Mathematical interclude: Stopping times and stopped processes

Final Assignment

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Stopped Times

In the realm of probability theory, particularly in the examination of stochastic processes, a stopping time, alternatively known as a Markov time, Markov moment, optional stopping time, or optional time, refers to a specific category of "random time." This random variable is construed as the moment at which a given stochastic process demonstrates a particular noteworthy behavior. Typically, a stopping time is determined by a stopping rule, a mechanism that assesses whether to continue or stop a process based on the current position and past occurrences. This rule generally results in a decision to halt at some definite point in time.

Stopping times are relevant in decision theory, and the optional stopping theorem holds significance in this domain. Moreover, in mathematical proofs, stopping times are frequently employed to "constrain the continuum of time," as articulated by Chung in his 1982 book.

Definition 1.1 (In fact definition-proposition)

Given a probability space $(\Omega, \mathcal{F}, P)$, a random variable $\tau : \Omega \to \mathbb{N} \cup \{\infty\}$ is a stopping time for a filtration $(\mathcal{F}_n)_{n\geq 0}$ if it satisfies any (and hence all) of the following equivalent properties:

(i) $\{\tau = n\} \in \mathcal{F}_n$ for all $n \in \mathbb{N}$.

(ii) $\{\tau \leq n\} \in \mathcal{F}_n$ for all $n \in \mathbb{N}$.

(iii) $\{\tau > n\} \in \mathcal{F}_n$ for all $n \in \mathbb{N}$.

In simpler terms, a stopping time is a concept where the decision to stop at a specific time n (not stopping beyond n, not stopping before n) is based on the information available up to time $\mathcal{F}_n$ (information "at time n"). Importantly, this decision is made without considering information from the future ("$future$" information) contained in subsequent σ-algebras of the filtration. The definition is presented in a triple form to provide greater flexibility for proofs. It's worth noting that we permit stopping times such as $\tau = 0$ (for instance, an investor exercising an option immediately upon purchase) and $\tau = \infty$ (for instance, an investor choosing not to exercise the option). However, the latter possibility is not pertinent to the stopping character of τ.

Proof of the equivalence of (i)–(iii)

Proof of (i) ⇒ (ii)

$$\{\tau \leq n\} = \bigcup_{k=0}^{n} \{\tau = k\} \quad (1.1.1)$$

By (i), each event on the right belongs to $\mathcal{F}_k \subset \mathcal{F}_n$, hence their union belongs to $\mathcal{F}_n$.

(ii) ⇒ (iii)

$$\{\tau > n\} = \{\tau \leq n\}^c \quad (1.1.2)$$

The event on the right is in $\mathcal{F}_n$ by (ii) and the fact that σ-algebras are closed under complements.

(iii) ⇒ (i)

$$\{\tau = n\} = \{\tau > n\} \setminus \{\tau > n-1\} \quad (1.1.3)$$

Both events on the right are in $\mathcal{F}_n$ by (iii) and the fact that $\mathcal{F}_{n-1} \subset \mathcal{F}_n$. Hence, so is their difference.

Example

Consider a two-period binary market, which only contains situations of H and T. Then $\Omega = \{(\omega_1, \omega_2) : \omega_i \in \{H, T\}\}$ and the filtration is formed only by $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2 = \mathcal{F}$. As there are only three times in this small universe — namely t_0, t_1, t_2 — then possible stopping times are functions $\tau : \Omega \to \{0, 1, 2, \infty\}$. Let us consider the functions in Table 6.1.1 below and decide whether they are or not stopping times.

τ	HH	HT	TH	TT
τ_1	1	1	1	1
τ_2	1	1	2	2
τ_3	∞	2	1	1
τ_4	1	2	2	2

Table 1.1.1: A few stopping policies for a 2-period binary market.

Let us first try to understand which of them are stopping times by describing, in words, the stopping strategy. This is not a proof (the proof will come later), but it helps to develop understanding. We complete the previous table.

After thinking a little, we conclude that the first three times use the information available at each instant, hence they seem to be stopping times. In contrast, the policy in τ_4 requires "looking into the future" to see if there is a second H coming up. This policy does not seem to define a stopping time.

Let us now see the rigorous analysis. We shall use the criterion $\{\tau = n\} \in \mathcal{F}_n$. We point out that only the cases $n = 0, 1$ matter because the condition $\{\tau = 2\} \in \mathcal{F}_2 = \mathcal{F}$ is automatically true because all events are $\mathcal{F}_2$-measurable. Let us recall that

$$\mathcal{F}_0 = \{\emptyset, \Omega\}, \quad \mathcal{F}_1 = \{\emptyset, \mathcal{C}_H, \mathcal{C}_T, \Omega\},$$

where $\mathcal{C}_H = \{(HH), (HT)\}$ and $\mathcal{C}_T = \{(TH), (TT)\}$.

$$\mathcal{F}_2 = \{\emptyset, \{(HH)\}, \{(HT)\}, \{(TH)\}, \{(TT)\}, \mathcal{C}_H, \mathcal{C}_T, \{(HT), (TH)\}, \{(TH), (HT)\}, \\ \{(HH), (HT), (TT)\}, \ldots, \Omega\} \tag{1}$$

τ	HH	HT	TH	TT	Policy
τ_1	1	1	1	1	Stop at t_1 no matter what
τ_2	1	1	2	2	Stop at the first H or at the end
τ_3	∞	2	1	1	Stop at the first T
τ_4	1	2	2	2	Stop at the first H only if followed by a second H, otherwise at the end

Table 1.1.2: Stopping policies in words

τ	HH	HT	TH	TT	$\{\tau = 0\} \in \mathcal{F}_0$	$\{\tau = 1\} \in \mathcal{F}_1$	Conclusion
τ_1	1	1	1	1	$\emptyset \in \mathcal{F}_0$	$\Omega \in \mathcal{F}_1$	Stopping time
τ_2	1	1	2	2	$\emptyset \in \mathcal{F}_0$	$\{(HH), (HT)\} = \mathcal{C}_H \in \mathcal{F}_1$	Stopping time
τ_3	∞	2	1	1	$\emptyset \in \mathcal{F}_0$	$\{(TH), (TT)\} = \mathcal{C}_T \in \mathcal{F}_1$	Stopping time
τ_4	1	2	2	2	$\emptyset \in \mathcal{F}_0$	$\{(HH)\} \notin \mathcal{F}_1$	Not a stopping time

Table 1.1.3: Rigorous verification of the stopping character

We see that, as we suspected, the first three times satisfy the conditions $\{\tau = 0\} \in \mathcal{F}_0$ and $\{\tau = 1\} \in \mathcal{F}_1$. The fact that τ_4 violates the second condition proves that it is not a stopping time.

Definition 1.2 Stopped Process

Consider a probability space $(\Omega, \mathcal{F}, P)$, a filtration $F = (F_n)_n$, and a stopping time τ. Given an adapted stochastic process $X = (X_n)_n$, the (τ)-stopped (X)-process (or frozen process) is the stochastic process X^τ defined by

$$X_n^\tau = \begin{cases} X_\tau, & \text{if } n \geq \tau \\ X_n, & \text{if } n \leq \tau \end{cases} = X_{min\{n,\tau\}} \quad (1.2.1)$$

Since $\{\tau = k\} \in \mathcal{F}_k \subset \mathcal{F}_n$ for $0 \leq k \leq n$, we can rewrite the stopped process X_n^τ as:

$$X_n^\tau = X_n \mathbb{1}_{[\tau > n]} + \sum_{k=0}^{n} X_k \mathbb{1}_{[\tau = k]} \quad (1.2.2)$$

Then, let Θ_n be denoted as $\Theta_n = \mathbb{1}_{\{\tau > n\}}$ and let X be an adapted process and τ a stopping time.

Then we have

$$X_n^\tau - X_{n-1}^\tau = \Theta_{n-1} \cdot (X_n - X_{n-1}) \quad (1.2.3)$$

thus

$$X_n^\tau = X_0 + \sum_{k=1}^{n} \Theta_{k-1} \cdot (X_k - X_{k-1}) \quad (1.2.4)$$

Proof for 1.2.3

Proof. According to equation (1.2.4),

$$\begin{aligned} X_n^\tau - X_{n-1}^\tau &= X_n \mathbb{1}_{[\tau > n]} + \sum_{k=0}^{n} X_k \mathbb{1}_{[\tau = k]} - (X_{n-1} \mathbb{1}_{[\tau > n-1]} + \sum_{k=0}^{n-1} X_k \mathbb{1}_{[\tau = k]}) \\ &= X_n \mathbb{1}_{[\tau \geq n]} - X_{n-1} \mathbb{1}_{[\tau > n-1]} \end{aligned} \quad (2)$$

Then we study separately the scenarios that $\tau \geq n$ and $\tau \leq n - 1$:

1. $\tau \geq n$

$$X_n^\tau - X_{n-1}^\tau = X_n * 1 - X_{n-1} * 1 = \Theta_{n-1} \cdot (X_n - X_{n-1})$$

2. $\tau \leq n - 1$

$$X_n^\tau - X_{n-1}^\tau = X_n * 0 - X_{n-1} * 0 = \Theta_{n-1} \cdot (X_n - X_{n-1})$$

We have that in both cases, equation (1.2.3) stands. □

Now, consider the stochastic process where $\underline{X} = (X_n)_n$ and τ is stopping time. We have that, if

$$\underline{X} \text{ is a} \begin{cases} \text{sub-martingale} \\ \text{martingale} \\ \text{super-martingale} \end{cases}$$

then,

a)

$$\underline{X^\tau} \text{ is also a} \begin{cases} \text{sub-martingale} \\ \text{martingale} \\ \text{super-martingale} \end{cases}$$

b)

$$E(\underline{X_n^\tau}) \begin{cases} \leq \\ = \\ \geq \end{cases} E(X_n)$$

Proof.

a) According to equation (1.2.3), we have that

$$X_n^\tau = X_{n-1}^\tau + \Theta_{n-1} \cdot (X_n - X_{n-1}) \quad (1.2.5).$$

Take conditional expectations for both sides of equation (1.2.5) and by TOWIK, $E(M_n \mid \mathcal{F}_n) = M_n$:

$$\begin{aligned} E\left[X_{n+1}^\tau \mid \mathcal{F}_n\right] &= E\left[X_n^\tau + \Theta_n \cdot (X_{n+1} - X_n) \mid \mathcal{F}_n\right] && (3) \\ &= X_n^\tau + \Theta_n \left[E(X_{n+1} \mid \mathcal{F}_n) - X_n\right]. && (4) \end{aligned}$$

We know that$\Theta_n \geq 0$, the sign of the second term on the right is the sign of the term $E(X_{n+1} \mid \mathcal{F}_n) - X_n$, thus we can divide this into 3 scenarios:

i. $\underline{X}$ is a sub-martingale:
We have that $E(X_{n+1} \mid \mathcal{F}_n) > X_n$ and $E(X_{n+1} \mid \mathcal{F}_n) - X_n > 0$. Thus, $E\left[X_{n+1}^\tau \mid \mathcal{F}_n\right] > X_n^\tau$, and $\underline{X^\tau}$ is also a sub-martingale.

ii. $\underline{X}$ is a martingale:
We have that $E(X_{n+1} \mid \mathcal{F}_n) = X_n$ and $E(X_{n+1} \mid \mathcal{F}_n) - X_n = 0$. Thus, $E\left[X_{n+1}^\tau \mid \mathcal{F}_n\right] = X_n^\tau$, and $\underline{X^\tau}$ is also a martingale.

iii. $\underline{X}$ is a super-martingale:
We have that $E(X_{n+1} \mid \mathcal{F}_n) < X_n$ and $E(X_{n+1} \mid \mathcal{F}_n) - X_n < 0$. Thus, $E\left[X_{n+1}^\tau \mid \mathcal{F}_n\right] < X_n^\tau$, and $\underline{X^\tau}$ is also a super-martingale.

b) Proof By Induction:

Base Case: For $n = 0$, $X_0^\tau = X_{min\{0,\tau\}} = X_0$. $E(\underline{X_n^\tau}) = E(X_0)$, the statement holds.

Induction: Assume the statement is true for $n = k$ $(k \geq 0)$. Then we by taking expectations for both sides of equation (1.2.5), we have:

$$\begin{aligned}
E(X_{k+1}^\tau) &= E(X_k^\tau + \Theta_k \cdot (X_{k+1} - X_k)) \\
&= E(X_k^\tau) + E(\Theta_k \cdot (X_{k+1} - X_k)) \\
&\begin{cases} \leq \\ = \quad E(X_k) + E(\Theta_k(X_{k+1} - X_k)) \\ \geq \end{cases} \\
&= E[\Theta_k X_{k+1} + (1 - \Theta_k) X_k] \qquad (5) \\
&\begin{cases} \leq \\ = \quad E(\Theta_k X_{k+1}) + E((1 - \Theta_k) E(X_{k+1} | \mathcal{F}_k)) \\ \geq \end{cases} \\
&= E(\Theta_k X_{k+1} + (1 - \Theta_k) X_{k+1}) \\
&= E(X_{k+1}).
\end{aligned}$$

Thus, $E(\underline{X_{k+1}^\tau}) \begin{cases} \leq \\ = \\ \geq \end{cases} E(X_{k+1})$, the statements holds for $n = k + 1$.

Conclusion: If $\underline{X}$ is a $\begin{cases} \text{sub-martingale} \\ \text{martingale} \\ \text{super-martingale} \end{cases}$, $E(\underline{X_n^\tau}) \begin{cases} \leq \\ = \\ \geq \end{cases} E(X_n)$

□