Assignment #2

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8/30/2018

Problem 1

Part a

Here our goal will be to minimize a to show that $a = E[\theta|y]$ is the unique Bayes estimate of θ:

$$\begin{aligned}
\frac{d}{da}E[L(a|y)] &= \frac{d}{da}\int L(\theta, a)p(\theta|y)d\theta \\
&= \frac{d}{da}\int (\theta - a)^2 p(\theta|y)d\theta \\
&= -2\int (\theta - a)p(\theta|y)d\theta \\
&= -2\Big[\int \theta p(\theta|y)d\theta - a\int p(\theta|y)d\theta\Big] \\
&= -2\big[E[\theta|y] - a\big]
\end{aligned}$$

$-2\big[E[\theta|y] - a\big] = 0$ when $a = E[\theta|y]$

To prove that it is a unique minimizing statistic, we must look at the second derivative:

$\frac{d}{da}(-2\big[E[\theta|y] - a\big]) = 2$

As $2 > 0$, this shows that it is a unique minimzing statistic.

Part b

Here our goal will be to show that for any median value of a, the derivative of $L(\theta, a)$ will evaluate to 0.

$$\begin{aligned}
\frac{d}{da}\big[E[L(a|y)]\big] &= \frac{d}{da}\Big[\int_{-\infty}^{a}(a-\theta)p(\theta|y)d\theta + \int_{a}^{\infty}(\theta - a)p(\theta|y)d\theta\Big] \\
&= \int_{-\infty}^{a}\frac{d}{da}(a-\theta)p(\theta|y)d\theta + \int_{a}^{\infty}\frac{d}{da}(\theta - a)p(\theta|y)d\theta \\
&= \int_{-\infty}^{a}p(\theta|y)d\theta + \int_{a}^{\infty}(-1)p(\theta|y)d\theta \\
&= \int_{-\infty}^{a}p(\theta|y)d\theta - \int_{a}^{\infty}p(\theta|y)d\theta \\
&= \frac{1}{2} - \frac{1}{2} \\
&= 0
\end{aligned}$$

As a result, it has been shown that any posterior median of θ is a Bayes estimate of θ.

Taking the second derivative we again get a positive number, thus again indicating that it is a minimizing statistic.

Part c

Here our goal will be to show that for any value of a, the derivative of $L(\theta, a)$ will evaluate to 0 where k_0 and k_1 are nonnegative numbers.

$$\begin{aligned}
\frac{d}{da}\big[E[L(a|y)]\big] &= \frac{d}{da}\left[\int_{-\infty}^{a} k_1(a-\theta)p(\theta|y)d\theta + \int_{a}^{\infty} k_0(\theta-a)p(\theta|y)d\theta\right] \\
&= \int_{-\infty}^{a} \frac{d}{da} k_1(a-\theta)p(\theta|y)d\theta + \int_{a}^{\infty} \frac{d}{da} k_0(\theta-a)p(\theta|y)d\theta \\
&= \int_{-\infty}^{a} k_1 p(\theta|y)d\theta + \int_{a}^{\infty} (-k_0)p(\theta|y)d\theta \\
&= \int_{-\infty}^{a} k_1 p(\theta|y)d\theta - \int_{a}^{\infty} k_0 p(\theta|y)d\theta \\
&= k_1\int_{-\infty}^{a} p(\theta|y)d\theta - k_0\int_{a}^{\infty} p(\theta|y)d\theta
\end{aligned}$$

Noting that: $k_0\int_a^{\infty} p(\theta|y)d\theta = k_0 - k_0\int_{-\infty}^{a} p(\theta|y)d\theta$

$$\begin{aligned}
k_1\int_{-\infty}^{a} p(\theta|y)d\theta - k_0\int_{a}^{\infty} p(\theta|y)d\theta &= k_1\int_{-\infty}^{a} p(\theta|y)d\theta - \left[k_0 - k_0\int_{-\infty}^{a} p(\theta|y)d\theta\right] \\
&= k_1\int_{-\infty}^{a} p(\theta|y)d\theta + k_0\int_{-\infty}^{a} p(\theta|y)d\theta - k_0 \\
&= (k_1+k_0)\int_{-\infty}^{a} p(\theta|y)d\theta - k_0
\end{aligned}$$

Now setting $\int_{-\infty}^{a} p(\theta|y)d\theta = \frac{k_0}{k_0+k_1}$ we get our result that any quantile is a Bayes estimate of θ.

Taking the second derivative we again get a positive number, thus again indicating that it is a minimizing statistic.

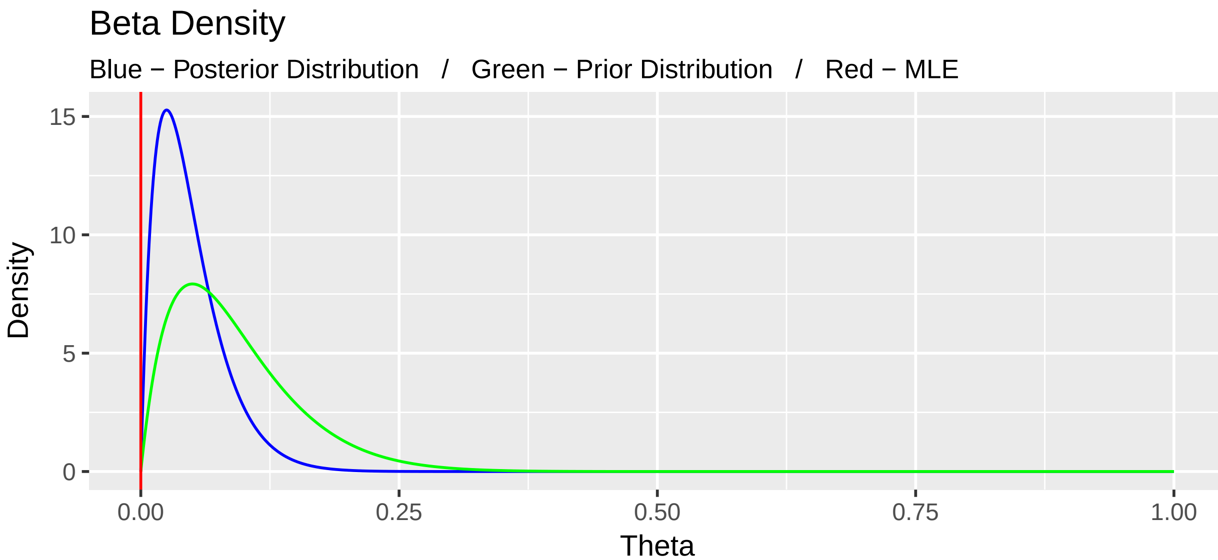
Problem 2

n = 20
Sampling Distribution: $y|\theta \sim Binomial(n = 20, \theta)$
Prior Distribution: $\theta \sim Beta(\alpha = 2, \beta = 20)$
Posterior Distribution:

$$\begin{aligned}
p(\theta|y) &= \binom{n}{y}\theta^{y}(1-\theta)^{n-y} \times \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1} \\
&= \binom{n}{y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1} \\
&\propto Beta(y+\alpha, (n-y)+\beta) \\
&= Beta(y+2, (20-y)+20)
\end{aligned}$$

Part i

$y = 0$



From this graph we can see that while the MLE, $\hat{y} = \frac{y}{n}$ is not properly represented the true distribution due to the fact that there were no positive cases in this hospital, the posterior distribution, via the affect of the prior distribution, is impacting our results enough to render a more accurate result.

Part ii

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## HPD Interval:  ( 0.00175054 , 0.1119731 )
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## Frequentist Interval:  ( 0 , 0 )
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From this result, we can see that the Frequestist result compares poorly ($(0, 0)$ is not really an interval at all) against the Bayesian posterior credible interval of θ. Again, the frequentist approach for this hospital is being adversely affected by the fact that there are no positive cases, and as such, we do not get a good representation of the true distribution of positive cases. Whereas the Bayesian approach factors in our prior beliefs on the number of positive cases, thus allowing us to interpret the results for this case.

Part iii

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## HPD Interval:  ( -4.875741 , -1.777612 )
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Problem 3

Part i

My methodology for selecting my α and β for my $\theta_1, \theta_2 \sim Gamma(\alpha, \beta)$ distribution is as follows. First, I estimated my λ parameter from my Poisson distribution by $\Sigma_{i=1}^{n} \frac{y_i}{n}$. Then, I generated a sample of 1000 data points from a Poisson distribution with my esimated λ parameter. I calculated the mean and variance of this sample, and using these values, ascertained the values of α and β of a Gamma distribution by using the formula for mean $(\frac{\alpha}{\beta})$ and variance $(\frac{\alpha}{\beta^2})$.

```
## Gamma Parameters
## Alpha Parameter:  1.8867
## Beta Parameter:  0.9993
```

Part ii

Posterior Distributions:

$$
\begin{aligned}
p(\theta|y_1) &\propto p(y_1|\theta)p(\theta) \\
&= \frac{\theta^{y_1}e^{-\theta}}{y_1!}\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta} \\
&\propto \theta^{y_1}e^{-\theta}\theta^{\alpha-1}e^{-\beta\theta} \\
&= \theta^{y_1+\alpha-1}e^{-(1+\beta)\theta} \\
&\sim Gamma(y_1+\alpha,\beta+1)
\end{aligned}
$$

$$
\begin{aligned}
p(\theta|y_2) &\propto p(y_2|\theta)p(\theta) \\
&= \frac{\theta^{y_2}e^{-\theta}}{y_2!}\frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta} \\
&\propto \theta^{y_2}e^{-\theta}\theta^{\alpha-1}e^{-\beta\theta} \\
&= \theta^{y_2+\alpha-1}e^{-(1+\beta)\theta} \\
&\sim Gamma(y_2+\alpha,\beta+1)
\end{aligned}
$$