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A HETEROCLINIC CONNECTION BETWEEN TWO SADDLE SLOW MANIFOLDS IN THE OLSEN MODEL

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Keywords: A list of 3–5 keywords are to be supplied.

1. Introduction

Slow-fast dynamical systems are characterized by a separation of variables into those that evolve on a fast time scale and those that evolve on a slower time scale. The separation of variables into fast and slow can be found in many systems in the world around us: chemical systems, neurons, electric circuits, lasers and predator-prey dynamics have been, among others, described by slow-fast models [Brøens & Bar-Eli, 1991; De Maesschalck & Wechselberger, 2015; van der Pol, 1927; Otto *et al.*, 2012; Piltz *et al.*, 2017]. By reason of their ubiquity, the various phenomena that arise from the multiple-time-scale nature of slow-fast systems are of significant interest. These have been be described for two- and three-dimensional systems by well-established theory [Benoît *et al.*, 1981; Benoît, 1982, 1985; Guckenheimer, 1985; Brøens *et al.*, 2006; Krupa *et al.*, 2008]. Small-amplitude limit cycles transitioning to larger-amplitude relaxation oscillations were studied in two dimensions, for example, the Van der Pol oscillator and Fitz-Hugh-Nagumo model [Benoît *et al.*, 1981; FitzHugh, 1955]. In three-dimensional systems, periodic orbits with epochs of localized small-amplitude oscillations (SAOs) and epochs of large-amplitude oscillations (LAOs) have been observed [Hudson *et al.*, 1979]. The mechanisms that cause SAOs of these appropriately named mixed-mode oscillations (MMOs) are described in [Desroches *et al.*, 2012]. We now investigate novel phenomena that arise in four-dimensional slow-fast systems which may provide insight into undiscovered mechanisms for MMOs.

We consider a prototypical four-dimensional slow-fast dynamical system which exhibits MMOs. We study the so-called Olsen model for peroxidase-oxidase reaction, first introduced by Lars F. Olsen in 1983 [Olsen, 1983], in a parameter regime corresponding to three fast and one slow variable. Mechanisms for MMOs in the Olsen model were previously investigated in [Desroches *et al.*, 2009] after an assumption was made to reduce the model to a three-dimensional system. Manifolds on which the flow progresses on the slower timescale were computed along with the manifolds consisting of trajectories that converged to them in forward and backwards time respectively. These were found to be very insightful into the formation of MMOs as well as the cause of their particular geometry. However, because of the assumptions used to reduce the model to a three-dimensional system, some of the computed manifolds were found to be of lower dimension than the corresponding manifolds in the full system. In this research, we develop techniques for computing the same manifolds in the full four-dimensional model in the interest of studying their geometry and interactions with each other. In particular, we focus on interactions between higher-dimensional manifolds that do not exist in systems of three dimensions or lower.

To better observe the separation between fast and slow variables, we use a change of coordinates described in [Kuehn & Szmolyan, 2015] and given by the system of ODEs

$$\begin{cases} \frac{dA}{dt} &= \mu - \alpha A - ABY, \\ \frac{dB}{dt} &= \varepsilon(1 - BX - ABY), \\ \frac{dX}{dt} &= \frac{1}{\eta}(BX - X^2 + 3ABY - \zeta X + \delta), \\ \frac{dY}{dt} &= \frac{\kappa}{\eta}(X^2 - Y - ABY), \end{cases} \tag{1}$$

where $(A, B, X, Y) \in \mathbb{R}^4$ are positive concentrations of chemicals. The system parameters given by Greek letters in Table 1 are functions of original system parameters given in [Olsen, 1983]. They were chosen to be the same as in [Kuehn & Szmolyan, 2015] with a minor modification. Note that for notational convenience, we have substituted ε_b and ε^2 [Kuehn & Szmolyan, 2015] with ε and η respectively.

Table 1. System parameters for system (1)

α	δ	ε	η	κ	μ	ζ
0.0912	1.2121e-04	0.0037	0.0540	3.7963	0.9697	0.9847

The classification of variables as either slow or fast is not straightforward for the Olsen model because the variables are not consistently slow or fast over all regions of phase space. In fact system (1) nominally has three different time scales. The time-scaling parameters ε and η depend on the original system parameter k_1. As suggested by [Kuehn & Szmolyan, 2015], we decrease k_1 past 0.16 to 0.1 so that there are only two time scales. We study a parameter regime corresponding to three fast variables, A, X and Y, and one slow variable, B.

2. The Main Text

Contributions are to be in English. Authors are encouraged to have their contribution checked for grammar. American spelling should be used. Abbreviations are allowed but should be spelt out in full when first used. Integers ten and below are to be spelt out. Italicize foreign language phrases (e.g. Latin, French).

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3.1. *Subheadings*

Subheadings should be typeset in bold italics, with the first letter of first word capitalized and the section number in boldface.

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Typeset in italics (section number to be in roman) and capitalize the first letter of the first word only.

3.2. *Numbering and spacing*

Sections, subsections and sub-subsections are numbered with Arabic numerals. Use double spacing after major and subheadings, and single spacing after sub-subheadings.

4. Lists of Items

Lists are broadly classified into four major categories that can randomly be used as desired by the author:

(a) Numbered list.
(b) Lettered list.
(c) Unnumbered list.
(d) Bulleted list.

4.1. *Numbered and lettered list*

(1) The `\begin{arabiclist}[]` command is used for the arabic number list (arabic numbers appearing within parenthesis), e.g., (1), (2), etc.
(2) The `\begin{romanlist}[]` command is used for the roman number list (roman numbers appearing within parenthesis), e.g., (i), (ii), etc.
(3) The `\begin{Romanlist}[]` command is used for the cap roman number list (cap roman numbers appearing within parenthesis), e.g., (I), (II), etc.
(4) The `\begin{alphlist}[]` command is used for the alphabetic list (alphabets appearing within parenthesis), e.g., (a), (b), etc.
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Note: For all the above mentioned lists (with the exception of alphabetic list), it is obligatory to enter the last entry's number in the list within the square bracket, to enable unit alignment.

4.2. *Bulleted and unnumbered list*

The `\begin{itemlist}` command is used for the bulleted list.

The `\begin{unnumlist}` command is used for creating the unnumbered list with the turnovers hangindent by 1 pica.

Lists may be laid out with each item marked by a dot:

- item one
- item two
- item three.

Items may also be numbered with lowercase Roman numerals:

(i) item one
(ii) item two
    (a) lists within lists can be numbered with lowercase Roman letters
    (b) second item.
(iii) item three
(iv) item four.

5. Theorems and Definitions

Input:

```
\begin{theorem}
We have $\# H^2 (M \supset N) < \infty$ for an inclusion ...
\end{theorem}
```

Output:

Theorem 1. *We have* $\# H^2(M \supset N) < \infty$ *for an inclusion* $M \supset N$ *of factors of finite index.*

Input:

```
\begin{theorem}[Longo, 1998]
For a given $Q$-system...
\[
N = \{x \in N; T x = \gamma (x) T, T x^* = \gamma (x^*) T\},
\]
and $E_\Xi (\cdot) = T^* \gamma (\cdot) T$ gives ...
\end{theorem}
```

Output:

Theorem 2 [Longo, 1998]. *For a given* Q*-system...*

$$N = \{x \in N; Tx = \gamma(x)T, Tx^* = \gamma(x^*)T\},$$

and $E_\Xi(\cdot) = T^*\gamma(\cdot)T$ *gives a conditional expectation onto* N*.*

5.1. *Proofs*

The WSPC document styles also provide a predefined proof environment for proofs. The proof environment produces the heading 'Proof' with appropriate spacing and punctuation. It also appends a 'Q.E.D.' symbol, ■, at the end of a proof, e.g.

```
\begin{proof}
This is just an example.
\end{proof}
```

to produce

Proof. This is just an example. ■

The proof environment takes an argument in curly braces, which allows you to substitute a different name for the standard 'Proof'. If you want to display, 'Proof of Lemma', then write e.g.

```
\begin{proof}[Proof of Lemma]
This is just an example.
\end{proof}
```

produces

Proof. [Proof of Lemma] This is just an example. ■

6. Equations

Displayed equations should be numbered consecutively in each section, with the number set flush right and enclosed in parentheses:

$$\mu(n,t) = \frac{\displaystyle\sum_{i=1}^{\infty} 1(d_i < t, N(d_i) = n)}{\displaystyle\int_{\sigma=0}^{t} 1(N(\sigma) = n)d\sigma} \,. \tag{2}$$

Equations should be referred to in abbreviated form, e.g. "Eq. (2)" or "(2)". In multiple-line equations, the number should be given on the last line.

Displayed equations are to be centered on the page width. Standard English letters like x are to appear as x (italicized) in the text if they are used as mathematical symbols. Punctuation marks are used at the end of equations as if they appeared directly in the text.[1]

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Figures are to be inserted in the text nearest their first reference. Please send one set of originals with copies. If the publisher is required to reduce the figures, ensure that the figures (including lettering and numbers) are large enough to be clearly seen after reduction.

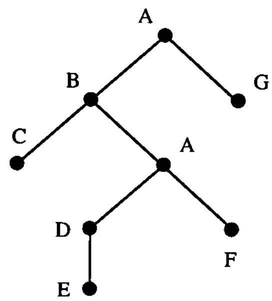

Fig. 1. Labeled tree T.

Figures are to be sequentially numbered with Arabic numerals. The caption must be placed below the figure. For those figures with multiple parts which appear on different pages, it is best to place the full caption below the first part, and have e.g. "Fig. 1 (*continued*)" below the last part. Typeset in 9 pt Times Roman with baselineskip of 11 pt. Use double spacing between a caption and the text that follows immediately.

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Very large figures and tables should be placed on a separate page by themselves. Landscape tables and figures can be typeset with the following environments:

- `sidewaystable` and
- `sidewaysfigure`.

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If tables need to extend over to a second page, the continuation of the table should be preceded by a caption, e.g. "Table 1 (*continued*)".

By using `\tbl` command in table environment, long captions will be justified to the table width while the short or single line captions are centered. `\tbl{table caption}{tabullar environment}`.

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[1]Sample footnote

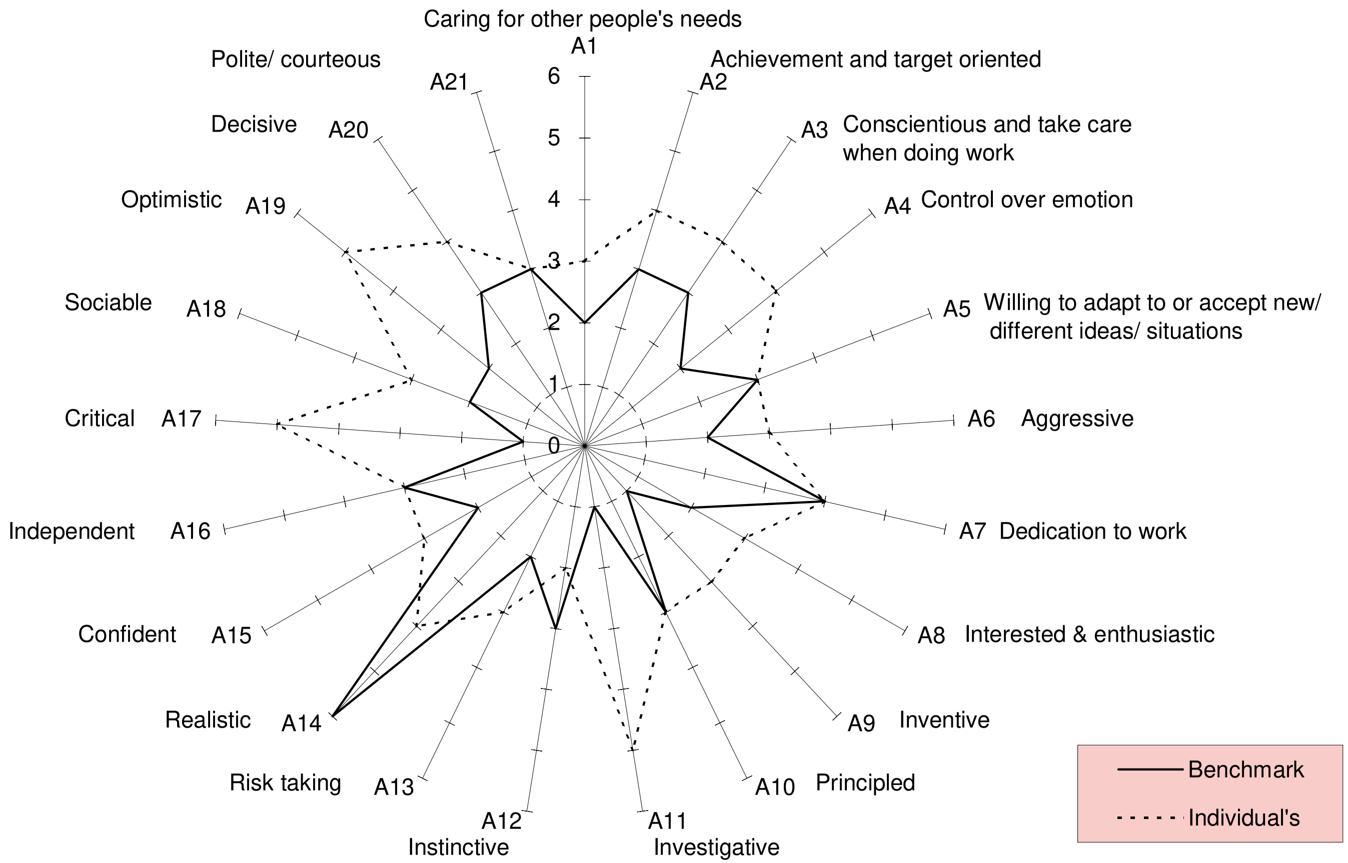

Fig. 2. The bifurcating response curves of system $\alpha = 0.5, \beta = 1.8; \delta = 0.2, \gamma = 0$: (a) $\mu = -1.3$; and (b) $\mu = 0.3$.

Table 2. Number of tests for WFF triple NA = 5, or NA = 8.

		3	4	8	10
	3	1200	2000	2500	3000
NC	5	2000	2200	2700	3400
	8	2500	2700	16000	22000
	10	3000	3400	22000	28000

toprule one rule at the top
colrule one rule separating column heads from data cells
botrule one bottom rule
Hline one thick rule at the top and bottom of the tables with multiple column heads

To avoid the rules sticking out at either end of the table, add `@{}` before the first and after the last descriptors, e.g. @llll@. Please avoid vertical rules in tables. But if you think the vertical rule is a must, you can use the standard LaTeX `tabular` environment.

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9. Cross-references

Use `\label` and `\ref` for cross-references to equations, figures, tables, sections, subsections, etc., instead of plain numbers. Every numbered part to which one wants to refer, should be labeled with the instruction `\label`. For example:

```
\begin{equation}
\mu(n, t)=\frac{\sum\limits^\infty_{i=1}1 (d_i < t, N(d_i)=n)}
{\int\limits^t_{\sigma=0}1 (N(\sigma)=n)d\sigma}.\label{aba:eq1}
\end{equation}
```

With the instruction `\ref` one can refer to a numbered part that has been labeled:

```
..., see also Eq. (\ref{aba:eq1})
```

The `\label` instruction should be typed

- immediately after (or one line below), but not inside the argument of a number-generating instruction such as `\section` or `\caption`, e.g.: `\caption{ ... caption ... }\label{aba:fig1}`.
- roughly in the position where the number appears, in environments such as an equation,
- labels should be unique, e.g., equation 1 can be labeled as `\label{aba:eq1}`, where '`aba`' is author's initial and '`eq1`' the equation number.

10. References

References cited in the text should be placed within square brackets and stated as [surname of author(s), year of publication], e.g.. If the reference reads as part of the sentence, the square brackets enclose only the year of publication, e.g., "According to Golub and Van Loan [1989], . . . "

Note Added

A note can be added before Acknowledgments.

Acknowledgments

This part should come before References. Funding information may also be included here.

Appendices

Appendices should be used only when absolutely necessary. They should come immediately before References.

Appendix A

If there is more than one appendix, number them alphabetically.

$$\mu(n,t) = \frac{\displaystyle\sum_{i=1}^{\infty} 1(d_i < t, N(d_i) = n)}{\displaystyle\int_{\sigma=0}^{t} 1(N(\sigma) = n) d\sigma} \,. \tag{A.1}$$

Number displayed equations occurring in the appendix as (A.1), (A.2), etc.

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