Assignment #3

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Problem 1

Sampling Distribution: $y_1, ..., y_n|\theta \overset{\text{iid}}{\sim} N(\theta, \sigma_y^2)$
Prior Distribution: $\theta \sim N(\mu_\theta, \sigma_\theta^2)$
Posterior Distribution: $\theta|y \sim N(Q_\theta^{-1}\ell_\theta, Q_\theta^{-1})$

Posterior Distribution:

$$\begin{aligned}
p(\theta|y) &\propto p(y|\theta)p(\theta) \\
&= \left[\frac{1}{\sqrt{2\pi\sigma_y^2}}\right]^n exp\left(-\frac{1}{2\sigma_y^2}\sum_{i=1}^n (y_i-\theta)^2\right) \times \frac{1}{\sqrt{2\pi\sigma_\theta^2}} exp\left(-\frac{1}{2\sigma_\theta^2}(\theta-\mu_\theta)^2\right) \\
&\propto exp\left(-\frac{1}{2\sigma_\theta^2}(\theta-\mu_\theta)^2\right) \times exp\left(-\frac{1}{2\sigma_y^2}\sum_{i=1}^n (y_i-\theta)^2\right) \\
&= exp\left(-\frac{\theta^2 - 2\theta\mu_\theta + \mu_\theta^2}{2\sigma_\theta^2}\right) \times exp\left(-\frac{\sum_{i=1}^n (y_i^2 - 2\theta y_i + \theta^2)}{2\sigma_y^2}\right) \\
&= exp\left(\frac{-\theta^2 + 2\theta\mu_\theta - \mu_\theta^2}{2\sigma_\theta^2} - \frac{\sum_{i=1}^n (y_i^2 - 2\theta y_i + \theta^2)}{2\sigma_y^2}\right) \\
&= exp\left(\frac{\sigma_y^2[-\theta^2 + 2\theta\mu_\theta - \mu_\theta^2] - \sigma_\theta^2[\sum_{i=1}^n (y_i^2 - 2\theta y_i + \theta^2)]}{2\sigma_\theta^2\sigma_y^2}\right) \\
&= exp\left(\frac{-\theta^2(\sigma_y^2 + n\sigma_\theta^2) + 2\theta(\mu_\theta\sigma_y^2 + \sigma_\theta^2 y_1 + ... + \sigma_\theta^2 y_n) - (\mu_\theta^2\sigma_y^2 + \sigma_\theta^2 y_1^2 + ... + \sigma_\theta^2 y_n^2)}{2\sigma_\theta^2\sigma_y^2}\right) \\
&= exp\left(\frac{-\theta^2 + 2\theta\frac{\mu_\theta\sigma_y^2 + \sum_{i=1}^n \sigma_\theta^2 y_i}{\sigma_y^2 + n\sigma_\theta^2} - \left(\frac{\mu_\theta\sigma_y^2 + \sum_{i=1}^n \sigma_\theta^2 y_i}{\sigma_y^2 + n\sigma_\theta^2}\right)}{2\frac{\sigma_\theta^2\sigma_y^2}{\sigma_y^2 + n\sigma_\theta^2}}\right) \times exp\left(-\frac{\mu_\theta^2\sigma_y^2 + \sum_{i=1}^n \sigma_\theta^2 y_i^2}{2\sigma_\theta^2\sigma_y^2}\right) \\
&= exp\left(-\frac{\left(\theta - \frac{\mu_\theta\sigma_y^2 + \sum_{i=1}^n \sigma_\theta^2 y_i}{\sigma_y^2 + n\sigma_\theta^2}\right)^2}{2\frac{\sigma_\theta^2\sigma_y^2}{\sigma_y^2 + n\sigma_\theta^2}}\right)
\end{aligned}$$

From this result, we can see the following result:

$$
\begin{aligned}
\theta|y &\sim N\left(\frac{\mu_\theta\sigma_y^2 + \sum_{i=1}^n \sigma_\theta^2 y_i}{\sigma_y^2 + n\sigma_\theta^2}, \frac{\sigma_y^2\sigma_\theta^2}{\sigma_y^2 + n\sigma_\theta^2}\right) \\
&\sim N\left(\frac{\mu_\theta\sigma_\theta^{-2} + \sigma_y^{-2}\sum_{i=1}^n y_i}{\sigma_\theta^{-2} + n\sigma_y^{-2}}, \frac{1}{\sigma_\theta^{-2} + n\sigma_y^{-2}}\right) \\
&\sim N(Q_\theta^{-1}\ell_\theta, Q_\theta^{-1})
\end{aligned}
$$

Where the following holds:

$Q_\theta = n\sigma_y^{-2} + \sigma_\theta^{-2}$
$\ell_\theta = \sigma_y^{-2}\sum_{i=1}^n y_i + \sigma_\theta^{-2}\mu_\theta$

Problem 2

Part i

Posterior Distribution:

$$
\begin{aligned}
p(\theta|y,\tau_y) &= \frac{1}{\sqrt{2\pi\sigma_y^2}} exp\left[-\frac{1}{2\sigma_y^2}\sum_{i=1}^n (y_i - \theta x_i)^2\right] \\
&\propto exp\left[-\frac{1}{2\sigma_y^2}\sum_{i=1}^n (y_i - \theta x_i)^2\right] \\
&= exp\left[-\frac{1}{2}\left(\frac{1}{\sigma_y^2}\sum_{i=1}^n (y_i^2 - 2\theta x_i y_i + \theta^2 x_i^2\right)\right] \\
&= exp\left[-\frac{1}{2}\left(\frac{1}{\sigma_y^2}\left[\sum_{i=1}^n y_i^2 - \sum_{n=1}^n 2\theta x_i y_i + \sum_{i=1}^n \theta^2 x_i^2\right]\right)\right] \\
&= exp\left[-\frac{1}{2}\left(\frac{1}{\sigma_y^2}\left[\sum_{i=1}^n y_i^2 - 2\theta\sum_{n=1}^n x_i y_i + \theta^2\sum_{i=1}^n x_i^2\right]\right)\right] \\
&= exp\left[-\frac{1}{2}\left(\sigma_y^{-2}\sum_{i=1}^n y_i^2 - 2\theta\sigma_y^{-2}\sum_{n=1}^n x_i y_i + \theta^2\sigma_y^{-2}\sum_{i=1}^n x_i^2\right)\right] \\
&\propto exp\left[-\frac{1}{2}\left(\theta^2\sigma_y^{-2}\sum_{i=1}^n x_i^2 - 2\theta\sigma_y^{-2}\sum_{n=1}^n x_i y_i\right)\right]
\end{aligned}
$$

Now, given the formula: $p(\theta|y) \propto exp\left[-\frac{1}{2}\left(Q_\theta\theta^2 - 2\ell_\theta\theta\right)\right]$

We get the results that:
$Q_\theta = \sigma_y^{-2}\sum_{i=1}^n x_i^2$
$\ell_\theta = \sigma_y^{-2}\sum_{n=1}^n x_i y_i$

$$
\begin{aligned}
p(\theta|y,\tau_y) &\sim N(Q_\theta^{-1}\ell_\theta, Q_\theta^{-1}) \\
&\sim N\left(\left(\sigma_y^{-2}\sum_{i=1}^n x_i^2\right)^{-1} \times \left(\sigma_y^{-2}\sum_{n=1}^n x_i y_i\right)^{-1}, \left(\sigma_y^{-2}\sum_{i=1}^n x_i^2\right)^{-1}\right)
\end{aligned}
$$

Problem 3

Sampling Distribution: $p(y|\theta) = \frac{\theta^y e^-\theta}{y!}$
Domain: $\theta > 0$

$$
\begin{aligned}
p(y|\theta) &= \prod_{i=1}^{n} \frac{\theta^{y_i} e^-\theta}{y_i!} \\
&= \theta^{\sum_{i=1}^{n} y_i} e^{-n\theta} \times \prod_{i=1}^{n} \frac{1}{y_i!} \\
log[p(y|\theta)] &= log\left[\theta^{\sum_{i=1}^{n} y_i} e^{-n\theta} \times \prod_{i=1}^{n} \frac{1}{y_i!}\right] \\
&= \left[\sum_{i=1}^{n} y_i\right] log(\theta) - n\theta - log\left[\sum_{i=1}^{n} y_i!\right] \\
\frac{d}{d\theta} log[p(y|\theta)] &= \frac{d}{d\theta}\left[\left[\sum_{i=1}^{n} y_i\right] log(\theta)\right] - \frac{d}{d\theta}(n\theta) + 0 \\
&= \left[\sum_{i=1}^{n} y_i\right] \frac{1}{\theta} - n \\
\frac{d^2}{d\theta^2} log[p(y|\theta)] &= \left[\sum_{i=1}^{n} \frac{d}{d\theta} \theta^{-1}\right] + 0 \\
&= \frac{-\sum_{i=1}^{n} y_i}{\theta^2} \\
E\left[\frac{d^2 log[p(y|\theta)]}{d\theta^2}\middle|\theta\right] &= \frac{-n\theta}{\theta^2} \\
&= \frac{-n}{\theta} \\
\sqrt{-E\left[\frac{d^2 log[p(y|\theta)]}{d\theta^2}\middle|\theta\right]} &= \sqrt{\frac{n}{\theta}} \\
&\propto \theta^{-\frac{1}{2}} \\
&\sim Gamma\left(\alpha = \frac{1}{2}, \beta = 0\right)
\end{aligned}
$$