Assignment #2

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Problem 1

Part a

Here our goal will be to minimize a to show that $a = E[\theta|y]$ is the unique Bayes estimate of θ:

$$\begin{aligned}
\frac{d}{da}E[L(a|y)] &= \frac{d}{da}\int L(\theta,a)p(\theta|y)d\theta \\
&= \frac{d}{da}\int (\theta-a)^2 p(\theta|y)d\theta \\
&= -2\int (\theta-a)p(\theta|y)d\theta \\
&= -2\Big[\int \theta p(\theta|y)d\theta - a\int p(\theta|y)d\theta\Big] \\
&= -2\big[E[\theta|y]-a\big]
\end{aligned}$$

$-2\big[E[\theta|y]-a\big] = 0$ when $a = E[\theta|y]$

To prove that it is a unique minimizing statistic, we must look at the second derivative:

$\frac{d}{da}(-2\big[E[\theta|y]-a\big]) = 2$

As $2 > 0$, this shows that it is a unique minimzing statistic.

Part b

Here our goal will be to show that for any median value of a, the derivative of $L(\theta, a)$ will evaluate to 0.

$$\begin{aligned}
\frac{d}{da}\big[E[L(a|y)]\big] &= \frac{d}{da}\Big[\int_{-\infty}^{a}(a-\theta)p(\theta|y)d\theta + \int_{a}^{\infty}(\theta-a)p(\theta|y)d\theta\Big] \\
&= \int_{-\infty}^{a}\frac{d}{da}(a-\theta)p(\theta|y)d\theta + \int_{a}^{\infty}\frac{d}{da}(\theta-a)p(\theta|y)d\theta \\
&= \int_{-\infty}^{a}p(\theta|y)d\theta + \int_{a}^{\infty}(-1)p(\theta|y)d\theta \\
&= \int_{-\infty}^{a}p(\theta|y)d\theta - \int_{a}^{\infty}p(\theta|y)d\theta \\
&= \frac{1}{2}-\frac{1}{2} \\
&= 0
\end{aligned}$$

As a result, it has been shown that any posterior median of θ is a Bayes estimate of θ.

Part c

Here our goal will be to show that for any value of a, the derivative of $L(\theta, a)$ will evaluate to 0 where k_0 and k_1 are nonnegative numbers.

$$\begin{aligned}
\frac{d}{da}\left[E[L(a|y)]\right] &= \frac{d}{da}\left[\int_{-\infty}^{a} k_1(a-\theta)p(\theta|y)d\theta + \int_{a}^{\infty} k_0(\theta-a)p(\theta|y)d\theta\right] \\
&= \int_{-\infty}^{a} \frac{d}{da} k_1(a-\theta)p(\theta|y)d\theta + \int_{a}^{\infty} \frac{d}{da} k_0(\theta-a)p(\theta|y)d\theta \\
&= \int_{-\infty}^{a} k_1 p(\theta|y)d\theta + \int_{a}^{\infty} (-k_0)p(\theta|y)d\theta \\
&= \int_{-\infty}^{a} k_1 p(\theta|y)d\theta - \int_{a}^{\infty} k_0 p(\theta|y)d\theta \\
&= k_1\int_{-\infty}^{a} p(\theta|y)d\theta - k_0\int_{a}^{\infty} p(\theta|y)d\theta
\end{aligned}$$

Noting that: $k_0\int_a^\infty p(\theta|y)d\theta = k_0 - k_0\int_{-\infty}^{a} p(\theta|y)d\theta$

$$\begin{aligned}
k_1\int_{-\infty}^{a} p(\theta|y)d\theta - k_0\int_{a}^{\infty} p(\theta|y)d\theta &= k_1\int_{-\infty}^{a} p(\theta|y)d\theta - \left[k_0 - k_0\int_{-\infty}^{a} p(\theta|y)d\theta\right] \\
&= k_1\int_{-\infty}^{a} p(\theta|y)d\theta + k_0\int_{-\infty}^{a} p(\theta|y)d\theta - k_0 \\
&= (k_1+k_0)\int_{-\infty}^{a} p(\theta|y)d\theta - k_0
\end{aligned}$$

Now setting $\int_{-\infty}^{a} p(\theta|y)d\theta = \frac{k_0}{k_0+k_1}$ we get our result that any quantile is a Bayes estimate of θ.

Taking the second derivative we again get a positive number, thus again indicating that it is a minimizing statistic.

Problem 2

n = 20
Sampling Distribution: $y|\theta \sim Binomial(n = 20, \theta)$
Prior Distribution: $\theta \sim Beta(\alpha = 2, \beta = 20)$
Posterior Distribution:

$$\begin{aligned}
p(\theta|y) &= \binom{n}{y}\theta^y(1-\theta)^{n-y} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1} \\
&= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1} \\
&\propto Beta(y+\alpha, (n-y)+\beta)
\end{aligned}$$

Part i