# Assignment #3

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## Problem 1

Sampling Distribution: $y_1, ..., y_n|\theta \overset{\text{iid}}{\sim} N(\theta, \sigma_y^2)$
Prior Distribution: $\theta \sim N(\mu_\theta, \sigma_\theta^2)$
Posterior Distribution: $\theta|y \sim N(Q_\theta^{-1}\ell_\theta, Q_\theta^{-1})$

Posterior Distribution:

$$\begin{aligned}
p(\theta|y) &\propto p(y|\theta)p(\theta) \\
&= \left[\frac{1}{\sqrt{2\pi\sigma_y^2}}\right]^n exp\left(-\frac{1}{2\sigma_y^2}\sum_{i=1}^n (y_i-\theta)^2\right) \times \frac{1}{\sqrt{2\pi\sigma_\theta^2}} exp\left(-\frac{1}{2\sigma_\theta^2}(\theta-\mu_\theta)^2\right) \\
&\propto exp\left(-\frac{1}{2\sigma_\theta^2}(\theta-\mu_\theta)^2\right) \times exp\left(-\frac{1}{2\sigma_y^2}\sum_{i=1}^n (y_i-\theta)^2\right) \\
&= exp\left(-\frac{\theta^2-2\theta\mu_\theta+\mu_\theta^2}{2\sigma_\theta^2}\right) \times exp\left(-\frac{\sum_{i=1}^n (y_i^2-2\theta y_i+\theta^2)}{2\sigma_y^2}\right) \\
&= exp\left(\frac{-\theta^2+2\theta\mu_\theta-\mu_\theta^2}{2\sigma_\theta^2} - \frac{\sum_{i=1}^n (y_i^2-2\theta y_i+\theta^2)}{2\sigma_y^2}\right) \\
&= exp\left(\frac{\sigma_y^2[-\theta^2+2\theta\mu_\theta-\mu_\theta^2] - \sigma_\theta^2[\sum_{i=1}^n (y_i^2-2\theta y_i+\theta^2)]}{2\sigma_\theta^2\sigma_y^2}\right) \\
&= exp\left(\frac{-\theta^2(\sigma_y^2+n\sigma_\theta^2) + 2\theta(\mu_\theta\sigma_y^2+\sigma_\theta^2 y_1+...+\sigma_\theta^2 y_n) - (\mu_\theta^2\sigma_y^2+\sigma_\theta^2 y_1^2+...+\sigma_\theta^2 y_n^2)}{2\sigma_\theta^2\sigma_y^2}\right) \\
&= exp\left(\frac{-\theta^2+2\theta\frac{\mu_\theta\sigma_y^2+\sum_{i=1}^n \sigma_\theta^2 y_i}{\sigma_y^2+n\sigma_\theta^2} - \left(\frac{\mu_\theta\sigma_y^2+\sum_{i=1}^n \sigma_\theta^2 y_i}{\sigma_y^2+n\sigma_\theta^2}\right)}{2\frac{\sigma_\theta^2\sigma_y^2}{\sigma_y^2+n\sigma_\theta^2}}\right) \times exp\left(-\frac{\mu_\theta^2\sigma_y^2+\sum_{i=1}^n \sigma_\theta^2 y_i^2}{2\sigma_\theta^2\sigma_y^2}\right) \\
&= exp\left(-\frac{\left(\theta-\frac{\mu_\theta\sigma_y^2+\sum_{i=1}^n \sigma_\theta^2 y_i}{\sigma_y^2+n\sigma_\theta^2}\right)^2}{2\frac{\sigma_\theta^2\sigma_y^2}{\sigma_y^2+n\sigma_\theta^2}}\right)
\end{aligned}$$

From this result, we can see the following result:

$$\begin{aligned}
\theta|y &\sim N\left(\frac{\mu_\theta\sigma_y^2 + \sum_{i=1}^n \sigma_\theta^2 y_i}{\sigma_y^2 + n\sigma_\theta^2}, \frac{\sigma_y^2\sigma_\theta^2}{\sigma_y^2 + n\sigma_\theta^2}\right) \\
&\sim N\left(\frac{\mu_\theta\sigma_\theta^{-2} + \sigma_y^{-2}\sum_{i=1}^n y_i}{\sigma_\theta^{-2} + n\sigma_y^{-2}}, \frac{1}{\sigma_\theta^{-2} + n\sigma_y^{-2}}\right) \\
&\sim N(Q_\theta^{-1}\ell_\theta, Q_\theta^{-1})
\end{aligned}$$

Where the following holds:

$Q_\theta = n\sigma_y^{-2} + \sigma_\theta^{-2}$
$\ell_\theta = \sigma_y^{-2}\sum_{i=1}^n y_i + \sigma_\theta^{-2}\mu_\theta$

# Problem 2

## Part i

Posterior Distribution:

$$\begin{aligned}
p(\theta|y,\tau_y) &= \frac{1}{\sqrt{2\pi\sigma_y^2}} exp\left[-\frac{1}{2\sigma_y^2}\sum_{i=1}^n (y_i - \theta x_i)^2\right] \\
&\propto exp\left[-\frac{1}{2\sigma_y^2}\sum_{i=1}^n (y_i - \theta x_i)^2\right] \\
&= exp\left[-\frac{1}{2}\left(\frac{1}{\sigma_y^2}\sum_{i=1}^n (y_i^2 - 2\theta x_i y_i + \theta^2 x_i^2\right)\right] \\
&= exp\left[-\frac{1}{2}\left(\frac{1}{\sigma_y^2}\left[\sum_{i=1}^n y_i^2 - \sum_{n=1}^n 2\theta x_i y_i + \sum_{i=1}^n \theta^2 x_i^2\right]\right)\right] \\
&= exp\left[-\frac{1}{2}\left(\frac{1}{\sigma_y^2}\left[\sum_{i=1}^n y_i^2 - 2\theta\sum_{n=1}^n x_i y_i + \theta^2\sum_{i=1}^n x_i^2\right]\right)\right] \\
&= exp\left[-\frac{1}{2}\left(\sigma_y^{-2}\sum_{i=1}^n y_i^2 - 2\theta\sigma_y^{-2}\sum_{n=1}^n x_i y_i + \theta^2\sigma_y^{-2}\sum_{i=1}^n x_i^2\right)\right] \\
&\propto exp\left[-\frac{1}{2}\left(\theta^2\sigma_y^{-2}\sum_{i=1}^n x_i^2 - 2\theta\sigma_y^{-2}\sum_{n=1}^n x_i y_i\right)\right]
\end{aligned}$$

Now, given the formula: $p(\theta|y) \propto exp\left[-\frac{1}{2}\left(Q_\theta\theta^2 - 2\ell_\theta\theta\right)\right]$

We get the results that:

$Q_\theta = \sigma_y^{-2}\sum_{i=1}^n x_i^2$
$\ell_\theta = \sigma_y^{-2}\sum_{n=1}^n x_i y_i$

$$\begin{aligned}
p(\theta|y,\tau_y) &\sim N(Q_\theta^{-1}\ell_\theta, Q_\theta^{-1}) \\
&\sim N\left((\sigma_y^{-2}\sum_{i=1}^n x_i^2)^{-1} \times \sigma_y^{-2}\sum_{n=1}^n x_i y_i, (\sigma_y^{-2}\sum_{i=1}^n x_i^2)^{-1}\right)
\end{aligned}$$

## Part ii

Prior Distribution: $\tau_y \sim Gamma(\alpha, \beta)$
Normal Likelihood: The derivation from Part i
Posterior Distribution: $[\tau_y|y] \sim Gamma(\alpha + \frac{n-1}{2}, \beta + \frac{(n-1)s^2}{2}$

Our Goal:

1. Remove terms that don't depend on $\theta$
2. Integrate with respect to $\theta$ (integral is proportional to Normal pdf)
3. See that result is proportional to Gamma pdf

$$
\begin{aligned}
p(\tau_t|y) &= \int_\theta p(\tau_y, \theta|y)p(\theta)d\theta \\
&= \int_\theta p(\tau_y)p(y|\theta, \tau_y)d\theta \\
&= \int_\theta \theta^{\alpha-1}e^{-\beta\theta} \times exp\Bigg[-\frac{1}{2}\Bigg(\theta^2\sigma_y^{-2}\sum_{i=1}^n x_i^2 - 2\theta\sigma_y^{-2}\sum_{i=1}^n x_i y_i\Bigg)\Bigg]d\theta \\
&= \int_\theta \theta^{\alpha-1} \times exp\Bigg[-\frac{1}{2}\Bigg(\theta^2\sigma_y^{-2}\sum_{i=1}^n x_i^2 - 2\theta\sigma_y^{-2}\sum_{i=1}^n x_i y_i\Bigg) - \beta\theta\Bigg]d\theta \\
&= \int_\theta \theta^{\alpha-1} \times exp\Bigg[-\Bigg(\frac{1}{2}\theta^2\sigma_y^{-2}\sum_{i=1}^n x_i^2 - \theta\sigma_y^{-2}\sum_{i=1}^n x_i y_i + \beta\Bigg)\theta\Bigg]d\theta
\end{aligned}
$$

I could not get passed this point as I was unable to get this quantity into the form required for it to represent a Gamma distribution. As you can see, I understand the general concept, but I was unable to get past this point in this problem. I have it in the Gamma form (with $\beta$ being equal to term in the parentheses), but am unsure how to proceed.

## Part iii

Below is a sample of 10 draws from my sample of $10^4$ draws of the posterior distribution, $[\theta, \tau_y|y]$. Please refer to my code (in the Code Appendix section) for the precise sampling strategy.

```
## Sample Draw:
##
##  76.6535 76.54076 76.55635 76.42906 76.63199 76.269 76.50727 76.597 76.61971 76.53184
```

## Part iv

Below is the 95% HPD interval for the age of the universe in years after the designated data transformations. Again, please refer to my code in the Code Appendix section for details on the algorithm used.

```
## HPD Interval:  ( 12763102059 , 12828149442 )
```

## Problem 3

Sampling Distribution: $p(y|\theta) = \frac{\theta^y e^-\theta}{y!}$
Domain: $\theta > 0$

$$\begin{aligned}
p(y|\theta) &= \prod_{i=1}^{n} \frac{\theta^{y_i} e^- \theta}{y_i!} \\
&= \theta^{\sum_{i=1}^{n} y_i} e^{-n\theta} \times \prod_{i=1}^{n} \frac{1}{y_i!} \\
log[p(y|\theta)] &= log\left[\theta^{\sum_{i=1}^{n} y_i} e^{-n\theta} \times \prod_{i=1}^{n} \frac{1}{y_i!}\right] \\
&= \left[\sum_{i=1}^{n} y_i\right] log(\theta) - n\theta - log\left[\sum_{i=1}^{n} y_i!\right] \\
\frac{d}{d\theta} log[p(y|\theta)] &= \frac{d}{d\theta}\left[\left[\sum_{i=1}^{n} y_i\right] log(\theta)\right] - \frac{d}{d\theta}(n\theta) + 0 \\
&= \left[\sum_{i=1}^{n} y_i\right] \frac{1}{\theta} - n \\
\frac{d^2}{d\theta^2} log[p(y|\theta)] &= \left[\sum_{i=1}^{n} \frac{d}{d\theta}\theta^{-1}\right] + 0 \\
&= \frac{-\sum_{i=1}^{n} y_i}{\theta^2} \\
E\left[\frac{d^2 log[p(y|\theta)]}{d\theta^2}\middle|\theta\right] &= \frac{-n\theta}{\theta^2} \\
&= \frac{-n}{\theta} \\
\sqrt{-E\left[\frac{d^2 log[p(y|\theta)]}{d\theta^2}\middle|\theta\right]} &= \sqrt{\frac{n}{\theta}} \\
&\propto \theta^{-\frac{1}{2}} \\
&\sim Gamma\left(\alpha = \frac{1}{2}, \beta = 0\right)
\end{aligned}$$

# Code Appendix

```
########## Problem 2 ##########




## Set options

options(scipen = 999)

## Load in the necessary packages

library(gamair)
library(coda)

## Load in the data

data("hubble")

x <- hubble$x
y <- hubble$y

## Setup the parameters

alpha <- 0.01
beta <- 0.01
n <- 24
theta_hat <- sum(x * y) / sum(x^2)
s_squared <- (1 / (n - 1)) * sum(y - (x * theta_hat))^2



##### Part iii #####



## Make 10,000 draws from [tau_y|y], tau^s

draws_1 <- rgamma(n = 10000, shape = alpha + ((n - 1) / 2),
                  rate = beta + (n - 1) * (s_squared / 2))

## Make 10,000 draws from [theta|y, tau_y = t^s]

var_y <- 1 / draws_1

draws_2 <- rnorm(n = 10000,
                 mean = ((1/var_y) * sum(x^2))^(-1) * (1/var_y) * sum(x * y),
                 sd = sqrt(((1/var_y) * sum(x^2))^(-1))
                 )

## Print out a sample of the draws
```

```r
draw_samp <- sample(x = draws_2, size = 10)

cat("Sample Draw:\n\n", draw_samp)



##### Part iv #####



## Construct the HPD interval

post_hub <- 1/draws_2*3.09e19/(60^2*24*365)

hpd_int <- as.vector(HPDinterval(as.mcmc(post_hub), prob = 0.95))

cat("HPD Interval: ", "(", hpd_int[1], ",", hpd_int[2], ")")
```