Assignment #1

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Problem 1

Part i

$f(x)$ is the pdf of any of the x_i, and since they are *iid*, they all follow one pdf.

The joint pdf is: $f_{X_1,...,X_n}(x_1,...,x_n) = f(x_1) \times ... \times f(x_n)$ because they are *iid*.

Since you can multiply the right-hand side in any order, this implies symmetry in regards to the left-hand side.

As a result, $X_1, ..., X_n$ are exchangeable.

Part ii

$$
\begin{aligned}
p(y_1,...,y_n) &= \int p(y_1,...,y_n|\theta)p(\theta)d(\theta) \\
&= \int \left(\prod_{i=1}^{n} p(y_i|\theta)\right) p(\theta)d(\theta)) \\
&= \int \left(\prod_{i=1}^{n} p(y_{\pi_i}|\theta)\right) p(\theta)d(\theta)) \\
&= \int p(y_{\pi_1,...,y_{\pi_n}}|\theta)p(\theta)d(\theta) \\
&= p(y_{\pi_1,...,y_{\pi_n}})
\end{aligned}
$$

Where:

- The first line is the definition of marginal probability
- The second line is because the Y_i's are conditionally *iid*
- The third line is because the product does not depend on order
- The fourth line we are converting back to the form used in the first line
- Finally, the last line is the definition of marginal probability

Problem 2

Part a

Prior Density: $p(\theta) = 1$
Sampling Distribution: $p(y|\theta) = \binom{n}{y}\theta^y(1-\theta)^{n-y}$

Posterior Density: $p(\theta|y) \propto p(y|\theta)p(\theta) = \binom{n}{y}\theta^y(1-\theta)^{n-y}$
Posterior Distribution: $\theta|y \sim Beta(y+1, (n-y)+1)$

$$
\begin{aligned}
\binom{n}{y}\frac{\Gamma(y+1)\Gamma(n-y+1)}{\Gamma((y+1)+(n-y+1))} &= \binom{n}{y}\frac{\Gamma(y+1)\Gamma(n-y+1)}{\Gamma(n+2)} \\
&= \frac{n!}{y!(n-y)!}\frac{y!(n-y)!}{(n+1)!} \\
&= \frac{n!}{(n+1)!} \\
&= \frac{1}{n+1}
\end{aligned}
$$

Part b

Prior Distribution: $\theta \sim Beta(\alpha, \beta)$
Sampling Distribution: $y|\theta \sim Binomial(n, \theta)$

Posterior Density:

$$
\begin{aligned}
p(\theta|y) &= \binom{n}{y}\theta^{y}(1-\theta)^{n-y} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1} \\
&= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1} \\
&\propto Beta(y+\alpha, (n-y)+\beta)
\end{aligned}
$$

Posterior Mean:

$$
\begin{aligned}
\frac{y+\alpha}{\alpha+\beta+n} &= \frac{y}{n} + \lambda\left(\frac{\alpha}{\alpha+\beta} - \frac{y}{n}\right) \\
\frac{y+\alpha}{\alpha+\beta+n} - \frac{y}{n} &= \lambda\left(\frac{\alpha}{\alpha+\beta} - \frac{y}{n}\right) \\
\frac{ny+n\alpha-\alpha y-\beta y-ny}{(\alpha+\beta+n)n} &= \lambda\left(\frac{n\alpha-\alpha y-\beta y}{(\alpha+\beta)n}\right) \\
\lambda &= \frac{\alpha+\beta}{\alpha+\beta+n}
\end{aligned}
$$

Since λ will always be between 0 and 1 the Posterior Mean will act as a weighted average between our Prior Mean, $\frac{y}{n}$, and the data.

Part c

Posterior Density: $\theta|y \sim Beta(y+\alpha, (n-y)+\beta)$
Prior Variance ($\alpha = 1, \beta = 1$): $\frac{1}{12}$
Posterior Variance ($\alpha = 1, \beta = 1$):

$$
\begin{aligned}
\frac{(y+1)(n-y+1)}{(n+2)^2(n+3)} &= \frac{ny-y^2+y+n-y+1}{(n^2+4n+4)(n+3)} \\
&= \frac{ny-y^2+y+n-y+1}{n^3+3n^2+4n^2+12n+4n+12} \\
&= \frac{-y^2+ny+n+1}{n^3+7n^2+16n+12}
\end{aligned}
$$

Now, we can deduce that smaller values of n will maximize this quantity; since $n \geq y$, we will set $n = y = 1$.

Posterior Variance ($n = y = 1$): $\frac{-y^2+ny+n+1}{n^3+7n^2+16n+12} = \frac{2}{36} = \frac{1}{18}$

Thus, the Posterior Variance, which we just maximized, is always less than the Prior Variance of $\frac{1}{12}$.

Part d

- $n = y = 1$
- $\alpha = 1$
- $\beta = 10$

Prior Variance: $\frac{(1)(10)}{(11)^2(12)} = \frac{10}{1452} = 0.0069$
Posterior Variance: $\frac{(2)(10)}{(12)^2(13)} = \frac{20}{1872} = 0.0107$

Problem 3

Part a

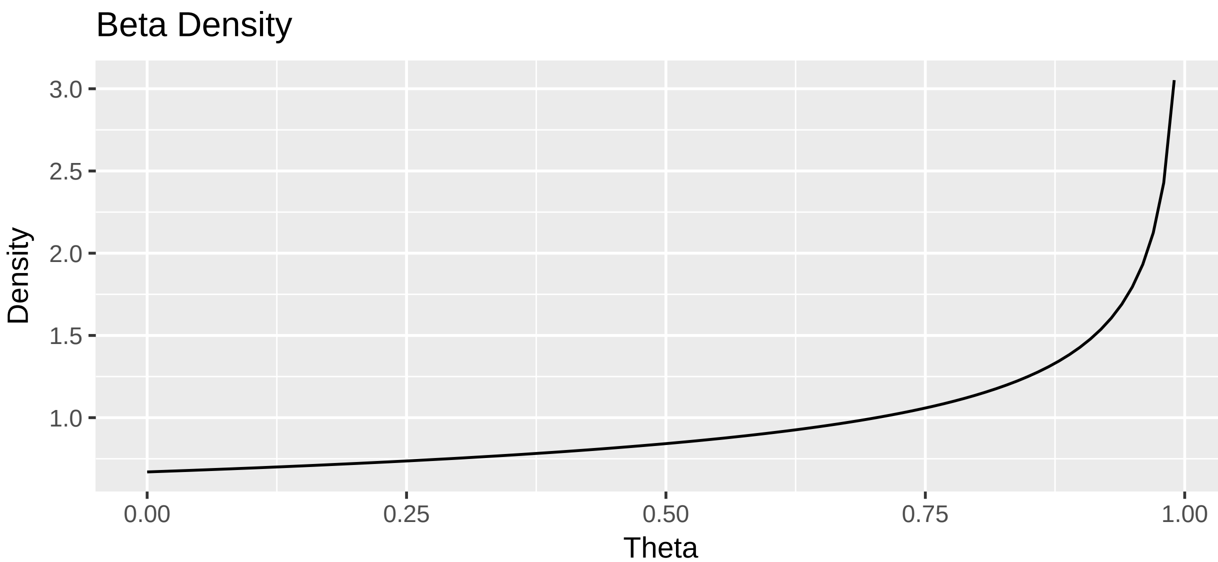
Prior Distribution: $\theta \sim Beta(\alpha, \beta)$
Prior Mean: 0.6
Prior Variance: 0.09

Sampling Distribution: $y|\theta \sim Binomial(n, \theta)$

$$\begin{aligned}
\mu &= \frac{\alpha}{\alpha+\beta} \\
(\alpha+\beta)\mu &= \alpha \\
\alpha\mu + \beta\mu &= \alpha \\
\beta\mu &= \alpha - \alpha\mu \\
\beta &= \alpha(\frac{1}{\mu} - 1)
\end{aligned}$$

$$\begin{aligned}
\sigma^2 &= \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)} \\
\alpha &= \left(\frac{1-\mu}{\sigma^2} - \frac{1}{\mu}\right)\mu^2
\end{aligned}$$

$\alpha = 1$
$\beta = 0.67$



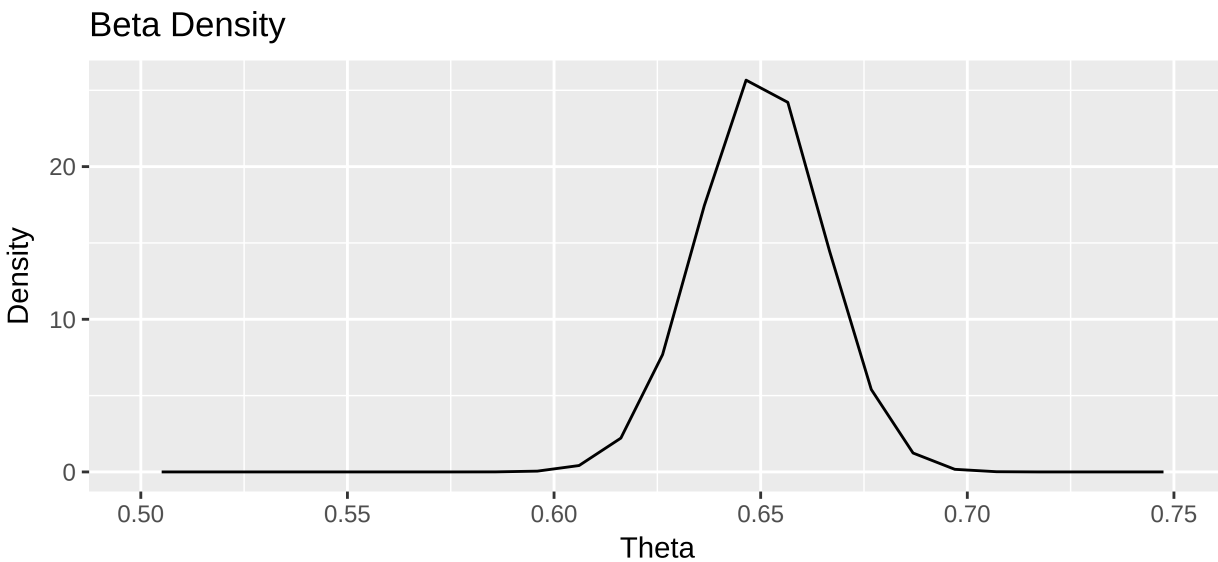
Part b

$n = 1000$
$y = 650$
Prior Distribution: $\theta \sim Beta(\alpha, \beta)$
Sampling Distribution: $y|\theta \sim Binomial(n, \theta)$

Posterior Density:

$$\begin{aligned} p(\theta|y) &= \binom{n}{y}\theta^y(1-\theta)^{n-y} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1} \\ &= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1} \\ &\propto Beta(y+\alpha, (n-y)+\beta) \end{aligned}$$

$\alpha_{post} = y + \alpha$
$\beta_{post} = (n - y) + \beta$
Posterior Mean: $\mu_{post} = \frac{y+\alpha}{(y+\alpha)+((n-y)+\beta)} = 0.6499$
Posterior Variance: $\sigma^2_{post} = \frac{(y+\alpha)((n-y)+\beta)}{((y+\alpha)+((n-y)+\beta))^2((y+\alpha)+((n-y)+\beta)+1)} = 0.0151$

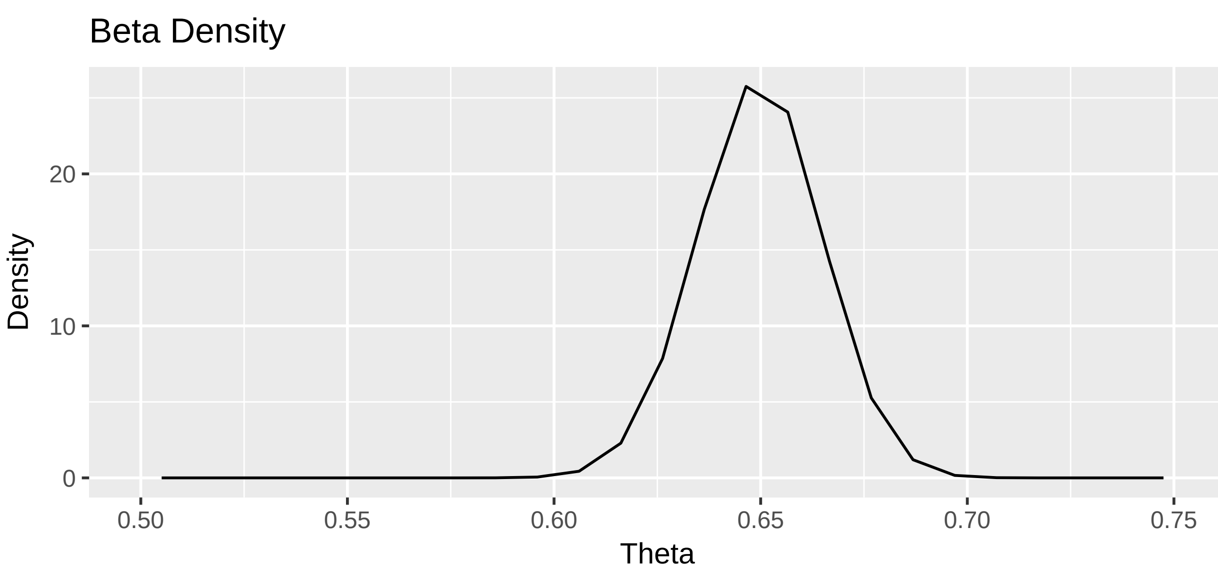


Part c

First Sensitivity Check - Uniform Prior

Prior Mean: $\frac{1}{2}$
Prior Variance: $\frac{1}{12}$
Prior Distribution: $\theta \sim Uniform(0, 1)$
Sampling Distribution: $y|\theta \sim Binomial(n, \theta)$
Posterior Density:

$$\begin{aligned}
p(\theta|y) &= \binom{n}{y}\theta^y(1-\theta)^{n-y} \times 1 \\
&= \binom{n}{y}\theta^y(1-\theta)^{n-y} \times 1 \\
&\propto Beta(y+1, (n-y)+1)
\end{aligned}$$

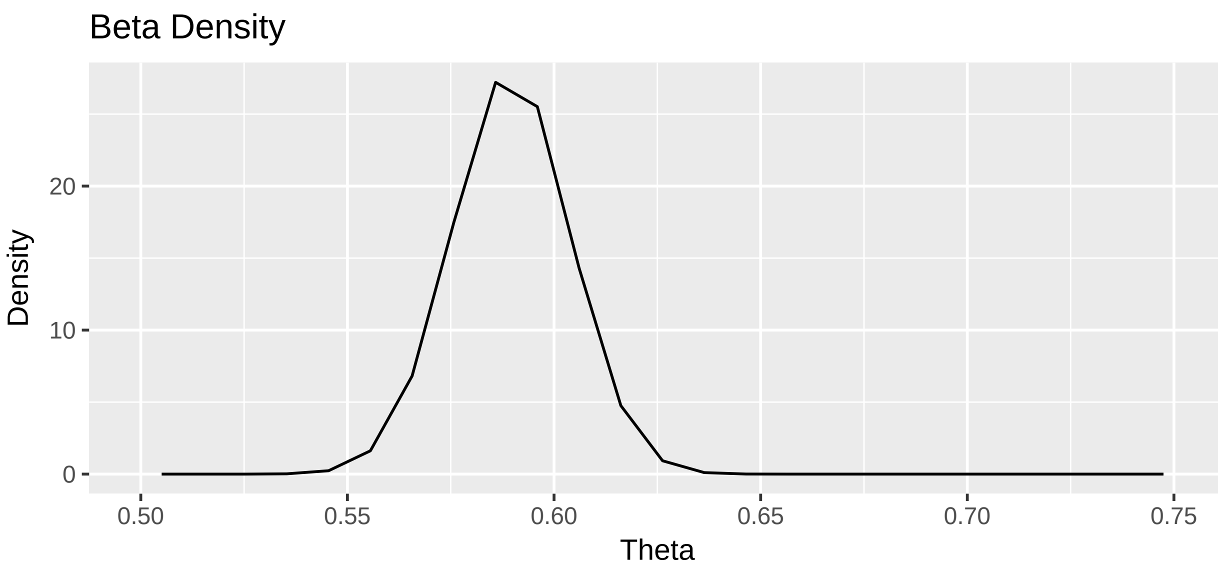


Second Sensitivity Check - Beta Prior

Prior Mean: 0.3
Prior Variance: 0.001
Prior Distribution: $\theta \sim Beta(62.7, 146.3)$
Sampling Distribution: $y|\theta \sim Binomial(n, \theta)$

Posterior Density:

$$\begin{aligned} p(\theta|y) &= \binom{n}{y} \theta^y (1-\theta)^{n-y} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1} \\ &= \binom{n}{y} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{y+\alpha-1} (1-\theta)^{n-y+\beta-1} \\ &\propto Beta(y+\alpha, (n-y)+\beta) \end{aligned}$$

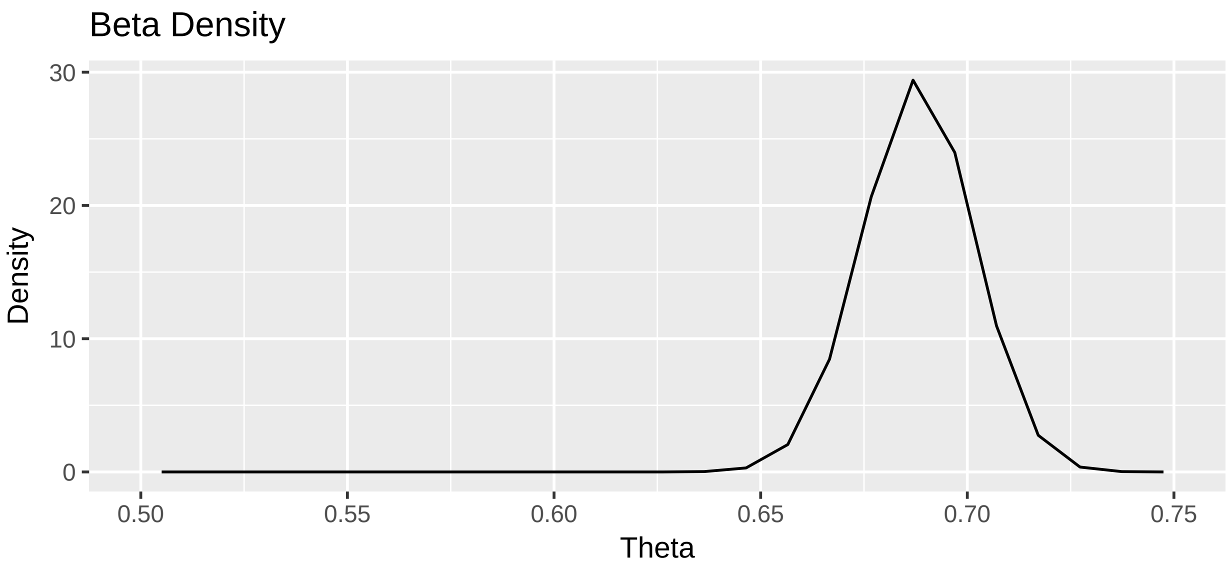


Third Sensitivity Check - Beta Prior

Prior Mean: 0.9
Prior Variance: 0.0005
Prior Distribution: $\theta \sim Beta(161.1, 17.9)$
Sampling Distribution: $y|\theta \sim Binomial(n, \theta)$

Posterior Density:

$$\begin{aligned} p(\theta|y) &= \binom{n}{y} \theta^y (1-\theta)^{n-y} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{\alpha-1} (1-\theta)^{\beta-1} \\ &= \binom{n}{y} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \theta^{y+\alpha-1} (1-\theta)^{n-y+\beta-1} \\ &\propto Beta(y+\alpha, (n-y)+\beta) \end{aligned}$$



Conclusion

We can see that a prior distribution such as $Uniform(0, 1)$ has a negligible affect on the posterior distribution, whereas, our $Beta(\alpha, \beta)$ distributions with prior means and variances that are substantially different enough from the observed data will have a noticeable affect on the posterior distribution. This shows us how our prior distribution may have a large affect on our posterior distribution, depending on the family, and parameters of said family, that we select.