Voronoi Diagrams

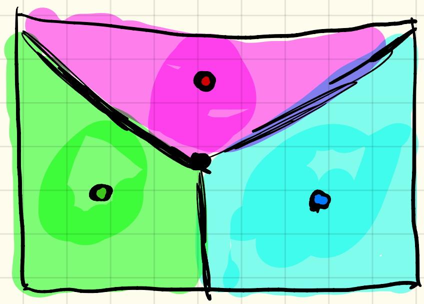
- Voronoi diagram: encoding of proximity (what is closest to what)

Let $P = \{p_1, \dots, p_n\}$ be a set of points in the plane called sites

$V(p_i)$ is the Voronoi cell of p_i
all the points $q \in \mathbb{R}^2$ closer to p_i than any other site in P

$$V(p_i) = \{ q \mid \delta(p_i, q) < \delta(p_j, q),\ \forall i \neq j,\ q \in \mathbb{R}^2 \}$$

$$\delta(p, q) = \left((p_x - q_x)^2 + (p_y - q_y)^2 \right)^{\frac{1}{2}}$$ ← Euclidean dist in 2d

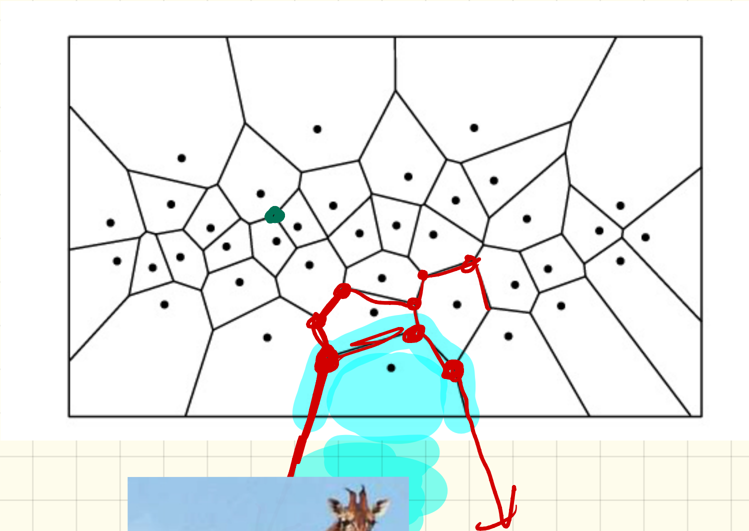


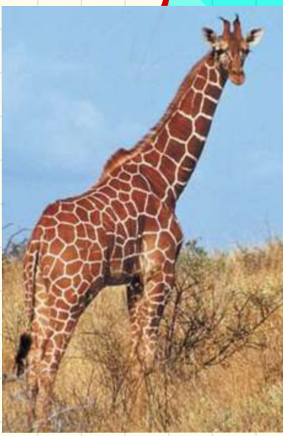
Alt def of $V(p_i)$

- for any 2 sites p_i, p_j
  set of points strictly closer to p_i is an open halfplane
- Bounding line is $\perp$ bisector of $\overline{p_i p_j}$
- Let $h(p_i, p_j)$ be the halfplane
  $q \in V(p_i) \iff q \in \bigcap_{i \neq j} h(p_i, p_j)$

$\Rightarrow V(p_i)$ is an intersection of halfplanes
$\Rightarrow V(p_i)$ is a (potentially unbounded) convex polygon

Voronoi diagram denoted as Vor(P)
is what is left after removing all open Voronoi cells
⇒ Voronoi diagram is a set of bounded (and unbounded) segments





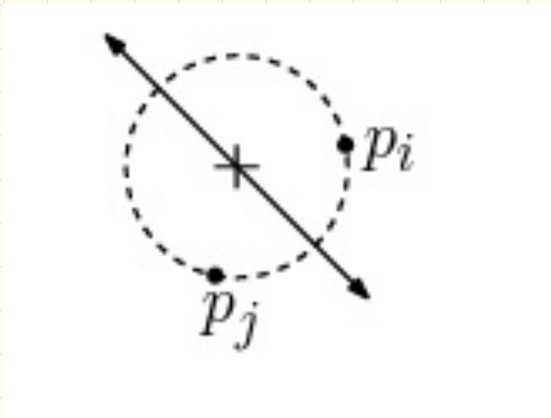
Encyclopedia Britanica

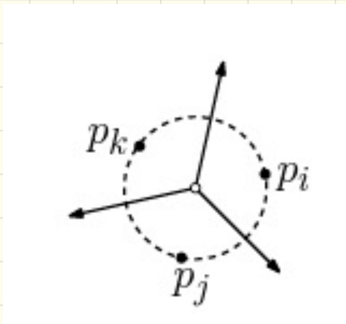
Prop of Voroni in the plane

- Voronoi complex

cell complex

- faces are convex (possibly unbounded) polygons
- points on edges they have 2 nearest neighbors in P, p_i, p_j
  $\Rightarrow$ circle centered on the point, through p_i and p_j
  no other site in P can be in this circle





Voronoi vertices

- points wher 3 voronoi cells "intersect"
  $V(v_i), V(v_j), V(v_k)$ $v_i, v_j, v_k \in P$

vertex is equidistant to all 3 sites

$\Rightarrow$ circle w/ center at vertex passing through 3 sites is free of other sites in P

G.P. assumption: no more than 3 sites are co-circular

What is the degree of each Voroni vertex ... 3

Voronoi & CH

cell is unbounded

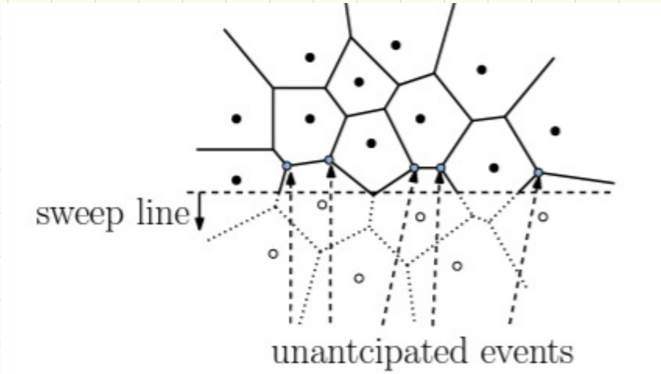
$\iff$ corresponding site is on $CH(P)$

- Size: n is # of sites, Voronoi diagram is planar
  - $\Rightarrow$ n faces
  - $\sim 2n$ verts
  - $\sim 3n$ edges

Eulers formula
$v - e + f = 2$ w/ $f = n$
$\Rightarrow v - e + n = 2$ w/ each vertex is deg 3
$\Rightarrow 3v = 2e = v - \frac{3}{2}v = 2$
$\Rightarrow v = 2n - 4$

$\Rightarrow$ in 2d size of $Vor(P)$ w/ $|P| = n$ is $O(n)$
in d-dim up to $O(n^{\lceil d/2 \rceil})$

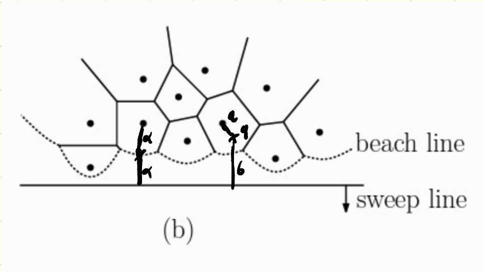
Compute:
- $O(n^2 \lg n)$ as each cell w/ halfplane intersection
- $O(n^2)$ historically known
- expected $O(n \lg n)$ w/ RIC
- Shamos & Hoye $O(n \lg n)$ D&C algo
- Fortune's algo $O(n \lg n)$ sweepline



Beach line:
Sweep 2 objects
- horizontal sweep line
- x-monotone curve called the beach line

Subdivide halfplane above the sweepline into:
1. points that are closer to sites than the sweep line
2. points closer to the sweep line than other sites

Props of boundary of beachline



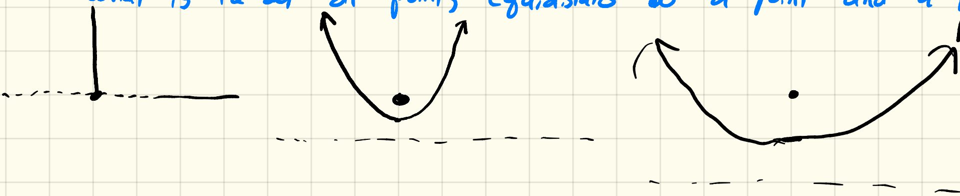
(b)

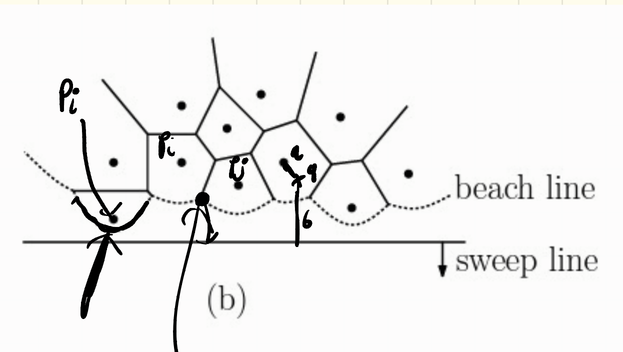
- points are equidistant between nearest site and the sweepline
- points are above the beachline closest site cannot change
  ⇒ anything above beachline is "safe"

What does the beachline look like:

What is the set of points equidistant to a point and a line?

parabola





(b)

break points – equidistant between p_i, p_j and sweep line
⇒ breakpoints are on Voronoi edges

Lemma: Beach line is an x-monotone curve
of parabolic arcs
and Breakpoints of the beach line lie on Voronoi edges of the dgm