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Linear Algebra

Applications in Digital Signal Processing

Fourier Series and the Fast Fourier Transform

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1 What is the Fourier Series

1.1 The Fourier Basis

The Fourier Series is an infinite series of sines and cosines that can accurately approximate any periodic function. It is a way of splitting up a function into a bunch of sine and cosine waves. Joseph Fourier discovered this when trying to solve the heat equation. The Fourier series can also be seen as a basis for a function space and it makes use of the orthogonality between sines and cosines.

1. The Fourier basis
   - $1, \sin(x), \cos(x), \sin(2x), \cos(2x), \cdots$
   - Fourier basis functions are all periodic, and repeat over every 2π interval.
   - This basis is orthogonal as every sine and cosine is orthogonal to every other sine and cosine.
   - This basis is also a great basis because sine and cosine are great for approximations (Taylor series expansion error bounds).

1.2 How to compute the Fourier Series for a function f(x)?

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(2\pi nxL\right)$$

In order to compute the coefficients, we will draw comparisons to vectors and vector spaces and then connect them to function spaces.

1.2.1 Review of Inner Products with functions and vectors

Similar to with Vectors, you can also take the inner product of two functions.

Example 1: Inner Product for Vectors
The inner product of the vectors x and y is $x^T y = x_1y_1 + x_2y_2 + ... + x_ny_n$

Additionally, $\|\vec{v}^2\| = \vec{v}^T\vec{v} = v_1^2 + v_1^2 + v_1^2 + \cdots$. The sum is finite because the length of a vector is finite and the inner product of a vector with itself is the length squared.

Example 2: Inner Product Functions
Inner product $= (\boldsymbol{f}, \boldsymbol{g}) = \int f(x)g(x)dx$
Complex inner product $= (\boldsymbol{f}, \boldsymbol{g}) = \int f(\bar{x})g(x)dx, \bar{f} =$ complex conjugate
Weighted inner product $= (\boldsymbol{f}, \boldsymbol{g})_w = \int w(x)f(\bar{x})g(x)dx, w(x) =$ weight function

Similarly, $\|f^2\| = (f, f) = \int_0^{2\pi} |f(x)|^2 dx$ must also be a finite integral and the inner product of a function with itself is equal to the length of the path is creates squared.

When two functions are orthogonal, their inner product is 0. This idea is crucial to the computing the Fourier Series for a function because all the terms are orthogonal to each other.

1.2.2 Computing the Fourier Series

The Fourier Series is a an infinite sum of orthogonal terms. This allows for easy computation of the Fourier coefficients a_k and b_k

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_1 \sin 2x + \cdots$$

$$(f, f) = \int_0^{2\pi} (a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_1 \sin 2x + \cdots)^2 dx$$

$$= \int_0^{2\pi} a_0^2 + a_1 \cos x^2 + b_1 \sin x^2 + a_2 \cos 2x^2 + b_1 \sin 2x^2 + \cdots dx$$

Every other term cancels out because every term in the Fourier series is orthogonal to each other. We can use this property to solve for the coefficients.

For example, in order to find a_1 simply find the inner product between $f(x)$ and $\cos x$. Every other term will cancel out except for the $\cos x^2$ term because they are orthogonal and their inner product (integral) is zero.

$$(f, \cos x) = \int_0^{2\pi} f(x) \cos x dx = \int_0^{2\pi} (a_0 + a_1 \cos x + b_1 \sin x + a_2 \cos 2x + b_1 \sin 2x + \cdots) \cos x dx$$

$$= \int_0^{2\pi} a_1 \cos x^2 dx = \pi a_1$$

$$a_1 = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos x dx$$

Using this same logic and solving generically we get the following relationships for Fourier coefficients a_k and b_k:

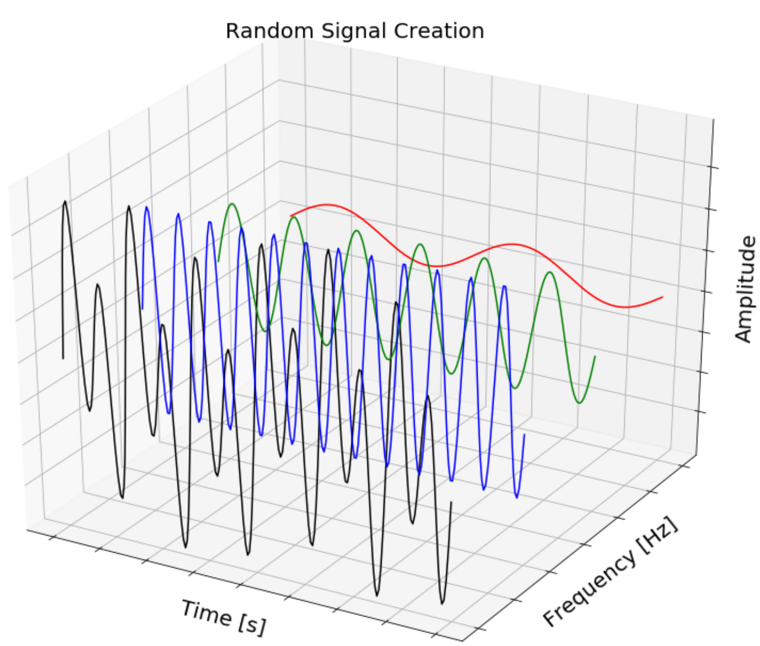
$$a_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos kx dx$$

$$b_k = \frac{1}{\pi} \int_0^{2\pi} f(x) \sin kx dx$$

2 Discrete Fourier Transform

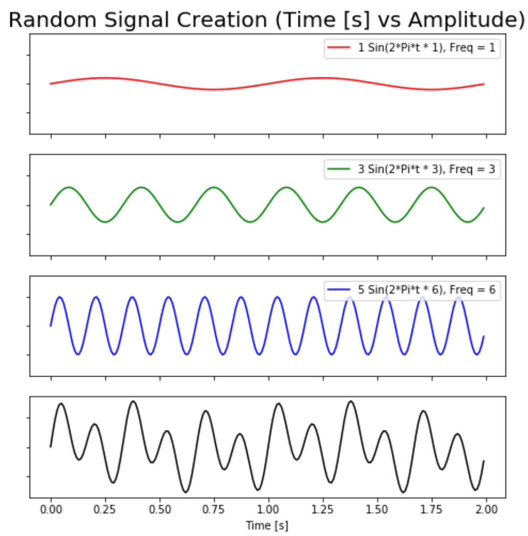
2.1 Signals in 3-Dimensions

One way to think about a signal is to picture it filling a three dimensional space. A complex signal can be decomposed in to the sines and cosines of the Fourier Series. Looking at a signal this way, it becomes clear that there can be two domains: The Time Domain $g(t)$ and the frequency domain $F(t)$



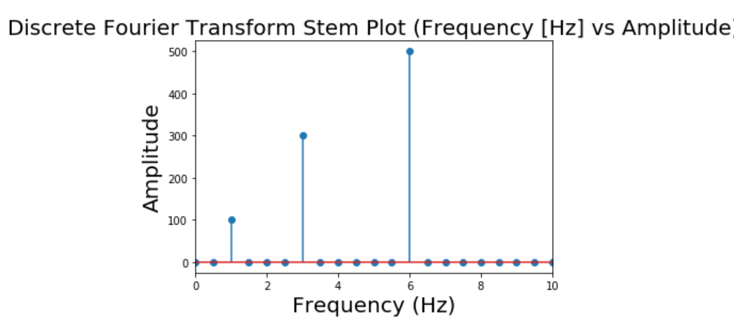
2.2 The Time Domain

In the time domain, we have a signal as function of time t. This signal can also be represented as a sum of simpler sine waves. As seen below in the image, these waves are the same as the ones seen in the 3D image. However, in this picture we are only looking at the amplitude of the wave as a function of time.



2.3 The Frequency Domain

In the frequency domain, we have a signal as function of time $frequency(Hz)$. This signal can be represented as a sum of waves with harmonic frequencies weighted differently. As seen below in the image, these waves are the same as the ones seen in the 3D image. However, in this picture we are only looking at the amplitude of the wave as a function of different frequencies. The frequency domain is most commonly known as the Fourier transform of.



2.4 The Fast Fourier Transform (FFT)

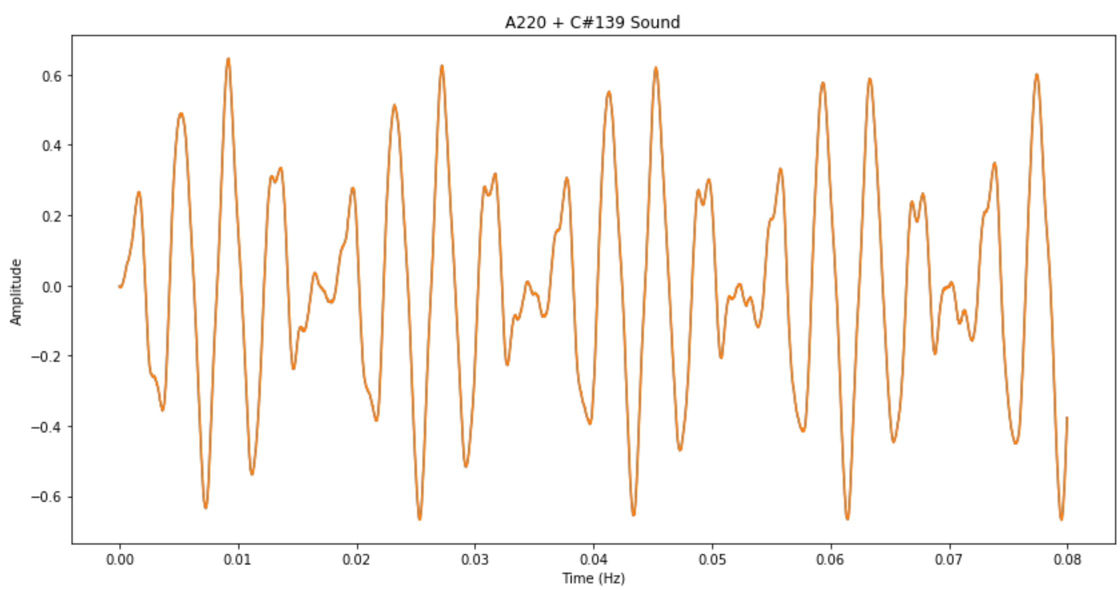
The Fast Fourier Transform takes a wave that is in the time domain (amplitude as a function of frequency) and transforms it into the frequency domain. Similarly, the Inverse of the Fast Fourier Transform takes a signal from the frequency domain to the time domain. Applying this transformation allows one to break up complex time signals into the sum of harmonic sine and cosine waves. Additionally, one can use the Inverse of the Fast Fourier Transform to synthesize harmonic frequencies into a complex wave. The Fast Fourier Transform simplifies finding the Fourier Series for a given signal.

3 Application in Digital Signal Processing

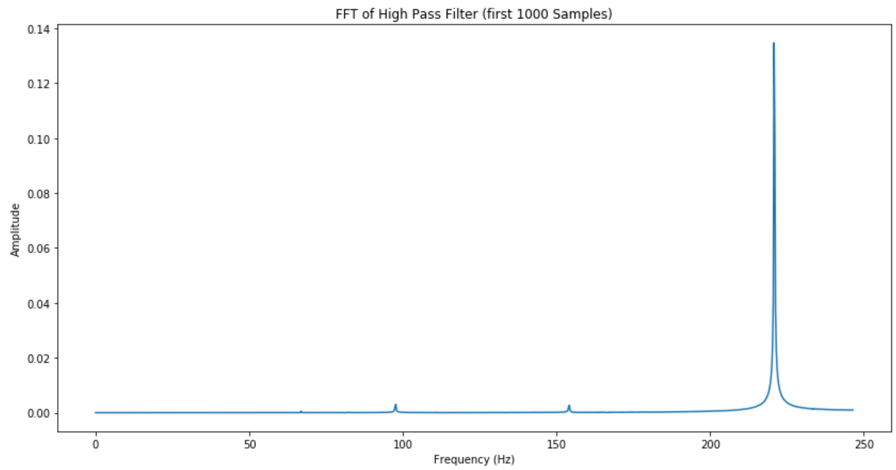
3.1 Audio Filtering

The Fourier Transform can be applied to audio filtering. By starting with a complex wave, one can apply the Fast Fourier Transform and transform the signal into the frequency domain. Then unwanted frequencies can be filtered out and the new wave synthesized.

Example: Take the following signal which is the sum of 2 simpler signals: A220 (220 Hz) and C139 (139 Hz).



Applying the FFT and filtering out all frequencies above 200 Hz returns a new signal with no frequencies below 200 Hz:

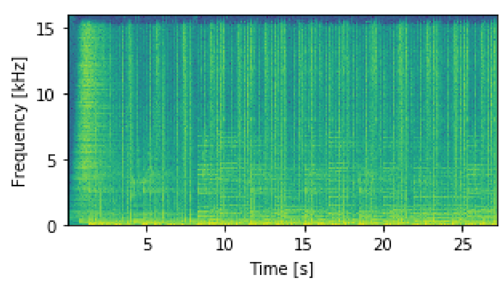


Looking at the Fourier Transform, we see that the primary frequency remaining is 220 Hz. The information gained from the Filtered Fourier Transform allows for a new signal to be synthesized that does not contain unwanted frequencies.

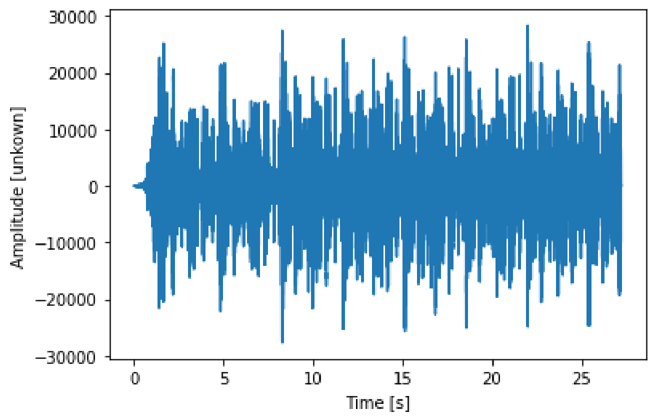
3.2 Speech Recognition

One interesting use of the Fourier Transform is for speech recognition. In speech recognition, the Discrete Fourier Transform can be used to analyze samples of a complex signal, which in this case these samples are of speech sounds. The output of the transformation is similar to the output of a spectrograph and makes it possible to identify phonetic features in the original signal while also determining the sequence.

Below is an example of a spectrogram of a signal: The x-axis is time, the y-axis is frequency, and the third dimension: color shows the strength of each frequency.



Below is the signal that the spectrogram represents



3.3 Other Applications

The Fourier Transform can be used in a wide variety of situations such as signal processing, encryption, finance (options pricing), solving differential equations, and more. It is one of the greatest mathematical discoveries ever. In this paper, the derivation of the Fourier Series was presented in a linear algebra context. Additionally, a practical example of audio filtering was examined.

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