Introduction to the Symplectic Groups

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(Do people know what a bilinear form is?) A bilinear form is alternating if $f(v, v) = 0$ for all $v \in V$. Any alternating bilinear form is skew symmetric $f(u, v) = -f(v, u)$. We write $V^\perp = \{w \in V : f(w, v) = 0 \text{ for all } v \in V\}$ - i.e. the set of v in V that are orthogonal to every other vector in V. This is called the radical of V and f is non-singular if this radical is 0.

The symplectic group $Sp_{2m}(q)$ is the isometry group of a non-singular alternating bilinear form f on V ($\cong$ to $(\mathbb{F}_q)^{2m}$). i.e. the subgroup of $GL_{2m}(q)$ consisting of those elements g s.t. $f(u^g, v^g) = f(u, v)$ for all u, $v \in V$. It is always of even dimension since the determinant of a skew symmetric form of odd dimension is zero and hence not in GL(V). $Sp_{2m}(q)$ is not dependent on f since all forms of this type are equivalent.

A linear map $\tau : V \to V$ is a transvection with fixed hyperplane W if $\tau|_W = Id|_W$ and $\tau x - x \in W$ for all $x \in V$. (Do people know what a hyperplane is?) Given a vector v and a scalar λ we can construct a transvection $T_v(\lambda) : x \to x + \lambda f(x, v)v$. This is a transvection since it fixes the hyperplane $v^\perp = \{w \in V : f(w, v) = 0\}$ and $T_v(\lambda)(x) - x = \lambda f(x, v)v$ is a scalar multiple of v and so is in $\{v\}^\perp$ since f is alternating.

Theorem 1.1 *$Sp_{2m}(q)$ is generated by the set of symplectic transvections. Let S be the group generated by the symplectic transvections.*

Proof: $S \leq \mathrm{Sp}_{2m}(q)$ is an easy exercise.

a) S is transitive on non-zero vectors.

Let v and w be two distinct non-zero vectors with (case 1) $f(v, w) = \lambda$ non-zero. Then $T_{(v-w)}(\lambda^{-1})$ sends v to w (easy exercise - stick the values into Rob Wilson's definition of a transvection). (case 2) If $f(v, w) = 0$ pick $x \in V$ s.t. $f(v, x)$ and $f(w, x)$ are non-zero. This can be done since there exists y and z s.t. $f(v, y) = f(w, z) = 0$ but $f(v, z)$ and $f(w, y)$ are non-zero since f is non-singular. Then we can take x to be a suitable linear combination of y and z. Now we can map v to x and x to w and hence S is transitive on non-zero vectors.

b) S is transitive on hyperbolic pairs (pairs (u, v) where $f(u, v) = 1$) i.e. there us a product of transvections mapping the hyperbolic pair (u_1, v_1) to (u_2, v_2). Part a) tells us there exists $\tau : u_1 \to u_2$. So $\tau : (u_1, v_1) \to (u_2, (v_1)^\tau = v_3)$. We want to construct σ s.t. $(u_2)^\sigma = u_2$ and $(v_3)^\sigma = v_2$.

i) If $f(v_3, v_2)$ is non-zero then the transvection $T_{(v_3-v_2)}(\frac{1}{f(v_3,v_2)})$ maps u_2 to u_2 and v_3 to v_2.

ii) If $f(v_3, v_2) = 0$, $f(v_3, u_2+v_3) = -1$. $f(u_2+v_3, v_2) = f(u_2, v_2)$, which is non-zero and $f(u_2, -u_2) = 0 = f(u_2, u_2 + v_3 - v_2)$. Now, $s_1 = T_{(-u_2)}(\frac{1}{f(v_3,u_2)})(v_3) = u_2 + v_3$ and $s_2 = T_{(u_2+v_3-v_2)}(\frac{1}{f(u_2+v_3,v_2)})(u_2 + v_3) = v_2$ and both s_1 and s_2 fix u_2. Hence, $s_2 s_1(u_1, v_1) = (u_2, v_2)$.

All calculations in part b were tested and are correct.

c) Induction on m. $m = 1$ gives $\mathrm{SL}(V) = \mathrm{Sp}(V)$:
$M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, $T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, a general element of $\mathrm{GL}(2, F)$.

Then $T \in \mathrm{Sp}(2, F)$ iff. $T^t M T = M$. (leave this out if you can: since $f(u, w) = f(\sum a_i v_i, \sum c_i, v_i) = \sum a_i f(v_i, v_j) c_j = u^t M w$, where the vs are the basis vectors for V and u is the vector with the as in it and w the vector with the cs in it.)

which implies that

$$\begin{pmatrix} 0 & ad - bc \\ -ad + bc & 0 \end{pmatrix}$$

$=$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

and so $\det(T) = 1$.

Now, $\mathrm{SL}(V)$ is generated by transvections (see Grove) so the result for $m = 1$ follows.

Now choose vectors u, v in V s.t. $f(u, v) = 1$ and let W be the plane that they span. $V = W \oplus W^\perp$. Take any σ in $\mathrm{Sp}(V)$. Then $f(u^\sigma, v^\sigma) = 1$ since σ is an isometry of f. Part b states that there is a τ in S s.t. $\tau\sigma u = u$ and $\tau\sigma v = v$ which implies that $\tau\sigma|_W = id_W$. $\tau\sigma|_{W^\perp}$ is in $\mathrm{Sp}(W^\perp)$ and so by induction $\tau\sigma|_{W^\perp}$ is a product of symplectic transvections on $W^\perp$ (since dim $W^\perp <$ dim V). Any transvection on $W^\perp$ can be extended to a transvection of V, which included W in its fixed hyperplane via the same formula for $\tau\sigma|_{W^\perp}$. ($T_v(\lambda) : x \to x + \lambda f(x, v) v$ where x is in W and v is in $W^\perp$ so $f(x, v) = 0$ and so W is indeed fixed.) Since $\tau\sigma|_W = id|_W$, $\tau\sigma \in S$ implies that σ is in S.

□

For many cases $\mathrm{Sp}'(V) = \mathrm{Sp}(V)$ e.g. characteristic of $F = 2$ and dim $V \geq 6$ and char $F = 3$, dim $V \geq 4$. We prove one of these: char $F \geq 4$.

Theorem 1.2 *If $F \geq 4$, $\mathrm{Sp}'(V) = \mathrm{Sp}(V)$.*

Proof: Fix non-zero u in V and a in F^*. Choose b in $F - \{0, 1, -1\}$ and set $c = a/(1 - b^2)$, $d = -(b^2)c$. Then $c + d = a$ and so $T_u(c)T_u(d) = T_u(a)$ - easy exercise. Choose σ in $\mathrm{Sp}(V)$ s.t. $\sigma u = bu$ (can be done sine action of $\mathrm{Sp}(V)$ on V is transitive). Then $\sigma(T_u(c))^{-1}\sigma^{-1} = \sigma T_u(-c)\sigma^{-1} = T_{\sigma u}(-c) = T_{bu}(-c) = T_u(-(b^2)c) = T_u(d)$.

($\sigma T_u(-c)\sigma^{-1} = T_{\sigma u}(-c)$ since: $\sigma T_u(-c)\sigma^{-1}(x) = \sigma T_u(-c)[\sigma^{-1}(x)] = \sigma[\sigma^{-1}(x) - cf(\sigma^{-1}(x), u)u] = x - cf(x, \sigma u)\sigma u = T_{\sigma u}(-c)$.

This implies that $T_u(c)\sigma(T_u(c))^{-1}\sigma^{-1} = T_u(c)T_u(d) = T_u(a)$ which is in $\mathrm{Sp}\prime(V)$. As $Sp(V)$ is generated by transvections and this can be done for all non-zero u and a, $\mathrm{Sp}\prime(V) = Sp(V)$. $\square$

Let $\mathbb{P} = \mathbb{P}_{n-1}(V)$ denote the set of all projective lines in V. That is, for each non-zero $v \in V$, the set or all $[v]$ where $[v]$ denotes the line through the origin $\mathbb{F}_q v$.

Theorem 1.3 $\mathrm{Sp}(V)$ *is primitive on* $\mathbb{P} = \mathbb{P}_{n-1}(V)$.

Proof:

We may assume that $n \geq 4$ as $\mathrm{Sp}(2, F_q) = \mathrm{SL}(2, F_q)$, which is doubly transitive. Suppose that S is a subset of $\mathbb{P}$, $|S| > 1$ and either $\sigma S = S$ or $\sigma S \cap S$ for each $\sigma \in \mathrm{Sp}(V)$. We show first that there exist $[u], [v] \in S$ with $B(u, v)$ non-zero. Suppose, for contradiction, that for all $[u], [v] \in S$, $B(u, v) = 0$. Choose $[u] \neq [v]$ in S and choose $f \in V^*$ (the dual of V) with $f(u) = 1, f(u) = 0$. There exists $x \in V$ with $B(u, x) = 1, B(v, x) = 0$ (by corollary 2.2 of Grove). Set $W = \mathrm{Span}(u, x)$ - a hyperbolic plane. Set $H = \{\sigma \in \mathrm{Sp}(V) : \sigma|_W = 1_W\} \leq \mathrm{Sp}(V)$. Since $W \oplus W^\perp$, it is clear that every $\sigma \in \mathrm{Sp}(W^\perp)$ extends trivially to $\sigma' \in \mathrm{Sp}(V)$ with $\sigma' = 1_W$ so in fact $\mathrm{Sp}(W^\perp) = \{\tau|_W^\perp : \tau \in H\}$. Choose a non-zero $w \in W^\perp$. Since $v \in W^\perp$, there is some $\tau \in H$ with $\tau v = w$, since the group generated by symplectic transvections is transitive on non-zero vectors. Since $\tau u = u$ (because $u \in W$ and τ is the identity on W), we have that $[u] \in S \cap \tau$, so $\tau S = S$. Also $[w] = \tau[v] \in S$ and w was an arbitary non-zero vector in $W^\perp$. Since $W^\perp \neq 0$, there is a hyperbolic pair (y, z) in $W^\perp$; hence $[y], [z] \in S$, but $B(y, z) = 1$ - a contradiction.

Thus we can choose $[u], [v] \in S$ with $B(u, v)$ non-zero so we can assume that $B(u, v) = 1$ and so (u, v) is a hyperbolic pair. Take any $[w] \in \mathbb{P}$. If $B(u, w) \neq 0$, we mat assume that (u, w) is a hyperbolic pair and so we can find $\sigma \in \mathrm{Sp}(V)$ with $\sigma u = u$ and $\sigma v = w$ by the fact that $\mathrm{Sp}(V)$ is transitive on hyperbolic pairs. Thus $[u] \in S \cap \sigma S$ and so $\sigma S = S$ and $[w] \in S$. On the other hand, if $B(u, w) = 0$, choose $f \in V^*$ with $f(u) = f(w) = 1$ and thereby obtain $x \in V$ with $B(u, x) = B(w, x) = 1$. As above, $[x] \in S$ and also there exists $\tau \in \mathrm{Sp}(V)$ with $\tau u = w$ and $\tau x = x$. Again $\tau S = S$, since $[x] \in S \cap \tau S$ and $[w] = \tau[u] \in S$. Thus $S = \mathbb{P}$. $\square$

Theorem 1.4 *Except for* $\mathrm{PSp}(2, 2), \mathrm{PSp}(2, 3)$ *and* $\mathrm{PSp}(4, 2)$ *every projective symplectic group* $\mathrm{PSp}(V)$ *is simple.*

Proof: We know that $\mathrm{PSp}(V)$ acts faithfully on $\mathbb{P}$ and the action is primitive by the above. With the exceptions of above, $\mathrm{PSp}(V)$ is equal to its derived group - one example has been done above. Fix $[u] \in \mathbb{P}$ and set $H = \mathrm{Stab}_{Sp(V)}([u])$, $\bar{H} = H/\{\pm 1\} = \mathrm{Stab}_{PSp(V)}([u])$. Set $K = \{\tau_{u,a} : a \in F\}$. K is normal in H and is isomorphic to F (exercise - since ${}^{\sigma}\tau_{u,a} = \tau_{\sigma u,a}$ (K normal) and $\tau_{u,a}\tau_{u,b} = \tau_{u,a+b}, \tau_{bu,a} = \tau_{u,ab^2}$ and $\tau^{-1}{}_{u,a}$ (H isom to F)) so K is abelian. If $\sigma \in \mathrm{Sp}(V)$ then ${}^{\sigma}K \supseteq \{\tau_{\sigma u,a} : a \in F\}$. This means that $\bigcup\{{}^{\sigma}K : \sigma \in \mathrm{Sp}(V)\}$ contains all symplectic transvections and hence generates $\mathrm{Sp}(V)$. Thus $\left\langle {}^{\bar{\sigma}}\bar{K} : \bar{\sigma} \in \mathrm{PSp}(V)\right\rangle = \mathrm{PSp}(V)$ and hence (bar the noted exceptions) is simple by Iwasawa's Lemma. (G faithful and primitive on S and equal to its derived group. $s \in S$ fixed $H = Stab_G(S)$. $K \triangleleft H$ and $G = \langle K^x : x \in G\rangle$. Then G is simple.) □