We are considering the natural representation only. We wish to kill an arbitrary element of $\Omega^-(d,q)$, where $q$ is even. In order to do this, we wish to embed $\Omega^+(d-2,q)$ into it. That is to say write the generators of $\Omega^+(d-2,q)$ in terms of those of $\Omega^-(d,q)$. There is only one generator of $\Omega^+(d-2,q)$ that we need to consider and that is $t$:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The following is the proof in the thesis of how to create this element. Can you think of any way of finding out $z$ without using logarithms? One of the problems I have here is that I don't have $\gamma + \gamma^q$ as a power of $\omega$ (notation explained below).

**Lemma 0.1** *Let $B(h) = (h^{v^2})^{-1}((h^\delta)^v)h^{v^2}$ and let $\omega^z$ be (4, 1) entry of $B(h)$, where $v$ and $\delta$ are generators for $\Omega^-(2d,q)$ as they appear in the above table, $h$ is some other generator and $\omega$ is the primitive element of the ground field. Then for even characteristic:*

- $t \in \Omega^+(2d-2,q)$ *is formed by* $(t^v B(t))^{\delta^z} \in \Omega^-(2d,q)$*;*
- $r \in \Omega^+(2d-2,q)$ *is formed by* $(r^v B(r))^{\delta^z} \in \Omega^-(2d,q)$*;*
- $t' \in \Omega^+(2d-2,q)$ *is formed by* $((t^v B(t))^{\delta^z})^s \in \Omega^-(2d,q)$*;*
- $r' \in \Omega^+(2d-2,q)$ *is formed by* $((r^v B(r))^{\delta^z})^s \in \Omega^-(2d,q)$*;*

Proof: Firstly, consider $(t^v)^\delta$. This gives a matrix of the following form:

$$\begin{pmatrix} 1 & \omega^{-2} & 0 & 0 & 0 & \dots & 0 & * & * \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & * & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix},$$

where the asterisks represent arbitrary elements of $\mathrm{GF}(q)$.

Now we conjugate by $t^{v^2}$, which only affects the top left $4 \times 4$ block. A simple calculation shows that this gives the following matrix:

$$B(t) = (t^{v^2})^{-1}(t^v)^\delta t^{v^2} = \begin{pmatrix} 1 & \omega^{-2} & 0 & x & 0 & \dots & 0 & * & * \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & x & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & * & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & * & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix},$$

where $x$ is an element of $\mathrm{GF}(q)$ and the asterisks represent the same arbitrary elements of $\mathrm{GF}(q)$ as in the first step.

We now need to work out how to set the (1, 2) entry to 0. By direct calculation, we can see that conjugating the above matrix by $\delta^{-1}$ gives:

$$\begin{pmatrix} 1 & 1 & 0 & y & 0 & \dots & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & y & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & \gamma+\gamma^q & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix},$$

where $y \in \mathrm{GF}(q)$ and $\gamma$ is the primitive element of $\mathrm{GF}(q^2)$. As the asterisked entries were not changed by conjugating by $t^{v^2}$, the portion of the matrix outside the $4 \times 4$ block will look like $t^v$, since $t^{v\delta\delta^{-1}} = t^v$.

Pre-multiplying by $t^v$ then gives:

$$\begin{pmatrix} 1 & 0 & 0 & y & 0 & \dots & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & y & 1 & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 \end{pmatrix}.$$

Hence conjugating the above element by $\delta^z$ gives the required matrix and so we're done.

The other three equations can be shown to hold by a similar method. $\square$