CSCI 338: Assignment 1 (6 points)

Feb 3, 2020

This assignment is due on **Monday, Feb 3, 11:30pm**. You will need to use Latex to generate a single pdf file and upload it under *Assignment 1* on D2L. There will be a penalty for not using Latex (to finish the assignment). This is **not** a group-assignment, so you must finish the assignment by yourself.

Problem 1

Prove that $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{1}{3}n(4n^2-1)$.

PROOF. By Induction

We show that the base case of $n = 1$ holds. The left hand side of the relation is $\sum_{i=1}^{1}(2*i-1)^2 = 1$, and the right hand side is $1/3*1*(4*1^2-1) = 1$. Therefore the base case holds.

Next we show that $n \to n+1$. We assume that the relation holds for some $k \in \mathbf{Z}$. Therefore we know that $\sum_{i=1}^{k}(2*i-1)^2 = 1/3*k*(4*k^2-1)$. Now we show that the relation also holds for $k+1$.

$$\sum_{i=1}^{k+1}(2*i-1)^2 = 1/3*(k+1)*(4*(k+1)^2-1)$$

$$(2k+1)^2 + \sum_{i=1}^{k}(2*i-1)^2 = 1/3*(k+1)*(4*(k^2+2k+1)-1)$$

$$(2k+1)^2 + \sum_{i=1}^{k}(2*i-1)^2 = 1/3*(k+1)*(4k^2+8k+3)$$

$$(2k+1)^2 + \sum_{i=1}^{k}(2*i-1)^2 = 1/3*((4k^3+8k^2+3k)+(4k^2+8k+3))$$

$$(4k^2+4k+1) + \sum_{i=1}^{k}(2*i-1)^2 = 1/3*(4k^3+12k^2+12k-k+3)$$

$$(4k^2+4k+1) + \sum_{i=1}^{k}(2*i-1)^2 = 1/3*(4k^3-k)+1/3*(12k^2+12k+3)$$

$$(4k^2+4k+1) + \sum_{i=1}^{k}(2*i-1)^2 = 1/3*k(4k^2-1)+(4k^2+4k+1)$$

$$\sum_{i=1}^{k}(2*i-1)^2 = 1/3*k(4k^2-1)$$

Which is true by the inductive assumption. Therefore the inductive step and base case hold and the claim is proven true by mathematical induction.

□

Problem 2

Given a planar graph $P = (V, E)$, we have Euler's formula: $|V| + |F| - |E| = 2$, where F is the set of faces of P and E is the set of edges of P. Let $|V| = n$, where V is the set of vertices of P. Prove that $|F|$ is at most $2n$.

Problem 3

Prove that in any simple graph there is a path from any vertex of odd degree to some other vertex of odd degree.

Problem 4

A fully binary tree T is a tree such that all internal nodes have two children. Prove that a fully binary tree with n internal nodes in total has $2n + 1$ nodes.

Problem 5

Given an undirected graph $G = (V, E)$, the breadth-first-search starting at $v \in V$ ($bfs(v)$ for short) is to generate a shortest path tree starting at vertex $v \in V$. The diameter of G is the longest of all shortest paths $\delta(u, v), u, v \in V$.

When G is a tree, the following algorithm is proposed to compute the diameter of G.

1. Run $bfs(w), w \in V$, and compute the vertex $x \in V$ furthest from w.

2. Run $bfs(x)$ and compute the vertex $y \in V$ furthest from x.

3. Return $\delta(x, y)$ as the diameter of G.

Prove that this algorithm is correct; i.e., $\delta(x, y)$ is in fact the longest among all the shortest paths between $u, v \in V$.