The Order of the Symplectic Groups

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Theorem 1.1 *If* $\mathbb{F}_q$ *is finite then* $|\mathrm{Sp}(2m,q)| = |q^{m^2}| \prod_{i=1}^{m}(q^{2i}-1)$ *and* $\mathrm{PSp}(2m,q) = \frac{|\mathrm{Sp}(2m,q)|}{(2,q-1)}$.

PROOF: Let $n = 2m$. Any one of the $q^n - 1$ non-zero vectors of V will do for the first vector u om a hyperbolic pair. Now, $|q^{\perp}| = q^{n-1}$ and so there are $q^n - q^{n-1}$ choices of vector v such that $f(u,v) \neq 0$. Thus, there are $(q^n - q^{n-1})/(q-1) = q^{n-1}$ choices of v such that $f(u,v) = 1$. Hence, there are $(q^n - 1)q^{n-1}$ distinct hyperbolic pairs (number of choices of u times number of choices of v). If (u,v) is a hyperbolic pair and W is the hyperbolic plane with $\{u,v\}$ as a basis then $\sigma \in \mathrm{Stab}_{Sp(V)}((u,v))$ if and only if $\sigma|_W = 1_W$. As we saw in the proof that $\mathrm{Sp}(V)$ is primitive on $\mathbb{P}_{n-1}(V)$, we have that $\mathrm{Stab}_{Sp(V)} \cong \mathrm{Sp}(W^{\perp})$. Thus $|\mathrm{Sp}(n,q)| = (q^n-1)q^{n-1}|\mathrm{Sp}(n-2,q)|$ by the Orbit-Stabilizer Theorem. We now proceed by induction. When m = 1 (and so n = 2) we have that $|\mathrm{Sp}(2,q)| = |\mathrm{SL}(2,q| = q(q^2-1)$. Assume inductively that $|\mathrm{Sp}(n-2,q)| = q^{(m-1)^2}\prod_{i=1}^{m-1}(q^{2i}-1)$.

Then, $|\mathrm{Sp}(n,q)| = q^{2m-1}(q^{2m}-1)q^{(m-1)^2}\prod_{i=1}^{m-1}(q^{2i}-1) = q^{m^2}\prod_{i=1}^{m}(q^{2i}-1)$.

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