

Assignment #4

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Problem 1

Problem 2

Part i

Prior Distribution:

$$\begin{aligned} p(\theta) &\propto \exp\left[-\frac{1}{2}(\theta - \mu_\theta)^T \Sigma_\theta^{-1}(\theta - \mu_\theta)\right] \\ &\propto \exp\left[-\frac{1}{2}(\theta^T \Sigma_\theta^{-1} \theta - 2\theta^T \Sigma_\theta^{-1} \mu_\theta + \mu_\theta^T \Sigma_\theta^{-1} \mu_\theta)\right] \\ &\propto \exp\left[-\frac{1}{2}(\theta^T \Sigma_\theta^{-1} \theta - 2\theta^T \Sigma_\theta^{-1} \mu_\theta + \mu_\theta^T \Sigma_\theta^{-1} \mu_\theta)\right] \\ &\propto \exp\left[-\frac{1}{2}(\theta^T \Sigma_\theta^{-1} \theta - 2\theta^T \Sigma_\theta^{-1} \mu_\theta)\right] \\ &\propto \exp\left[-\frac{1}{2}(\theta^T Q_{\theta_1} \theta - 2\theta^T \ell_{\theta_1})\right] \end{aligned}$$

Where:

$$\begin{aligned} Q_{\theta_1} &= \Sigma_\theta^{-1} \\ \ell_{\theta_1} &= \Sigma_\theta^{-1} \mu_\theta \end{aligned}$$

Likelihood:

$$\begin{aligned} p(y|\theta) &\propto \exp\left[-\frac{1}{2}\left(\sum_{i=1}^n y_i^T \Sigma^{-1} y_i - 2\sum_{i=1}^n \theta^T \Sigma^{-1} y_i + \sum_{i=1}^n \theta^T \Sigma^{-1} \theta\right)\right] \\ &\propto \exp\left[-\frac{1}{2}\left(\theta^T n \Sigma^{-1} \theta - 2\theta^T \Sigma^{-1} n \bar{y}\right)\right] \\ &\propto \exp\left[-\frac{1}{2}\left(\theta^T Q_{\theta_2} \theta - 2\theta^T \ell_{\theta_2}\right)\right] \end{aligned}$$

Where:

$$\begin{aligned} Q_{\theta_2} &= n \Sigma^{-1} \\ \ell_{\theta_2} &= \Sigma^{-1} n \bar{y} \end{aligned}$$

Posterior:

$$\begin{aligned}
p(\theta|y) &\propto \exp\left[-\frac{1}{2}(\theta^T Q_{\theta_1} \theta - 2\theta^T \ell_{\theta_1})\right] \times \exp\left[-\frac{1}{2}(\theta^T Q_{\theta_2} \theta - 2\theta^T \ell_{\theta_2})\right] \\
&\propto \exp\left[-\frac{1}{2}\theta^T Q_{\theta_1} \theta + \theta^T \ell_{\theta_1} - \frac{1}{2}\theta^T Q_{\theta_2} \theta + \theta^T \ell_{\theta_2}\right] \\
&\propto \exp\left[\theta^T (\ell_{\theta_1} + \ell_{\theta_2}) - \frac{1}{2}\theta^T (Q_{\theta_1} + Q_{\theta_2}) \theta\right] \\
&\propto \exp\left[-\frac{1}{2}(\theta^T Q_{\theta} \theta - 2\theta^T \ell_{\theta})\right]
\end{aligned}$$

Where:

$$\begin{aligned}
Q_{\theta} &= Q_{\theta_1} + Q_{\theta_2} = \Sigma_{\theta}^{-1} + n\Sigma_{\theta}^{-1} \\
\ell_{\theta} &= \ell_{\theta_1} + \ell_{\theta_2} = \Sigma_{\theta}^{-1}\mu_{\theta} + \Sigma^{-1}n\bar{y}
\end{aligned}$$

Thus:

$$\theta|y \sim MVN(Q_{\theta}^{-1}\ell_{\theta}, Q_{\theta}^{-1})$$

Part ii

Posterior Mean, $Q_{\theta}^{-1}\ell_{\theta}$:

$$\begin{pmatrix} \frac{1}{\sigma_{\theta_1}^2} & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{\sigma_{\theta_2}^2} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{\sigma_{\theta_3}^2} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{\sigma_{\theta_4}^2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} + \begin{pmatrix} \frac{n}{\sigma_1^2} & 0 & 0 & 0 & \dots \\ 0 & \frac{n}{\sigma_2^2} & 0 & 0 & \dots \\ 0 & 0 & \frac{n}{\sigma_3^2} & 0 & \dots \\ 0 & 0 & 0 & \frac{n}{\sigma_4^2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} \times \begin{pmatrix} \frac{1}{\sigma_{\theta_1}^2} & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{\sigma_{\theta_2}^2} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{\sigma_{\theta_3}^2} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{\sigma_{\theta_4}^2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \mu_{\theta} + \begin{pmatrix} \frac{n\bar{y}}{\sigma_1^2} & 0 & 0 & 0 & \dots \\ 0 & \frac{n\bar{y}}{\sigma_2^2} & 0 & 0 & \dots \\ 0 & 0 & \frac{n\bar{y}}{\sigma_3^2} & 0 & \dots \\ 0 & 0 & 0 & \frac{n\bar{y}}{\sigma_4^2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Posterior Covariance, Q_{θ}^{-1} :

$$\begin{pmatrix} \frac{1}{\sigma_{\theta_1}^2} & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{\sigma_{\theta_2}^2} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{\sigma_{\theta_3}^2} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{\sigma_{\theta_4}^2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} + \begin{pmatrix} \frac{n}{\sigma_1^2} & 0 & 0 & 0 & \dots \\ 0 & \frac{n}{\sigma_2^2} & 0 & 0 & \dots \\ 0 & 0 & \frac{n}{\sigma_3^2} & 0 & \dots \\ 0 & 0 & 0 & \frac{n}{\sigma_4^2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1}$$

Part iii

Posterior Mean, $Q_{\theta}^{-1}\ell_{\theta}$:

$$\begin{pmatrix} \frac{1}{\sigma_\theta^2} & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{\sigma_\theta^2} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{\sigma_\theta^2} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{\sigma_\theta^2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} + \begin{pmatrix} \frac{n}{\sigma^2} & 0 & 0 & 0 & \dots \\ 0 & \frac{n}{\sigma^2} & 0 & 0 & \dots \\ 0 & 0 & \frac{n}{\sigma^2} & 0 & \dots \\ 0 & 0 & 0 & \frac{n}{\sigma^2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} \times \begin{pmatrix} \frac{1}{\sigma_\theta^2} & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{\sigma_\theta^2} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{\sigma_\theta^2} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{\sigma_\theta^2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \mu_\theta + \begin{pmatrix} \frac{n\bar{y}}{\sigma^2} & 0 & 0 & 0 & \dots \\ 0 & \frac{n\bar{y}}{\sigma^2} & 0 & 0 & \dots \\ 0 & 0 & \frac{n\bar{y}}{\sigma^2} & 0 & \dots \\ 0 & 0 & 0 & \frac{n\bar{y}}{\sigma^2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Posterior Covariance, Q_θ^{-1} :

$$\begin{pmatrix} \frac{1}{\sigma_\theta^2} & 0 & 0 & 0 & \dots \\ 0 & \frac{1}{\sigma_\theta^2} & 0 & 0 & \dots \\ 0 & 0 & \frac{1}{\sigma_\theta^2} & 0 & \dots \\ 0 & 0 & 0 & \frac{1}{\sigma_\theta^2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1} + \begin{pmatrix} \frac{n}{\sigma^2} & 0 & 0 & 0 & \dots \\ 0 & \frac{n}{\sigma^2} & 0 & 0 & \dots \\ 0 & 0 & \frac{n}{\sigma^2} & 0 & \dots \\ 0 & 0 & 0 & \frac{n}{\sigma^2} & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}^{-1}$$

Problem 3

Solving for a in equation (b) yields:

$$\begin{aligned} La &= \ell_\theta \\ a &= \frac{\ell_\theta}{L} \end{aligned}$$

Plugging in our result for a into equation (c) yields:

$$\begin{aligned} L^T \theta^* &= a + e \\ &= \frac{\ell_\theta}{L} + e \\ LL^T \theta^* &= \ell_\theta + Le \\ Q_\theta \theta^* &= \ell_\theta + Le \\ \theta^* &= \frac{\ell_\theta + Le}{Q_\theta} \\ &= Q_\theta^{-1}(\ell_\theta + Le) \\ &= Q_\theta^{-1} \ell_\theta + Q_\theta^{-1} Le \\ &= Q_\theta^{-1} \ell_\theta + N[0, (Q_\theta^{-1} L)^T I_d (Q_\theta^{-1} L)] \\ &= Q_\theta^{-1} \ell_\theta + N[0, L^T Q_\theta^{-1 T} Q_\theta^{-1} L] \\ &= Q_\theta^{-1} \ell_\theta + N[0, \frac{1}{Q_\theta}] \\ &= Q_\theta^{-1} \ell_\theta + N[0, Q_\theta^{-1}] \\ &\sim N(Q_\theta^{-1} \ell_\theta, Q_\theta^{-1}) \end{aligned}$$

Thus we see that $\theta^* \sim N(Q_\theta^{-1} \ell_\theta, Q_\theta^{-1})$.