CSCI 338: Assignment 1 (6 points)

Feb 3, 2020

This assignment is due on **Monday, Feb 3, 11:30pm**. You will need to use Latex to generate a single pdf file and upload it under *Assignment 1* on D2L. There will be a penalty for not using Latex (to finish the assignment). This is **not** a group-assignment, so you must finish the assignment by yourself.

Problem 1

Prove that $1^2 + 3^2 + 5^2 + \cdots + (2n-1)^2 = \frac{1}{3}n(4n^2-1)$.

PROOF. By Induction

We show that the base case of $n = 1$ holds. The left hand side of the relation is $\sum_{i=1}^{1}(2*i-1)^2 = 1$, and the right hand side is $1/3*1*(4*1^2-1) = 1$. Therefore the base case holds.

Next we show that $n \to n+1$. We assume that the relation holds $\forall n \leq k$ for $k \in \mathbf{Z}$. Therefore we know that $\sum_{i=1}^{k}(2*i-1)^2 = 1/3*k*(4*k^2-1)$. Now we show that the relation also holds for $k+1$.

$$\sum_{i=1}^{k+1}(2*i-1)^2 = 1/3*(k+1)*(4*(k+1)^2-1)$$

$$(2k+1)^2+\sum_{i=1}^{k}(2*i-1)^2 = 1/3*(k+1)*(4*(k^2+2k+1)-1)$$

$$(2k+1)^2+\sum_{i=1}^{k}(2*i-1)^2 = 1/3*(k+1)*(4k^2+8k+3)$$

$$(2k+1)^2+\sum_{i=1}^{k}(2*i-1)^2 = 1/3*((4k^3+8k^2+3k)+(4k^2+8k+3))$$

$$(4k^2+4k+1)+\sum_{i=1}^{k}(2*i-1)^2 = 1/3*(4k^3+12k^2+12k-k+3)$$

$$(4k^2+4k+1)+\sum_{i=1}^{k}(2*i-1)^2 = 1/3*(4k^3-k)+1/3*(12k^2+12k+3)$$

$$(4k^2+4k+1)+\sum_{i=1}^{k}(2*i-1)^2 = 1/3*k(4k^2-1)+(4k^2+4k+1)$$

$$\sum_{i=1}^{k}(2*i-1)^2 = 1/3*k(4k^2-1)$$

Which is true by the inductive assumption. Therefore the inductive step and base case hold and the claim is proven true by mathematical induction.

$\square$

Problem 2

Given a planar graph $P = (V, E)$, we have Euler's formula: $|V| + |F| - |E| = 2$, where F is the set of faces of P and E is the set of edges of P. Let $|V| = n$, where V is the set of vertices of P. Prove that $|F|$ is at most $2n$.

PROOF. If we count the edges in P face by face we count $2|E|$ edges. Since each face must consist of at least 3 edges then $2|E| \geq 3|F|$ which is equivalent to $|E| \geq 3/2|F|$.

Then by Euler's Formula $|V| + |F| - 3/2|F| \geq 2$ solving for $|F|$ we find $2|V| - 2 \geq |F|$ $\square$

Problem 3

Prove that in any simple graph there is a path from any vertex of odd degree to some other vertex of odd degree.

PROOF.

Consider vertex of odd degree, x, we show there must be a path from x to another vertex of odd degree. The total degree of a graph must be even. Therefore if we consider the subgraph G connected to a vertex of odd degree, the total degree of G must also be even. Therefore there must be another vertex of odd degree connected to x. Thus there is a path from x to another vertex of odd degree and the claim is proven true.

$\square$

Problem 4

A fully binary tree T is a tree such that all internal nodes have two children. Prove that a fully binary tree with n internal nodes in total has $2n + 1$ nodes.

Because T is a full binary tree we know that there is exactly 2^x nodes in layer x. Therefore the total number of nodes in the first k layers is: $\sum_{i=0}^{k} 2^i$. We let $k =$ the total depth of the tree. Therefore $n = \sum_{i=0}^{k-1} 2^i$. We can express the problem as $\sum_{i=0}^{k} 2^i = 2n + 1$.

PROOF. By induction. We show $\sum_{i=0}^{k} 2^i = 2n + 1$.

We show that the base case of $k = 1$ holds. In this case there is 1 internal node. $\sum_{i=0}^{1} 2^i = 2(1) + 1 = 3$

We assume that the relation holds $\forall n \leq k$ for $k \in \mathbf{Z}$.

Next we show that the relation is true for $k + 1$. Thus $\sum_{i=0}^{k} 2^i = 2n + 1 = 2\left(\sum_{i=0}^{k-1} 2^i\right) + 1$

$$\begin{aligned}
\sum_{i=0}^{k+1} 2^i &= 2n + 1 \\
2^{k+1} + \sum_{i=0}^{k} 2^i &= 2(\sum_{i=0}^{k} 2^i) + 1 \\
2^{k+1} + \left[2\left(\sum_{i=0}^{k-1} 2^i\right) + 1\right] &= 2\left(2^k + \sum_{i=0}^{k-1} 2^i\right) + 1 \\
2^{k+1} + 2 * \sum_{i=0}^{k-1} 2^i + 1 &= 2^{k+1} + 2 * \sum_{i=0}^{k-1} 2^i + 1
\end{aligned} \tag{1}$$

Thus the base case and inductive step hold and the claim is proven true by mathematical induction. □

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Problem 5

Given an undirected graph $G = (V, E)$, the breadth-first-search starting at $v \in V$ ($bfs(v)$ for short) is to generate a shortest path tree starting at vertex $v \in V$. The diameter of G is the longest of all shortest paths $\delta(u, v), u, v \in V$.

When G is a tree, the following algorithm is proposed to compute the diameter of G.

1. Run $bfs(w), w \in V$, and compute the vertex $x \in V$ furthest from w.

2. Run $bfs(x)$ and compute the vertex $y \in V$ furthest from x.

3. Return $\delta(x, y)$ as the diameter of G.

Prove that this algorithm is correct; i.e., $\delta(x, y)$ is in fact the longest among all the shortest paths between $u, v \in V$.