Assignment #1

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Problem 1

Part i

$f(x)$ is the pdf of any of the x_i, and since they are *iid*, they all follow one pdf.

The joint pdf is: $f_{X_1,...,X_n}(x_1,...,x_n) = f(x_1) \times ... \times f(x_n)$ because they are *iid*.

Since you can multiply the right-hand side in any order, this implies symmetry in regards to the left-hand side.

As a result, $X_1, ..., X_n$ are exchangeable.

Part ii

$$\begin{aligned}
p(y_1, ..., y_n) &= \int p(y_1, ..., y_n|\theta)p(\theta)d(\theta) \\
&= \int \left(\prod_{i=1}^{n} p(y_i|\theta)\right) p(\theta)d(\theta)) \\
&= \int \left(\prod_{i=1}^{n} p(y_{\pi_i}|\theta)\right) p(\theta)d(\theta)) \\
&= \int p(y_{\pi_1,...,y_{\pi_n}}|\theta)p(\theta)d(\theta) \\
&= p(y_{\pi_1,...,y_{\pi_n}})
\end{aligned}$$

Problem 2

Part a

Prior Density: $p(\theta) = 1$
Sampling Distribution: $p(y|\theta) = \binom{n}{y}\theta^y(1-\theta)^{n-y}$

Posterior Density: $p(\theta|y) \propto p(y|\theta)p(\theta) = \binom{n}{y}\theta^y(1-\theta)^{n-y}$
Posterior Distribution: $\theta|y \sim Beta(y+1, (n-y)+1)$

$$\begin{aligned}\binom{n}{y}\frac{\Gamma(y+1)\Gamma(n-y+1)}{\Gamma((y+1)+(n-y+1))} &= \binom{n}{y}\frac{\Gamma(y+1)\Gamma(n-y+1)}{\Gamma(n+2)}\\ &= \frac{n!}{y!(n-y)!}\frac{y!(n-y)!}{(n+1)!}\\ &= \frac{n!}{(n+1)!}\\ &= \frac{1}{n+1}\end{aligned}$$

Part b

Prior Distribution: $\theta \sim Beta(\alpha, \beta)$
Sampling Distribution: $y|\theta \sim Binomial(n, \theta)$

Posterior Density:

$$\begin{aligned}p(\theta|y) &= \binom{n}{y}\theta^{y}(1-\theta)^{n-y} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}\\ &= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1}\\ &\propto Beta(y+\alpha, (n-y)+\beta)\end{aligned}$$

Posterior Mean:

$$\begin{aligned}\frac{y+\alpha}{\alpha+\beta+n} &= \frac{y}{n} + \lambda\left(\frac{\alpha}{\alpha+\beta} - \frac{y}{n}\right)\\ \frac{y+\alpha}{\alpha+\beta+n} - \frac{y}{n} &= \lambda\left(\frac{\alpha}{\alpha+\beta} - \frac{y}{n}\right)\\ \frac{ny+n\alpha-\alpha y-\beta y-ny}{(\alpha+\beta+n)n} &= \lambda\left(\frac{n\alpha-\alpha y-\beta y}{(\alpha+\beta)n}\right)\\ \lambda &= \frac{\alpha+\beta}{\alpha+\beta+n}\end{aligned}$$

Since λ will always be between 0 and 1 the Posterior Mean will act as a weighted average between our Prior Mean, $\frac{y}{n}$, and the data.

Part c

Posterior Density: $\theta|y \sim Beta(y+\alpha, (n-y)+\beta)$
Prior Variance ($\alpha = 1, \beta = 1$): $\frac{1}{12}$
Posterior Variance ($\alpha = 1, \beta = 1$):

$$\begin{aligned}\frac{(y+1)(n-y+1)}{(n+2)^2(n+3)} &= \frac{ny-y^2+y+n-y+1}{(n^2+4n+4)(n+3)}\\ &= \frac{ny-y^2+y+n-y+1}{n^3+3n^2+4n^2+12n+4n+12}\\ &= \frac{-y^2+ny+n+1}{n^3+7n^2+16n+12}\end{aligned}$$

Now, we can deduce that smaller values of n will maximize this quantity; since $n \geq y$, we will set $n = y = 1$.

Posterior Variance ($n = y = 1$): $\frac{-y^2+ny+n+1}{n^3+7n^2+16n+12} = \frac{2}{36} = \frac{1}{18}$

Thus, the Posterior Variance, which we just minimized, is always less than the Prior Variance of $\frac{1}{12}$.

Part d

- $n = y = 1$
- $\alpha = 1$
- $\beta = 10$

Prior Variance: $\frac{(1)(10)}{(11)^2(12)} = \frac{10}{1452} = 0.0069$
Posterior Variance: $\frac{(2)(10)}{(12)^2(13)} = \frac{20}{1872} = 0.0107$

Problem 3

Part a