Assignment #2

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Problem 1

Part a

Here our goal will be to minimize a to show that $a = E[\theta|y]$ is the unique Bayes estimate of θ:

$$
\begin{aligned}
\frac{d}{da}E[L(a|y)] &= \frac{d}{da}\int L(\theta, a)p(\theta|y)d\theta \\
&= \frac{d}{da}\int (\theta - a)^2 p(\theta|y)d\theta \\
&= -2\int (\theta - a)p(\theta|y)d\theta \\
&= -2\Big[\int \theta p(\theta|y)d\theta - a\int p(\theta|y)d\theta\Big] \\
&= -2\big[E[\theta|y] - a\big]
\end{aligned}
$$

$-2\big[E[\theta|y] - a\big] = 0$ when $a = E[\theta|y]$

To prove that it is a unique minimizing statistic, we must look at the second derivative:

$\frac{d}{da}(-2\big[E[\theta|y] - a\big]) = 2$

As $2 > 0$, this shows that it is a unique minimzing statistic.

Part b

Here our goal will be to show that for any median value of a, the derivative of $L(\theta, a)$ will evaluate to 0.

$$
\begin{aligned}
\frac{d}{da}\big[E[L(a|y)]\big] &= \frac{d}{da}\Big[\int_{-\infty}^{a}(a-\theta)p(\theta|y)d\theta + \int_{a}^{\infty}(\theta - a)p(\theta|y)d\theta\Big] \\
&= \int_{-\infty}^{a}\frac{d}{da}(a-\theta)p(\theta|y)d\theta + \int_{a}^{\infty}\frac{d}{da}(\theta - a)p(\theta|y)d\theta \\
&= \int_{-\infty}^{a}p(\theta|y)d\theta + \int_{a}^{\infty}(-1)p(\theta|y)d\theta \\
&= \int_{-\infty}^{a}p(\theta|y)d\theta - \int_{a}^{\infty}p(\theta|y)d\theta \\
&= \frac{1}{2} - \frac{1}{2} \\
&= 0
\end{aligned}
$$

As a result, it has been shown that any posterior median of θ is a Bayes estimate of θ.

Part c

Here our goal will be to show that for any value of a, the derivative of $L(\theta, a)$ will evaluate to 0 where k_0 and k_1 are nonnegative numbers.

$$\begin{aligned}
\frac{d}{da}\left[E[L(a|y)]\right] &= \frac{d}{da}\left[\int_{-\infty}^{a} k_1(a-\theta)p(\theta|y)d\theta + \int_{a}^{\infty} k_0(\theta-a)p(\theta|y)d\theta\right] \\
&= \int_{-\infty}^{a} \frac{d}{da} k_1(a-\theta)p(\theta|y)d\theta + \int_{a}^{\infty} \frac{d}{da} k_0(\theta-a)p(\theta|y)d\theta \\
&= \int_{-\infty}^{a} k_1 p(\theta|y)d\theta + \int_{a}^{\infty} (-k_0)p(\theta|y)d\theta \\
&= \int_{-\infty}^{a} k_1 p(\theta|y)d\theta - \int_{a}^{\infty} k_0 p(\theta|y)d\theta \\
&= k_1\int_{-\infty}^{a} p(\theta|y)d\theta - k_0\int_{a}^{\infty} p(\theta|y)d\theta
\end{aligned}$$

Noting that: $k_0\int_a^\infty p(\theta|y)d\theta = k_0 - k_0\int_{-\infty}^{a} p(\theta|y)d\theta$

$$\begin{aligned}
k_1\int_{-\infty}^{a} p(\theta|y)d\theta - k_0\int_{a}^{\infty} p(\theta|y)d\theta &= k_1\int_{-\infty}^{a} p(\theta|y)d\theta - \left[k_0 - k_0\int_{-\infty}^{a} p(\theta|y)d\theta\right] \\
&= k_1\int_{-\infty}^{a} p(\theta|y)d\theta + k_0\int_{-\infty}^{a} p(\theta|y)d\theta - k_0 \\
&= (k_1+k_0)\int_{-\infty}^{a} p(\theta|y)d\theta - k_0
\end{aligned}$$

Now setting $\int_{-\infty}^{a} p(\theta|y)d\theta = \frac{k_0}{k_0+k_1}$ we get our result that any quantile is a Bayes estimate of θ.

Taking the second derivative we again get a positive number, thus again indicating that it is a minimizing statistic.

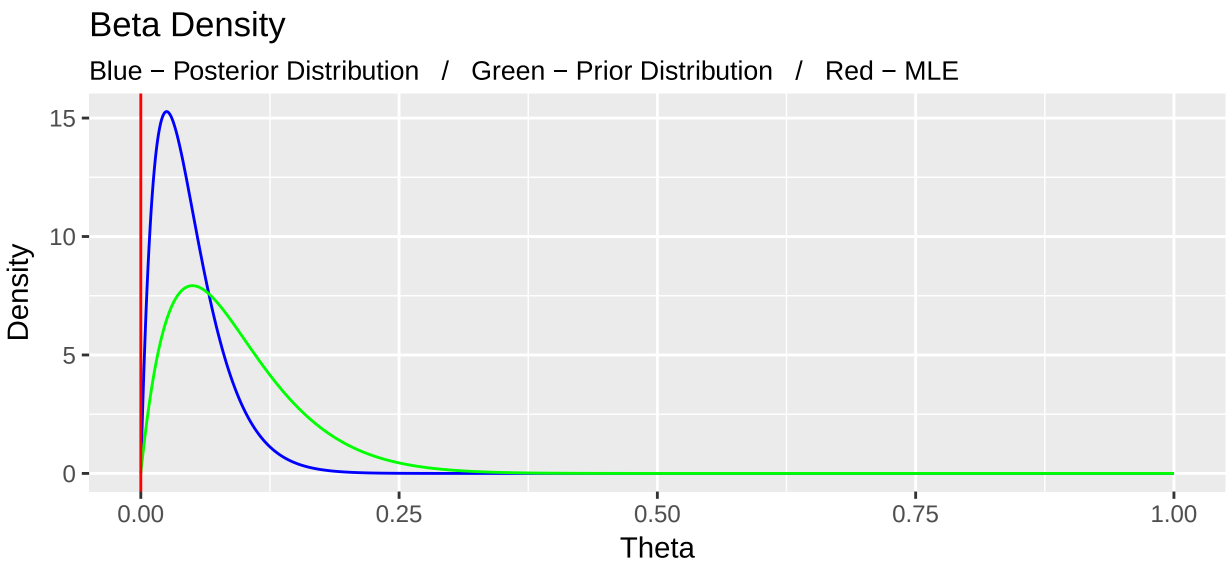
Problem 2

n = 20
Sampling Distribution: $y|\theta \sim Binomial(n = 20, \theta)$
Prior Distribution: $\theta \sim Beta(\alpha = 2, \beta = 20)$
Posterior Distribution:

$$\begin{aligned}
p(\theta|y) &= \binom{n}{y}\theta^{y}(1-\theta)^{n-y} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1} \\
&= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1} \\
&\propto Beta(y+\alpha, (n-y)+\beta) \\
&= Beta(y+2, (20-y)+20)
\end{aligned}$$

Part i

$y = 0$



Part ii

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## HPD Interval:  ( 0.0009688175 , 0.1098091 )
## Frequentist Interval:  ( 0 , 0 )
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Part iii

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## HPD Interval:  ( -4.896443 , -1.805408 )
```

Problem 3

Part i

My methodology for selecting my α and β for my $\theta_1, \theta_2 \sim Gamma(\alpha, \beta)$ distribution is as follows. First, I estimated my λ parameter from my Poisson distribution by $\Sigma_{i=1}^{n} \frac{y_i}{n}$. Then, I generated a sample of 1000 data points from a Poisson distribution with my esimated λ parameter. I calculated the mean and variance of this sample, and using these values, ascertained the values of α and β of a Gamma distribution by using the formula for mean $(\frac{\alpha}{\beta})$ and variance $(\frac{\alpha}{\beta^2})$.

```
## Gamma Parameters
## Alpha Parameter:  1.8301
## Beta Parameter:  0.9898
```

Part ii

Posterior Distributions:

$$
\begin{aligned}
p(\theta|y_1) &\propto p(y_1|\theta)p(\theta) \\
&= \frac{\theta^{y_1}e^{-\theta}}{y_1!}\frac{\beta^{\alpha}}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta} \\
&\propto \theta^{y_1}e^{-\theta}\theta^{\alpha-1}e^{-\beta\theta} \\
&= \theta^{y_1+\alpha-1}e^{-(1+\beta)\theta} \\
&\sim Gamma(y_1+\alpha, \beta+1)
\end{aligned}
$$

$$
\begin{aligned}
p(\theta|y_2) &\propto p(y_2|\theta)p(\theta) \\
&= \frac{\theta^{y_2}e^{-\theta}}{y_2!}\frac{\beta^{\alpha}}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta} \\
&\propto \theta^{y_2}e^{-\theta}\theta^{\alpha-1}e^{-\beta\theta} \\
&= \theta^{y_2+\alpha-1}e^{-(1+\beta)\theta} \\
&\sim Gamma(y_2+\alpha, \beta+1)
\end{aligned}
$$