

Assignment #2

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Problem 1

Part a

Here our goal will be to minimize a to show that $a = E[\theta|y]$ is the unique Bayes estimate of θ :

$$\begin{aligned}\frac{d}{da}E[L(a|y)] &= \frac{d}{da} \int L(\theta, a)p(\theta|y)d\theta \\ &= \frac{d}{da} \int (\theta - a)^2 p(\theta|y)d\theta \\ &= -2 \int (\theta - a)p(\theta|y)d\theta \\ &= -2 \left[\int \theta p(\theta|y)d\theta - a \int p(\theta|y)d\theta \right] \\ &= -2[E[\theta|y] - a]\end{aligned}$$

$$-2[E[\theta|y] - a] = 0 \text{ when } a = E[\theta|y]$$

To prove that it is a unique minimizing statistic, we must look at the second derivative:

$$\frac{d}{da}(-2[E[\theta|y] - a]) = 2$$

As $2 > 0$, this shows that it is a unique minimizing statistic.

Part b

Here our goal will be to show that for any median value of a , the derivative of $L(\theta, a)$ will evaluate to 0.

$$\begin{aligned}\frac{d}{da}[E[L(a|y)]] &= \frac{d}{da} \left[\int_{-\infty}^a (a - \theta)p(\theta|y)d\theta + \int_a^{\infty} (\theta - a)p(\theta|y)d\theta \right] \\ &= \int_{-\infty}^a \frac{d}{da}(a - \theta)p(\theta|y)d\theta + \int_a^{\infty} \frac{d}{da}(\theta - a)p(\theta|y)d\theta \\ &= \int_{-\infty}^a p(\theta|y)d\theta + \int_a^{\infty} (-1)p(\theta|y)d\theta \\ &= \int_{-\infty}^a p(\theta|y)d\theta - \int_a^{\infty} p(\theta|y)d\theta \\ &= \frac{1}{2} - \frac{1}{2} \\ &= 0\end{aligned}$$

As a result, it has been shown that any posterior median of θ is a Bayes estimate of θ .

Part c

Here our goal will be to show that for any value of a , the derivative of $L(\theta, a)$ will evaluate to 0 where k_0 and k_1 are nonnegative numbers.

$$\begin{aligned}
\frac{d}{da} [E[L(a|y)]] &= \frac{d}{da} \left[\int_{-\infty}^a k_1(a - \theta)p(\theta|y)d\theta + \int_a^{\infty} k_0(\theta - a)p(\theta|y)d\theta \right] \\
&= \int_{-\infty}^a \frac{d}{da} k_1(a - \theta)p(\theta|y)d\theta + \int_a^{\infty} \frac{d}{da} k_0(\theta - a)p(\theta|y)d\theta \\
&= \int_{-\infty}^a k_1 p(\theta|y)d\theta + \int_a^{\infty} (-k_0)p(\theta|y)d\theta \\
&= \int_{-\infty}^a k_1 p(\theta|y)d\theta - \int_a^{\infty} k_0 p(\theta|y)d\theta \\
&= k_1 \int_{-\infty}^a p(\theta|y)d\theta - k_0 \int_a^{\infty} p(\theta|y)d\theta
\end{aligned}$$

Noting that: $k_0 \int_a^{\infty} p(\theta|y)d\theta = k_0 - k_0 \int_{-\infty}^a p(\theta|y)d\theta$

$$\begin{aligned}
k_1 \int_{-\infty}^a p(\theta|y)d\theta - k_0 \int_a^{\infty} p(\theta|y)d\theta &= k_1 \int_{-\infty}^a p(\theta|y)d\theta - \left[k_0 - k_0 \int_{-\infty}^a p(\theta|y)d\theta \right] \\
&= k_1 \int_{-\infty}^a p(\theta|y)d\theta + k_0 \int_{-\infty}^a p(\theta|y)d\theta - k_0 \\
&= (k_1 + k_0) \int_{-\infty}^a p(\theta|y)d\theta - k_0
\end{aligned}$$

Now setting $\int_{-\infty}^a p(\theta|y)d\theta = \frac{k_0}{k_0 + k_1}$ we get our result that any quantile is a Bayes estimate of θ .

Taking the second derivative we again get a positive number, thus again indicating that it is a minimizing statistic.

Problem 2

$n = 20$

Sampling Distribution: $y|\theta \sim \text{Binomial}(n = 20, \theta)$

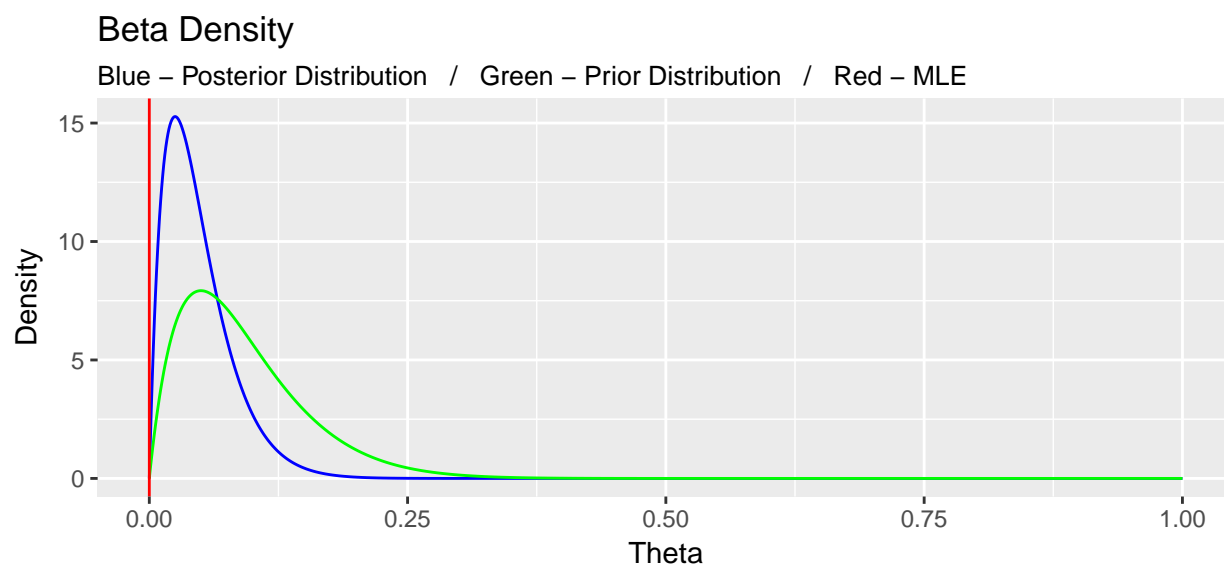
Prior Distribution: $\theta \sim \text{Beta}(\alpha = 2, \beta = 20)$

Posterior Distribution:

$$\begin{aligned}
p(\theta|y) &= \binom{n}{y} \theta^y (1 - \theta)^{n-y} \times \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (1 - \theta)^{\beta-1} \\
&= \binom{n}{y} \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{y+\alpha-1} (1 - \theta)^{n-y+\beta-1} \\
&\propto \text{Beta}(y + \alpha, (n - y) + \beta) \\
&= \text{Beta}(y + 2, (20 - y) + 20)
\end{aligned}$$

Part i

$y = 0$



Part ii

HPD Interval: (0.001957683 , 0.1110038)

Frequentist Interval: (0 , 0)

Part iii

HPD Interval: (-4.872338 , -1.766281)