

CSCI 338: Assignment 1 (6 points)

Feb 3, 2020

This assignment is due on **Monday, Feb 3, 11:30pm**. You will need to use Latex to generate a single pdf file and upload it under *Assignment 1* on D2L. There will be a penalty for not using Latex (to finish the assignment). This is **not** a group-assignment, so you must finish the assignment by yourself.

Problem 1

Prove that $1^2 + 3^2 + 5^2 + \cdots + (2n - 1)^2 = \frac{1}{3}n(4n^2 - 1)$.

PROOF. By Induction

We show that the base case of $n = 1$ holds. The left hand side of the relation is $\sum_{i=1}^1 (2 * i - 1)^2 = 1$, and the right hand side is $1/3 * 1 * (4 * 1^2 - 1) = 1$. Therefore the base case holds.

Next we show that $n \rightarrow n + 1$. We assume that the relation holds $\forall n \leq k$ for $k \in \mathbf{Z}$. Therefore we know that $\sum_{i=1}^k (2 * i - 1)^2 = 1/3 * k * (4 * k^2 - 1)$. Now we show that the relation also holds for $k + 1$.

COMPUTER SCIENCE THEORY: ASSIGNMENT 1

$$\begin{aligned}
 \sum_{i=1}^{k+1} (2 * i - 1)^2 &= 1/3 * (k + 1) * (4 * (k + 1)^2 - 1) \\
 (2k + 1)^2 + \sum_{i=1}^k (2 * i - 1)^2 &= 1/3 * (k + 1) * (4 * (k^2 + 2k + 1) - 1) \\
 (2k + 1)^2 + \sum_{i=1}^k (2 * i - 1)^2 &= 1/3 * (k + 1) * (4k^2 + 8k + 3) \\
 (2k + 1)^2 + \sum_{i=1}^k (2 * i - 1)^2 &= 1/3 * ((4k^3 + 8k^2 + 3k) + (4k^2 + 8k + 3)) \\
 (4k^2 + 4k + 1) + \sum_{i=1}^k (2 * i - 1)^2 &= 1/3 * (4k^3 + 12k^2 + 12k - k + 3) \\
 (4k^2 + 4k + 1) + \sum_{i=1}^k (2 * i - 1)^2 &= 1/3 * (4k^3 - k) + 1/3 * (12k^2 + 12k + 3) \\
 (4k^2 + 4k + 1) + \sum_{i=1}^k (2 * i - 1)^2 &= 1/3 * k(4k^2 - 1) + (4k^2 + 4k + 1) \\
 \sum_{i=1}^k (2 * i - 1)^2 &= 1/3 * k(4k^2 - 1)
 \end{aligned}$$

Which is true by the inductive assumption. Therefore the inductive step and base case hold and the claim is proven true by mathematical induction.

□

Problem 2

Given a planar graph $P = (V, E)$, we have Euler's formula: $|V| + |F| - |E| = 2$, where F is the set of faces of P and E is the set of edges of P . Let $|V| = n$, where V is the set of vertices of P . Prove that $|F|$ is at most $2n$.

Problem 3

Prove that in any simple graph there is a path from any vertex of odd degree to some other vertex of odd degree.

Problem 4

A fully binary tree T is a tree such that all internal nodes have two children. Prove that a fully binary tree with n internal nodes in total has $2n + 1$ nodes.

Because T is a full binary tree we know that there is exactly 2^x nodes in layer x . Therefore the total number of nodes in the first k layers is: $\sum_{i=0}^k 2^i$. We let k = the total depth of the tree. Therefore $n = \sum_{i=0}^{k-1} 2^i$. We can express the problem as $\sum_{i=0}^k 2^i = 2n + 1$.

PROOF. By induction. We show $\sum_{i=0}^k 2^i = 2n + 1$.

We show that the base case of $k = 1$ holds. In this case there is 1 internal node. $\sum_{i=0}^1 2^i = 2(1) + 1 = 3$

We assume that the relation holds $\forall n \leq k$ for $k \in \mathbf{Z}$.

Next we show that the relation is true for $k + 1$. Thus $\sum_{i=0}^k 2^i = 2n + 1 = 2 \left(\sum_{i=0}^{k-1} 2^i \right) + 1$

$$\begin{aligned}
 \sum_{i=0}^{k+1} 2^i &= 2n + 1 \\
 2^{k+1} + \sum_{i=0}^k 2^i &= 2 \left(\sum_{i=0}^{k-1} 2^i \right) + 1 \\
 2^{k+1} + \left[2 \left(\sum_{i=0}^{k-1} 2^i \right) + 1 \right] &= 2 \left(2^k + \sum_{i=0}^{k-1} 2^i \right) + 1 \\
 2^{k+1} + 2 * \sum_{i=0}^{k-1} 2^i + 1 &= 2^{k+1} + 2 * \sum_{i=0}^{k-1} 2^i + 1
 \end{aligned} \tag{1}$$

Thus the base case and inductive step hold and the claim is proven true by mathematical induction. \square

Problem 5

Given an undirected graph $G = (V, E)$, the breadth-first-search starting at $v \in V$ ($bfs(v)$ for short) is to generate a shortest path tree starting at vertex $v \in V$. The diameter of G is the longest of all shortest paths $\delta(u, v)$, $u, v \in V$.

When G is a tree, the following algorithm is proposed to compute the diameter of G .

1. Run $bfs(w)$, $w \in V$, and compute the vertex $x \in V$ furthest from w .
2. Run $bfs(x)$ and compute the vertex $y \in V$ furthest from x .
3. Return $\delta(x, y)$ as the diameter of G .

Prove that this algorithm is correct; i.e., $\delta(x, y)$ is in fact the longest among all the shortest paths between $u, v \in V$.