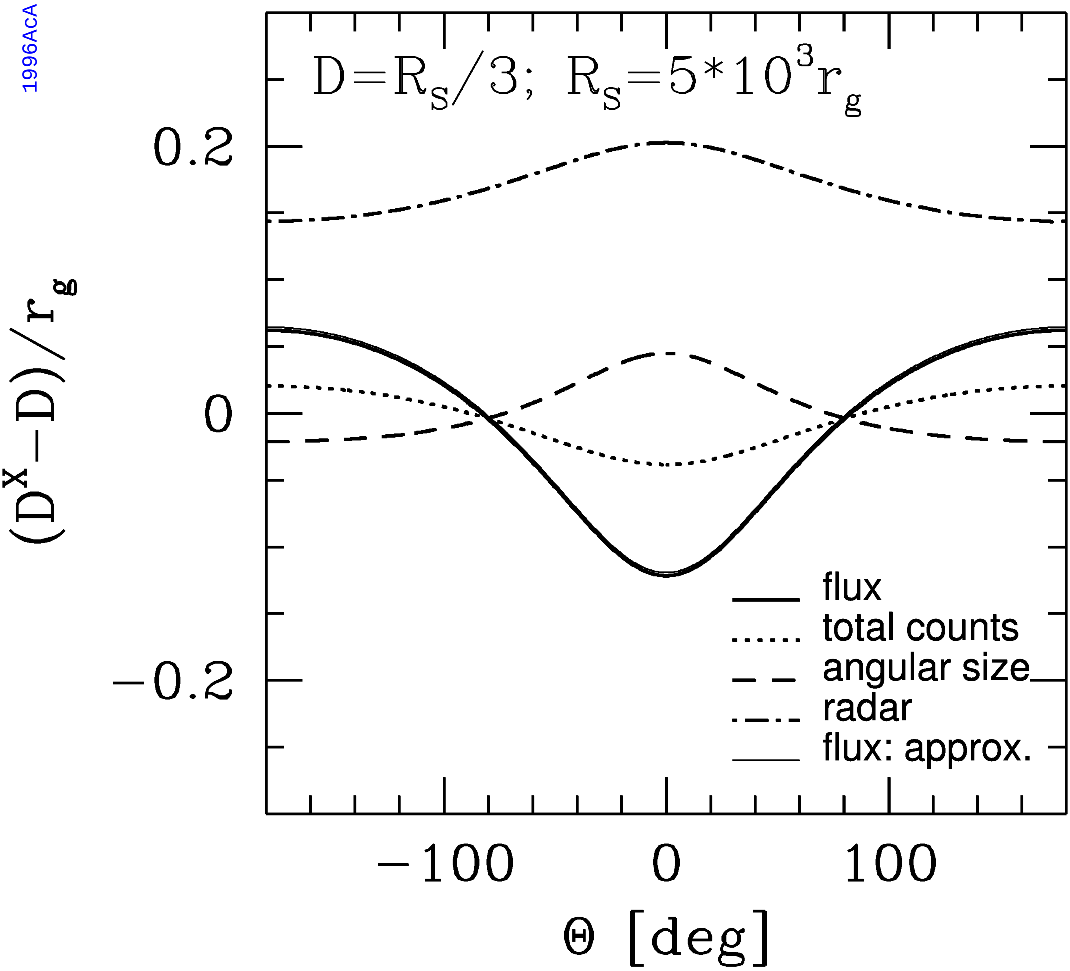

Fig. 4. The same as in Fig. 3, but for the mass outside the observers' "sphere". One may notice much smaller amplitude of the effects as compared with the previous case. The first order calculation gives a very accurate description for all observers positions and the corresponding curves (thick and thin solid lines) are impossible to distinguish on the plot.

As can be seen on the plots, the situation when the observer and the mass are on the opposite sides of the source is not singular: the dependence of distance measures on on the angle remains smooth and finite, even if $\alpha \to \pi$. Substituting $\alpha = \pi - \epsilon$, where ϵ is assumed to be small, and expanding in RHS of Eqs. (28, A3), one gets:

$$D^N(\alpha = \pi) = D + \frac{1}{2} r_g \left(\frac{D}{R_S} - \ln \frac{R_S + D}{R_S} \right) . \tag{31}$$

This formula shows explicitly the influence of mass passing behind the source on its apparent characteristics. In particular, after correcting by the redshift factor, one can get the estimate of the induced change in the source brightness. If the observer is farther away from the source than the mass ($D \geq R_S$ or even $D \gg R_S$), the relative change in the apparent distance and apparent luminosity is of the order of r_g/R_S. Unless a black hole is passing behind the source at the distance of a few gravitational radii, the effect is not measurable. In the opposite case ($D \leq R_S$) the effect is even weaker and for $D \ll R_S$ becomes a second order effect.

6. Discussion

We have considered highly idealized situation, taking into account only one isolated mass acting as a "lens" and treating both the source and the observers as mass less bodies. In the "true lensing" case, where the lens is close to the line joining the source and the observer, the influence of other masses is relatively unimportant and consideration of one massive body is fully justified. In the opposite case (mass far away) this is not so: usually there are many massive bodies which are at distances comparable to the source – observer distance. But in this case (as we have shown) the linear approximation is fully adequate and the total effect can be calculated as a superposition of small contributions from all masses, calculated to the first order. In the limit of high number of massive bodies and their homogeneous distribution in space the approach of Section 3 assuming the spherical symmetry becomes adequate.

The mass of the source, if taken into account, influences all the rays in the same manner: all the photons are redshifted by the source's gravity. Since photons initially move radially, the bending of rays by the source becomes possible only after the deflection by some other mass, which can only be a high order correction. The influence of the local gravitational field and the velocity of any of the observers can be removed by means of data reduction. The influence of other observers on the observations made by one of them can be treated in the same manner as the influence of other masses. And finally one can think of the observers' "sphere" as of the possible locations, where an observer can be placed. As we see there are many ways to justify our idealized approach.

The far masses (*i.e.*, masses, which are at greater distances from the source than the observer), on average, have no influence on the observations, but can slightly influence individual observers. Since the rays are deflected toward the mass, observers closer to the mass than the source gain more photons and the photons are blueshifted. In this case one can say that the photons "stolen" from the other hemisphere increase the observed flux and the total photon counts of the closer half of observers.

The masses inside the observers' "sphere" (close masses) do influence the observations on average. The inspection of Fig. 3 shows that only one of the considered measures changes in the same manner regardless of the observer position: the angular diameter distance which becomes a little smaller under the influence of the mass. Thus all the observers at a given metric distance from the source would observe a slight increase in its angular dimensions under the influence of the "lens".

The total number of photons registered after an outburst by observers at a given metric distance from the source is on average larger if the close mass (masses) are present. This is possible because the surface area of the observers' "sphere" shrinks while the detectors area remains the same. The observers who are opposing the mass, however, would register fewer photons as compared to the case without the lens. In a sense the photons are "stolen" from some observers and go to the observers closer to the optical axis.

On average close masses cause redshift of the photons. The redshift is smaller (it may be a blueshift) for observers closer to the optical axis. As a result the observers who receive fewer photons, have them also more redshifted. The flux of photon energy is "stolen" from some observers, and the observers close to the optical axis get some extra photon energy. Inspection of Eqs. 16, 17 and 19 shows, however, than on average the flux distance D^F increases if a close mass is present. That means the decrease of the averaged flux of energy, so the "losses" of some observers are not compensated by the "gains" of others.

It must be stressed here, that all the effect described above are very small and practically unmeasurable. In a typical situation any of the distance measures, which we consider, changes by a few Schwarzschild radii under the influence of the mass, except for a true lensing case. For a stellar mass lens the effect could be compared to moving the telescope by a few kilometers. In particular, the passing of a massive body behind the source has no measurable effect, unless the passage is very close. In the latter case the source and the mass should be relativistic objects of stellar mass, or the lens has to be a supermassive black hole, in order to keep their distance so small as to make a significant effect.

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8. Appendix

The shape of the observers "sphere" in the coordinate system with the origin at the source is given by Eq. (6). From Fig. 1 one can see that:

$$R(r) = \sqrt{r^2 + R_S^2 - 2rR_S \cos\alpha}\,. \tag{A1}$$

If $D < R_S$ or when $\alpha \gg r_g/R_S$ the integral in Eq. (6) can be calculated after changing the integration variable to $s = R_S \cos\alpha - r$ giving:

$$r(\alpha) = D - \frac{1}{2} r_g \ln \frac{R_S(1+\cos\alpha)}{R_O(1+\cos\beta)} \tag{A2}$$

where

$$R_O = \sqrt{R_S^2 + D^2 - 2R_S D \cos\alpha}\,, \quad \cos\beta = \frac{R_S \cos\alpha - D}{R_O}\,. \tag{A3}$$

The above formulae work regardless of the signs of $\cos\alpha$ and $\cos\beta$.

To calculate the surface area of the observers' "sphere" we do not use the open form of the Eq. (A2) but the original form of Eq. (7). We have:

$$S = 2\pi \int_{-1}^{+1} \mathrm{d}\mu A^2(R) r^2(\mu) \tag{A4}$$

where $\mu \equiv \cos\alpha$. To the first order in r_g/R we obtain:

$$S = 4\pi D^2 \left(1 + \frac{1}{2} \int_{-1}^{+1} \mathrm{d}\mu \frac{r_g}{R} - \frac{1}{2}\frac{r_g}{D} \int_{-1}^{+1} \mathrm{d}\mu \int_{0}^{D} \frac{\mathrm{d}r}{R} \right) \tag{A5}$$

where R under the double integral is given by Eq. (A1) with $\cos\alpha = \mu$, and under the single integral – by the same expression with D substituted for r. Integration over μ in the double integral leaves $2/\max(r, R_S)$ to be integrated over r. If $D < R_S$ we finally get

$$S = 4\pi D^2 \left(1 + \frac{r_g}{R_S} - \frac{r_g}{D}\frac{D}{R_S} \right) \equiv 4\pi D^2 \tag{A6}$$

and with the opposite relation, $D > R_S$ we get

$$S = 4\pi D^2 \left(1 + \frac{r_g}{D} - \frac{r_g}{D}(1 + \ln \frac{D}{R_S}) \right)\,. \tag{A7}$$

The above formulae show that the surface area remains unchanged when the mass is outside the "sphere", and it shrinks when the mass is located inside.