Finite automata and formal languages (DIT323, TMV029)

Nils Anders Danielsson,
partly based on slides by Ana Bove

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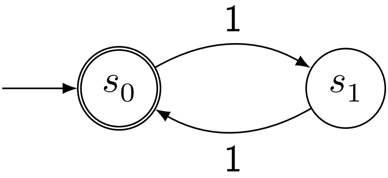
Today

- Deterministic finite automata.

DFAs

DFAs

Recall from the first lecture:



- A DFA specifies a language.
- In this case the language $\{ 11 \}^* = \{ \varepsilon, 11, 1111, \ldots \}$.
- DFAs are for instance used to implement regular expression matching.

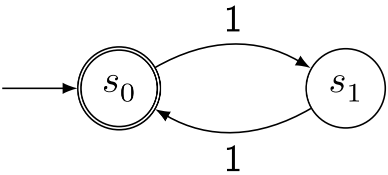
DFAs

A DFA can be given by a 5-tuple $(Q, \Sigma, \delta, q_0, F)$:

- A finite set of states (Q).
- An alphabet (Σ).
- A transition function ($\delta \in Q \times \Sigma \to Q$).
- A start state ($q_0 \in Q$).
- A set of accepting states ($F \subseteq Q$).

DFAs

The diagram



corresponds to the 5-tuple

$$Even = (\{ s_0, s_1 \}, \{ 1 \}, \delta, s_0, \{ s_0 \}),$$

where δ is defined in the following way:

$$\begin{aligned}
&\delta \in \{ s_0, s_1 \} \times \{ 1 \} \to \{ s_0, s_1 \} \\
&\delta(s_0, 1) = s_1 \\
&\delta(s_1, 1) = s_0
\end{aligned}$$

Which of the following 5-tuples can be seen as DFAs?

1. $(\mathbb{N}, \{ 0, 1 \}, \delta, 0, \{ 13 \})$, where $\delta(n, m) = n + m$.
2. $(\{ 0, 1 \}, \emptyset, \delta, 0, \{ 1 \})$, where $\delta(n, \_) = n$.
3. $(\{ q_0, q_1 \}, \{ 0, 1 \}, \delta, q_0, \{ 1 \})$, where $\delta(\_, \_) = q_0$.
4. $(\{ q_0, q_1 \}, \{ 0, 1 \}, \delta, q_0, \{ q_0 \})$, where $\delta(q, \_) = q$.
5. $(\{ q_0, q_1 \}, \{ 0, 1 \}, \delta, q_0, \{ q_0 \})$, where $\delta(\_, \_) = 0$.

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Semantics

The language of a DFA

The language $L(A)$ of a DFA $A = (Q, \Sigma, \delta, q_0, F)$ is defined in the following way:

- A transition function for strings is defined by recursion:

$$
\begin{aligned}
&\hat{\delta} \in Q \times \Sigma^* \to Q \\
&\hat{\delta}(q,\, \varepsilon) \quad = q \\
&\hat{\delta}(q,\, aw) = \hat{\delta}(\delta(q, a), w)
\end{aligned}
$$

- The language is $\left\{\, w \in \Sigma^* \;\middle|\; \hat{\delta}(q_0, w) \in F \,\right\}$.

The language of a DFA

For Even:

$$\begin{aligned}
\hat{\delta}(s_0, 11) &= \\
\hat{\delta}(\delta(s_0, 1), 1) &= \\
\hat{\delta}(s_1, 1) &= \\
\hat{\delta}(\delta(s_1, 1), \varepsilon) &= \\
\hat{\delta}(s_0, \varepsilon) &= \\
s_0 &
\end{aligned}$$

Which strings are members of the language of $(\{ s_0, s_1, s_2, s_3 \}, \{ a, b \}, \delta, s_0, \{ s_0 \})$? Here δ is defined in the following way:

$$\begin{aligned} \delta(s_0, a) &= s_1 & \delta(s_0, b) &= s_2 \\ \delta(s_1, a) &= s_0 & \delta(s_2, b) &= s_0 \\ \delta(\_, \_) &= s_3 & &\text{(In all other cases.)} \end{aligned}$$

1. ε.
2. aab.
3. aba.
4. $aabbaa$.
5. $abbaab$.
6. $bbaaaa$.

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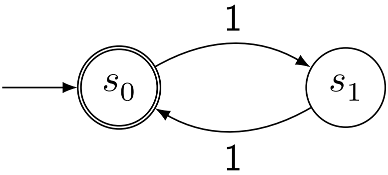
Transition diagrams

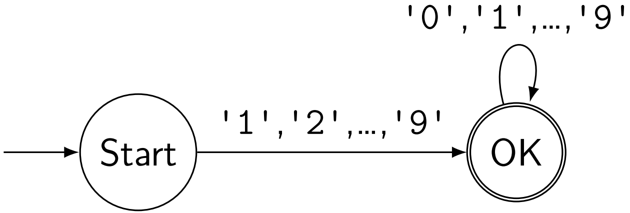
Transition diagrams



- One node per state.
- An arrow "from nowhere" to the start state.
- Double circles for accepting states.
- For every transition $\delta(s_1, a) = s_2$, an arrow marked with a from s_1 to s_2.
  - Multiple arrows can be combined.

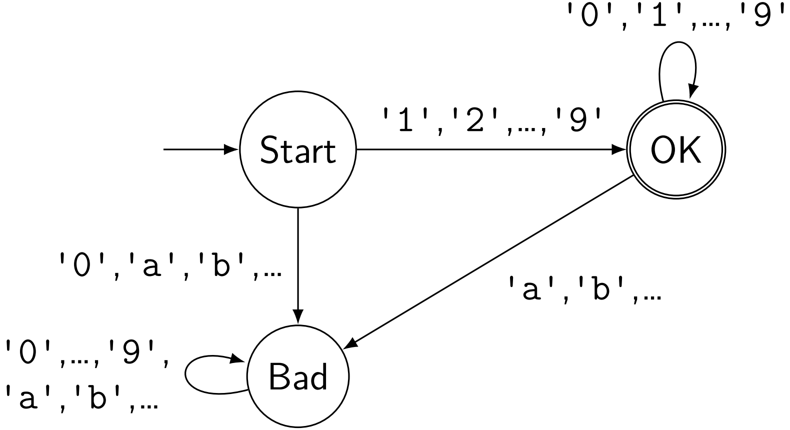
A variant

Diagrams with “missing transitions”:



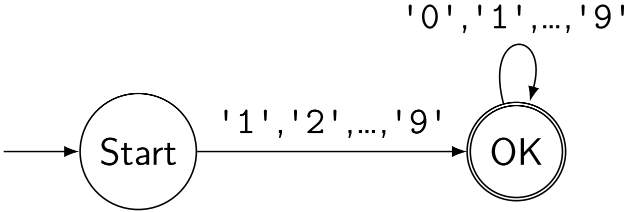
A variant

Every missing transition goes to a new state (that is not accepting):



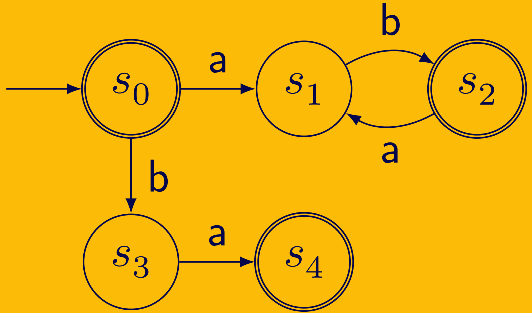
A variant

Note that diagrams with missing transitions do not define the alphabet unambiguously:



The alphabet must be a (finite) superset of { '0', '1', ..., '9' }, but which one?

Which strings are members of the language of the DFA defined by the following transition diagram? The alphabet is $\{ a, b \}$.



1. ε.
2. aa.
3. ab.
4. ba.
5. $abab$.
6. $baba$.

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Transition tables

Transition tables

	0	1
$\rightarrow *s_0$	s_2	s_1
s_1	s_2	s_0
s_2	s_2	s_2

- States: Left column.
- Alphabet: Upper row.
- Start state: Arrow.
- Accepting states: Stars.
- Transition function: Table.

Which strings are members of the language of the DFA defined by the following transition table?

	0	1
$\rightarrow s_0$	s_2	s_1
$*s_1$	s_2	s_0
$*s_2$	s_2	s_2

1. ε.
2. 0.
3. 1.
4. 11.
5. 111.
6. 1010.

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Constructions

Complement

Given a DFA $A = (Q, \Sigma, \delta, q_0, F)$ we can construct a DFA $\overline{A}$ that satisfies the following property:

$$L(\overline{A}) = \overline{L(A)} := \Sigma^* \setminus L(A).$$

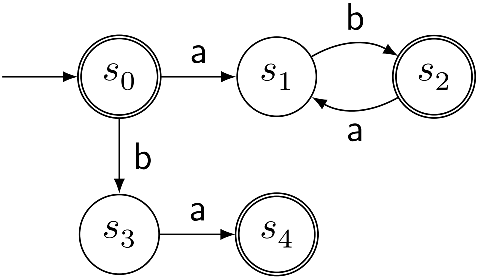
Construction:

$$(Q, \Sigma, \delta, q_0, Q \setminus F).$$

We accept if the original automaton doesn't.

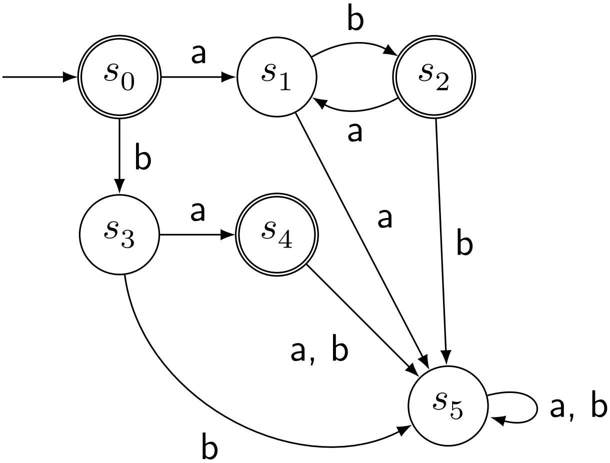
Complement

$A =$



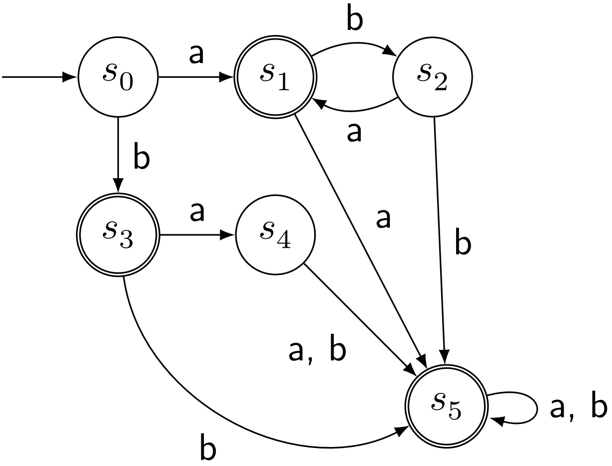
Complement

$A =$



Complement

$\overline{A} =$



Product

Given two DFAs $A_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$ and $A_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$ with the same alphabet we can construct a DFA $A_1 \otimes A_2$ that satisfies the following property:

$$L(A_1 \otimes A_2) = L(A_1) \cap L(A_2).$$

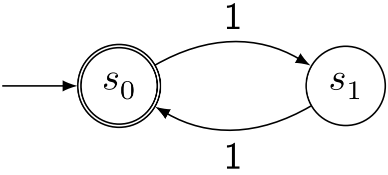
Construction:

$(Q_1 \times Q_2, \Sigma, \delta, (q_{01}, q_{02}), F_1 \times F_2)$, where
$\delta((s_1, s_2), a) = (\delta_1(s_1, a), \delta_2(s_2, a))$.

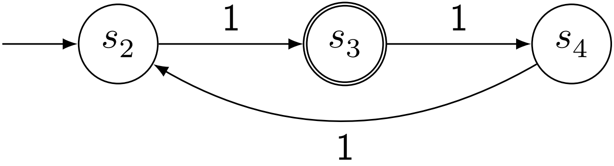
We basically run the two automatons in parallel and accept if both accept.

Product

$\{\, 2n \mid n \in \mathbb{N} \,\} \cap \{\, 1 + 3n \mid n \in \mathbb{N} \,\}$
(in unary notation, with ε standing for 0):

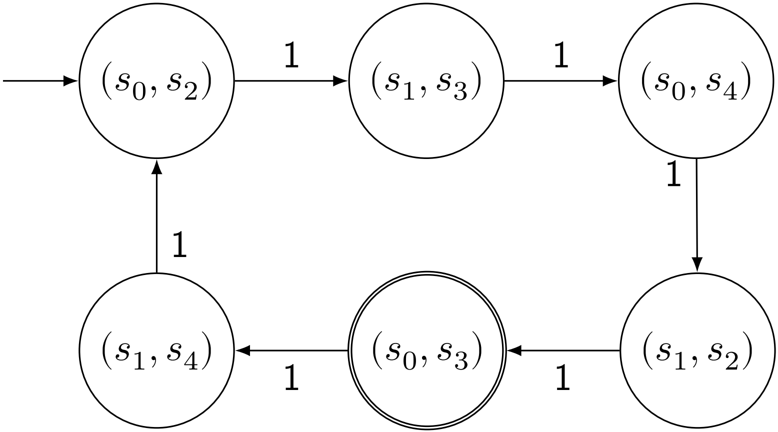


$\otimes$



Product

$\{ 4 + 6n \mid n \in \mathbb{N} \}$:



We can also construct a DFA $A_1 \oplus A_2$ that satisfies the following property:

$$L(A_1 \oplus A_2) = L(A_1) \cup L(A_2).$$

The construction is basically that of $A_1 \otimes A_2$, but with a different set of accepting states. Which one?

1. $F_1 \cup F_2$.
2. $F_1 \cap F_2$.
3. $Q_1 \times Q_2$.
4. $F_1 \times Q_2 \ \cup \ Q_1 \times F_2$.
5. $F_1 \times Q_2 \ \cap \ Q_1 \times F_2$.

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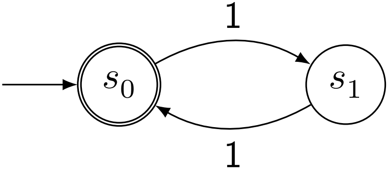
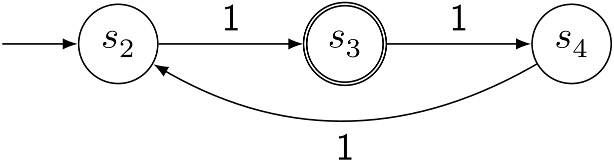
Sum

$\{\, 2n \mid n \in \mathbb{N} \,\} \cup \{\, 1 + 3n \mid n \in \mathbb{N} \,\}$:

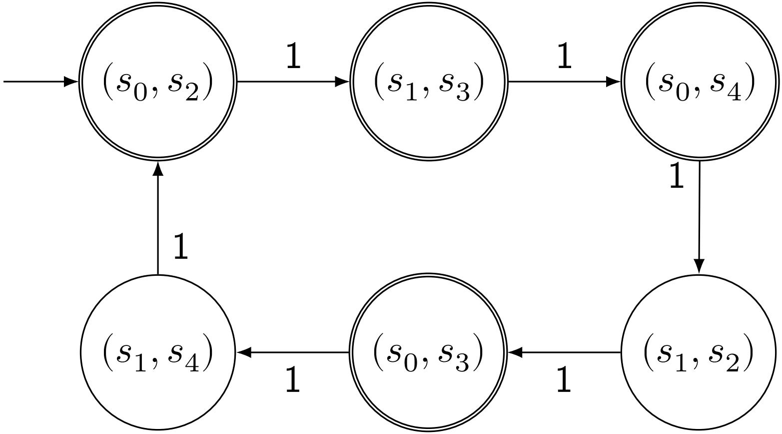


$\oplus$



Sum

$\{ 2n \mid n \in \mathbb{N} \} \cup \{ 1 + 6n \mid n \in \mathbb{N} \}$:



Accessible states

- Let $A = (Q, \Sigma, \delta, q_0, F)$ be a DFA.
- The set $Acc(q) \subseteq Q$ of states that are accessible from $q \in Q$ can be defined in the following way:

$$Acc(q) = \left\{ \hat{\delta}(q, w) \;\middle|\; w \in \Sigma^* \right\}$$

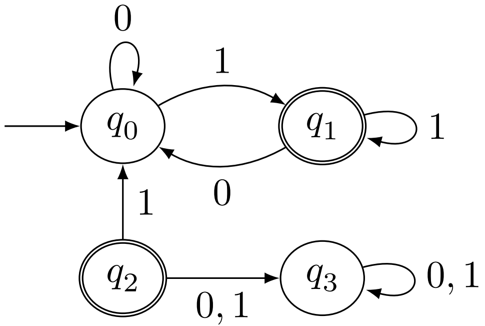
- A possibly smaller DFA:

$$A' = (Acc(q_0), \Sigma, \delta', q_0, F \cap Acc(q_0))$$
$$\delta'(q, a) = \delta(q, a)$$

- We have $L(A') = L(A)$.

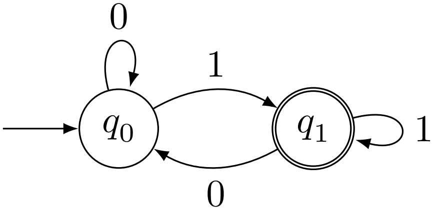
Accessible states

Note that some states cannot be reached from the start state:



Accessible states

The following DFA defines the same language:



Regular languages

Regular languages

- A language $M \subseteq \Sigma^*$ is *regular* if there is some DFA A with alphabet Σ such that $L(A) = M$.
- Note that if M and N are regular, then $M \cap N$, $M \cup N$ and $\overline{M}$ are also regular.
- We will see later that if M and N are regular, then MN is regular.

Which of the following languages are regular? ($\Sigma = \{ 0, 1 \}$.)

1. $\{ w \in \Sigma^* \mid |w| \leq 7 \}$.
2. $\{ w \in \Sigma^* \mid |w| > 7 \}$.
3. $\Sigma^* \{ 11 \} \Sigma^*$.
4. $\{ w \in \Sigma^* \mid \exists u, v \in \Sigma^*.\, w = u11v \}$.
5. $\{ w \in \Sigma^* \mid |w| \leq 7 \vee \exists u, v \in \Sigma^*.\, w = u11v \}$.
6. $\{ w \in \Sigma^* \mid |w| > 7 \wedge \nexists u, v \in \Sigma^*.\, w = u11v \}$.

Today

Deterministic finite automata:

- 5-tuples.
- Semantics.
- Transition diagrams.
- Transition tables.
- Constructions.
- Regular languages.

Next lecture

- Nondeterministic finite automata (NFAs).
- The subset construction
  (turns NFAs into DFAs).