Assignment #3

Elliot Smith

9/9/2018

Problem 1

Sampling Distribution: $y_1, ..., y_n|\theta \overset{\text{iid}}{\sim} N(\theta, \sigma_y^2)$
Prior Distribution: $\theta \sim N(\mu_\theta, \sigma_\theta^2)$
Posterior Distribution: $\theta|y \sim N(Q_\theta^{-1}\ell_\theta, Q_\theta^{-1})$

Posterior Distribution:

$$\begin{aligned}
p(\theta|y) &\propto p(y|\theta)p(\theta) \\
&= \left[\frac{1}{\sqrt{2\pi\sigma_y^2}}\right]^n exp\left(-\frac{1}{2\sigma_y^2}\sum_{i=1}^n (y_i-\theta)^2\right) \times \frac{1}{\sqrt{2\pi\sigma_\theta^2}} exp\left(-\frac{1}{2\sigma_\theta^2}(\theta-\mu_\theta)^2\right) \\
&\propto exp\left(-\frac{1}{2\sigma_\theta^2}(\theta-\mu_\theta)^2\right) \times exp\left(-\frac{1}{2\sigma_y^2}\sum_{i=1}^n (y_i-\theta)^2\right) \\
&= exp\left(-\frac{\theta^2-2\theta\mu_\theta+\mu_\theta^2}{2\sigma_\theta^2}\right) \times exp\left(-\frac{\sum_{i=1}^n(y_i^2-2\theta y_i+\theta^2)}{2\sigma_y^2}\right) \\
&= exp\left(\frac{-\theta^2+2\theta\mu_\theta-\mu_\theta^2}{2\sigma_\theta^2} - \frac{\sum_{i=1}^n(y_i^2-2\theta y_i+\theta^2)}{2\sigma_y^2}\right) \\
&= exp\left(\frac{\sigma_y^2[-\theta^2+2\theta\mu_\theta-\mu_\theta^2] - \sigma_\theta^2[\sum_{i=1}^n(y_i^2-2\theta y_i+\theta^2)]}{2\sigma_\theta^2\sigma_y^2}\right) \\
&= exp\left(\frac{-\theta^2(\sigma_y^2+n\sigma_\theta^2)+2\theta(\mu_\theta\sigma_y^2+\sigma_\theta^2 y_1+...+\sigma_\theta^2 y_n)-(\mu_\theta^2\sigma_y^2+\sigma_\theta^2 y_1^2+...+\sigma_\theta^2 y_n^2)}{2\sigma_\theta^2\sigma_y^2}\right) \\
&= exp\left(\frac{-\theta^2+2\theta\frac{\mu_\theta\sigma_y^2+\sum_{i=1}^n\sigma_\theta^2 y_i}{\sigma_y^2+n\sigma_\theta^2} - \left(\frac{\mu_\theta\sigma_y^2+\sum_{i=1}^n\sigma_\theta^2 y_i}{\sigma_y^2+n\sigma_\theta^2}\right)}{2\frac{\sigma_\theta^2\sigma_y^2}{\sigma_y^2+n\sigma_\theta^2}}\right) \times exp\left(-\frac{\mu_\theta^2\sigma_y^2+\sum_{i=1}^n\sigma_\theta^2 y_i^2}{2\sigma_\theta^2\sigma_y^2}\right) \\
&= exp\left(-\frac{\left(\theta-\frac{\mu_\theta\sigma_y^2+\sum_{i=1}^n\sigma_\theta^2 y_i}{\sigma_y^2+n\sigma_\theta^2}\right)^2}{2\frac{\sigma_\theta^2\sigma_y^2}{\sigma_y^2+n\sigma_\theta^2}}\right)
\end{aligned}$$

From this result, we can see the following result:

$$\begin{aligned}
\theta|y &\sim N\left(\frac{\mu_\theta\sigma_y^2 + \sum_{i=1}^n \sigma_\theta^2 y_i}{\sigma_y^2 + n\sigma_\theta^2}, \frac{\sigma_y^2\sigma_\theta^2}{\sigma_y^2 + n\sigma_\theta^2}\right) \\
&\sim N\left(\frac{\mu_\theta{\sigma_\theta^{-2}} + \sum_{y=1}^n y_i{\sigma_y^{-2}}}{{\sigma_\theta^{-2}} + n{\sigma_y^{-2}}}, \frac{1}{{\sigma_\theta^{-2}} + n{\sigma_y^{-2}}}\right) \\
&\sim N(Q_\theta^{-1}\ell_\theta, Q_\theta^{-1})
\end{aligned}$$

Where the following holds:

$Q_\theta = n{\sigma_y^{-2}} + {\sigma_\theta^{-2}}$
$\ell_\theta = {\sigma_y^{-2}} \sum_{i=1}^n y_i + {\sigma_\theta^{-2}}\mu_\theta$

Problem 2

Problem 3