

Assignment #6

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Problem 1

Part i

$$p(\beta|y, \lambda_1, \lambda_2, k) \propto p(y|\lambda_1, \lambda_2, k, \beta) \times p(\lambda_1|y, \lambda_2, k, \beta) \times p(\lambda_2|y, \lambda_1, k, \beta) \times p(k|y, \lambda_1, \lambda_2, \beta) \times p(\beta)$$

Since $p(y|\lambda_1, \lambda_2, k, \beta)$ and $p(k|y, \lambda_1, \lambda_2, \beta)$ don't depend on β , we may ignore them and our equation becomes:

$$p(\beta|y, \lambda_1, \lambda_2, k) \propto p(\lambda_1|y, \lambda_2, k, \beta) \times p(\lambda_2|y, \lambda_1, k, \beta) \times p(\beta)$$

Where:

$$\begin{aligned} p(\lambda_1|y, \lambda_2, k, \beta) &= p(y|\lambda_1, \lambda_2, k)p(\lambda_1) \\ &\sim \text{Gamma}\left(\alpha + \sum_{t=1}^k y_t, \beta + k\right) \end{aligned}$$

$$\begin{aligned} p(\lambda_2|y, \lambda_1, k, \beta) &= p(y|\lambda_1, \lambda_2, k)p(\lambda_2) \\ &\sim \text{Gamma}\left(\alpha + \sum_{t=k+1}^n y_t, \beta + (n - k)\right) \end{aligned}$$

So:

$$\begin{aligned} p(\beta|y, \lambda_1, \lambda_2, k) &\propto p(\lambda_1|y, \lambda_2, k, \beta) \times p(\lambda_2|y, \lambda_1, k, \beta) \times p(\beta) \\ &\propto \beta^\alpha \lambda_1^{\alpha-1+\sum_{t=1}^k y_t} e^{-(\beta+k)\lambda_1} \beta^\alpha \lambda_2^{\alpha-1+\sum_{t=k+1}^n y_t} e^{-(\beta+(n-k))\lambda_2} \beta^{a_\beta-1} e^{-b_\beta\beta} \\ &\propto \beta^{2\alpha} e^{-(\beta+k)\lambda_1 - (\beta+(n-k))\lambda_2} \beta^{a_\beta-1} e^{-b_\beta\beta} \\ &\propto \beta^{2\alpha+a_\beta-1} e^{-\beta\lambda_1 - \beta\lambda_2 - b_\beta\beta} \\ &\propto \beta^{2\alpha+a_\beta-1} e^{-(\lambda_1+\lambda_2+b_\beta)\beta} \\ &\sim \text{Gamma}(2\alpha + a_\beta, \lambda_1 + \lambda_2 + b_\beta) \end{aligned}$$

Part ii

Please refer to the Code Appendix section for the implementation of the Gibbs sampler.

Part iii

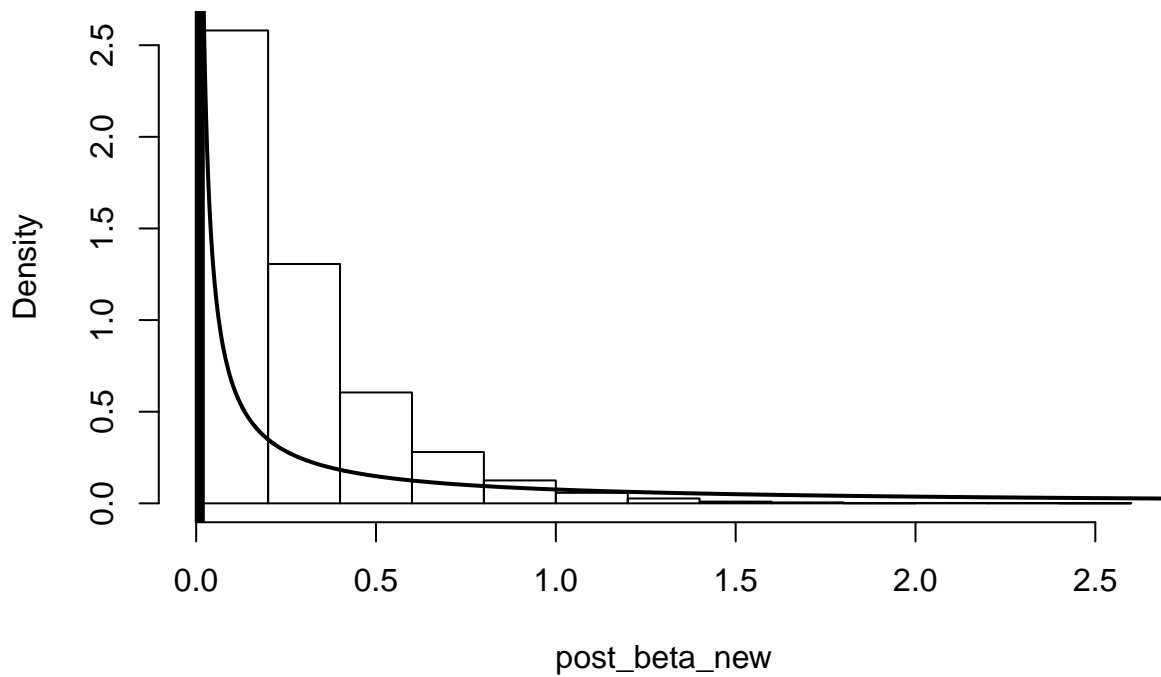
```
## HPD Interval Lambda_1 with Beta = 0.01: ( 2.575887 , 3.704449 )
## HPD Interval Lambda_2 with Beta = 0.01: ( 0.7083599 , 1.169247 )
## HPD Interval k + 1850 with Beta = 0.01: ( 1885 , 1894 )
```

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## HPD Interval Lambda_1 with Beta Hyperprior: ( 2.540048 , 3.690528 )  
## HPD Interval Lambda_2 with Beta Hyperprior: ( 0.6948146 , 1.146471 )  
## HPD Interval k + 1850 with Beta Hyperprior: ( 1884 , 1894 )
```

There does not appear to be substantial difference between the posterior credible intervals of the two models.

Part iv

Histogram of post_beta_new



No. According to the results and the density representation there is not evidence to suggest the the posterior distribution $[\beta|y]$ differs substantially from our previous choice of $\beta = 0.01$

Problem 2

Part i

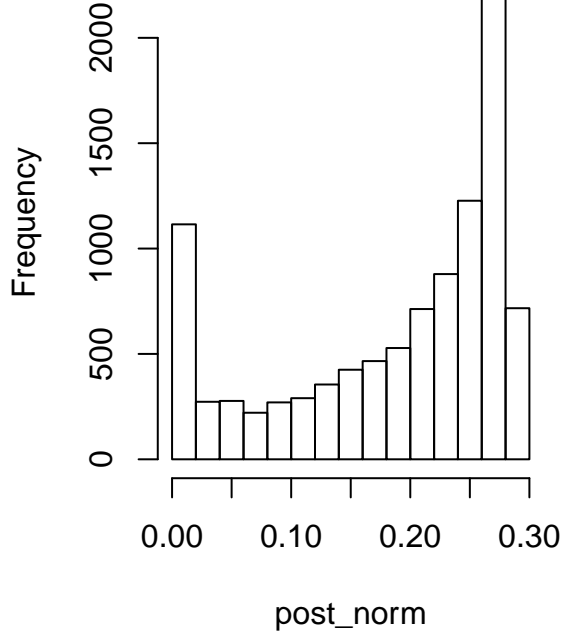
$$\begin{aligned}
 p(y|\mu, \alpha, \beta) &= \int \frac{\beta^\alpha}{\Gamma(\alpha)} \tau^{\alpha-1} e^{-\beta\tau} \left(\frac{\tau}{2\pi} \right)^{\frac{1}{2}} e^{-\frac{\tau}{2}(y-\mu)^2} d\tau \\
 &= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{(2\pi)^{\frac{1}{2}}} \int \tau^{\alpha-\frac{1}{2}} e^{-\beta\tau} e^{-\frac{\tau}{2}(y-\mu)^2} d\tau \\
 &= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{(2\pi)^{\frac{1}{2}}} \int \tau^{\alpha-\frac{1}{2}} e^{-(\beta+\frac{1}{2}(y-\mu)^2)\tau} d\tau \\
 &= \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{1}{(2\pi)^{\frac{1}{2}}} \frac{\Gamma(\alpha+\frac{1}{2})}{(\beta+\frac{1}{2}(y-\mu)^2)^{\alpha+\frac{1}{2}}} \\
 &= \frac{\Gamma(\alpha+\frac{1}{2})}{\Gamma(\alpha)} \frac{1}{(2\pi\beta)^{\frac{1}{2}}} \frac{1}{(1+\frac{1}{2\beta}(y-\mu)^2)^{\alpha+\frac{1}{2}}}
 \end{aligned}$$

And now substituting in for z_i and our Gamma parameters for α and β we get the following result:

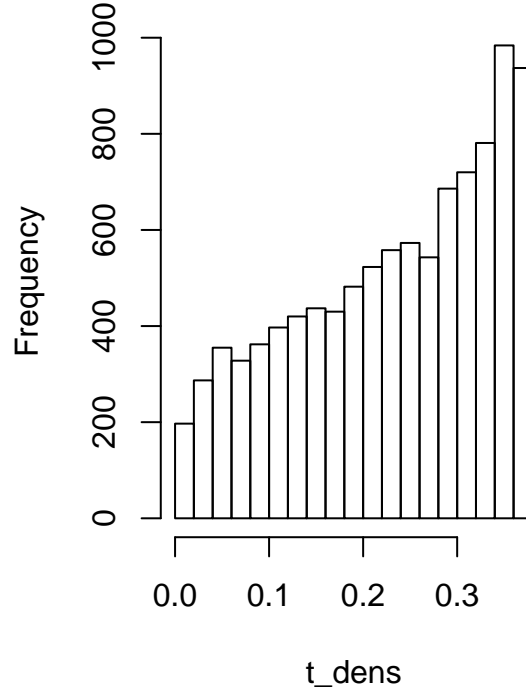
$$p(z_i) = \frac{\Gamma(\frac{\nu+1}{2})}{\sqrt{\pi\nu}\Gamma(\frac{\nu}{2})} \frac{1}{\left(1+\frac{z_i^2}{\nu}\right)^{\frac{\nu+1}{2}}}$$

Part ii

Histogram of post_norm



Histogram of t_dens



As we can see from our result, we get similar density when comparing our results from Part i with the $t_\nu(0, 1)$ distribution.

Part iii

Yes, this result does make sense, because as $\nu \rightarrow \infty$ the $Gamma(\frac{\nu}{2}, \frac{\nu}{2})$ will approach the Normal distribution.

Problem 3

Part i

$$p(\theta|y, \tau_y, \xi_i) \propto p(\theta)p(y|\theta, \tau_y, \xi_i)$$

$$p(\tau_y|y, \theta, \xi_i) \propto p(\tau_y)p(y|\theta, \tau_y, \xi_i)$$

$$p(\xi_i|y, \theta, \tau_y) \propto p(\xi_i)p(y|\theta, \tau_y, \xi_i)$$

Part ii

Part iii

Part iv

Part v

Code Appendix