

# Assignment #3

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## Problem 1

Sampling Distribution:  $y_1, \dots, y_n | \theta \stackrel{\text{iid}}{\sim} N(\theta, \sigma_y^2)$

Prior Distribution:  $\theta \sim N(\mu_\theta, \sigma_\theta^2)$

Posterior Distribution:  $\theta | y \sim N(Q_\theta^{-1} \ell_\theta, Q_\theta^{-1})$

Posterior Distribution:

$$\begin{aligned}
 p(\theta | y) &\propto p(y | \theta) p(\theta) \\
 &= \left[ \frac{1}{\sqrt{2\pi\sigma_y^2}} \right]^n \exp\left(-\frac{1}{2\sigma_y^2} \sum_{i=1}^n (y_i - \theta)^2\right) \times \frac{1}{\sqrt{2\pi\sigma_\theta^2}} \exp\left(-\frac{1}{2\sigma_\theta^2} (\theta - \mu_\theta)^2\right) \\
 &\propto \exp\left(-\frac{1}{2\sigma_\theta^2} (\theta - \mu_\theta)^2\right) \times \exp\left(-\frac{1}{2\sigma_y^2} \sum_{i=1}^n (y_i - \theta)^2\right) \\
 &= \exp\left(-\frac{\theta^2 - 2\theta\mu_\theta + \mu_\theta^2}{2\sigma_\theta^2}\right) \times \exp\left(-\frac{\sum_{i=1}^n (y_i^2 - 2\theta y_i + \theta^2)}{2\sigma_y^2}\right) \\
 &= \exp\left(\frac{-\theta^2 + 2\theta\mu_\theta - \mu_\theta^2}{2\sigma_\theta^2} - \frac{\sum_{i=1}^n (y_i^2 - 2\theta y_i + \theta^2)}{2\sigma_y^2}\right) \\
 &= \exp\left(\frac{\sigma_y^2[-\theta^2 + 2\theta\mu_\theta - \mu_\theta^2] - \sigma_\theta^2[\sum_{i=1}^n (y_i^2 - 2\theta y_i + \theta^2)]}{2\sigma_\theta^2\sigma_y^2}\right) \\
 &= \exp\left(\frac{-\theta^2(\sigma_y^2 + n\sigma_\theta^2) + 2\theta(\mu_\theta\sigma_y^2 + \sigma_\theta^2 y_1 + \dots + \sigma_\theta^2 y_n) - (\mu_\theta^2\sigma_y^2 + \sigma_\theta^2 y_1^2 + \dots + \sigma_\theta^2 y_n^2)}{2\sigma_\theta^2\sigma_y^2}\right) \\
 &= \exp\left(\frac{-\theta^2 + 2\theta\frac{\mu_\theta\sigma_y^2 + \sum_{i=1}^n \sigma_\theta^2 y_i}{\sigma_y^2 + n\sigma_\theta^2} - \left(\frac{\mu_\theta\sigma_y^2 + \sum_{i=1}^n \sigma_\theta^2 y_i}{\sigma_y^2 + n\sigma_\theta^2}\right)^2}{2\frac{\sigma_\theta^2\sigma_y^2}{\sigma_y^2 + n\sigma_\theta^2}}\right) \times \exp\left(-\frac{\mu_\theta^2\sigma_y^2 + \sum_{i=1}^n \sigma_\theta^2 y_i^2}{2\sigma_\theta^2\sigma_y^2}\right) \\
 &= \exp\left(-\frac{\left(\theta - \frac{\mu_\theta\sigma_y^2 + \sum_{i=1}^n \sigma_\theta^2 y_i}{\sigma_y^2 + n\sigma_\theta^2}\right)^2}{2\frac{\sigma_\theta^2\sigma_y^2}{\sigma_y^2 + n\sigma_\theta^2}}\right)
 \end{aligned}$$

From this result, we can see the following result:

$$\begin{aligned}
\theta|y &\sim N\left(\frac{\mu_\theta\sigma_y^2 + \sum_{i=1}^n \sigma_\theta^2 y_i}{\sigma_y^2 + n\sigma_\theta^2}, \frac{\sigma_y^2\sigma_\theta^2}{\sigma_y^2 + n\sigma_\theta^2}\right) \\
&\sim N\left(\frac{\mu_\theta\sigma_\theta^{-2} + \sigma_y^{-2} \sum_{i=1}^n y_i}{\sigma_\theta^{-2} + n\sigma_y^{-2}}, \frac{1}{\sigma_\theta^{-2} + n\sigma_y^{-2}}\right) \\
&\sim N(Q_\theta^{-1}\ell_\theta, Q_\theta^{-1})
\end{aligned}$$

Where the following holds:

$$\begin{aligned}
Q_\theta &= n\sigma_y^{-2} + \sigma_\theta^{-2} \\
\ell_\theta &= \sigma_y^{-2} \sum_{i=1}^n y_i + \sigma_\theta^{-2} \mu_\theta
\end{aligned}$$

## Problem 2

### Part i

Posterior Distribution:

$$\begin{aligned}
p(\theta|y, \tau_y) &= \frac{1}{\sqrt{2\pi\sigma_y^2}} \exp\left[-\frac{1}{2\sigma_y^2} \sum_{i=1}^n (y_i - \theta x_i)^2\right] \\
&\propto \exp\left[-\frac{1}{2\sigma_y^2} \sum_{i=1}^n (y_i - \theta x_i)^2\right] \\
&= \exp\left[-\frac{1}{2} \left(\frac{1}{\sigma_y^2} \sum_{i=1}^n (y_i^2 - 2\theta x_i y_i + \theta^2 x_i^2)\right)\right] \\
&= \exp\left[-\frac{1}{2} \left(\frac{1}{\sigma_y^2} \left[\sum_{i=1}^n y_i^2 - \sum_{n=1}^n 2\theta x_i y_i + \sum_{i=1}^n \theta^2 x_i^2\right]\right)\right] \\
&= \exp\left[-\frac{1}{2} \left(\frac{1}{\sigma_y^2} \left[\sum_{i=1}^n y_i^2 - 2\theta \sum_{n=1}^n x_i y_i + \theta^2 \sum_{i=1}^n x_i^2\right]\right)\right] \\
&= \exp\left[-\frac{1}{2} \left(\sigma_y^{-2} \sum_{i=1}^n y_i^2 - 2\theta \sigma_y^{-2} \sum_{n=1}^n x_i y_i + \theta^2 \sigma_y^{-2} \sum_{i=1}^n x_i^2\right)\right] \\
&\propto \exp\left[-\frac{1}{2} \left(\theta^2 \sigma_y^{-2} \sum_{i=1}^n x_i^2 - 2\theta \sigma_y^{-2} \sum_{n=1}^n x_i y_i\right)\right]
\end{aligned}$$

Now, given the formula:  $p(\theta|y) \propto \exp\left[-\frac{1}{2} \left(Q_\theta \theta^2 - 2\ell_\theta \theta\right)\right]$

We get the results that:

$$\begin{aligned}
Q_\theta &= \sigma_y^{-2} \sum_{i=1}^n x_i^2 \\
\ell_\theta &= \sigma_y^{-2} \sum_{n=1}^n x_i y_i
\end{aligned}$$

$$\begin{aligned}
p(\theta|y, \tau_y) &\sim N(Q_\theta^{-1}\ell_\theta, Q_\theta^{-1}) \\
&\sim N\left(\left(\sigma_y^{-2} \sum_{i=1}^n x_i^2\right)^{-1} \times \sigma_y^{-2} \sum_{n=1}^n x_i y_i, \left(\sigma_y^{-2} \sum_{i=1}^n x_i^2\right)^{-1}\right)
\end{aligned}$$

## Part ii

Prior Distribution:  $\tau_y \sim \text{Gamma}(\alpha, \beta)$

Normal Likelihood: The derivation from Part i

Posterior Distribution:  $[\tau_y|y] \sim \text{Gamma}(\alpha + \frac{n-1}{2}, \beta + \frac{(n-1)s^2}{2})$

Our Goal:

1. Remove terms that don't depend on  $\theta$
2. Integrate with respect to  $\theta$  (integral is proportional to Normal pdf)
3. See that result is proportional to Gamma pdf

$$\begin{aligned} p(\tau_t|y) &= \int_{\theta} p(\tau_y, \theta|y) p(\theta) d\theta \\ &= \int_{\theta} p(\tau_y) p(y|\theta, \tau_y) d\theta \\ &= \int_{\theta} \theta^{\alpha-1} e^{-\beta\theta} \times \exp \left[ -\frac{1}{2} \left( \theta^2 \sigma_y^{-2} \sum_{i=1}^n x_i^2 - 2\theta \sigma_y^{-2} \sum_{i=1}^n x_i y_i \right) \right] d\theta \\ &= \int_{\theta} \theta^{\alpha-1} \times \exp \left[ -\frac{1}{2} \left( \theta^2 \sigma_y^{-2} \sum_{i=1}^n x_i^2 - 2\theta \sigma_y^{-2} \sum_{i=1}^n x_i y_i \right) - \beta\theta \right] d\theta \end{aligned}$$

I could not get passed this point as I was unable to get this quantity into the form required for it to represent a Gamma distribution. As you can see, I understand the general concept, but I was unable to get past this point in this problem.

## Part iii

Below is a sample of 10 draws from my sample of  $10^4$  draws of the posterior distribution,  $[\theta, \tau_y|y]$ . Please refer to my code (in the Code Appendix section) for the precise sampling strategy.

```
## Sample Draw:
##
## 76.75128 76.56818 76.64999 76.54518 76.49867 76.6138 76.42144 76.61013 76.56337 76.44815
```

## Part iv

Below is the 95% HPD interval for the age of the universe in years after the designated data transformations. Again, please refer to my code in the Code Appendix section for details on the algorithm used.

```
## HPD Interval: ( 12763875951 , 12827667930 )
```

### Problem 3

Sampling Distribution:  $p(y|\theta) = \frac{\theta^y e^{-\theta}}{y!}$

Domain:  $\theta > 0$

$$\begin{aligned}
 p(y|\theta) &= \prod_{i=1}^n \frac{\theta^{y_i} e^{-\theta}}{y_i!} \\
 &= \theta^{\sum_{i=1}^n y_i} e^{-n\theta} \times \prod_{i=1}^n \frac{1}{y_i!} \\
 \log[p(y|\theta)] &= \log \left[ \theta^{\sum_{i=1}^n y_i} e^{-n\theta} \times \prod_{i=1}^n \frac{1}{y_i!} \right] \\
 &= \left[ \sum_{i=1}^n y_i \right] \log(\theta) - n\theta - \log \left[ \sum_{i=1}^n y_i! \right] \\
 \frac{d}{d\theta} \log[p(y|\theta)] &= \frac{d}{d\theta} \left[ \left[ \sum_{i=1}^n y_i \right] \log(\theta) \right] - \frac{d}{d\theta} (n\theta) + 0 \\
 &= \left[ \sum_{i=1}^n y_i \right] \frac{1}{\theta} - n \\
 \frac{d^2}{d\theta^2} \log[p(y|\theta)] &= \left[ \sum_{i=1}^n \frac{d}{d\theta} \theta^{-1} \right] + 0 \\
 &= \frac{-\sum_{i=1}^n y_i}{\theta^2} \\
 E \left[ \frac{d^2 \log[p(y|\theta)]}{d\theta^2} \middle| \theta \right] &= \frac{-n\theta}{\theta^2} \\
 &= \frac{-n}{\theta} \\
 \sqrt{-E \left[ \frac{d^2 \log[p(y|\theta)]}{d\theta^2} \middle| \theta \right]} &= \sqrt{\frac{n}{\theta}} \\
 &\propto \theta^{-\frac{1}{2}} \\
 &\sim \text{Gamma} \left( \alpha = \frac{1}{2}, \beta = 0 \right)
 \end{aligned}$$

## Code Appendix

```
##### Problem 2 #####

## Set options

options(scipen = 999)

## Load in the necessary packages

library(gamair)
library(coda)

## Load in the data

data("hubble")

x <- hubble$x
y <- hubble$y

## Setup the parameters

alpha <- 0.01
beta <- 0.01
n <- 24
theta_hat <- sum(x * y) / sum(x^2)
s_squared <- (1 / (n - 1)) * sum(y - (x * theta_hat))^2

##### Part iii #####

## Make 10,000 draws from  $[\tau_y|y]$ ,  $\tau^s$ 

draws_1 <- rgamma(n = 10000, shape = alpha + ((n - 1) / 2),
                 rate = beta + (n - 1) * (s_squared / 2))

## Make 10,000 draws from  $[\theta|y, \tau_y = \tau^s]$ 

var_y <- 1 / draws_1

draws_2 <- rnorm(n = 10000,
                mean = ((1/var_y) * sum(x^2))^-1 * (1/var_y) * sum(x * y),
                sd = sqrt(((1/var_y) * sum(x^2))^-1))
                )

## Print out a sample of the draws
```

```

draw_samp <- sample(x = draws_2, size = 10)

cat("Sample Draw:\n\n", draw_samp)

##### Part iv #####

## Construct the HPD interval

post_hub <- 1/draws_2*3.09e19/(60^2*24*365)

hpd_int <- as.vector(HPDinterval(as.mcmc(post_hub), prob = 0.95))

cat("HPD Interval: ", "(", hpd_int[1], ",", hpd_int[2], ")")

```