Project 2

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1. Bivariate-Normal

Input : a

Output: -0.163864

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The simulated correlation rho is: -0.1638640937610348
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2. Evaluate the following expected values by using Monte Carlo simulation:

Input: ρ = 0.6

Output: 1.5372813821388835

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The Expected Value with 0.6 correlation is: 1.5372813821388835
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3.

(a) The estimated value by simulation:

	A1	A3	A5	B1	B3	B5
Expected Value	1.077572	3.141554	5.195038	0.985553	0.981528	0.937837
Variance	2.759930	20.988268	57.130256	0.567152	9.081625	77.199072

(b)

The B(t) values of t=1,3,5 are related that they are all around 1 and the reason is the equation is a maritingale.

For the equation : $e^{\frac{t}{2}}cos(W_t)$, if we apply Ito's lemma $f(t, B_t) = \frac{\partial f}{\partial t}d_t + \frac{\partial f}{\partial x}dB_t + \frac{1}{2}\frac{\partial^2 f}{\partial x^2}(dB_t)^2$

Let $Y(t, B_t) = e^{\frac{t}{2}}cos(W_t)$, we have

$dy_t = \frac{1}{2}e^{\frac{1}{2}}cos(W_t)dt + e^{\frac{1}{2}}(-sin(W_t))dW_t + \frac{1}{2}e^{\frac{1}{2}}(-cos(W_t))(dW_t)^2$

= $\frac{1}{2}e^{\frac{1}{2}}cos(W_t)dt + e^{\frac{1}{2}}(-sin(W_t))dW_t - \frac{1}{2}e^{\frac{1}{2}}cos(W_t)d_t$

= $e^{\frac{1}{2}}(-sin(W_t))dW_t$ = $-e^{\frac{1}{2}}sin(W_t)dW_t$

$Y_T - Y_0 = \int_0^T -e^{\frac{1}{2}}sin(W_t)dW_t$, thus: $Y_T = Y_0 - \int_0^T e^{\frac{1}{2}}sin(W_t)dW_t$

$Y_0 = e^0 cos(0) = 1$

$Y_T = 1 - \int_0^T e^{\frac{1}{2}}sin(W_t)dW_t$, which is a martingale.

Finally, if we take the expectation of Y_T, we see the expectation is always 1.

$E[Y_T] = E[1 - \int_0^T e^{\frac{1}{2}}sin(W_t)dW_t] = 1 - 0 = 1$

(c) Expected value after variance control:

Here I apply the Antithetic Variates Reduction Method.

As we can see for A, the variance changes from 57.130 to 55.678, which does improve a little bit, but it cannot be considered truly effective. However, the variance of B5 decreases a lot, from 77.199 to 55.666, which can be seen as a big improvement.

	A5	A5_after	B5	B5_after
Expected Value	5.195038	5.375014	0.937837	0.992073
Variance	57.130256	55.677837	77.199072	55.666178

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(a) The call price estimate call price by Monte Carlo simulation is 17.981594283190127.

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The estimate the price of this European Call option is: 17.981594283190127
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(b) The call price calculated by Black-Scholes is 18.28376570485581.

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The Black-Scholes price of this European Call option is: 18.28376570485581
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(c) The call option price after using the Antithetic Variates reduction technique is 18.213360.

- The standard deviation before variance control is 34.29
- The standard deviation after variance control is 21.78
- We can see that we reduce the variance significantly. At the same time, this method improves the accuracy of simulation, it approaches the value calculated by Black-Scholes 18.284

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The price of this European Call option is with variance control is: 18.213359908553063
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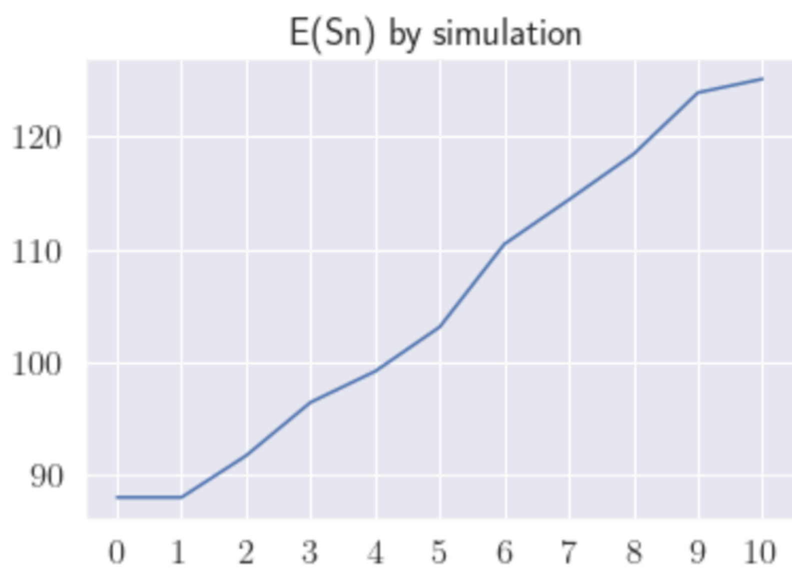
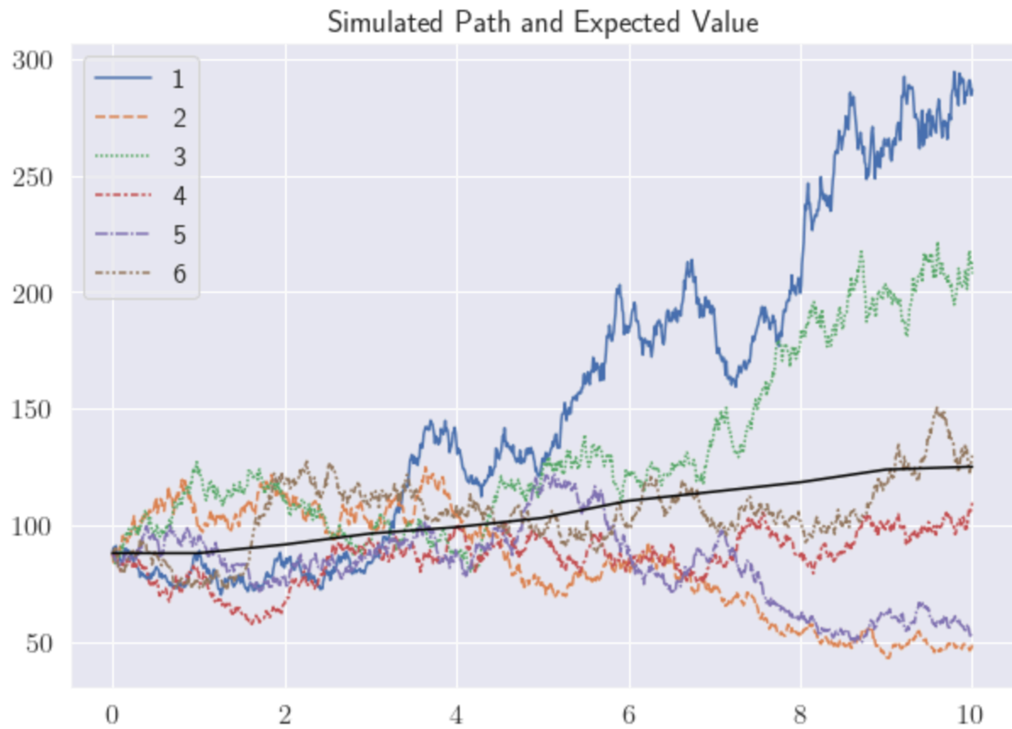
	Before Variance Control	After Variance Control
Price	17.981594	18.213360
Stdev	34.291620	21.777919

5.

(a) Plot of $E(S_n)$

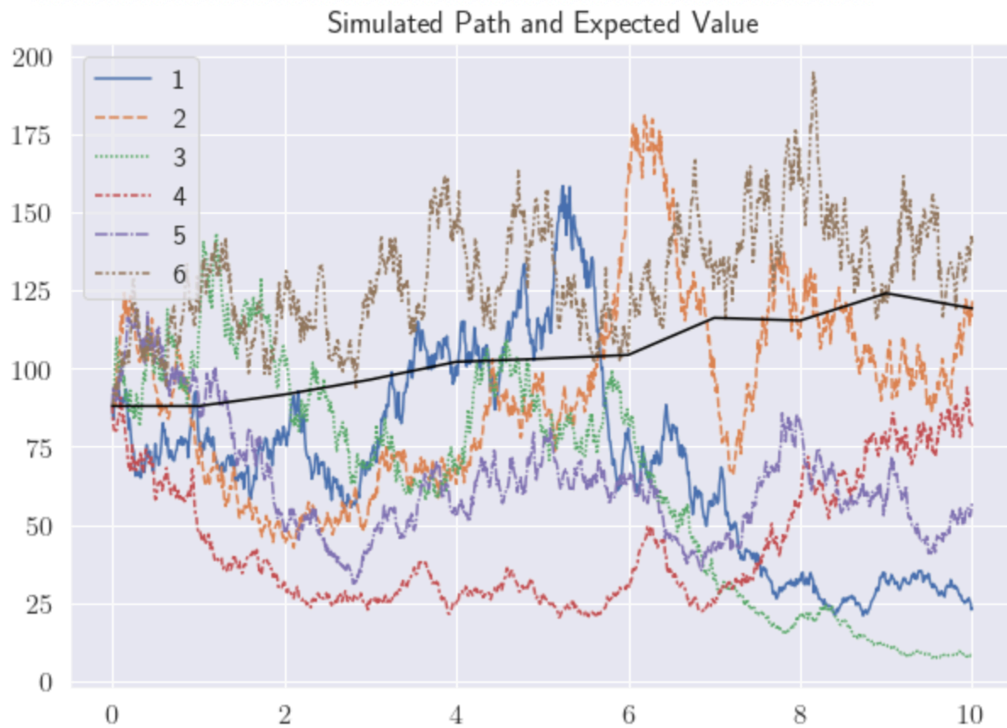


(b)(c) Plot of 6 paths and $E(Sn)$



(d) Plot of 6 paths and $E(Sn)$ If we change σ from 18% to 35%.

- As we change the volatility from 18% to 35%, the expected value theoretically would not change, but the graph shows it actually becomes more fluctuated
- The variance of simulated 6 paths are way more than smaller volatility ones.



6.

(a)Euler Discretization

$\frac{dy}{dx} = f(y, t), andy(0) = y_0$, we can divide $[0, T]$ to N equal parts so that $\tau = \frac{T}{N}$ as step.

For this equation, we have : $4[\int_0^1 \sqrt{1-x^2}\, dx]$ and for simplicity we let y = $4\sqrt{1-x^2}$

According to the formula $y_t = y_o \ + [\int_{t_0}^{t} \ f(y_\tau, \tau)d_\tau]$, we can know

$y_{t_1} = y_o \ + [\int_{t_0}^{t_1} \ f(y_\tau, \tau)d_\tau]$

$y_{t_1} \approx y_1 = y_0 + f(y_0, t_0)\tau$

$y_{n+1} = y_n + f(y_n, t_n)\tau,\ n = 0, 1, 2 \cdots N - 1$

We now implement this idea to get the estimated value: 3.14355

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Estimate value by using the Euler’s discretization scheme is: 3.143555466911028.
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(b) Estimate the integral by Monte Carlo simulation.

The estimated value by Monte Carlo simulation is : 3.13555

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Estimate value by using the Monte Carlo simulation is: 3.1355512852508935.
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(c) Importance Sampling Method

$\theta = 4[\int_0^1 \sqrt{1 - x^2}\, dx] = \pi$

$= E\left[4\sqrt{1 - u^2}\right]$

Let $x = U,\ g(x) = 4\sqrt{1 - x^2}$

$t(\cdot) = \begin{cases} \frac{1 - ax^2}{1 - \frac{a}{3}} & \text{for } [0, 1] \\ 0 & else \end{cases}$

$E_{t(\cdot)}\left[\frac{4\sqrt{1-y^2}}{t(y)}\right] \approx \frac{1}{N} \sum_{i=1}^{N} \frac{4\sqrt{1-y^2}}{t(y)}$

1. First, I try to get the alpha value by finding the minimum variance, and the result is 0.76, slightly different form the true optimal value of 0.74, therefore, I apply 0.74 in the following calculation.
2. Second, I use the acceptance - rejection method to simulate random variable with uniform distribution with t(·)
3. Third, I reestimate with the sampling random variable generated from step 2.
4. Finally, I compare the expected value and variance, with and without variance control.

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   Estimate value by with importance sampling is: 3.149794491445853.
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   The estimated value by with importance sampling is : 3.149794491445853.

   - The variance before importance sampling is 0.856790
   - The variance after importance sampling is 0.191652
   - We can observe that the variance has decreased significantly with variance control. However,

in practice, the accuracy after importance sampling seems not appealing. If we keep generating different samples with accept-reject method, we might get values like 3.16, 3.15, even though the variance still remain small.

	Expected Value	Variance
Before Importance Sampling	3.135551	0.856790
After Importance Sampling	3.149794	0.191652