Mine Depth Measurement Report

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Introduction

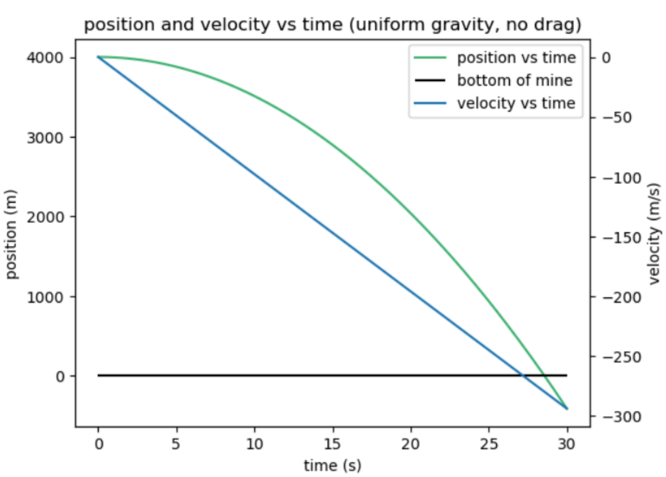
This report will begin by discussing a method for measuring the depth of a mine by dropping a test mass and measuring the time required to reach the bottom. It will consider varying assumptions about drag, the uniformity of gravity, the Coriolis force, and the density profile of Earth. It will then discuss the feasibility of this measurement; in particular about whether the Coriolis force would cause the ball to hit the wall before reaching the bottom of the mine. Then, the report will consider a theoretical scenario involving dropping a ball into a mine which passes through the center of the Earth. The report will conclude by discussing potential future work needed to make more accurate calculations.

Calculation of fall time

This section will discuss the time needed for a 1 kg test mass to reach the bottom of the mineshaft under varying conditions. Gravity is one of the primary forces which affects fall time. The gravitational force can be calculated as $F = G * m_1 * m_2 / r^2$, where G is the gravitational constant, $6.67*10^{-11}$, m_1 and m_2 are the masses of the 2 objects involved, and r is the distance between the objects.

Considering the case of no drag and uniform gravity we can model the depth of a test mass using the expression, $y = -0.5 * g * t^2$, where y represents the vertical displacement in meters, g is the gravitational acceleration on earth, and t is time. The numerical differential equation solving capabilities of solve_ivp, a module of the scipy library for Python (essentially an add-on for the programming language) were also used to determine the time it would take to reach the bottom of the shaft. Under both calculation methods, assuming the depth of the shaft is 4km, it would take 28.6 seconds to reach the bottom. A plot of distance and velocity vs time is shown to the left. All figures in this report were generated using MatPlotLib, a Library for Python, and all calculations were done using Quad and Solve_ivp from the Scipy library, as well as the numpy library from Python.

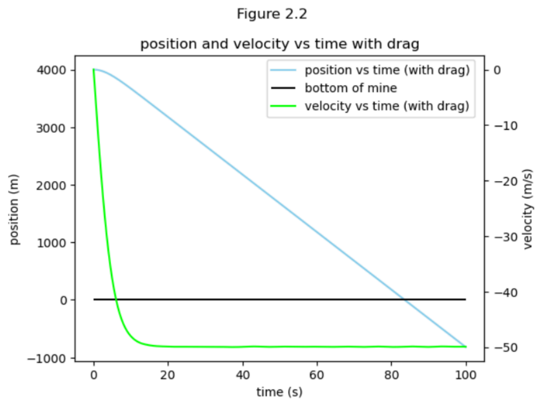
Figure 1



From this, we can see that velocity decreases linearly with time, while position decreases quadratically with time.

In the case of no drag and gravity which increases linearly with distance from the Earth's center, the time to fall to the bottom was again 28.6 seconds. The values were different at the third decimal place; however this difference is insignificant compared to likely errors with a timing apparatus or the difference in time when considering drag, which will be discussed next.

When drag and non-uniform, linear gravity are considered, it would take 83.6 seconds to fall to the bottom. This is nearly 3 times longer than the fall time without drag, so drag is a major factor when considering the fall time to the bottom of the mine. The drag coefficient was set to be 0.00393. This value was manually calibrated as follows: first, we know the terminal velocity of the object should be 50 meters per second, and we know the speed dependence of drag is 2, thus drag increases like velocity squared, or



that, if speed were doubled, drag would be quadrupled. Then, we can test values for the drag coefficient until we obtain one which has the test mass reach a terminal velocity of 50 meters per second. This value was 0.00393.

When considering the case with drag, once the object reaches terminal velocity, its velocity remains constant with time as its position decreases linearly. This is shown in the plot to the left.

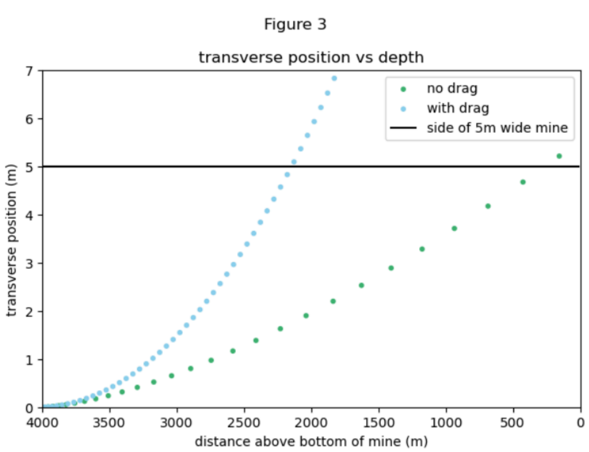
Increasing gravity would cause the acceleration and terminal velocity to be greater, and thus cause the test mass to reach the bottom of the mine faster. Increasing the drag coefficient would have the opposite effect, causing the test mass to fall slower. Specifically, increasing gravity by a factor of 2 causes the terminal velocity to increase by a factor of $\sqrt{2}$.

Feasibility of Depth Measurement Approach

This report will now discuss whether the previously mentioned depth measurement would be feasible, particularly considering the Coriolis force. The Coriolis force is an effect which deflects an object, and is observed when using a rotating reference frame, such as the Earth. The Coriolis force affects the vertical and horizontal motion of the test mass. The Coriolis force can be represented in the following manner: $Fc_x = +2 * m * \Omega * v_y$ $\quad$ $Fc_y = -2 * m * \Omega * v_x$

Where Fc_y represents the vertical component of the Coriolis force, and Fc_x represents the horizontal component of the coriolis force, V_y and V_x represent the vertical and horizontal components of the velocity respectively, m represents the mass of the object, and Ω is the rotation rate of the Earth.

When including the Coriolis force and considering both the vertical and horizontal displacement of a dropped test mass, we find that, assuming the shaft is 5 meters wide, the object will hit the side of the shaft before hitting the bottom. In the case without drag, the object would hit the wall 263 meters from the bottom of the mine after 27.6 seconds. When considering drag, the test mass would hit the wall 2146.6 meters from the bottom, after 40.6 seconds. The ultimate outcome as to whether the object hits the wall or the bottom first is not affected by drag, however the distance from the bottom at which it hits the wall is affected by drag, as the object has about 8 times the distance remaining in the case with drag.



The plot to the left shows that the object would hit the wall before reaching the bottom of the shaft. Given this, it would not be recommended to use this method as a way to measure the depth of the mine, as, if we were measuring a sound or vibration, we would hear it hit the side first. Also, after the object hits the side, its motion becomes difficult to predict, so a measurement of the time it takes for the object to stop making sounds / vibrations, which would occur after it reaches the bottom, would not help determine the depth.

Calculation of Crossing times for Homogeneous and Nonhomogeneous Earth

Now, the report will discuss a hypothetical mine running through the center of the Earth and out the other side, as well as the same scenario on the moon, ignoring drag and Coriolis forces in both cases. An important factor to take into account in these scenarios is the distribution of mass throughout the bodies. The previous calculations using non-uniform gravity assumed that mass is evenly distributed over volume, however, in reality, more mass is concentrated at the center. Using a simple model for density vs radius, we can calculate that density = $p_n*(1 - (r/R)^2)^n$, where R is the radius of the Earth/Moon, r is the distance from the center of the Earth/Moon, n is an exponent describing the distribution of mass versus distance, and p_n is a normalizing constant. In particular, a value of n = 0 indicates uniform density, while n = ∞ would indicate that all the mass is concentrated at the center. A value of 9 indicates that the density at 2/3rds of the radius of the object is very close to 0 compared to the density at the center of the object.

The following values for n were tested: 0, 1, 2, and 9. Of these values, n = 2 is the closest to the true density distribution of the Earth. Under these varying conditions, a dropped object would reach the center after 1267.3, 1096.5, 1035.1, and 943.8 seconds, respectively. The test mass would be travelling at -7905, -10457, -12182, and -18370 meters per second, respectively.

For a mine through the center of the Moon, similar calculations were performed, for n = 0, 1, 2, and 9. The fall time to the center was 1625.1, 1406.1, 1327.3, and 1210.2 seconds, respectively, and the test mass would be travelling at -1680, -2222, -2589, and -3904 meters per second, respectively.

From these calculations, it is clear that the value for n does have a considerable effect on the fall time and especially on fall speed. For both the Earth and the Moon, the object would take about 25% less time to reach the center with n = 0 compared to n = 9, and would be travelling more than twice as fast with n = 9 compared to n = 0.

We can also compare the time it takes for the object to pass through the center of the Earth to the time it takes to orbit the Earth, assuming a low-Earth orbit. For n = 0, it would take 1266.6 seconds to reach the center, while it would take $4.1*10^6$ seconds for the object to orbit the Earth, so the orbital period is more than 3200 times the crossing time. This means that if a mine through the center of the Earth existed, it would be 800 times quicker to bring an object to the other side of the Earth by dropping it through this shaft rather than sending it in orbit around Earth. It would be 800 times quicker and not 3200 because the object would have to fall in and fall out again, which would double the time, and because the object would only have to orbit around half the earth, so the orbit time would be cut in half, resulting in a total change to be 1/4th the ratio between the crossing time and the orbital period. Still, a factor of 800 is a significant amount of time savings.

Discussion and Future Work

In summary, a test mass dropped down the mine would take 28.6 seconds to reach the bottom in a case with no drag, and 83.6 seconds in a case with drag, neglecting the Coriolis force for both. However, testing the depth of the mine by measuring this falling time would be infeasible, because—when including the Coriolis force—the object would hit the side of the shaft before the bottom. Considering a body-wide mine on the Earth and the Moon, and different density profiles for each, an object would take on the order of 1000 seconds and 1500 seconds to reach the center of Earth and the Moon, respectively.

Now, the report will consider sources of error, and future work that would make the calculations more accurate. One source of error in the calculations is that we approximated the drag coefficient by assuming a terminal velocity of 50m/s, as that is the terminal velocity reached by skydivers without an open parachute. Drag coefficients and terminal velocities can vary greatly because of an object's geometry and density, so a physical test to determine the terminal velocity and therefore the drag coefficient for the test mass would improve the accuracy of the calculations. Also, for the density profile of Earth, an exponent of 2 is a reasonable estimate, however a more accurate value would make calculations in the crossing times section more accurate.