Assignment #1

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Problem 1

Part i

$f(x)$ is the pdf of any of the x_i, and since they are *iid*, they all follow one pdf.

The joint pdf is: $f_{X_1,...,X_n}(x_1,...,x_n) = f(x_1) \times ... \times f(x_n)$ because they are *iid*.

Since you can multiply the right-hand side in any order, this implies symmetry in regards to the left-hand side.

As a result, $X_1, ..., X_n$ are exchangeable.

Part ii

$$\begin{aligned}
p(y_1, ..., y_n) &= \int p(y_1, ..., y_n|\theta)p(\theta)d(\theta) \\
&= \int \left(\prod_{i=1}^{n} p(y_i|\theta)\right) p(\theta)d(\theta)) \\
&= \int \left(\prod_{i=1}^{n} p(y_{\pi_i}|\theta)\right) p(\theta)d(\theta)) \\
&= \int p(y_{\pi_1,...,y_{\pi_n}}|\theta)p(\theta)d(\theta) \\
&= p(y_{\pi_1,...,y_{\pi_n}})
\end{aligned}$$

Where:

- The first line is the definition of marginal probability
- The second line is because the Y_i's are conditionally *iid*
- The third line is because the product does not depend on order
- The fourth line we are converting back to the form used in the first line
- Finally, the last line is the definition of marginal probability

Problem 2

Part a

Prior Density: $p(\theta) = 1$
Sampling Distribution: $p(y|\theta) = \binom{n}{y}\theta^y(1-\theta)^{n-y}$

Posterior Density: $p(\theta|y) \propto p(y|\theta)p(\theta) = \binom{n}{y}\theta^y(1-\theta)^{n-y}$
Posterior Distribution: $\theta|y \sim Beta(y+1, (n-y)+1)$

$$\begin{aligned}
\binom{n}{y}\frac{\Gamma(y+1)\Gamma(n-y+1)}{\Gamma((y+1)+(n-y+1))} &= \binom{n}{y}\frac{\Gamma(y+1)\Gamma(n-y+1)}{\Gamma(n+2)} \\
&= \frac{n!}{y!(n-y)!}\frac{y!(n-y)!}{(n+1)!} \\
&= \frac{n!}{(n+1)!} \\
&= \frac{1}{n+1}
\end{aligned}$$

Part b

Prior Distribution: $\theta \sim Beta(\alpha, \beta)$
Sampling Distribution: $y|\theta \sim Binomial(n, \theta)$

Posterior Density:

$$\begin{aligned}
p(\theta|y) &= \binom{n}{y}\theta^y(1-\theta)^{n-y} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1} \\
&= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1} \\
&\propto Beta(y+\alpha, (n-y)+\beta)
\end{aligned}$$

Posterior Mean:

$$\begin{aligned}
\frac{y+\alpha}{\alpha+\beta+n} &= \frac{y}{n} + \lambda\left(\frac{\alpha}{\alpha+\beta} - \frac{y}{n}\right) \\
\frac{y+\alpha}{\alpha+\beta+n} - \frac{y}{n} &= \lambda\left(\frac{\alpha}{\alpha+\beta} - \frac{y}{n}\right) \\
\frac{ny+n\alpha-\alpha y-\beta y-ny}{(\alpha+\beta+n)n} &= \lambda\left(\frac{n\alpha-\alpha y-\beta y}{(\alpha+\beta)n}\right) \\
\lambda &= \frac{\alpha+\beta}{\alpha+\beta+n}
\end{aligned}$$

Since λ will always be between 0 and 1 the Posterior Mean will act as a weighted average between our Prior Mean, $\frac{y}{n}$, and the data.

Part c

Posterior Distribution: $\theta|y \sim Beta(y+\alpha, (n-y)+\beta)$
Prior Variance $(\alpha = 1, \beta = 1)$: $\frac{1}{12}$
Posterior Variance $(\alpha = 1, \beta = 1)$:

$$\begin{aligned}
\frac{(y+1)(n-y+1)}{(n+2)^2(n+3)} &= \frac{ny-y^2+y+n-y+1}{(n^2+4n+4)(n+3)} \\
&= \frac{ny-y^2+y+n-y+1}{n^3+3n^2+4n^2+12n+4n+12} \\
&= \frac{-y^2+ny+n+1}{n^3+7n^2+16n+12}
\end{aligned}$$

Now, we can deduce that smaller values of n will maximize this quantity; since $n \geq y$, we will set $n = y = 1$.

Posterior Variance ($n = y = 1$): $\frac{-y^2+ny+n+1}{n^3+7n^2+16n+12} = \frac{2}{36} = \frac{1}{18}$

Thus, the Posterior Variance, which we just maximized, is always less than the Prior Variance of $\frac{1}{12}$.

Part d

- $n = y = 1$
- $\alpha = 1$
- $\beta = 10$

Prior Distribution: $\theta \sim Beta(\alpha, \beta)$
Posterior Distribution: $\theta | y \sim Beta(y + \alpha, (n - y) + \beta)$

Prior Variance: $\frac{(1)(10)}{(11)^2(12)} = \frac{10}{1452} = 0.0069$
Posterior Variance: $\frac{(2)(10)}{(12)^2(13)} = \frac{20}{1872} = 0.0107$

Problem 3

Part a

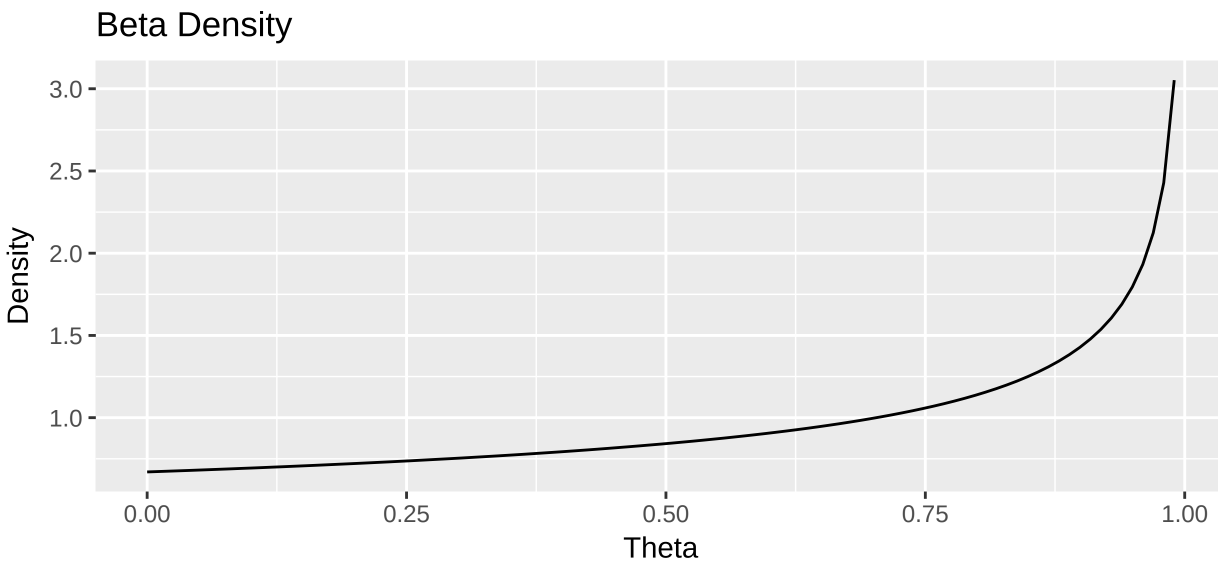
Prior Distribution: $\theta \sim Beta(\alpha, \beta)$
Prior Mean: 0.6
Prior Variance: 0.09

Sampling Distribution: $y|\theta \sim Binomial(n, \theta)$

$$\begin{aligned}
\mu &= \frac{\alpha}{\alpha + \beta} \\
(\alpha + \beta)\mu &= \alpha \\
\alpha\mu + \beta\mu &= \alpha \\
\beta\mu &= \alpha - \alpha\mu \\
\beta &= \alpha(\frac{1}{\mu} - 1)
\end{aligned}$$

$$\begin{aligned}
\sigma^2 &= \frac{\alpha\beta}{(\alpha + \beta)^2(\alpha + \beta + 1)} \\
\alpha &= \left(\frac{1 - \mu}{\sigma^2} - \frac{1}{\mu}\right)\mu^2
\end{aligned}$$

$\alpha = 1$
$\beta = 0.67$



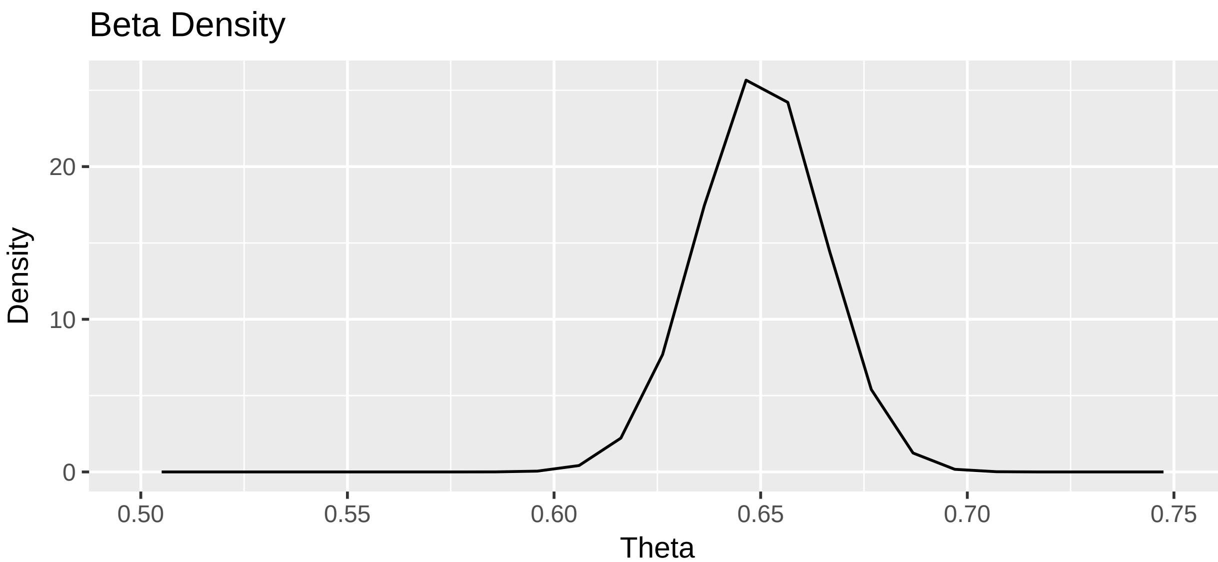
Part b

$n = 1000$
$y = 650$
Prior Distribution: $\theta \sim Beta(\alpha, \beta)$
Sampling Distribution: $y|\theta \sim Binomial(n, \theta)$

Posterior Density:

$$\begin{aligned} p(\theta|y) &= \binom{n}{y}\theta^y(1-\theta)^{n-y} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1} \\ &= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1} \\ &\propto Beta(y+\alpha, (n-y)+\beta) \end{aligned}$$

$\alpha_{post} = y + \alpha$
$\beta_{post} = (n - y) + \beta$
Posterior Mean: $\mu_{post} = \frac{y+\alpha}{(y+\alpha)+((n-y)+\beta)} = 0.6499$
Posterior Variance: $\sigma^2_{post} = \frac{(y+\alpha)((n-y)+\beta)}{((y+\alpha)+((n-y)+\beta))^2((y+\alpha)+((n-y)+\beta)+1)} = 0.0151$

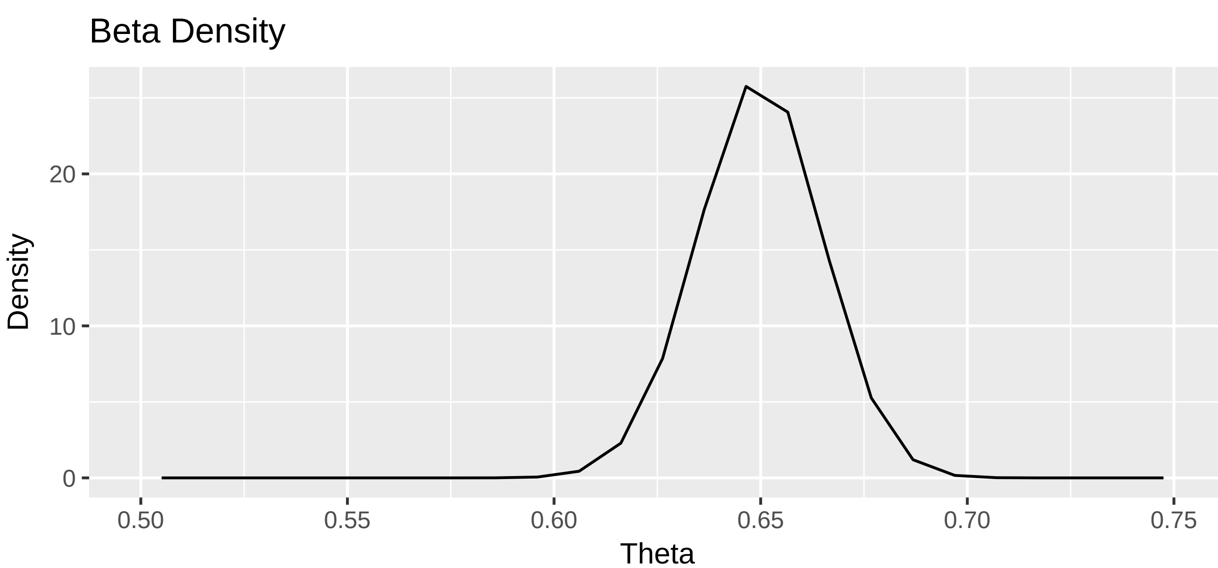


Part c

First Sensitivity Check - Uniform Prior

Prior Mean: $\frac{1}{2}$
Prior Variance: $\frac{1}{12}$
Prior Distribution: $\theta \sim Uniform(0, 1)$
Sampling Distribution: $y|\theta \sim Binomial(n, \theta)$
Posterior Density:

$$\begin{aligned} p(\theta|y) &= \binom{n}{y}\theta^y(1-\theta)^{n-y} \times 1 \\ &= \binom{n}{y}\theta^y(1-\theta)^{n-y} \times 1 \\ &\propto Beta(y+1, (n-y)+1) \end{aligned}$$

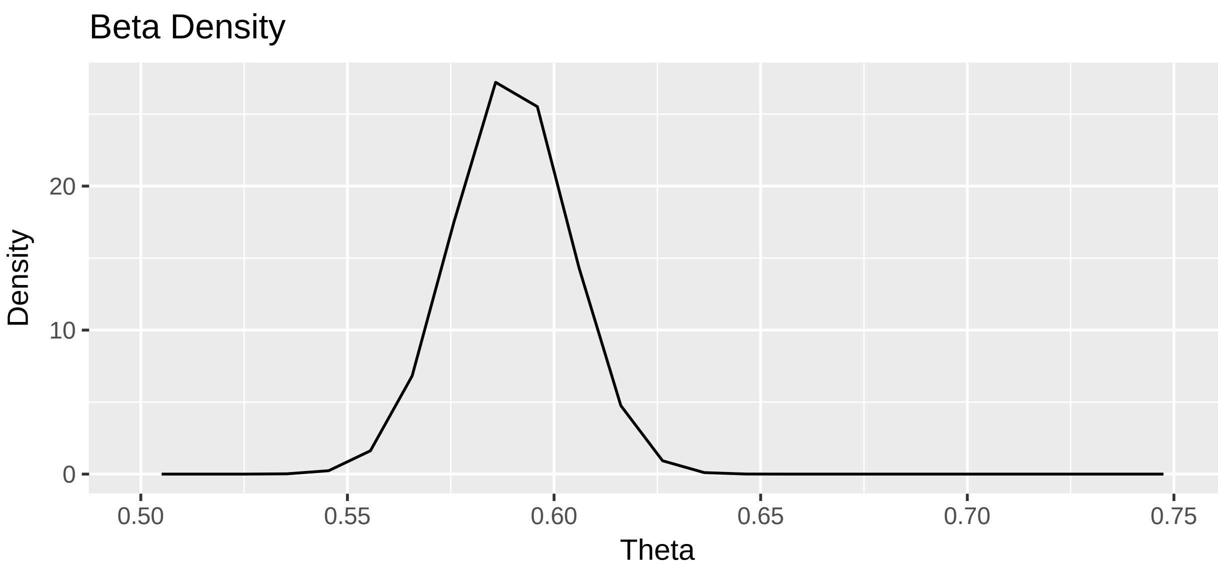


Second Sensitivity Check - Beta Prior

Prior Mean: 0.3
Prior Variance: 0.001
Prior Distribution: $\theta \sim Beta(62.7, 146.3)$
Sampling Distribution: $y|\theta \sim Binomial(n, \theta)$

Posterior Density:

$$\begin{aligned} p(\theta|y) &= \binom{n}{y}\theta^y(1-\theta)^{n-y} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1} \\ &= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1} \\ &\propto Beta(y+\alpha, (n-y)+\beta) \end{aligned}$$

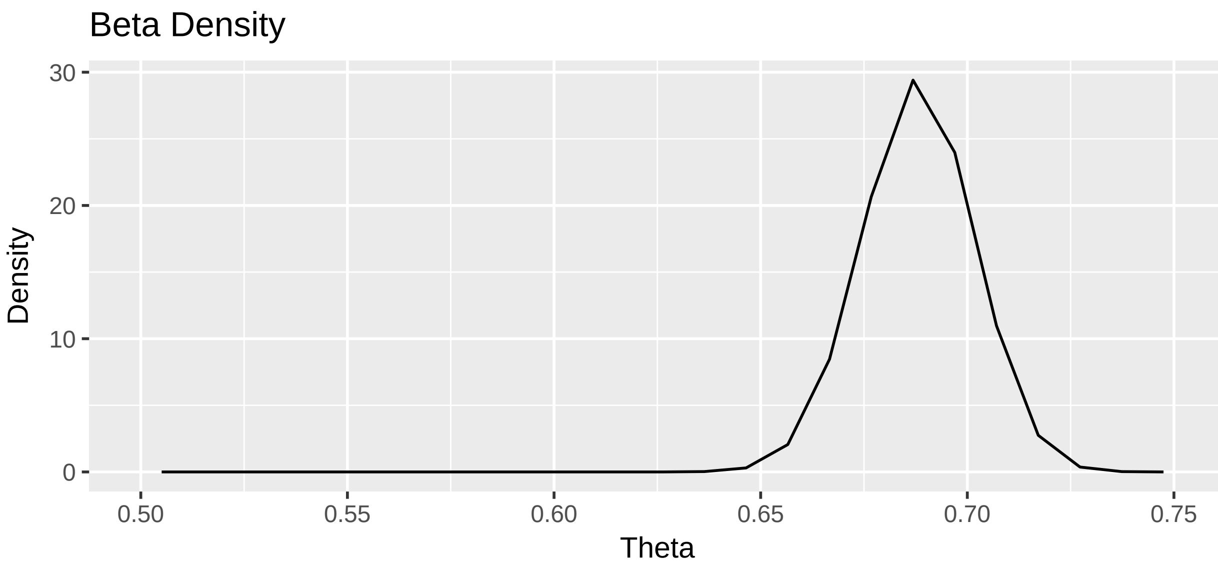


Third Sensitivity Check - Beta Prior

Prior Mean: 0.9
Prior Variance: 0.0005
Prior Distribution: $\theta \sim Beta(161.1, 17.9)$
Sampling Distribution: $y|\theta \sim Binomial(n, \theta)$

Posterior Density:

$$\begin{aligned} p(\theta|y) &= \binom{n}{y}\theta^y(1-\theta)^{n-y} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1} \\ &= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1} \\ &\propto Beta(y+\alpha, (n-y)+\beta) \end{aligned}$$



Conclusion

We can see that a prior distribution such as $Uniform(0,1)$ has a negligible affect on the posterior distribution, whereas, our $Beta(\alpha,\beta)$ distributions with prior means and variances that are substantially different enough from the observed data will have a noticeable affect on the posterior distribution. This shows us how our prior distribution may have a large affect on our posterior distribution, depending on the family, and parameters of said family, that we select.

Code Appendix

```
## Load in necessary packages

library(ggplot2)



##### Problem 3 #####



## Set mean and variance

mu <- 0.6
var <- 0.09



##### Part a



## Estimate the beta parameters

est_beta_params <- function(mu, var) {

    alpha <- (((1 - mu) / var) - (1 / mu)) * mu^2
    beta <- alpha * ((1 / mu) - 1)
    return(round(c(alpha, beta), 2))

}

beta_params <- est_beta_params(mu, var)

## Establish the x-axis values

theta <- seq(0, 1, length.out = 100)

## Set the beta density

beta_density <- dbeta(x = theta,
                      shape1 = beta_params[1],
                      shape2 = beta_params[2])

beta_density <- beta_density[-length(beta_density)]
theta <- theta[-length(theta)]

## Build the density plot

ggplot() +
    geom_line(aes(x = theta, y = beta_density)) +
    labs(x = "Theta", y = "Density", title = "Beta Density")
```

```r
##### Part b

## Set the sample size

n <- 1000

## Set the number of yes votes

y <- n * 0.65

## Set post alpha and beta

post_alpha <- y + beta_params[1]
post_beta <- (n - y) + beta_params[2]

## Posterior mean and variance

post_mean <- round(post_alpha / (post_alpha + post_beta), 4)
post_var <- round(sqrt((post_alpha * post_beta) /
                           ((post_alpha + post_beta)^2 * (post_alpha + post_beta + 1))), 4)

## Calculate the posterior density

post_density <- dbeta(x = theta,
                      shape1 = post_alpha,
                      shape2 = post_beta)

## Build the density plot

ggplot() +
    geom_line(aes(x = theta, y = post_density)) +
    labs(x = "Theta", y = "Density", title = "Beta Density") +
    xlim(c(0.5, 0.75))

##### Part c

## First sensitivity check

## Set mean and variance

mu <- 0.5
var <- (1/12)
```

```r
## Set post alpha and beta

post_alpha <- y + 1
post_beta <- (n - y) + 1

## Establish the x-axis values

theta <- seq(0, 1, length.out = 100)

## Set the beta density

beta_density <- dbeta(x = theta,
                      shape1 = post_alpha,
                      shape2 = post_beta)

beta_density <- beta_density[-length(beta_density)]
theta <- theta[-length(theta)]

## Build the density plot

ggplot() +
    geom_line(aes(x = theta, y = beta_density)) +
    labs(x = "Theta", y = "Density", title = "Beta Density") +
    xlim(c(0.5, 0.75))



## Second sensitivity check



## Set mean and variance

mu <- 0.3
var <- 0.001

## Estimate the beta parameters

beta_params <- est_beta_params(mu, var)

## Set post alpha and beta

post_alpha <- y + beta_params[1]
post_beta <- (n - y) + beta_params[2]

## Establish the x-axis values

theta <- seq(0, 1, length.out = 100)

## Set the beta density

beta_density <- dbeta(x = theta,
                      shape1 = post_alpha,
```

```r
                     shape2 = post_beta)

beta_density <- beta_density[-length(beta_density)]
theta <- theta[-length(theta)]

## Build the density plot

ggplot() +
    geom_line(aes(x = theta, y = beta_density)) +
    labs(x = "Theta", y = "Density", title = "Beta Density") +
    xlim(c(0.5, 0.75))



## Third sensitivity check



## Set mean and variance

mu <- 0.9
var <- 0.0005

## Estimate the beta parameters

beta_params <- est_beta_params(mu, var)

## Set post alpha and beta

post_alpha <- y + beta_params[1]
post_beta <- (n - y) + beta_params[2]

## Establish the x-axis values

theta <- seq(0, 1, length.out = 100)

## Set the beta density

beta_density <- dbeta(x = theta,
                     shape1 = post_alpha,
                     shape2 = post_beta)

beta_density <- beta_density[-length(beta_density)]
theta <- theta[-length(theta)]

## Build the density plot

ggplot() +
    geom_line(aes(x = theta, y = beta_density)) +
    labs(x = "Theta", y = "Density", title = "Beta Density") +
    xlim(c(0.5, 0.75))
```