Assignment #4

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Problem 1

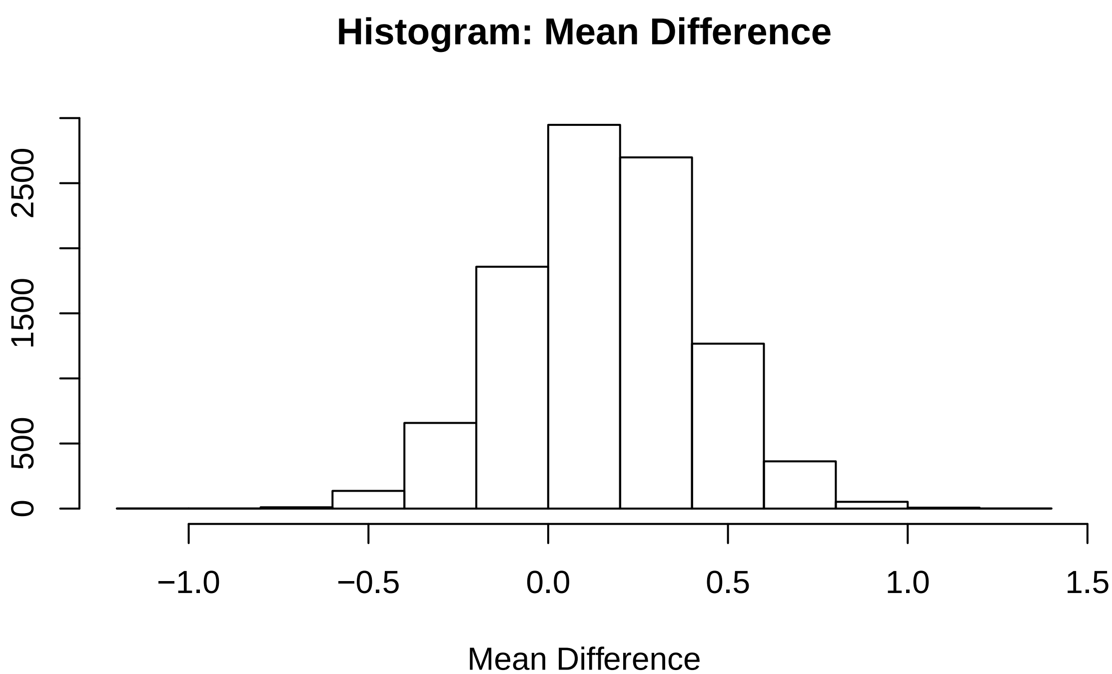
Part i

Please see the Code Appendix for the sampling technique.

Part ii

Please see the Code Appendix for the sampling technique.

Part iii



Part iv

```
## HPD Interval:  ( 0.7911925 , 1.476375 )
```

Since out interval does not contain zero, we can be certain that the treatment group mean is larger than the control group mean, thus implying that the treatment does have a material affect as compared to the control.

Problem 2

Part i

Prior Distribution:

$$
\begin{aligned}
p(\theta) &\propto exp\Big[-\frac{1}{2}\big(\theta-\mu_\theta\big)^T\Sigma_\theta^{-1}(\theta-\mu_\theta)\Big] \\
&\propto exp\Big[-\frac{1}{2}\big(\theta^T\Sigma_\theta^{-1}\theta-2\theta^T\Sigma_\theta^{-1}\mu_\theta+\mu_\theta^T\Sigma_\theta^{-1}\mu_\theta\Big] \\
&\propto exp\Big[-\frac{1}{2}\big(\theta^T\Sigma_\theta^{-1}\theta-2\theta^T\Sigma_\theta^{-1}\mu_\theta+\mu_\theta^T\Sigma_\theta^{-1}\mu_\theta\big)\Big] \\
&\propto exp\Big[-\frac{1}{2}\big(\theta^T\Sigma_\theta^{-1}\theta-2\theta^T\Sigma_\theta^{-1}\mu_\theta\big)\Big] \\
&\propto exp\Big[-\frac{1}{2}\big(\theta^TQ_{\theta_1}\theta-2\theta^T\ell_{\theta_1}\big)\Big]
\end{aligned}
$$

Where:

$Q_{\theta_1}=\Sigma_\theta^{-1}$
$\ell_{\theta_1}=\Sigma_\theta^{-1}\mu_\theta$

Likelihood:

$$
\begin{aligned}
p(y|\theta) &\propto exp\Bigg[-\frac{1}{2}\Bigg(\sum_{i=1}^n y_i^T\Sigma^{-1}y_i-2\sum_{i=1}^n\theta^T\Sigma^{-1}y_i+\sum_{i=1}^n\theta^T\Sigma^{-1}\theta\Bigg)\Bigg] \\
&\propto exp\Bigg[-\frac{1}{2}\Bigg(\theta^Tn\Sigma^{-1}\theta-2\theta^T\Sigma^{-1}n\bar{y}\Bigg)\Bigg] \\
&\propto exp\Bigg[-\frac{1}{2}\Bigg(\theta^TQ_{\theta_2}\theta-2\theta^T\ell_{\theta_2}\Bigg)\Bigg]
\end{aligned}
$$

Where:

$Q_{\theta_2}=n\Sigma^{-1}$
$\ell_{\theta_2}=\Sigma^{-1}n\bar{y}$

Posterior:

$$
\begin{aligned}
p(\theta|y) &\propto exp\Big[-\frac{1}{2}\big(\theta^TQ_{\theta_1}\theta-2\theta^T\ell_{\theta_1}\big)\Big]\times exp\Bigg[-\frac{1}{2}\Bigg(\theta^TQ_{\theta_2}\theta-2\theta^T\ell_{\theta_2}\Bigg)\Bigg] \\
&\propto exp\Big[-\frac{1}{2}\theta^TQ_{\theta_1}\theta+\theta^T\ell_{\theta_1}-\frac{1}{2}\theta^TQ_{\theta_2}\theta+\theta^T\ell_{\theta_2}\Big] \\
&\propto exp\Big[\theta^T\big(\ell_{\theta_1}+\ell_{\theta_2}\big)-\frac{1}{2}\theta^T\big(Q_{\theta_1}+Q_{\theta_2}\big)\theta\Big] \\
&\propto exp\Big[-\frac{1}{2}\big(\theta^TQ_\theta\theta-2\theta^T\ell_\theta\big)\Big]
\end{aligned}
$$

Where:

$Q_\theta = Q_{\theta_1} + Q_{\theta_2} = \Sigma_\theta^{-1} + n\Sigma_\theta^{-1}$
$\ell_\theta = \ell_{\theta_1} + \ell_{\theta_2} = \Sigma_\theta^{-1}\mu_\theta + \Sigma^{-1}n\bar{y}$

Thus:

$\theta|y \sim MVN(Q_\theta^{-1}\ell_\theta, Q_\theta^{-1})$

Part ii

Posterior Mean, $Q_\theta^{-1}\ell_\theta$:

$$\begin{pmatrix} \frac{1}{\sigma^2_{\theta_1}} & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{\sigma^2_{\theta_2}} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{\sigma^2_{\theta_3}} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{\sigma^2_{\theta_4}} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} + \begin{pmatrix} \frac{n}{\sigma^2_1} & 0 & 0 & 0 & \dots \\ 0 & \frac{n}{\sigma^2_2} & 0 & 0 & \dots \\ 0 & 0 & \frac{n}{\sigma^2_3} & 0 & \dots \\ 0 & 0 & 0 & \frac{n}{\sigma^2_4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} \times \begin{pmatrix} \frac{1}{\sigma^2_{\theta_1}} & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{\sigma^2_{\theta_2}} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{\sigma^2_{\theta_3}} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{\sigma^2_{\theta_4}} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \mu_\theta + \begin{pmatrix} \frac{n\bar{y}}{\sigma^2_1} & 0 & 0 & 0 & \dots \\ 0 & \frac{n\bar{y}}{\sigma^2_2} & 0 & 0 & \dots \\ 0 & 0 & \frac{n\bar{y}}{\sigma^2_3} & 0 & \dots \\ 0 & 0 & 0 & \frac{n\bar{y}}{\sigma^2_4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Posterior Covariance, Q_θ^{-1}:

$$\begin{pmatrix} \frac{1}{\sigma^2_{\theta_1}} & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{\sigma^2_{\theta_2}} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{\sigma^2_{\theta_3}} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{\sigma^2_{\theta_4}} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} + \begin{pmatrix} \frac{n}{\sigma^2_1} & 0 & 0 & 0 & \dots \\ 0 & \frac{n}{\sigma^2_2} & 0 & 0 & \dots \\ 0 & 0 & \frac{n}{\sigma^2_3} & 0 & \dots \\ 0 & 0 & 0 & \frac{n}{\sigma^2_4} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1}$$

Part iii

Posterior Mean, $Q_\theta^{-1}\ell_\theta$:

$$\begin{pmatrix} \frac{1}{\sigma^2_{\theta}} & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{\sigma^2_{\theta}} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{\sigma^2_{\theta}} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{\sigma^2_{\theta}} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} + \begin{pmatrix} \frac{n}{\sigma^2} & 0 & 0 & 0 & \dots \\ 0 & \frac{n}{\sigma^2} & 0 & 0 & \dots \\ 0 & 0 & \frac{n}{\sigma^2} & 0 & \dots \\ 0 & 0 & 0 & \frac{n}{\sigma^2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} \times \begin{pmatrix} \frac{1}{\sigma^2_{\theta}} & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{\sigma^2_{\theta}} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{\sigma^2_{\theta}} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{\sigma^2_{\theta}} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \mu_\theta + \begin{pmatrix} \frac{n\bar{y}}{\sigma^2} & 0 & 0 & 0 & \dots \\ 0 & \frac{n\bar{y}}{\sigma^2} & 0 & 0 & \dots \\ 0 & 0 & \frac{n\bar{y}}{\sigma^2} & 0 & \dots \\ 0 & 0 & 0 & \frac{n\bar{y}}{\sigma^2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Posterior Covariance, Q_θ^{-1}:

$$\begin{pmatrix} \frac{1}{\sigma^2_{\theta}} & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{\sigma^2_{\theta}} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{\sigma^2_{\theta}} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{\sigma^2_{\theta}} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} + \begin{pmatrix} \frac{n}{\sigma^2} & 0 & 0 & 0 & \dots \\ 0 & \frac{n}{\sigma^2} & 0 & 0 & \dots \\ 0 & 0 & \frac{n}{\sigma^2} & 0 & \dots \\ 0 & 0 & 0 & \frac{n}{\sigma^2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1}$$

Problem 3

Solving for a in equation (b) yields:

$$
\begin{aligned}
La &= \ell_\theta \\
a &= \frac{\ell_\theta}{L}
\end{aligned}
$$

Plugging in our result for a into equation (c) yields:

$$
\begin{aligned}
L^T\theta^* &= a + e \\
&= \frac{\ell_\theta}{L} + e \\
LL^T\theta^* &= \ell_\theta + Le \\
Q_\theta\theta^* &= \ell_\theta + Le \\
\theta^* &= \frac{\ell_\theta + Le}{Q_\theta} \\
&= Q_\theta^{-1}(\ell_\theta + Le) \\
&= Q_\theta^{-1}\ell_\theta + Q_\theta^{-1}Le \\
&= Q_\theta^{-1}\ell_\theta + N\big[0, (Q_\theta^{-1}L)^T I_d (Q_\theta^{-1}L)\big] \\
&= Q_\theta^{-1}\ell_\theta + N\big[0, L^T {Q_\theta^{-1}}^T Q_\theta^{-1}L\big] \\
&= Q_\theta^{-1}\ell_\theta + N\big[0, \frac{1}{Q_\theta}\big] \\
&= Q_\theta^{-1}\ell_\theta + N\big[0, Q_\theta^{-1}\big] \\
&\sim N(Q_\theta^{-1}\ell_\theta, Q_\theta^{-1})
\end{aligned}
$$

Thus we see that $\theta^* \sim N(Q_\theta^{-1}\ell_\theta, Q_\theta^{-1})$.

Code Appendix

```
## Load necessary packages

library(coda)
library(asbio)



########## Problem 1 ##########



##### Part i #####



num_draws <- 10^4
```

```r
mean_c <- 1.013
mean_t <- 1.173

## Draw from the scaled inverse chi-square

n_1 <- 32
df <- n_1 - 1

samp_i_1 <- rinvchisq(num_draws, df, scale=1/df)

## Draw from the normal distribution

samp_i_2 <- rnorm(num_draws, mean = mean_c, sqrt(samp_i_1))



##### Part ii #####

## Draw from the scaled inverse chi-square

n_2 <- 36
df <- n_2 - 1

samp_ii_1 <- rinvchisq(num_draws, df, scale=1/df)

## Draw from the normal distribution

samp_ii_2 <- rnorm(num_draws, mean = mean_t, sqrt(samp_ii_1))



##### Part iii #####

## Generate the histogram of mean differences

hist(samp_ii_2 - samp_i_2,
     xlab = "Mean Difference",
     ylab = "",
     main ="Histogram: Mean Difference")



##### Part iv #####

## Construct the HPD interval of mean differences

interval <- as.vector(HPDinterval(as.mcmc(samp_ii_2 - samp_ii_1), prob = 0.95)[1, 1:2])

## Output the result

cat("HPD Interval: ", "(", interval[1], ",", interval[2], ")")
```