Geometric Objects

4TH WEEK, 2021

Basic Elements

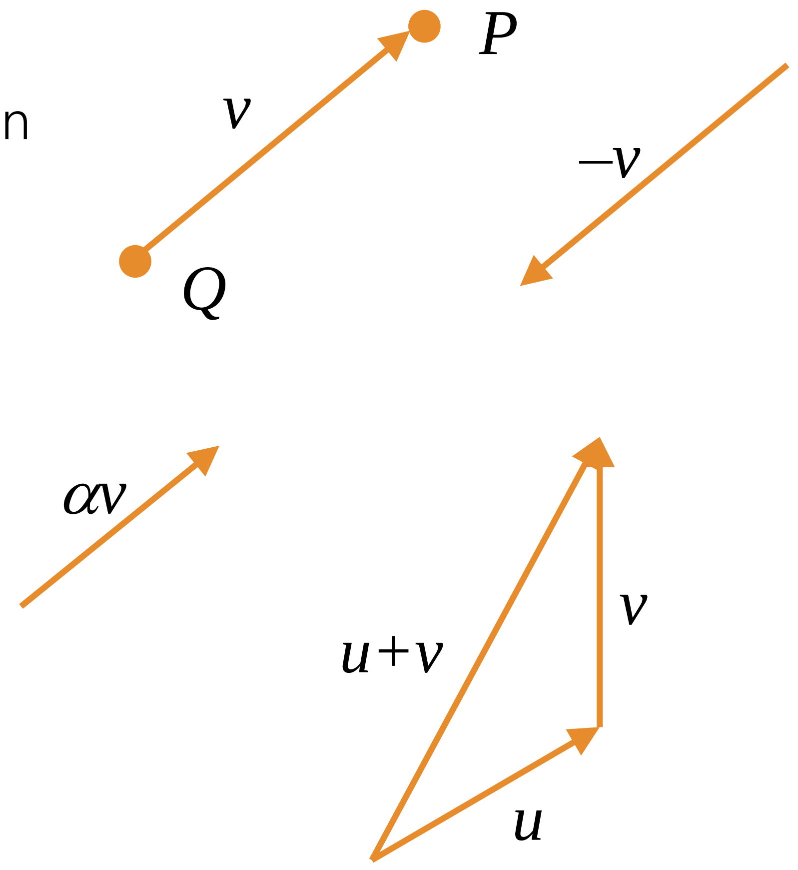
- Geometry – the study of the relationships among objects in an n-dimensional space
  - In computer graphics, objects exist in three dimensions
- Minimum set of primitives from which we can build more sophisticated objects
- Three basic elements
  - Points
  - Scalars
  - Vectors

Scalars, Points, and Vectors

- Points
  - Position in space
- Scalars
  - To specify quantities such as the distance between two points
  - Ex) real numbers, complex numbers
- Vectors
  - Any quantity with direction and magnitude
  - Ex) velocity, force

Vector Operations

- Every vector has an inverse
  - Same magnitude but points in opposite direction
- Every vector can be multiplied by a scalar
- There is a zero vector
  - Zero magnitude, undefined orientation
- The sum of any two vectors is a vector
  - Using head-to-tail axiom



Mathematical View (1)

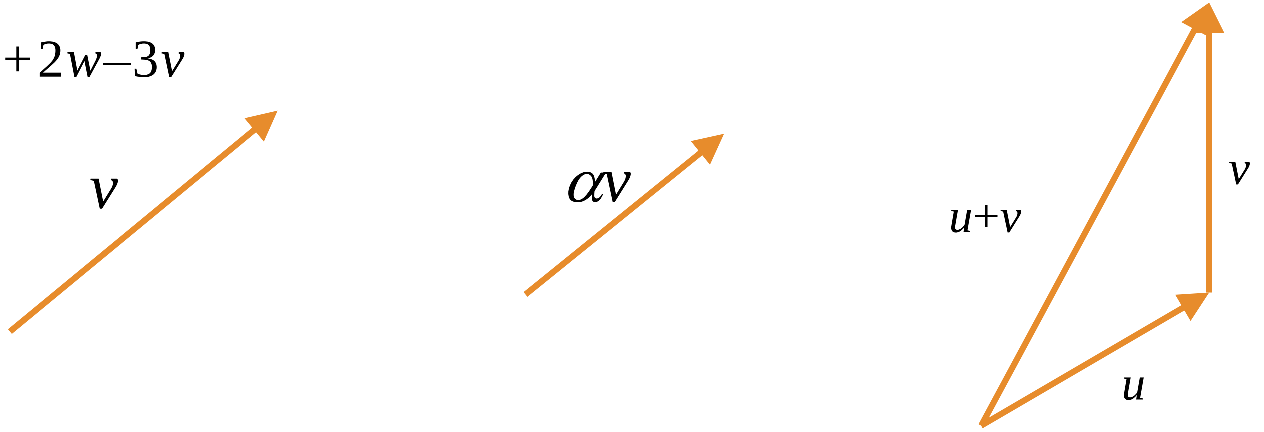
- Scalar field
  - Entities: scalars
  - Operations: addition and multiplication
- (Linear) Vector space
  - Entities: vectors and scalars
  - Operations: vector-vector addition, vector-scalar multiplication
- Euclidean space
  - Entities: vectors and scalars
  - Operations: vector-vector addition, vector-scalar multiplication, measure of size or distance

Mathematical View (2)

- Affine space
  - Entities: vectors, scalars, and points
  - Operations: vector-vector addition, vector-scalar multiplication, vector-point addition, point-point subtraction

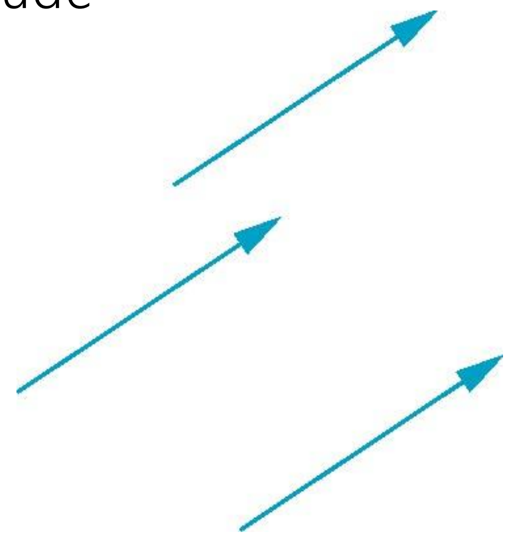
Linear Vector Spaces

- Mathematical system for manipulating vectors
- Operations
  - Scalar-vector multiplication: $u=\alpha v$
  - Vector-vector addition: $w=u+v$
- Expression
  - Ex) $v=u+2w-3v$



Vector Lack Position

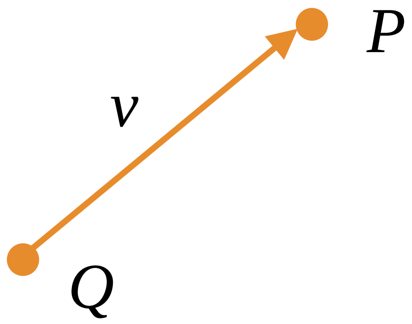
- These vectors are identical
  - Same direction and magnitude



- Vector spaces are insufficient for geometry
  - Need points

Points

- Location in space
- Operations allowed between points and vectors
  - Point-point subtraction yields a vector
  - Equivalent to point-vector addition



$$v = P - Q$$

$$P = v + Q$$

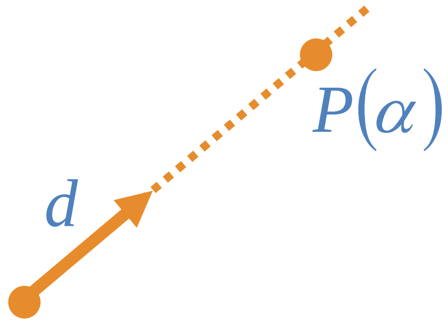
Affine Spaces

- Vector space + points
- Operations
  - Scalar-scalar operations
  - Scalar-vector multiplication
  - Vector-vector addition
  - Vector-point addition
  - Point-point subtraction

Lines

- Considering all points of the form
  - $P(\alpha)=P_0+\alpha d$
  - Set of all points that pass through P_0 in the direction of the vector d

$$\boxed{P(\alpha)=P_0+\alpha d}$$



Affine Sums

- In affine space
  - O: vector-vector addition, vector-scalar multiplication, vector-point addition
  - X: <u>addition</u> of two points, point-<u>scalar</u> multiplication
- <u>Affine addition</u>

$$P(\alpha) = Q + \alpha v$$

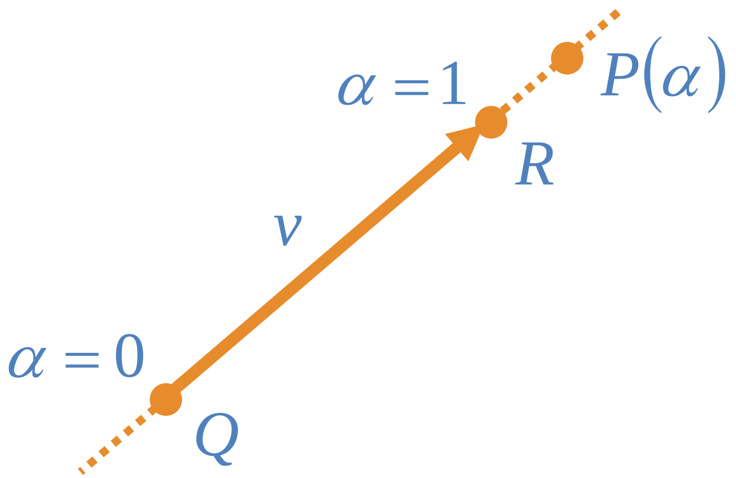
$$v = R - Q$$

$$P(\alpha) = Q + \alpha(R - Q)$$

$$= \alpha R + (1-\alpha)Q$$

$$\boxed{P = \alpha_1 R + \alpha_2 Q}$$

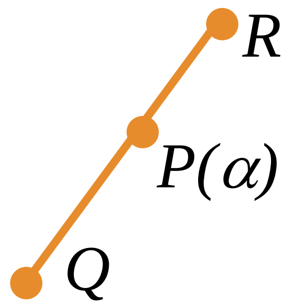
$$\text{where } \alpha_1 + \alpha_2 = 1$$



Convexity

- Convex object
  - Any point lying on the line segment connecting any two points in the object is also in the object
- Using affine sums
  - Convex object such as line segment

$$P(\alpha) = \alpha R + (1-\alpha)Q \quad \text{where} \quad 0 \le \alpha \le 1$$

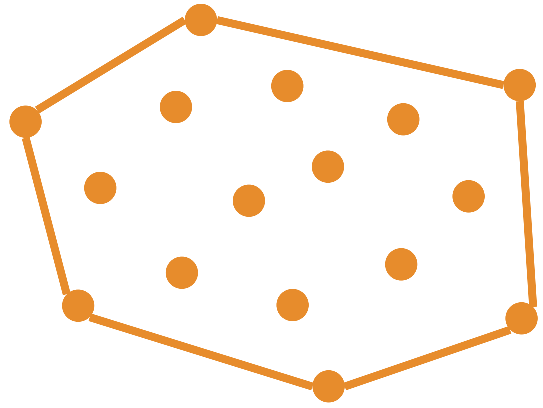


  - Convex hull

$$P = \alpha_1 P_1 + \alpha_2 P_2 + \ldots + \alpha_n P_n$$

$$\alpha_1 + \alpha_2 + \ldots + \alpha_n = 1$$

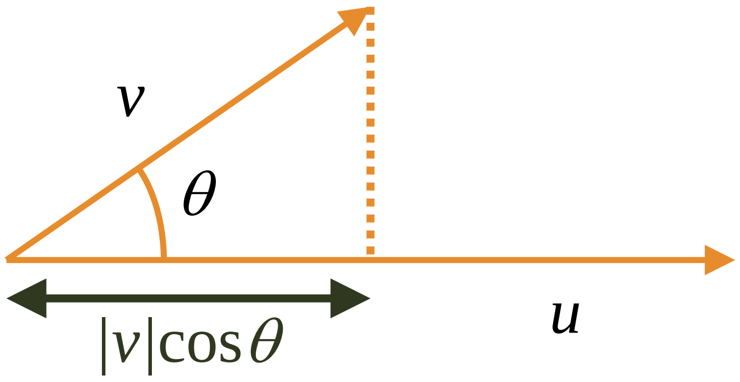
$$\alpha_i \ge 0, \quad i = 1, 2, \ldots, n$$



Dot Products

- Inner products – $u \cdot v$
  - Orthogonal: $u \cdot v = 0$
  - Magnitude of a vector: $|u|^2 = u \cdot u$
  - Angle between two vectors: $\cos\theta = \dfrac{u \cdot v}{|u||v|}$
  - Orthogonal projection: $|v|\cos\theta = u \cdot v / |u|$

$$\begin{pmatrix} x_1 & y_1 & z_1 \end{pmatrix} \cdot \begin{pmatrix} x_2 & y_2 & z_2 \end{pmatrix} = x_1 x_2 + y_1 y_2 + z_1 z_2$$

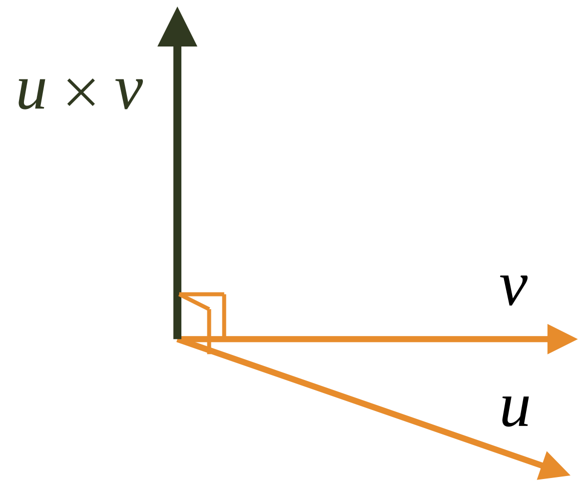


Cross Products

- Outer products – $u \times v = n$
  - Right-handed coordinates system
    - Direction of the thumb of the right hand

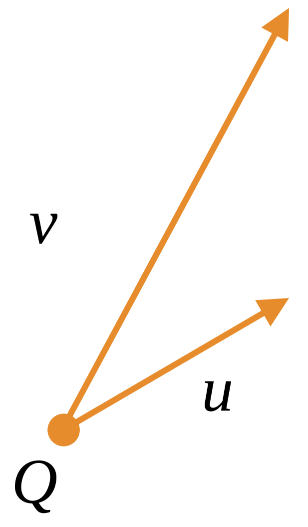
$$\left(x_1 \quad y_1 \quad z_1\right) \times \left(x_2 \quad y_2 \quad z_2\right) = \left(y_1 z_2 - y_2 z_1 \quad z_1 x_2 - z_2 x_1 \quad x_1 y_2 - x_2 y_1\right)$$

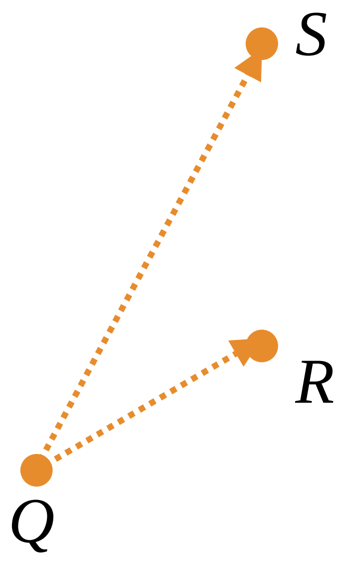
$$|\sin\theta| = \frac{|u \times v|}{|u||v|}$$



Planes

- Defined by a point and two vectors or by three points



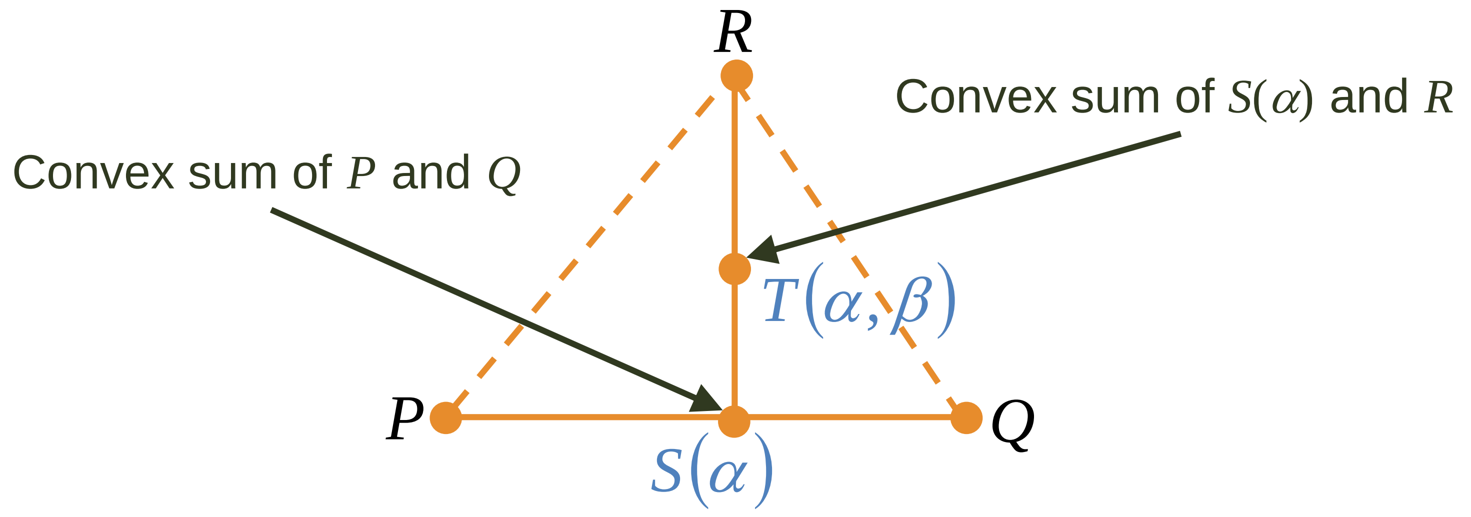


$P(\alpha,\beta) = Q + \alpha u + \beta v$

$P(\alpha,\beta) = Q + \alpha(S - Q) + \beta(R - Q)$

Triangles

$$T(\alpha,\beta) = P_0 + \alpha u + \beta v, \quad 0 \le \alpha, \beta \le 1$$



$$S(\alpha) = \alpha P + (1-\alpha)Q, \quad 0 \le \alpha \le 1$$

$$T(\beta) = \beta S + (1-\beta)R, \quad 0 \le \beta \le 1$$

$$T(\alpha,\beta) = \beta[\alpha P + (1-\alpha)Q] + (1-\beta)R$$

$$T(\alpha,\beta) = P + \beta(1-\alpha)(Q-P) + (1-\beta)(R-P)$$

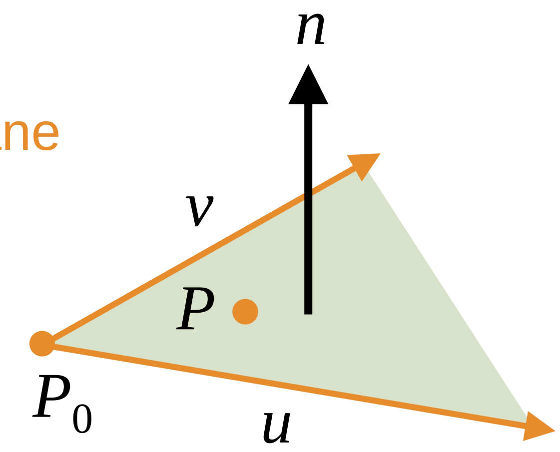
Normals

- Every plane a vector n <u>normal</u> (perpendicular, orthogonal) to it
- From two vectors which form $P(\alpha,\beta)=Q+\alpha u+\beta v$, we can use the <u>cross</u> product

$$P - P_0 = \alpha u + \beta v$$

$$n = u \times v$$ normal to the plane

$$\boxed{n \cdot (P - P_0) = 0}$$



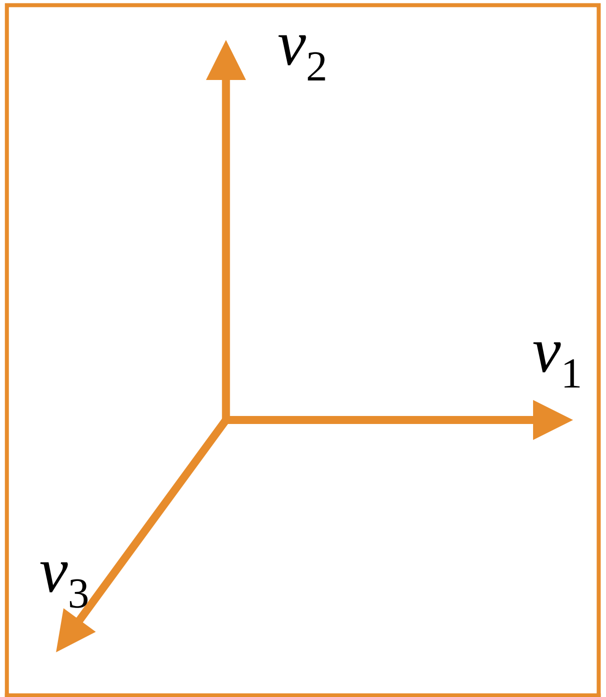
Coordinate Systems (1)

- 3D vector space

$$v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

  - Scalar component : $\alpha_1, \alpha_2, \alpha_3$
  - Basis vector : v_1, v_2, v_3
    - Defining a coordinate system
    - The origin: fixed reference point
  - Representation: column matrix

$$\mathbf{a} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} = \begin{bmatrix} \alpha_1 & \alpha_2 & \alpha_3 \end{bmatrix}^T$$



Coordinate System

Coordinate Systems (2)

- Example: $v = 2v_1 + 3v_2 + 4v_3$

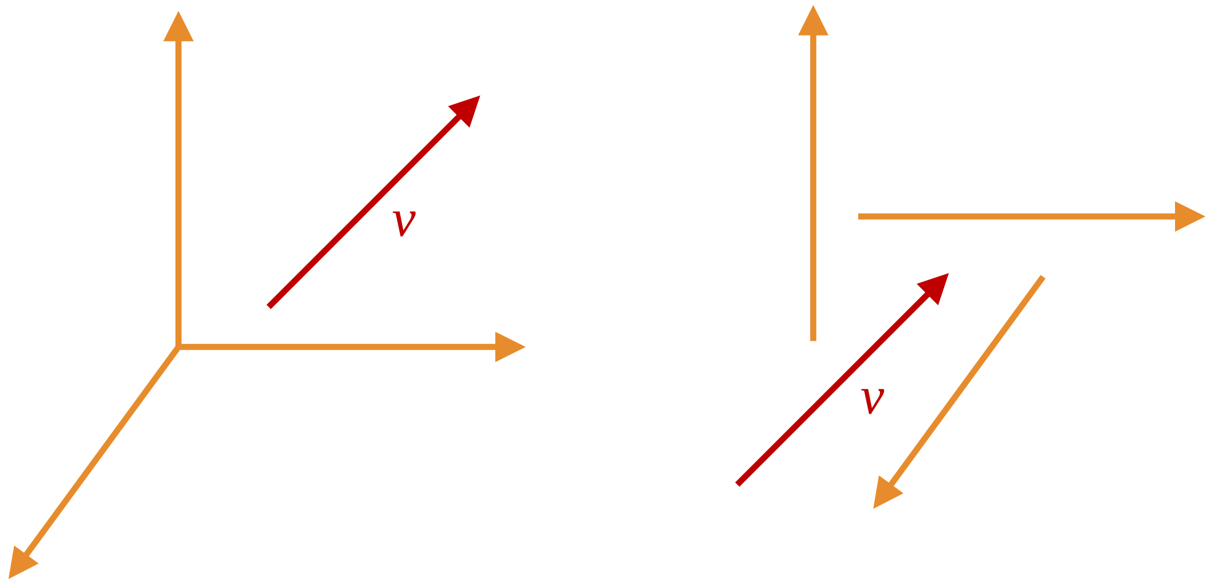
  $\mathbf{a} = \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}^T$

  - Note that this representation is with respect to a particular basis
  - In OpenGL, we start by representing vectors using object basis but later the system needs a representation in terms of the camera or eye basis

    ➔ "Change of Basis"

Coordinate Systems (3)

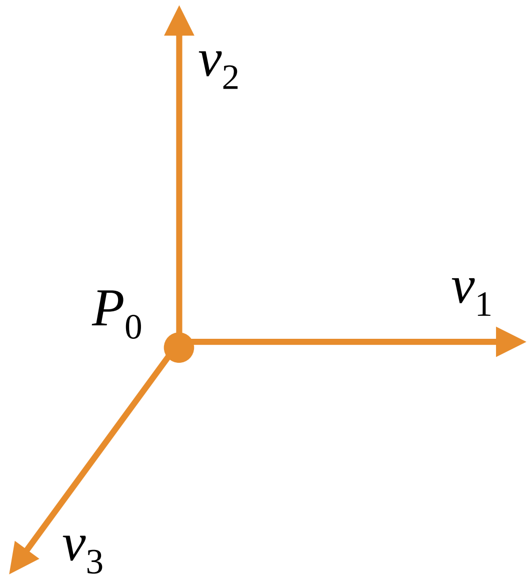
- Which is correct?



- Both are because vectors have no fixed location

Frames (1)

- A coordinate system is insufficient to represent points
- In affine space, we can add a single point, the origin, to the basis vectors to form a frame



Frames (2)

- Basis set of vectors and a particular point P_0
  - More general representation
  - Fixing the origin at P_0

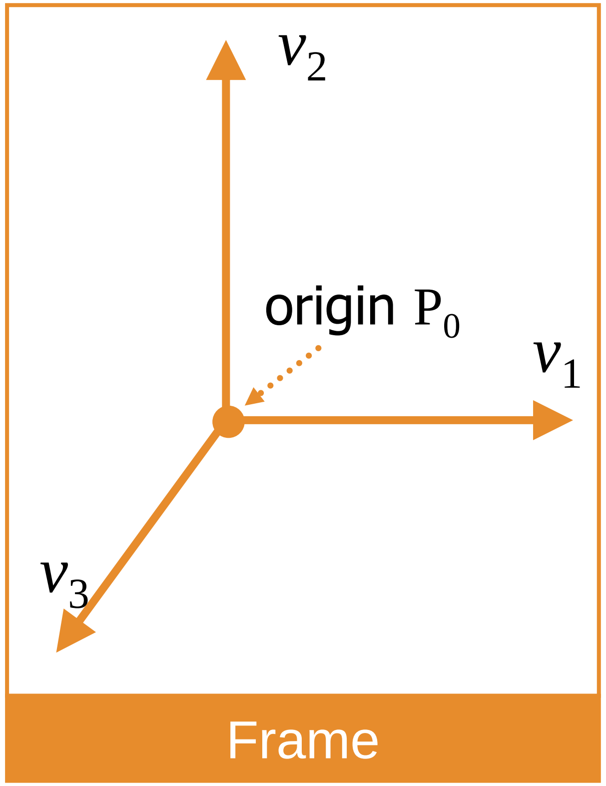
    vector : $v = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$

    point : $P = P_0 + \beta_1 v_1 + \beta_2 v_2 + \beta_3 v_3$

  - Homogeneous coordinates

    vector : $v = [\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad 0]^T$

    point : $P = [\beta_1 \quad \beta_2 \quad \beta_3 \quad 1]^T$



Homogeneous Coordinates (1)

$$P = \begin{bmatrix} x_w & y_w & z_w & w \end{bmatrix}^T$$

- If $w=0$, the representation is that of a vector
- Otherwise ($w \neq 0$), we return a three dimensional point by

$$P = \begin{bmatrix} x_w \\ y_w \\ z_w \\ w \end{bmatrix} = \begin{bmatrix} x_w/w \\ y_w/w \\ z_w/w \\ w/w \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

- To represent points and vectors with matrices but maintain a distinction between points and vectors

Homogeneous Coordinates (2)

- Homogeneous coordinates are key to all computer graphics systems
  - All standard transformation (rotation, translation, scaling) can be implemented with multiplications of $\underline{4} \times \underline{4}$ matrices
  - Hardware pipeline works with $\underline{4}$ dimensional representations
  - For orthographic viewing, we can maintain $w=0$ for <u>vector</u>s and $w=1$ for <u>point</u>s
  - For perspective viewing, we need a <u>perspective division</u>

Changes of Coordinate Systems (1)

- Two basis: { v_1, v_2, v_3 }, { u_1, u_2, u_3 }

$$u_1 = \gamma_{11} v_1 + \gamma_{12} v_2 + \gamma_{13} v_3$$
$$u_2 = \gamma_{21} v_1 + \gamma_{22} v_2 + \gamma_{23} v_3$$
$$u_3 = \gamma_{31} v_1 + \gamma_{32} v_2 + \gamma_{33} v_3$$

$\Downarrow$ ***matrix***

$$M = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} \\ \gamma_{21} & \gamma_{22} & \gamma_{23} \\ \gamma_{31} & \gamma_{32} & \gamma_{33} \end{bmatrix}$$

$$\mathbf{u} = M\ \mathbf{v}$$

Changes of Coordinate Systems (2)

- Vector: w

$$w = \alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3$$

$$w = \mathbf{a}^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}, \text{ where } \mathbf{a} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \Rightarrow w = \mathbf{b}^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}, \text{ where } \mathbf{b} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix}$$

$$w = \mathbf{b}^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \mathbf{b}^T M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \mathbf{a}^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$$

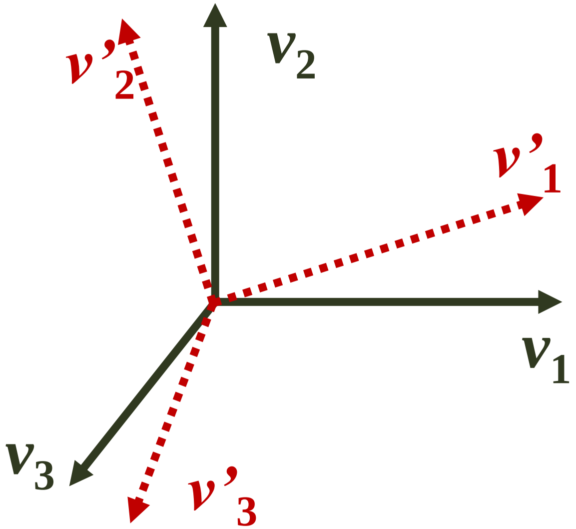
$$\mathbf{a} = M^T \mathbf{b}$$

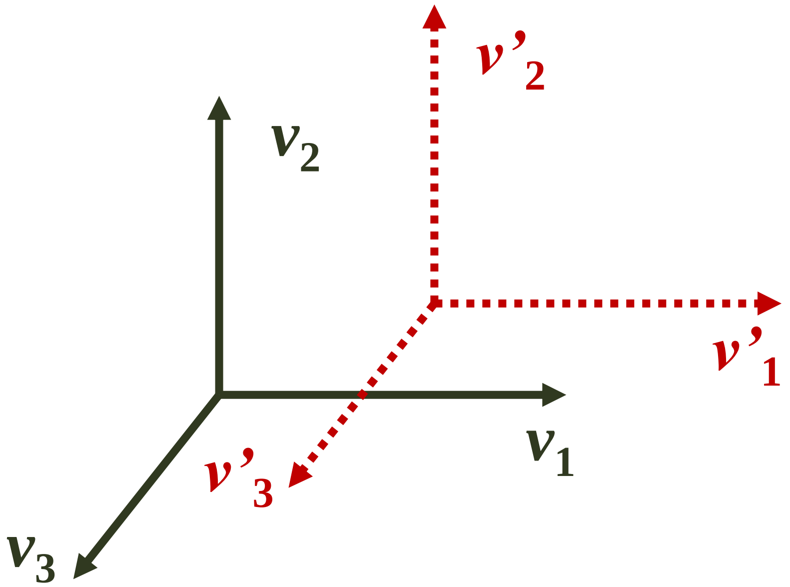
$$\mathbf{b} = \left(M^T\right)^{-1} \mathbf{a}$$

Changes of Basis

- Origin unchanged
  - Rotation and scaling of a set of basis vectors
- Origin changed
  - Translation of the origin, or change of frame

➔ <u>Homogeneous</u> coordinates





Example of Change Basis (1)

- Suppose a vector: w 🡐 $\mathbf{a} = [\ 1\ \ 2\ \ 3\ ]^T$
  - Three basis vectors : v_1, v_2, v_3

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$w = v_1 + 2v_2 + 3v_3$$

  - New basis : u_1, u_2, u_3

$$u_1 = v_1$$
$$u_2 = v_1 + v_2$$
$$u_3 = v_1 + v_2 + v_3$$

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

Example (2)

- Change of basis

$$\mathbf{b} = \left(M^T\right)^{-1}\mathbf{a}$$

$$= \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}$$

$$w = -u_1 - u_2 + 3u_3$$

Homogeneous Coordinates (1)

- Confusion between points and vectors !!

$$P = [x \quad y \quad z]^T, \quad w = [\delta_1 \quad \delta_2 \quad \delta_3]^T$$

- Point P and vector w in frame (v_1, v_2, v_3, P_0)

$$P = P_0 + xv_1 + yv_2 + zv_3 \qquad w = \delta_1 v_1 + \delta_2 v_2 + \delta_3 v_3$$

$$P = [x \quad y \quad z \quad 1]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix} \qquad w = [\delta_1 \quad \delta_2 \quad \delta_3 \quad 0]\begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}$$

$$\boxed{P = [x \quad y \quad z \quad 1]^T} \qquad \boxed{w = [\delta_1 \quad \delta_2 \quad \delta_3 \quad 0]^T}$$

Homogeneous Coordinates (2)

- Change of frames (v_1, v_2, v_3, P_0), (u_1, u_2, u_3, Q_0)

$$
\begin{aligned}
u_1 &= \gamma_{11}v_1 + \gamma_{12}v_2 + \gamma_{13}v_3 \\
u_2 &= \gamma_{21}v_1 + \gamma_{22}v_2 + \gamma_{23}v_3 \\
u_3 &= \gamma_{31}v_1 + \gamma_{32}v_2 + \gamma_{33}v_3 \\
Q_0 &= \gamma_{41}v_1 + \gamma_{42}v_2 + \gamma_{43}v_3 + P_0
\end{aligned}
\Rightarrow
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}
$$

$$
M = \begin{bmatrix} \gamma_{11} & \gamma_{12} & \gamma_{13} & 0 \\ \gamma_{21} & \gamma_{22} & \gamma_{23} & 0 \\ \gamma_{31} & \gamma_{32} & \gamma_{33} & 0 \\ \gamma_{41} & \gamma_{42} & \gamma_{43} & 1 \end{bmatrix}
$$

$$
\mathbf{b}^T \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ Q_0 \end{bmatrix} = \mathbf{b}^T M \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix} = \mathbf{a}^T \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ P_0 \end{bmatrix}
$$

$$
\mathbf{b} = \left(\mathbf{M}^T\right)^{-1} \mathbf{a}
$$

Example of Change in Frames

- Change of frames (v_1, v_2, v_3, P_0), (u_1, u_2, u_3, Q_0)

$$u_1 = v_1$$
$$u_2 = v_1 + v_2$$
$$u_3 = v_1 + v_2 + v_3$$
$$Q_0 = P_0$$

$$\Rightarrow \quad M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

- Point $P = [1\ \ 2\ \ 3\ \ 1]^T \rightarrow P' = [-1\ -1\ \ 3\ \ 1]^T$

$$\begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \\ 1 \end{bmatrix}$$

Another Example

- Change of frames (v_1, v_2, v_3, P_0), (u_1, u_2, u_3, Q_0)

$$u_1 = v_1$$
$$u_2 = v_1 + v_2$$
$$u_3 = v_1 + v_2 + v_3$$
$$Q_0 = P_0 + v_1 + 2v_2 + 3v_3$$

$$\Rightarrow \quad M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 2 & 3 & 1 \end{bmatrix}$$

- Point $P = [1 \;\; 2 \;\; 3 \;\; 1]^T \rightarrow P' = [0 \;\; 0 \;\; 0 \;\; 1]^T$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

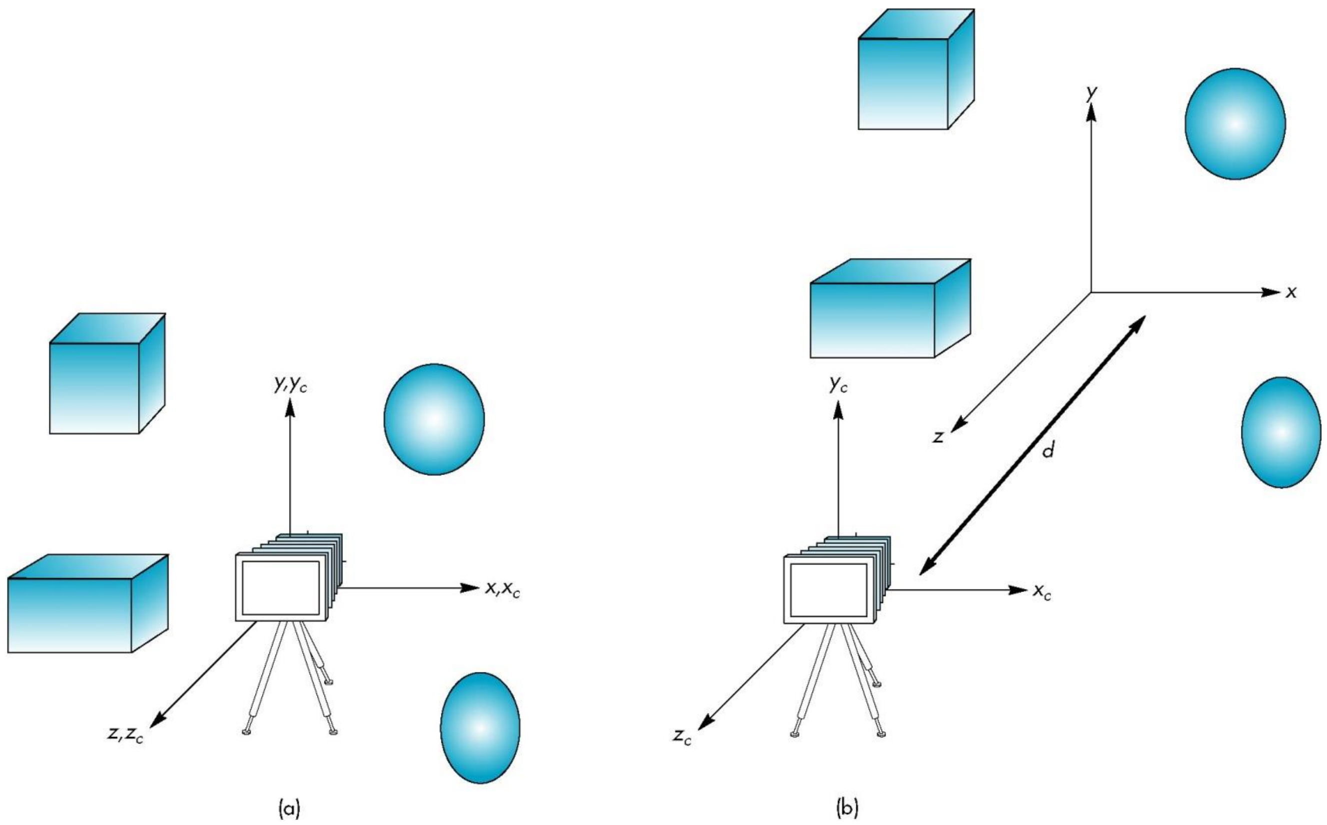
Frames in WebGL (1)

- Six representations embedded in the WebGL pipeline
  - Object (or model) coordinates
  - World coordinates
  - Eye (or camera) coordinates
  - Clip coordinates
  - Normalized devices coordinates
  - Window (or screen) coordinates
- Change in frames are defined by 4×4 matrices
  - Sequence of transformations

Frames in WebGL (2)

- Moving the camera frame relative to the object frame

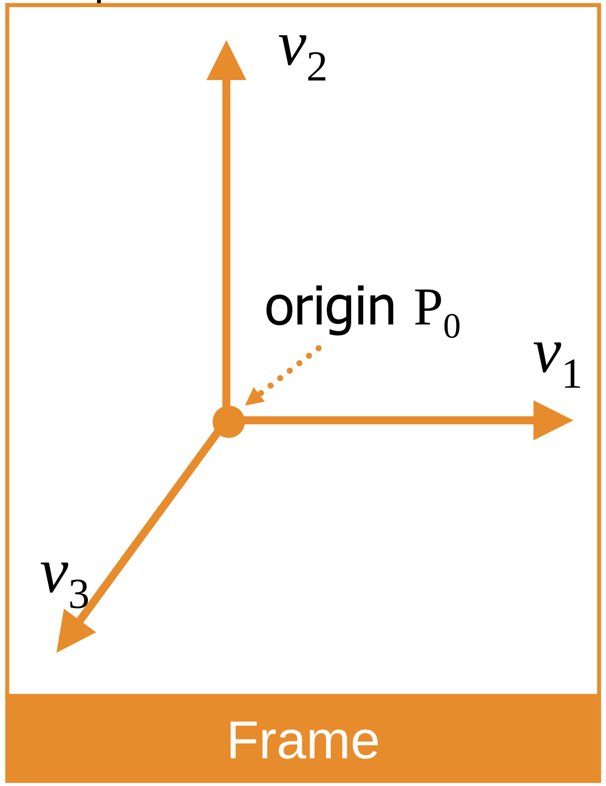
$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$



Summary

- Basic elements – points, scalars, vectors
- Scalar fields, linear vector spaces, Euclidean spaces, affine spaces
  - Lines – $P(\alpha) = P_0 + \alpha d$
  - Planes – $P(\alpha, \beta) = Q + \alpha u + \beta v$
- Dot product – $\cos\theta = \dfrac{u \cdot v}{|u||v|}$, Cross product – $|\sin\theta| = \dfrac{|u \times v|}{|u||v|}$
- Frames = basis + origin
  - Homogenous coordinates ➔ ***vector*** : $v = [\alpha_1 \quad \alpha_2 \quad \alpha_3 \quad 0]^T$

    point : $P = [\beta_1 \quad \beta_2 \quad \beta_3 \quad 1]^T$



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