A Value at Risk Framework for Kelly Gambling

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Abstract

We design a framework and a closed-form expression which allows a bettor to adjust their invested stake on a repeated gambling game subject to a prescribed value at risk. Given a betting opportunity, which the gambler can repeat a given number of times, or a collection of distinct betting opportunities at different moments in time, the bettor can calculate exactly what fraction of their capital, in a variation of the Kelly criterion, to stake in order to have a desired probability of exceeding losses of a certain magnitude.

1 Introduction

Value at risk (Jorion, 2007) is an immensely popular risk-management framework amongst financial practitioners (Diebold et al., 2010), whereby an investment is made such that there is a certain theoretical probability of losses exceeding a predefined threshold over a given investment horizon. Used primarily with respect to conventional financial investments, in this paper we develop an expression for value-at-risk staking for a repeated gambling game. A bettor can wager a certain amount of their capital on each wager, which is repeated a given number of times, and they seek to maximise growth of capital subject to their risk tolerance, in the form of value at risk. We evaluate our expression, demonstrating some practical limitations, and then expand the analysis to consider a related problem, where a gambler seeks to maintain a certain value at risk over a collection of distinct wagers, rather than a repetition of the same gamble. The fractional staking strategy is a variant of the Kelly criterion for gambling games (Kelly, 1956). Other researchers have published similar extensions of the Kelly criterion, within different gambling contexts, such as Baker and McHale (2013) and Busseti et al. (2016). Our paper is a contribution with similar practical relevance for bettors, who can adjust their staking to maximise growth while maintaining a desirable level of risk.

2 Derivation

We have a gamble on an event with probability p of occurring, offering profit to a winning bet of b. If we place this bet n times, winning w times, our return β is

$$\beta = (1 + fb)^w (1 - f)^{n-w} - 1 \tag{1}$$

where f is the Kelly fraction: the proportion of current capital staked on each bet. We want to model, after these n bets have been placed, the distribution of the return β. Since bets are independent and p is constant, we get a binomial distribution for the number of wins, $w \sim \mathrm{B}(n, p)$. We can rearrange (1) to obtain

$$w_\beta = \frac{\ln(1 + \beta) - n \ln(1 - f)}{\ln(1 + fb) - \ln(1 - f)} \tag{2}$$

For a given bet with Kelly fraction f, probability p and profit b, made n times, given a specified return of β, equation (2) gives the required number of won bets to achieve this return, w_β. Thus, to calculate the probability of cumulative returns less than or equal to β, we have that $P(B \leq \beta) = P(w \leq w_\beta)$, where B is the stochastic variable for returns after these n bets, and w is the binomially-distributed stochastic variable. Hence,

$$P(B \leq \beta) = \sum_{i=0}^{\lfloor w_\beta \rfloor} \binom{n}{i} p^i (1-p)^{n-i} \tag{3}$$

where $\lfloor w_\beta \rfloor$ is the floor of w_β, the greatest integer less than or equal to it. To avoid this rounding of w_β, assuming that the number of bets n under consideration is large, we can use a normal approximation to the binomial, giving

$$P(B \leq \beta) = \phi\left(\frac{w_\beta - np}{\sqrt{np(1-p)}}\right) \tag{4}$$

where ϕ is the cumulative distribution function of the standard normal distribution, and np and $\sqrt{np(1-p)}$ are the mean and standard deviation, respectively, of w. Since w_β is continuous, there is no need for a continuity correction. In general, the cumulative distribution function integral is

$$\phi(x) = \frac{1}{2}\left(1 + \text{erf}\left(\frac{x}{\sqrt{2}}\right)\right) \tag{5}$$

where erf is the error function

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt \tag{6}$$

and, in our case,

$$\frac{x}{\sqrt{2}} = \frac{w_\beta - np}{\sqrt{2np(1-p)}} = u \tag{7}$$

with u a simplifying substitution. Our optimisation problem is to find a Kelly fraction f such that we have a probability of at most α of experiencing cumulative returns of β or worse. Thus,

$$P(B \leq \beta) = \frac{1}{2}\left(1 + \text{erf}(u)\right) = \alpha \implies \text{erf}(u) = 2\alpha - 1 \tag{8}$$

A standard approximation for erf(u) (Winitzki, 2003; Zeng and Chen, 2015) is

$$\text{erf}(u) \approx \text{sgn}(u)\sqrt{1 - \exp\left(-u^2 \frac{\frac{4}{\pi} + au^2}{1 + au^2}\right)} \tag{9}$$

where $a = \frac{8(\pi-3)}{3\pi(4-\pi)}$. Substituting in $2\alpha - 1$ from equation (8), equation (9) can be rearranged to a biquartic equation in u, as

$$
\begin{aligned}
&Au^4 + Bu^2 + C = 0 \\
&\text{with} \\
&A = a \\
&B = \frac{4}{\pi} + aC \\
&C = \ln(1 - (2\alpha - 1)^2) \\
&\text{giving} \\
&u = \pm\sqrt{\frac{-B \pm \sqrt{B^2 - 4AC}}{2A}}
\end{aligned}
\tag{10}
$$

which depends entirely on α as an input parameter. We, thus, can find the number of wins w_α equivalent to the α-th percentile of the distribution of w. Given the domain $\alpha \in [0, 1]$, we can reason about the $\pm$ in the expression for u. Over this domain, C is negative, the discriminant $\Delta = B^2 - 4AC$ is strictly positive, and $\sqrt{\Delta} \geq B$. Therefore, the inner $\pm$ must be $+$, given that we want real solutions. For the outer $\pm$, from (7), assuming that this risk-management system is applied for minimising the probability of losses, we get that $u < 0$, meaning that the outer $\pm$ is $-$. Thus,

$$
\begin{aligned}
&u = \frac{w_\alpha - np}{\sqrt{2np(1-p)}} = -\sqrt{\frac{-B + \sqrt{B^2 - 4AC}}{2A}} \\
&\implies \\
&w_\alpha = np - \sqrt{np(1-p)\frac{-B + \sqrt{B^2 - 4AC}}{A}} = \frac{\ln(1+\beta) - n\ln(1-f)}{\ln(1+fb) - \ln(1-f)}
\end{aligned}
\tag{11}
$$

with the second equality coming from equating w_α with w_β from (2).

We now want to isolate f for a closed-form expression for the Kelly fraction given a desire to have at most a probability of α of seeing losses of β or worse through n repeated bets. This cannot be done directly; however, using the fact that for small x, $\ln(1+x) \approx x - \frac{x^2}{2} + \frac{x^3}{3} - \ldots$, and $\ln(1-x) \approx -x - \frac{x^2}{2} - \frac{x^3}{3} - \ldots$, via Taylor series, we can approximate $\ln(1-f) \approx -f - \frac{f^2}{2}$ and $\ln(1+fb) \approx fb - \frac{f^2b^2}{2}$, taking only the second-degree expansion. We thus get

$$
w_\beta \approx \frac{\ln(1+\beta) + nf(1+\frac{f}{2})}{fb(1-\frac{fb}{2}) + f(1+\frac{f}{2})}
\tag{12}
$$

which rearranges to the quadratic in f,

$$
\begin{aligned}
&Af^2 + Bf + C = 0 \\
&\text{with} \\
&A = \frac{n}{2} + \frac{w_\alpha}{2}(b^2 - 1) \\
&B = n - w_\alpha(1+b) \\
&C = \ln(1+\beta) \\
&\text{giving} \\
&k = \frac{w_\alpha(1+b) - n \pm \sqrt{(n - w_\alpha(1+b))^2 - 2\ln(1+\beta)(n + w_\alpha(b^2-1))}}{n + w_\alpha(b^2-1)}
\end{aligned}
\tag{13}
$$

where, given that $n > w_\alpha(1+b)$ in most cases and that $f > 0$, we can replace the $\pm$ with $+$.

Thus, we get the final closed-form expression for f:

$$
\begin{aligned}
f &= \frac{w_\alpha(1+b) - n + \sqrt{(n - w_\alpha(1+b))^2 - 2\ln(1+\beta)(n + w_\alpha(b^2-1))}}{n + w_\alpha(b^2-1)} \\
w_\alpha &= np - \sqrt{np(1-p)\frac{-B + \sqrt{B^2 - 4AC}}{A}} \\
A &= \frac{8(\pi-3)}{3\pi(4-\pi)} \\
B &= \frac{4}{\pi} + AC \\
C &= \ln(1 - (2\alpha - 1)^2)
\end{aligned}
\tag{14}
$$

which takes arguments p, b, n, α, β.

3 Single-Gamble Evaluation

To evaluate the accuracy of our formula, we perform simulations. For specified values of the input parameters, we calculate f and then create a $100,000 \times n$ matrix of random probabilities $q \sim \mathrm{U}(0,1)$. For each element in this matrix, we compare q to p, and in the corresponding cell in another $100,000 \times n$ matrix, we insert $1 + fb$ if $q \leq p$ and $1 - f$ otherwise. We then calculate the cumulative product along the rows of this second matrix, giving 100,000 final returns from the final column, finding the proportion $\hat{\alpha}$ of which are lower than β and comparing this value to α. We do this 1,000 times, drawing $p \sim \mathrm{U}(0.1, 0.9)$; the bet expected value $EV \sim \mathrm{U}(0.005, 0.05)$, giving b from $EV = p(1+b) - 1$; and $n = 10^u$, where $u \sim \mathrm{U}(2,3)$, and n is rounded to the nearest integer. We similarly have $\alpha \sim \mathrm{U}(0.01, 0.1)$ and $\beta \sim \mathrm{U}(-0.1, -0.01)$.

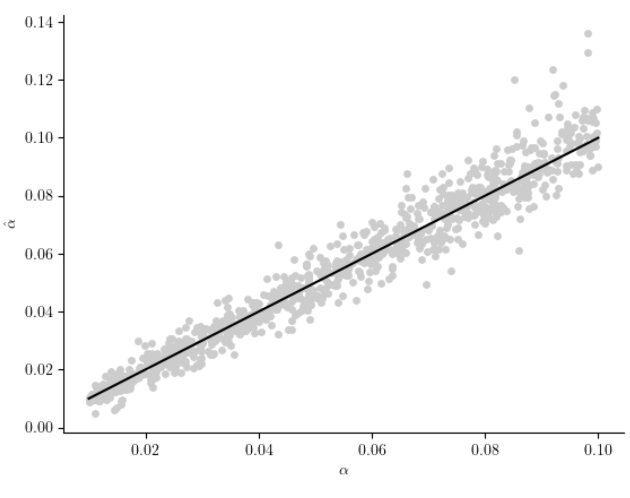

Figure 1: For a gamble with parameterisation n, p, b, α, β, we compare α to the proportion of runs with final returns less than or equal to β. The black line is $\hat{\alpha} = \alpha$, not a trend line, with our expression therefore an excellent fit.

Worthy of note is the discrete nature of $\hat{\alpha}$. In (14), we use the analytical value for the constant $A = \frac{8(\pi-3)}{3\pi(4-\pi)}$. If we instead investigate how $\hat{\alpha}$ varies for a fixed set of parameters (p, b, n, α, β) when we change the constant A, we find that $\hat{\alpha}$ does not respond to small changes in A. It instead displays a discontinuous stepwise relationship, shifting to a lower level when A increases sufficiently. This makes intuitive sense. The number of bets won in an iteration is determined solely by p and n in a binomial distribution. Since the cumulative multiplication process of Kelly capital is commutative, the sequencing of won and lost bets does not matter. Therefore, for a given value of f, we can imagine the range of won bets $w \in [0, \lfloor w_\beta \rfloor]$

which gives returns less than β, and then the range of won bets $[\lceil w_\beta \rceil, n]$ which does not. With this distribution of wins unchanged, small movements in f occasioned by nudges to A will change cumulative returns, but not sufficiently to shift the value of w_β into the domain between the next pair of integers. Only when this occurs, when one more or less won bet is required to fall below β, does $\hat{\alpha}$ shift. We can therefore conclude that $\hat{\alpha}$ can never equal α exactly, as it is binary not continuous. In the simulation studies, then, we cannot expect these $\hat{\alpha}$ to fit the α perfectly as the inputs vary, but instead hope to see a strong relationship between them, sensitive to changes in all inputs. As such, the results in Figure 1 are particularly impressive given this property.

We perform the simulation outlined above for all combinations of the parameters shown in Table 1. Across these 320 iterations, $Corr(\hat{\alpha}, \alpha) = 0.982$, with a mean absolute percentage error of $\hat{\alpha}$ with respect to α of 12.65%, impressive given that $\hat{\alpha}$ is discrete. For $\alpha = 0.01$, the mean absolute error was 0.00199, with values of 0.00334, 0.00567 and 0.00604 for $\alpha = 0.025$, 0.05, and 0.1 respectively. Additionally, 45% of iterations saw $\hat{\alpha} < \alpha$, close enough to the expected 50%. Dissecting further, if we use the notation $P(\hat{\alpha} < \alpha \mid n)$, for instance, to represent this proportion for those iterations with a specific value of n, we find that $E[P(\hat{\alpha} < \alpha \mid n)] = P(\hat{\alpha} < \alpha)$, $E[P(\hat{\alpha} < \alpha \mid \alpha)] = P(\hat{\alpha} < \alpha)$ and $E[P(\hat{\alpha} < \alpha \mid \beta)] = P(\hat{\alpha} < \alpha)$, that is, there are no significant trends in the data suggesting that the levels of n, α and β, respectively, influence the calibration of the calculation of f to the desired risk level.

n	100	250	500	1000	
α	0.01	0.025	0.05	0.1	
β	-0.01	-0.025	-0.05	-0.1	
(p, b)	(0.1,9.5)	(0.25,3.15)	(0.5, 1.05)	(0.75, 0.35)	(0.9,0.115)

Table 1: Parameter values for n, α, β, (p, b) in the simulation, giving $4 \cdot 4 \cdot 4 \cdot 5 = 320$ combinations.

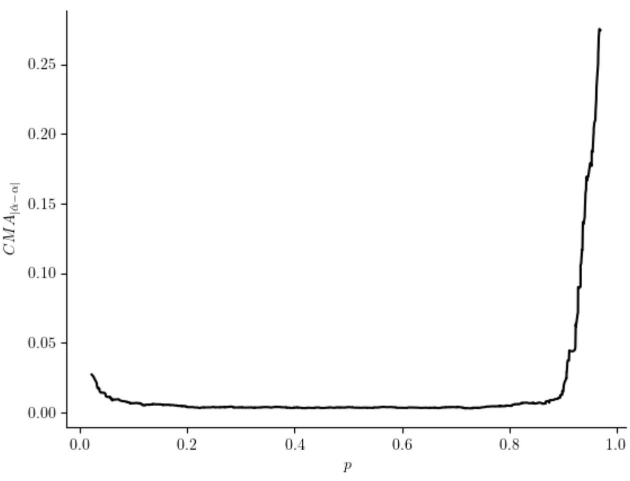

Figure 2: Central moving average across 25 preceding and succeeding points of the absolute difference between $\hat{\alpha}$ and α. Calibration decreases when probabilities become extreme.

However, while the correlation between $\hat{\alpha}$ and α remains almost perfect when controlling for p; at low probabilities, $\hat{\alpha}$ is lower than α, while the opposite is true at high values of p. This suggests that f is exaggerated for large probabilities and understated for low-probability bets. However, this is unsurprising. As outlined above, with $\hat{\alpha}$ being discrete, when considering the ranges $[0, \lfloor w_\beta \rfloor]$ and $[\lceil w_\beta \rceil, n]$ of won bets w giving returns less than respectively greater than β, when probabilities are extreme, the difference between α and the cumulative probability of the floor of w_β increases. From (11), we see that w_β depends on n, p and α as input. We therefore take the values $p = 0.1, 0.5, 0.9$ and for each of the combinations of these with the four values of n and α in Table 1, we calculate the difference between α and the cumulative probability of $\lfloor w_\beta \rfloor$, from the Binomial with n and p as parameters. In this simulation, the mean value (note, not mean absolute value) of $P(w \leq \lfloor w_\beta \rfloor) - \alpha$ was -0.0027,

-0.00025 and 0.0050, respectively, for the three values of p. In our equation for f, we calculate the cumulative probability of w_β by approximating it as continuous; however, in the simulations, it is discrete. At low levels of p, the discrete probability is lower than α, which is why we see a large proportion of $\hat{\alpha} < \alpha$ at low probabilities, and vice versa for high probabilities. Thus, the equation remains calibrated in p as well, the distorted values of $P(\hat{\alpha} < \alpha \mid p)$ simply a result of the discrete nature of the simulations.

We have limited ourselves to calculations where $p \in [0.1, 0.9]$. This is due to the deterioration in the calibration of $\hat{\alpha}$ to α as the probabilities move towards 0 or 1. Figure 2 plots the absolute difference between the two versus p, where b and f are calculated as in the simulations above. Plotted is a central moving average, at each point p the average absolute difference between $\hat{\alpha}$ and α across the previous and immediately succeeding twenty-five points. As p becomes increasingly extreme, the error increases, especially for large probabilities, meaning, on average, that the calibration of our risk-management is superior for more moderate probabilities. The points 0.1 and 0.9 used above are largely arbitrary, but seem to be roughly where the rate of deterioration increases drastically.

4 Elaboration for Portfolios of Distinct Gambles

Above we have seen that the equation for f is accurate across a broad range of the input parameters. The analysis presented a single bet with probability p and profit b, and optimised the Kelly fraction such that, over n repeated wagers, the probability of experiencing returns worse than β equals α. In many practical scenarios, however, a wager is available as a one-time opportunity, not being able to be repeated arbitrarily. We therefore explore the possibilities of using our calculation for f to create a general risk-management strategy, in the sense that, if we take n separate wagers, each optimised according to the same value-at-risk profile defined by n, α and β, then for these n distinct bets, combined, they will achieve the same value at risk. The intuition is simple. Take the two wagers $(p, b) = (0.25, 3.15), (0.75, 0.35)$, calculating f optimised over the same n, α and β. Repeating each bet $n-1$ times, adding the n-th bet, in each case, gives a probability α of experiencing returns less than or equal to β. Therefore, it seems reasonable that if we instead take the $n-1$ bets of each wager, but then add the n-th bet from the other, the same value-at-risk risk profile will be achieved. We can then use this reasoning inductively, with the commutativity of the Kelly gambling returns, to create a portfolio of n separate wagers with a well-defined value at risk.

Therefore, we seek to optimise f with respect to p. We do so by adjusting the constant A

$$A = \frac{8(\pi - 3)}{3\pi(4-\pi)} + \theta_0 + \theta_1 p + \theta_2 p^2 + \theta_3(p(1+b) - 1) \tag{15}$$

where the θ are parameters to be optimised, as an adjustment to the intercept, the probability and its quadratic term, and the wager expected value. The simulation performed is as follows, for a chosen risk profile n, α and β, we for a given set of parameters $\boldsymbol{\theta} = (\theta_0, \theta_1, \theta_2, \theta_3)$, for each of 100,000 runs (as above), draw n values $EV \sim \mathrm{U}(0, 0.025)$ and $p \sim \mathrm{U}(p_l, p_u)$, where we have eight pairs $(p_l, p_u) = (0.1, 0.2), (0.2, 0.3), \ldots (0.8, 0.9)$. We calculate the adjusted f with the input parameters, and hence get a value for $\hat{\alpha}$, as above. For each pair (p_l, p_u) we repeat this process five times, obtaining five values for $\hat{\alpha}$. For each pair, we select the median value of $\mid \hat{\alpha} - \alpha \mid$, and then across the eight medians, return the maximum value. We thus seek the parameters in $\boldsymbol{\theta}$ which minimises this maximum median value. The rationale is that we want the model to be calibrated across the entire domain of p, and hence split it into these segments of width 0.1, while considering only the worst value.

(p_l, p_u)	$(0.1, 0.2)$	$(0.2, 0.3)$	$(0.3, 0.4)$	$(0.4, 0.5)$	$(0.5, 0.6)$	$(0.6, 0.7)$	$(0.7, 0.8)$	$(0.8, 0.9)$
0	0.04777	0.0487	0.05101	0.05119	0.0528	0.0534	0.05705	0.07518
$\boldsymbol{\theta}$	0.05275	0.04928	0.0479	0.04586	0.04504	0.04486	0.04608	0.05799

Table 2: For eight ranges of (p_l, p_u), a comparison of $\hat{\alpha}$ for the median-error case between the optimised and unoptimised models.

For the value at risk profile $(n, \alpha, \beta) = (500, 0.05, -0.05)$, in our simulation we found that $\boldsymbol{\theta} = (-0.4, 0.933, 1.55, 0.3833)$ gives the best fit, with a representative distribution of $\hat{\alpha}$ values corresponding to the median value of $\mid \hat{\alpha} - \alpha \mid$ for each of the eight segments being displayed in Table 2, compared to the same values without optimisation (with parameters from the zero vector **0**). A couple of comments are that, first, here the model error is related to p, in a convex quadratic effect. For moderate values of p, the expression value for f will be too conservative, while it will be munificent for extreme values. Second, we have a rather conservative range of expected values, these ranging from 0 to 0.025. When the upper limit of this range is increased, the model calibration deteriorates rapidly (also true for the single-gamble case above). This is a clear limitation, with our expression only applicable for wagers which are only slightly favourable; of course, if a bettor can systematically identify extremely favourable betting opportunities, then their need for a conservative risk-management system will be decreased. Furthermore, these parameters are results which hold only for $(n, \alpha, \beta) = (500, 0.05, -0.05)$. When attempting to optimise $\boldsymbol{\theta}$ while also varying n, α and β, the calibration again exploded. Finally, in the majority of segments, the basic model actually performs reasonably well, sometimes even better than the optimised model; however, it is evident that, when the parameters are fit, the worst case is significantly better. As such, for this particular value at risk specification, and for the given ranges of probabilities and expected values, we get the final closed-form expression for f:

$$\begin{aligned}
k &= \frac{w_\beta(1+b) - n + \sqrt{(n - w_\beta(1+b))^2 - 2\ln(1+\beta)(n + w_\beta(b^2-1))}}{n + w_\beta(b^2-1)} \\
w_\beta &= np - \sqrt{np(1-p)\frac{-B + \sqrt{B^2 - 4AC}}{A}} \\
A &= \frac{8(\pi - 3)}{3\pi(4-\pi)} + \theta_0 + \theta_1 p + \theta_2 p^2 + \theta_3(p(1+b) - 1) \\
B &= \frac{4}{\pi} + AC \\
C &= \ln(1 - (2\alpha - 1)^2)
\end{aligned} \tag{16}$$

which takes arguments p, b, n, α, β, and $\boldsymbol{\theta}$, with $(\theta_0, \theta_1, \theta_2, \theta_3) = (-0.4, 0.933, 1.55, 0.3833)$ for $(n, \alpha, \beta) = (500, 0.05, -0.05)$. Of course, if an individual bettor has a certain class of wagers, in terms of the typical values for p and b that they can systematically identify, then they can customise $\boldsymbol{\theta}$ accordingly. We have here merely demonstrated that it is possible to create a reasonably well-calibrated model for the broadest range of parameters possible.

5 Conclusion

In this paper, we have demonstrated that our closed-form expression for the optimal Kelly fraction, subject to a desired value at risk, is well-calibrated across a broad range of gambling scenarios. We have also seen that a bettor can use an adjusted version of our expression when they stake on n distinct wagers, rather than the same wager repeated n times, to similarly control their downside. Although not applicable in all situations, our framework is useful for practitioners, who want to maximise their capital growth in the spirit of Kelly-criterion investing, while simultaneously managing risk.

6 References

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