Assignment #4

Elliot Smith

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Problem 1

Part i

BLAH

Part ii

BLAH

Part iii

BLAH

Part iv

BLAH

Problem 2

Part i

Prior Distribution:

$$
\begin{aligned}
p(\theta) &\propto exp\bigg[-\frac{1}{2}\big(\theta-\mu_\theta\big)^T\Sigma_\theta^{-1}(\theta-\mu_\theta)\bigg]\\
&\propto exp\bigg[-\frac{1}{2}\big(\theta^T\Sigma_\theta^{-1}\theta-2\theta^T\Sigma_\theta^{-1}\mu_\theta+\mu_\theta^T\Sigma_\theta^{-1}\mu_\theta\bigg]\\
&\propto exp\bigg[-\frac{1}{2}\big(\theta^T\Sigma_\theta^{-1}\theta-2\theta^T\Sigma_\theta^{-1}\mu_\theta+\mu_\theta^T\Sigma_\theta^{-1}\mu_\theta\big)\bigg]\\
&\propto exp\bigg[-\frac{1}{2}\big(\theta^T\Sigma_\theta^{-1}\theta-2\theta^T\Sigma_\theta^{-1}\mu_\theta\big)\bigg]\\
&\propto exp\bigg[-\frac{1}{2}\big(\theta^TQ_{\theta_1}\theta-2\theta^T\ell_{\theta_1}\big)\bigg]
\end{aligned}
$$

Where:

$Q_{\theta_1}=\Sigma_\theta^{-1}$
$\ell_{\theta_1}=\Sigma_\theta^{-1}\mu_\theta$

Likelihood:

$$\begin{aligned}
p(y|\theta) &\propto exp\Big[-\frac{1}{2}\Big(\sum_{i=1}^{n} y_i^T\Sigma^{-1}y_i - 2\sum_{i=1}^{n}\theta^T\Sigma^{-1}y_i + \sum_{i=1}^{n}\theta^T\Sigma^{-1}\theta\Big)\Big] \\
&\propto exp\Big[-\frac{1}{2}\Big(\theta^T n\Sigma^{-1}\theta - 2\theta^T\Sigma^{-1}n\bar{y}\Big)\Big] \\
&\propto exp\Big[-\frac{1}{2}\Big(\theta^T Q_{\theta_2}\theta - 2\theta^T\ell_{\theta_2}\Big)\Big]
\end{aligned}$$

Where:

$Q_{\theta_2} = n\Sigma^{-1}$
$\ell_{\theta_2} = \Sigma^{-1}n\bar{y}$

Posterior:

$$\begin{aligned}
p(\theta|y) &\propto exp\Big[-\frac{1}{2}\big(\theta^T Q_{\theta_1}\theta - 2\theta^T\ell_{\theta_1}\big)\Big] \times exp\Big[-\frac{1}{2}\Big(\theta^T Q_{\theta_2}\theta - 2\theta^T\ell_{\theta_2}\Big)\Big] \\
&\propto exp\Big[-\frac{1}{2}\theta^T Q_{\theta_1}\theta + \theta^T\ell_{\theta_1} - \frac{1}{2}\theta^T Q_{\theta_2}\theta + \theta^T\ell_{\theta_2}\Big] \\
&\propto exp\Big[\theta^T\big(\ell_{\theta_1} + \ell_{\theta_2}\big) - \frac{1}{2}\theta^T\big(Q_{\theta_1} + Q_{\theta_2}\big)\theta\Big] \\
&\propto exp\Big[-\frac{1}{2}\big(\theta^T Q_\theta\theta - 2\theta^T\ell_\theta\big)\Big]
\end{aligned}$$

Where:

$Q_\theta = Q_{\theta_1} + Q_{\theta_2} = \Sigma_\theta^{-1} + n\Sigma_\theta^{-1}$
$\ell_\theta = \ell_{\theta_1} + \ell_{\theta_2} = \Sigma_\theta^{-1}\mu_\theta + \Sigma^{-1}n\bar{y}$

Thus:

$\theta|y \sim MVN(Q_\theta^{-1}\ell_\theta, Q_\theta^{-1})$

Part ii

Posterior Mean, $Q_\theta^{-1}\ell_\theta$:

$$\begin{pmatrix} \frac{1}{\sigma_{\theta_1}^2} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{\sigma_{\theta_2}^2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{\sigma_{\theta_3}^2} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{\sigma_{\theta_4}^2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} + \begin{pmatrix} \frac{n}{\sigma_1^2} & 0 & 0 & 0 & \cdots \\ 0 & \frac{n}{\sigma_2^2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{n}{\sigma_3^2} & 0 & \cdots \\ 0 & 0 & 0 & \frac{n}{\sigma_4^2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} \times \begin{pmatrix} \frac{1}{\sigma_{\theta_1}^2} & 0 & 0 & 0 & \cdots \\ 0 & \frac{1}{\sigma_{\theta_2}^2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{1}{\sigma_{\theta_3}^2} & 0 & \cdots \\ 0 & 0 & 0 & \frac{1}{\sigma_{\theta_4}^2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \mu_\theta + \begin{pmatrix} \frac{n\bar{y}}{\sigma_1^2} & 0 & 0 & 0 & \cdots \\ 0 & \frac{n\bar{y}}{\sigma_2^2} & 0 & 0 & \cdots \\ 0 & 0 & \frac{n\bar{y}}{\sigma_3^2} & 0 & \cdots \\ 0 & 0 & 0 & \frac{n\bar{y}}{\sigma_4^2} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Posterior Covariance, Q_θ^{-1}:

$$\begin{pmatrix} \frac{1}{\sigma^2_{\theta_1}} & 0 & 0 & 0 & ... \\ 0 & \frac{1}{\sigma^2_{\theta_2}} & 0 & 0 & ... \\ 0 & 0 & \frac{1}{\sigma^2_{\theta_3}} & 0 & ... \\ 0 & 0 & 0 & \frac{1}{\sigma^2_{\theta_4}} & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} + \begin{pmatrix} \frac{n}{\sigma^2_1} & 0 & 0 & 0 & ... \\ 0 & \frac{n}{\sigma^2_2} & 0 & 0 & ... \\ 0 & 0 & \frac{n}{\sigma^2_3} & 0 & ... \\ 0 & 0 & 0 & \frac{n}{\sigma^2_4} & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1}$$

Part iii

Posterior Mean, $Q_\theta^{-1}\ell_\theta$:

$$\begin{pmatrix} \frac{1}{\sigma^2_{\theta}} & 0 & 0 & 0 & ... \\ 0 & \frac{1}{\sigma^2_{\theta}} & 0 & 0 & ... \\ 0 & 0 & \frac{1}{\sigma^2_{\theta}} & 0 & ... \\ 0 & 0 & 0 & \frac{1}{\sigma^2_{\theta}} & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} + \begin{pmatrix} \frac{n}{\sigma^2} & 0 & 0 & 0 & ... \\ 0 & \frac{n}{\sigma^2} & 0 & 0 & ... \\ 0 & 0 & \frac{n}{\sigma^2} & 0 & ... \\ 0 & 0 & 0 & \frac{n}{\sigma^2} & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} \times \begin{pmatrix} \frac{1}{\sigma^2_{\theta}} & 0 & 0 & 0 & ... \\ 0 & \frac{1}{\sigma^2_{\theta}} & 0 & 0 & ... \\ 0 & 0 & \frac{1}{\sigma^2_{\theta}} & 0 & ... \\ 0 & 0 & 0 & \frac{1}{\sigma^2_{\theta}} & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \mu_\theta + \begin{pmatrix} \frac{n\bar{y}}{\sigma^2} & 0 & 0 & 0 & ... \\ 0 & \frac{n\bar{y}}{\sigma^2} & 0 & 0 & ... \\ 0 & 0 & \frac{n\bar{y}}{\sigma^2} & 0 & ... \\ 0 & 0 & 0 & \frac{n\bar{y}}{\sigma^2} & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Posterior Covariance, Q_θ^{-1}:

$$\begin{pmatrix} \frac{1}{\sigma^2_{\theta}} & 0 & 0 & 0 & ... \\ 0 & \frac{1}{\sigma^2_{\theta}} & 0 & 0 & ... \\ 0 & 0 & \frac{1}{\sigma^2_{\theta}} & 0 & ... \\ 0 & 0 & 0 & \frac{1}{\sigma^2_{\theta}} & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} + \begin{pmatrix} \frac{n}{\sigma^2} & 0 & 0 & 0 & ... \\ 0 & \frac{n}{\sigma^2} & 0 & 0 & ... \\ 0 & 0 & \frac{n}{\sigma^2} & 0 & ... \\ 0 & 0 & 0 & \frac{n}{\sigma^2} & ... \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1}$$

Problem 3

Solving for a in equation (b) yields:

$$La = \ell_\theta$$
$$a = \frac{\ell_\theta}{L}$$

Plugging in our result for a into equation (c) yields:

$$
\begin{aligned}
L^T\theta^* &= a + e \\
&= \frac{\ell_\theta}{L} + e \\
LL^T\theta^* &= \ell_\theta + Le \\
Q_\theta\theta^* &= \ell_\theta + Le \\
\theta^* &= \frac{\ell_\theta + Le}{Q_\theta} \\
&= Q_\theta^{-1}(\ell_\theta + Le) \\
&= Q_\theta^{-1}\ell_\theta + Q_\theta^{-1}Le \\
&= Q_\theta^{-1}\ell_\theta + N\big[0, (Q_\theta^{-1}L)^T I_d (Q_\theta^{-1}L)\big] \\
&= Q_\theta^{-1}\ell_\theta + N\big[0, L^T {Q_\theta^{-1}}^T Q_\theta^{-1}L\big] \\
&= Q_\theta^{-1}\ell_\theta + N\big[0, \frac{1}{Q_\theta}\big] \\
&= Q_\theta^{-1}\ell_\theta + N\big[0, Q_\theta^{-1}\big] \\
&\sim N(Q_\theta^{-1}\ell_\theta, Q_\theta^{-1})
\end{aligned}
$$

Thus we see that $\theta^* \sim N(Q_\theta^{-1}\ell_\theta, Q_\theta^{-1})$.