Finite automata and formal languages (DIT323, TMV029)

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Today

- NFAs with ε-transitions.
- Exponential blowup.

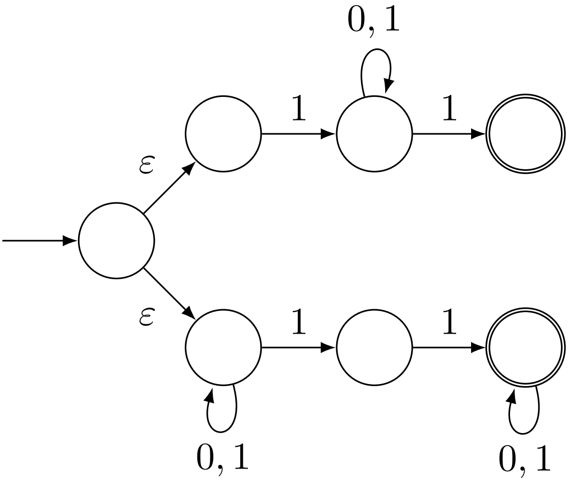
ε-NFAs

ε-NFAs

- Like NFAs, but with ε-transitions:
  The automaton can "spontaneously" make a transition from one state to another.
- Can be used to convert regular expressions to finite automata.

ε-NFAs

Strings over $\{ 0, 1 \}$ that start and end with a one, or that contain two consecutive ones:



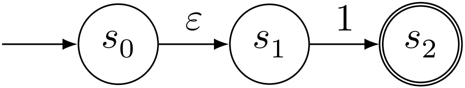
ε-NFAs

An ε-NFA can be given by a 5-tuple $(Q, \Sigma, \delta, q_0, F)$:

- A finite set of states (Q).
- An alphabet (Σ with $\varepsilon \notin \Sigma$).
- A transition function ($\delta \in Q \times (\Sigma \cup \{ \varepsilon \}) \to \wp(Q)$).
- A start state ($q_0 \in Q$).
- A set of accepting states ($F \subseteq Q$).

ε-NFAs

If the alphabet is $\{ 1 \}$, then the diagram



corresponds to the 5-tuple

$$One = (\{ s_0, s_1, s_2 \}, \{ 1 \}, \delta, s_0, \{ s_2 \}),$$

where δ is defined in the following way:

$$\begin{array}{lll}
\delta \in \{ s_0, s_1, s_2 \} \times \{ \varepsilon, 0, 1 \} \rightarrow \wp(\{ s_0, s_1, s_2 \}) & & \\
\delta(s_0, \varepsilon) = \{ s_1 \} & \delta(s_1, \varepsilon) = \emptyset & \delta(s_2, _) = \emptyset \\
\delta(s_0, 1) = \emptyset & \delta(s_1, 1) = \{ s_2 \} &
\end{array}$$

Transition diagrams

As for NFAs, but arrows can be labelled with ε.

Transition tables

As for NFAs, but with one column for ε.

ε-closure

ε-closure

The ε-closure of a state q consists of those states that one can reach from q by following zero or more ε-transitions.

ε-closure

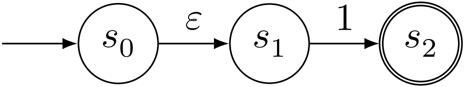
Given an ε-NFA $A = (Q, \Sigma, \delta, q_0, F)$ one can, for each state $q \in Q$, define the ε-closure of q (a subset of Q) inductively in the following way:

$$\frac{}{q \in \varepsilon\text{-}closure(q)}$$

$$\frac{q' \in \varepsilon\text{-}closure(q) \qquad q'' \in \delta(q', \varepsilon)}{q'' \in \varepsilon\text{-}closure(q)}$$

ε-closure

Consider the following ε-NFA again:



The set $\varepsilon\text{-}closure(s_0)$ contains two states:

$$\frac{}{s_0 \in \varepsilon\text{-}closure(s_0)}$$

$$\frac{\dfrac{}{s_0 \in \varepsilon\text{-}closure(s_0)} \qquad \dfrac{}{s_1 \in \delta(s_0, \varepsilon)}}{s_1 \in \varepsilon\text{-}closure(s_0)}$$

Some notation

The ε-closure of a set $S \subseteq Q$:

$$\varepsilon\text{-}closure(S) = \bigcup_{s \in S} \varepsilon\text{-}closure(s)$$

Transition functions applied to a set $S \subseteq Q$:

$$\delta(S, a) = \bigcup_{s \in S} \delta(s, a)$$

$$\hat{\delta}(S, w) = \bigcup_{s \in S} \hat{\delta}(s, w)$$

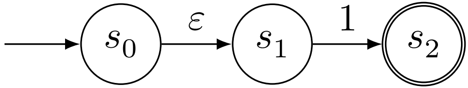
Computing the ε-closure

The ε-closure of q can be computed (perhaps not very efficiently) in the following way:

- Initialise C to $\{ q \}$.
- Repeat until $\delta(C, \varepsilon) \subseteq C$:
  - Set C to $C \cup \delta(C, \varepsilon)$.
- Return C.

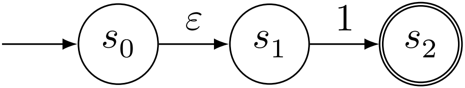
Computing the ε-closure

Let us compute $\varepsilon\text{-}closure(s_0)$ for the following ε-NFA:



Computing the ε-closure

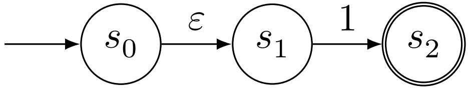
Let us compute $\varepsilon\text{-}closure(s_0)$ for the following ε-NFA:



- Initialise C to $\{ s_0 \}$.

Computing the ε-closure

Let us compute $\varepsilon\text{-}closure(s_0)$ for the following ε-NFA:



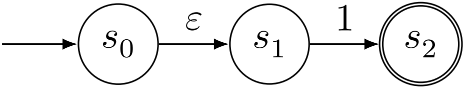
- We have $\delta(C, \varepsilon) \not\subseteq C$:

  $$\delta(C, \varepsilon) = \delta(\{\ s_0\ \}, \varepsilon) = \delta(s_0, \varepsilon) = \{\ s_1\ \}\ = C.$$

- Set C to $C \cup \delta(C, \varepsilon) = \{\ s_0, s_1\ \}$.

Computing the ε-closure

Let us compute $\varepsilon\text{-}closure(s_0)$ for the following ε-NFA:

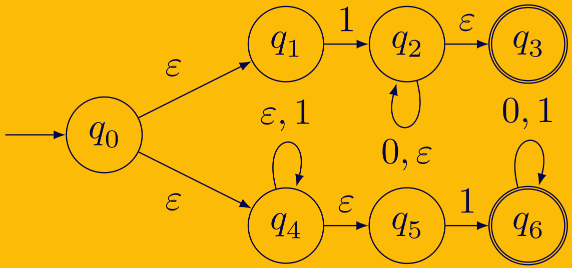


- We have $\delta(C, \varepsilon) \subseteq C$:

$$\begin{array}{lll} \delta(C,\varepsilon) & = \delta(\{\, s_0, s_1 \,\}, \varepsilon) & = \\ \delta(s_0,\varepsilon) \cup \delta(s_1,\varepsilon) & = \{\, s_1 \,\} \cup \emptyset & = \\ \{\, s_1 \,\} & \subseteq \{\, s_0, s_1 \,\} & = C. \end{array}$$

- Return C.

Which of the following propositions hold for the following ε-NFA over $\{ 0, 1 \}$?



1. $q_0 \in \varepsilon\text{-}closure(q_0)$.
2. $q_5 \in \varepsilon\text{-}closure(q_0)$.
3. $\varepsilon\text{-}closure(q_4) \subseteq \varepsilon\text{-}closure(q_0)$.
4. $q_6 \in \varepsilon\text{-}closure(q_0)$.
5. $q_3 \in \varepsilon\text{-}closure(q_1)$.
6. $\varepsilon\text{-}closure(q_4) \subseteq \varepsilon\text{-}closure(q_5)$.

Respond at `https://pingo.coactum.de/729558`.

Semantics

The language of an ε-NFA

The language $L(A)$ of an ε-NFA $A = (Q, \Sigma, \delta, q_0, F)$ is defined in the following way:

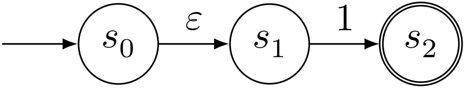
- A transition function for strings is defined by recursion:

$$\begin{aligned} &\hat{\delta} \in Q \times \Sigma^* \to \wp(Q) \\ &\hat{\delta}(q, \varepsilon) \quad = \varepsilon\text{-}closure(q) \\ &\hat{\delta}(q, aw) = \hat{\delta}(\delta(\varepsilon\text{-}closure(q), a), w) \end{aligned}$$

- The language is

$$\left\{\, w \in \Sigma^* \;\middle|\; \hat{\delta}(q_0, w) \cap F \neq \emptyset \,\right\}.$$

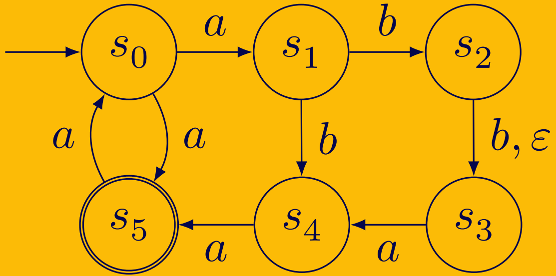
The language of an ε-NFA



For One:

$$
\begin{aligned}
&\hat{\delta}(s_0, 1) &=\\
&\hat{\delta}(\delta(\varepsilon\text{-}closure(s_0), 1), \varepsilon) &=\\
&\hat{\delta}(\delta(\{\ s_0, s_1\ \}, 1), \varepsilon) &=\\
&\hat{\delta}(\{\ s_2\ \}, \varepsilon) &=\\
&\hat{\delta}(s_2, \varepsilon) &=\\
&\varepsilon\text{-}closure(s_2) &=\\
&\{\ s_2\ \}
\end{aligned}
$$

Which strings are members of the language of the following ε-NFA over $\{ a, b, c \}$?



1. $abba$.
2. $abbaca$.
3. $aaabaa$.
4. $aaabaaa$.
5. $aaaabaa$.
6. $abbaaaabaa$.

Respond at `https://pingo.coactum.de/729558`.

Which of the following propositions are valid?

1. $\varepsilon\text{-}closure(\varepsilon\text{-}closure(q)) = \varepsilon\text{-}closure(q)$.
2. $\hat{\delta}(q, w) = \hat{\delta}(\varepsilon\text{-}closure(q), w)$.
3. $\hat{\delta}(\delta(\varepsilon\text{-}closure(q), a), w) =$
   $\hat{\delta}(\varepsilon\text{-}closure(\delta(q, a)), w)$.

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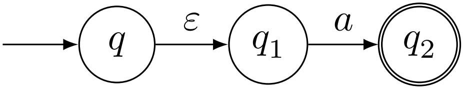
Which of the following propositions are valid?

3. $\hat{\delta}(\delta(\varepsilon\text{-}closure(q), a), w) = \hat{\delta}(\varepsilon\text{-}closure(\delta(q, a)), w)$.

No. Counterexample:



Denote the transition function by δ.

$$\hat{\delta}(\delta(\varepsilon\text{-}closure(q), a), \varepsilon) = \{\ q_2\ \} \neq \emptyset = \hat{\delta}(\varepsilon\text{-}closure(\delta(q, a)), \varepsilon)$$

Constructions

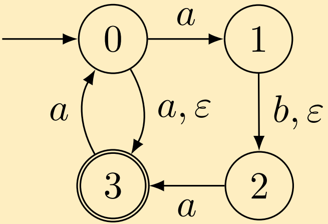
Subset construction

Given an ε-NFA $N = (Q, \Sigma, \delta, q_0, F)$ we can define a DFA D with the same alphabet in such a way that $L(N) = L(D)$:

$$
\begin{aligned}
&D = (\wp(Q), \Sigma, \delta', \varepsilon\text{-}closure(q_0), F') \\
&\delta'(S, a) = \varepsilon\text{-}closure(\delta(S, a)) \\
&F' = \{\, S \subseteq Q \mid S \cap F \neq \emptyset \,\}
\end{aligned}
$$

Every accessible state S is ε-*closed*
(i.e. $S = \varepsilon\text{-}closure(S)$).

If the subset construction is used to build a DFA corresponding to the following ε-NFA over $\{ a, b \}$, and inaccessible states are removed, how many states are there in the resulting DFA?



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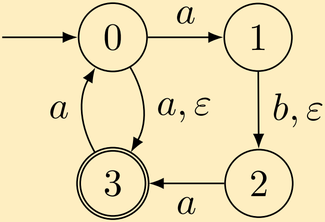
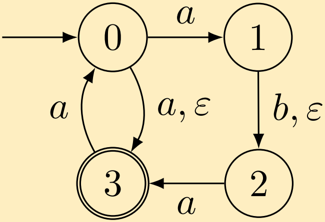


	a	b
$\rightarrow * \{\,0,3\,\}$	$\{\,0,1,2,3\,\}$	$\emptyset$
$* \{\,0,1,2,3\,\}$	$\{\,0,1,2,3\,\}$	$\{\,2\,\}$
$\emptyset$	$\emptyset$	$\emptyset$
$\{\,2\,\}$	$\{\,3\,\}$	$\emptyset$
$* \{\,3\,\}$	$\{\,0,3\,\}$	$\emptyset$



	a	b
$\rightarrow *A$	B	C
$*B$	B	D
C	C	C
D	E	C
$*E$	A	C

Regular languages

- Recall that a language $M \subseteq \Sigma^*$ is regular if there is some DFA (or NFA) A with alphabet Σ such that $L(A) = M$.
- For alphabets Σ with $\varepsilon \notin \Sigma$ a language $M \subseteq \Sigma^*$ is also regular if and only if there is some ε-*NFA* A with alphabet Σ such that $L(A) = M$.

Union

Recall:

- One can use ε-NFAs to convert regular expressions to finite automata.

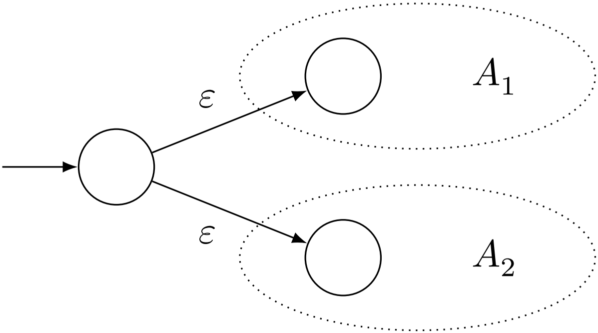
Union

Given two ε-NFAs A_1 and A_2 with the same alphabet we can construct an ε-NFA $A_1 \oplus A_2$ that satisfies the following property:

$$L(A_1 \oplus A_2) = L(A_1) \cup L(A_2).$$

Union

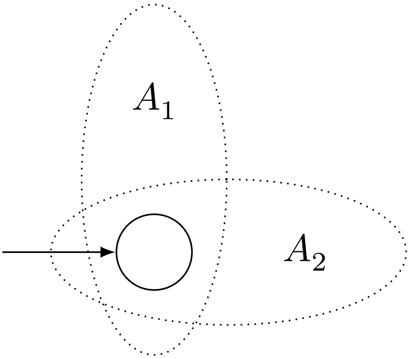
Construction:



- The transitions go to the start states.
- States are renamed if the state sets overlap.

Union

Can one do something similar for NFAs by "merging" the start states?



Given two NFAs $A_1 = (Q_1, \Sigma, \delta_1, q_{01}, F_1)$ and $A_2 = (Q_2, \Sigma, \delta_2, q_{02}, F_2)$ satisfying $Q_1 \cap Q_2 = \emptyset$ and $q_0 \notin Q_1 \cup Q_2$, is the language of the NFA

$$(f(Q_1 \cup Q_2), \Sigma, \delta, q_0, f(F_1 \cup F_2)), \text{ where}$$
$$f(S) = (S \setminus \{\, q_{01}, q_{02} \,\}) \cup \{\, q_0 \mid q_{01} \in S \vee q_{02} \in S \,\},$$
$$\delta(s, a) = \begin{cases} f(\delta_1(q_{01}, a) \cup \delta_2(q_{02}, a)), & \text{if } s = q_0, \\ f(\delta_1(s, a)), & \text{if } s \in Q_1, \\ f(\delta_2(s, a)), & \text{if } s \in Q_2 \end{cases}$$

equal to $L(A_1) \cup L(A_2)$?

1. Yes, always.
2. No, never.
3. No, not always, but sometimes.

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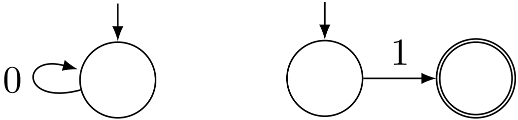
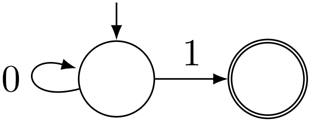
Can one do something similar for NFAs by “merging” the start states?

- Sometimes. For instance if F_1 and F_2 are empty.
- Not always. The following NFAs over $\{\, 0, 1 \,\}$ accept $\emptyset$ and $\{\, 1 \,\}$:



The combination accepts $\{\, 0^n 1 \mid n \in \mathbb{N} \,\}$:



Exponential blowup

Exponential blowup

Consider the following family of languages:

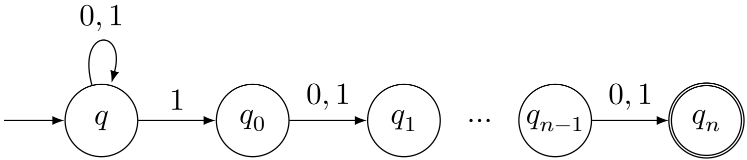
$$A \in \mathbb{N} \to \wp(\{\,0,1\,\}^*)$$
$$A(n) = \left\{\, u1v \;\middle|\; u,v \in \{\,0,1\,\}^*, |v| = n \,\right\}$$

Exponential blowup

The family:

$$A(n) = \{\, u1v \mid u, v \in \{\, 0, 1 \,\}^*, |v| = n \,\}$$

For every $n \in \mathbb{N}$ the NFAs for $A(n)$ with the least number of states have at most $n + 2$ states:



Exponential blowup

Furthermore one can prove:

- For every $n \in \mathbb{N}$ the DFAs for $A(n)$ with the least number of states have at least 2^{n+1} states.

A key part of the proof in the course text book uses the pigeonhole principle:

- A DFA over $\{ 0, 1 \}$ with less than 2^k states has to end up in the same state for at least two distinct k-bit strings.

Exponential blowup

Thus it might be inefficient to check if a string belongs to a language represented by an NFA (or ε-NFA) by using the following method:

- Translate the NFA to a corresponding DFA.
- Use the DFA to check if the string belongs to the language.

Exponential blowup

- This method is used in practice by some tools.
- It seems to work fine in many practical cases.
- Exercise (optional):
  Make such a tool “blow up” by giving it a short piece of carefully crafted input.

Today

- ε-NFAs.
- ε-closure.
- Semantics.
- Constructions.
- Exponential blowup.

Next lecture

- Regular expressions.
- Translation from finite automata to regular expressions.