Least Squares Explanation

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1 Least Squares

To compute the least squares method in our project, we used the mathematical model,

$$Y = X\beta + \epsilon, \tag{1}$$

where Y is the vector of tidal heights, X is the design matrix with our predictor variables, β is the vector of unknowns (h_0,a and b) and ϵ is the error term.

For our tidal model with 16 observations, for each of the ports, (t_i,h_i), we write,

$$\begin{bmatrix} h_1 \\ h_2 \\ \vdots \\ h_{16} \end{bmatrix} = \begin{bmatrix} 1 & \cos(\Omega t_1) & \sin(\Omega t_1) \\ 1 & \cos(\Omega t_2) & \sin(\Omega t_2) \\ & \vdots & \\ 1 & \cos(\Omega t_{16}) & \sin(\Omega t_{16}) \end{bmatrix} \begin{bmatrix} h_0 \\ a \\ b \end{bmatrix} + \begin{bmatrix} \epsilon_1 \\ \vdots \\ \epsilon_{16} \end{bmatrix}. \tag{2}$$

The least squares method minimises the residual errors and gives us the estimate for β,

$$\hat{\beta} = (X^T X)^{-1} XY, \tag{3}$$

where X' denotes the transpose of matrix X and $(X^T X)^{-1}$ denotes the inverse of $X^T X$. Implementing this method with data from TideTimes for Donaghadee and Bristol we obtained the following estimates of h_0,a and b as:

$$\text{Bristol} = \begin{bmatrix} h_0 \\ a \\ b \end{bmatrix} = \begin{bmatrix} 7.2309 \\ -1.4130 \\ -5.1554 \end{bmatrix}, \tag{4}$$

and,

$$\text{Donaghadee} = \begin{bmatrix} h_0 \\ a \\ b \end{bmatrix} = \begin{bmatrix} 2.4116 \\ 1.8208 \\ -3.0158 \end{bmatrix} \tag{5}$$

We can then use these estimates in Equations (4) and (5) to create a mathematical model for the tidal height in both locations. The following is the model for Bristol:

$$h_B(t) = 7.2309 - 1.4130\cos(\Omega t) - 5.1554\sin(\Omega t) \tag{6}$$

We can also create the model for predicting Donaghadee's tidal heights. The following is the model for Donaghadee:

$$h_B(t) = 2.4116 + 1.8208\cos(\Omega t) - 3.0158\sin(\Omega t) \tag{7}$$

From Equation (4) and (5), we can initially see that Bristol's average tide height (h_0) is significantly bigger than Donaghadee's average tide height (h_0). This may suggest that either Donaghadee's or Bristol's tidal behaviour is an outlier.