Homework 4

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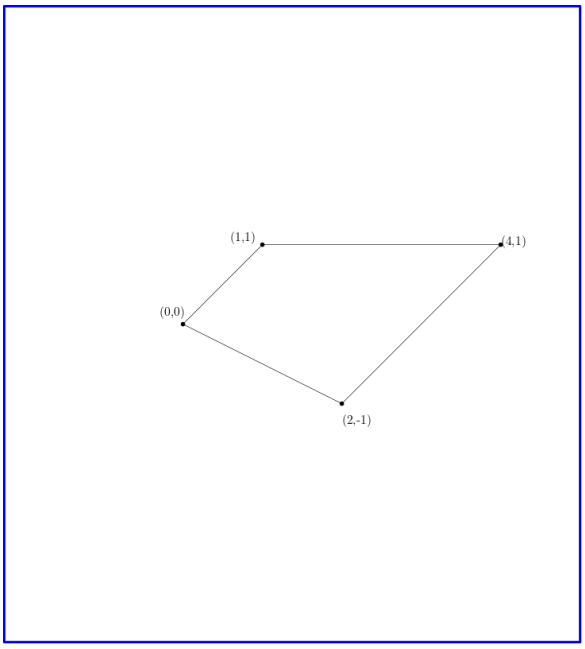
18 March 2021

Problem 1

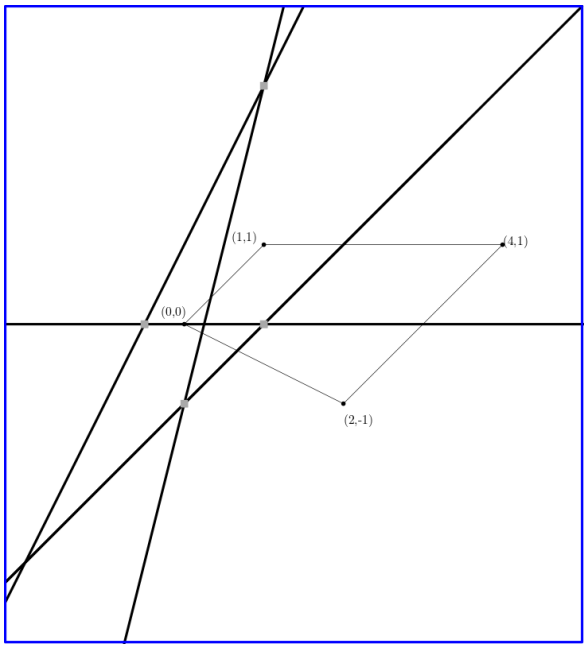
Draw a polygon with at least 4 vertices and give each vertex coordinates. Using the duality discussed in class, draw the dual of the polygon. In the dual,

- shade black the duals of the points that are on the vertices in the primal
- shade grey the duals of the points that are on the edges in the primal
- shade red the duals of the points that are inside the polygon in the primal

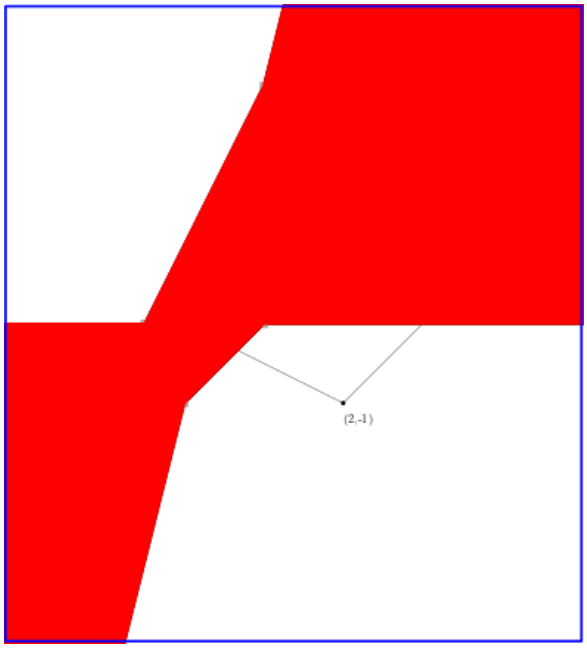
You should draw the primal and dual to scale. You may draw the figure by hand, but, I suggest using a tool like Ipe or Inkscape.



(a) Primal



(b) Vertices and Edges of polygon



(c) Interior region of polygon

Figure 1: Primal and Duals

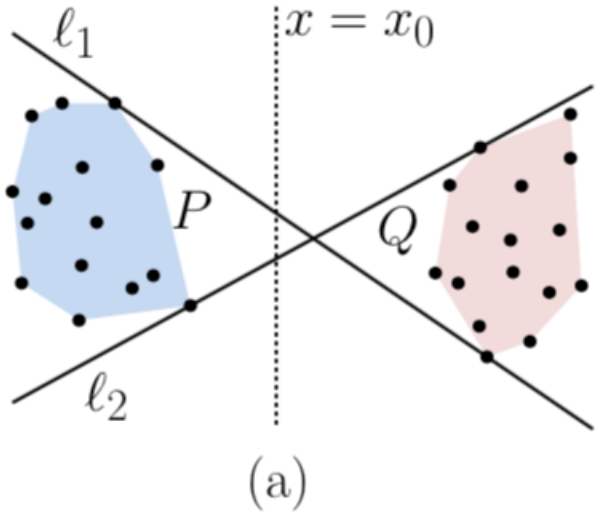
Problem 2

Explain how to solve each of the following problems in linear (expected) time.

Each can be modeled as a linear programming (LP) problem, perhaps with some additional pre- and/or post- processing. In each case, explain how the problem is converted into an LP instance and how the answer to the LP instance is used/interpreted to solve the stated problem.

Problem 2.1

You are given two point sets $P = \{p_1, \ldots, p_n\}$ and $Q = \{q_1, \ldots, q_m\}$ in the plane, and you are told that they are separated by a vertical line $x = x_0$, with P to the left and Q to the right (see Fig. a). Compute the line equations of the two "crossing tangents," that is, the lines ℓ_1 and ℓ_2 that are both supporting lines for conv(P) and conv(Q) such that P lies below ℓ_1 and above ℓ_2 and the reverse holds for Q. (Note that you are not given the hulls, just the point sets.) Your algorithm should run in time $O(n + m)$.



(a)

We use two LP's in succesion (one to find each line).

Our first line:

Objective:

$$\max m$$

Subject To:

$$p_i.x \cdot m + b \geq p_i.y \qquad \forall p_i \in P$$
$$q_i.x \cdot m + b \leq q_i.y \qquad \forall q_i \in Q$$

The first constraint ensures that every point in P is below the line ($y = mx + b$) Since $(x, y) \leq \ell_1 \implies y \leq \ell_1(x) = mx + b$. The second constant ensures that every point in Q is above the line. This LP finds the negative sloping line (ℓ_1), so we want to choose the one with the least negative slope, aka maximize the slope.

The second LP is almost identical except for flipping inequalities

Objective:

$$\min m$$

Subject To:

$$p_i.x \cdot m + b \leq p_i.y \qquad \forall p_i \in P$$
$$q_i.x \cdot m + b \geq q_i.y \qquad \forall q_i \in Q$$

The first constraint ensures that every point in P is above the line. and the second constant ensures that every point in Q is below the line. This LP finds the positive sloping line (ℓ_2), so we want the line with the least positive slope, aka minimizing the slope.

We note that m, b are unknowns, so the search-space is m, b plane in $\mathbb{R}^2$. So the resulting optimal point from our LP's is a point (m, b), which we translate into line $y = mx + b$

Running Time: The (expected) running time is determined by the number of constraints. There are n different constraints of the form: $p_i.x \cdot m + b \geq p_i.y$ one for each p_i, and similarly m different constraints of the form: $q_i.x \cdot m + b \leq q_i.y$ one for each q_i. Thus there are a total of $n + m$ constraints. So we can solve each LP in expected $O(n + m)$ time.

Correctness:

We only prove the correctness of the first LP as the second follows from the proof of the first.

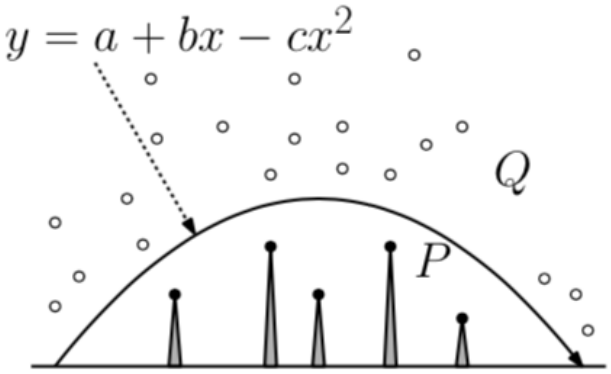
Proof. By Contradiction

Suppose not, suppose that the line (ℓ_1) is not a supporting tangent. By our constraints, the line returned must be supporting. Otherwise it violates constraint 1 or 2. Then there must be another line ℓ' such that $\ell' = m'x + b'$ and $m' > m$. A contradiction, since we found the line with maximum slope. □

Thus we find two supporting tangents of P, Q and our algorithm is correct.

Problem 2.2

You have a cannon in $\mathbb{R}^2$. It has three controls labeled "a" "b," and "c". A projectile shot from this cannon travels along the parabolic arc $y = a+bx-cx^2$. You are asked to determine whether it is possible to adjust the controls so that the projectile travels above a set of n building tops, represented by a point set $P = \{p_1, \ldots, p_n\}$ and beneath a set of m floating balloons, represented by a point set $Q = \{q_1, \ldots, q_m\}$ (see Fig. b)). Your algorithm should run in time $O(n+m)$. (I do not care where the cannon is actually located. If your solution is based on some assumption about the cannon's location, please state this.)



(b)

We first do some pre-processing of the points to map them into $\mathbb{R}^4$. $p = (u, v) \in P \to p' = (x, y, z, v) \in P'$ where $p' = (1, u, u^2, v)$.
And similarly for Q $q = (u, v) \in Q \to q' = (x, y, z, v) \in Q'$ where $q' = (1, u, u^2, v)$.

We then build linear program:

Objective:

$$\min 1$$

Subject To:

$$p_i.x \cdot a + p_i.y \cdot b - p_i.z \cdot c \geq p_i.v \qquad \forall p_i \in P'$$
$$q_i.x \cdot a + q_i.y \cdot b - q_i.z \cdot c \leq q_i.v \qquad \forall q_i \in Q'$$

Where each constraint is a halfplane in $\mathbb{R}^3$ with variables (a, b, c). The first constraint ensures that $P \leq parabola$ and the second constraint ensures that $Q \geq parabola$. We show this $P \leq parabola \implies v \leq parabola(u) = a \cdot (1) + b \cdot u - c \cdot u^2$. Which by our mapping is $p.v \leq a \cdot p.x + b \cdot p.y - c \cdot p.z$.

We do not care which parabola we choose, so our objective function is just a placeholder that will choose any feasible solution. The search space is $(a, b, c) \in \mathbb{R}^3$ which correspond to parabola $y = a + bx - cx^2$.

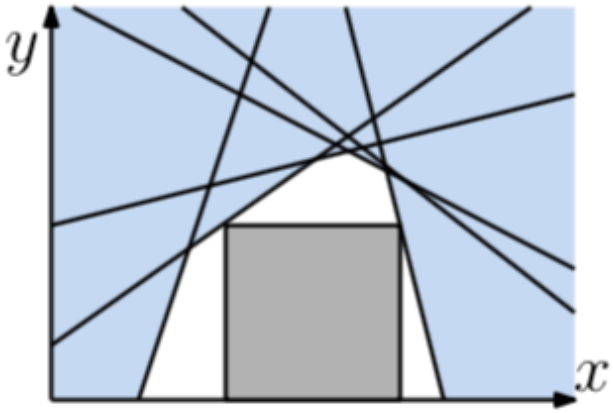
If the LP fails (empty feasible region) then we know that it is not possible to avoid all the balloons and buildings. If it returns a value, then the feasible region is non-empty and it is possible to avoid hitting balloons and buildings by setting proper a, b, c parameters. It should be noted, that since the algorithm returns values on the edges of the search space, the point returned will just barely clip a balloon/building. We can change the parameters slightly to avoid this, unless the feasible region is just a single point (lower bound of Q and upper of P lie on one parabola).

Running Time: There are a total of $m + n$ constraints, since there is one for each point in P and Q. Thus we can solve the LP in expected $O(n)$ time.

Correctness: We already showed that by our mapping a point is below the parabola if $p.v \leq a \cdot p.x + b \cdot p.y - c \cdot p.z$ (and opposite for above). Constraint 1 ensures that all $P \leq parabola$ and constraint 2 ensures that all $Q \geq parabola$. Thus if the algorithm finds a solution, it must be a valid solution that avoids all the balloons and buildings.

Problem 2.3

You are given a set of n halfplanes $H = \{h_1, \ldots, h_n\}$, where h_i is given as a pair (a_i, b_i) and it consists of all the points of the plane that lie on or beneath the line $y = a_i x + b_i$. Compute the axis-parallel square of the largest side length whose lower edge lies on the x-axis (see Fig. c). If no such square exists, your algorithm should indicate this.



(c)

Objective:

$$\max \ell$$

Subject To:

$$a_i \cdot s + a_i \cdot \ell + b_i \leq \ell \qquad \forall h_i = a_i \cdot x + b_i$$
$$a_i \cdot s + b_i \leq \ell \qquad \forall h_i = a_i \cdot x + b_i$$

In this case ℓ is the side length of our square (hence why we want to maximize it), and s is the horizontal shift (s is the location of left edge of box). The first constraint checks if the upper right corner is in the box (constraint $1 = (s + \ell, \ell) \leq h_i$) and the second constraint checks if the upper left corner is in

the box.

Since we always want the lower halfplane we do not need to check if the bottom corners are within the feasible region. Since they are directly below the two upper corners.

If the algorithm fails (empty feasible region) then the algorithm returns no square. Otherwise it returns (ℓ, s), where the square is defined by vertices: $(s, 0), (s + \ell, 0), (s + \ell, \ell), (s, \ell)$

Running time There are a total of $2n$ constraints (2 for each halfplane: one upper right corner, one lower right corner) So the total running time is $O(2n) = O(n)$ expected.

Correctness: The algorithm finds the square of maximum side-length that fits within the feasible region. If there is a bigger one, then the linear program did not find the max $\implies$ contradiction.