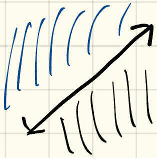
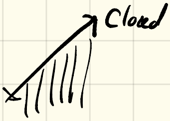
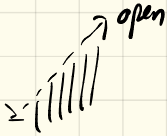
Half Plane Intersections → Point line duality

half plane



open if don't consider line
closed if do consider line





(today halfplanes are all closed)

Rep: GP assumption (no vertical lines)

$y = ax - b$

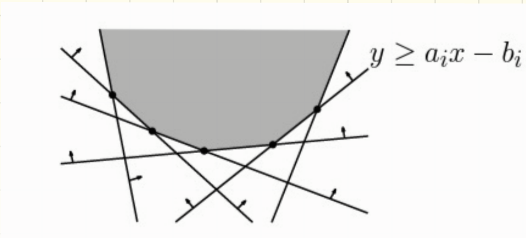
slope ↗ ; ↖ negative y intercept

lower (closed) halfplane: all points satisfying $y \leq ax - b$
upper (closed) halfpale: all points satisfying $y \geq ax - b$

Halfplane intersection problem

Given n closed halfplanes $H = \{h_1, \ldots, h_n\}$
compute the intersection

Note: intersection is convex because
intersection of finite # of convex sets is convex



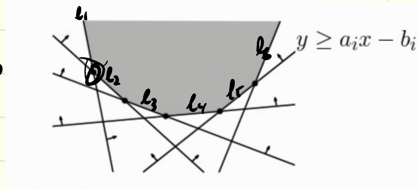
unbounded

$\ell \| m$

$H_1 \cap H_2 = \emptyset$

Output:

list of lines bounding the intersection region
CCW order



Worst case how many lines can bound the output?

n (because of convexity)

How fast can we compute?

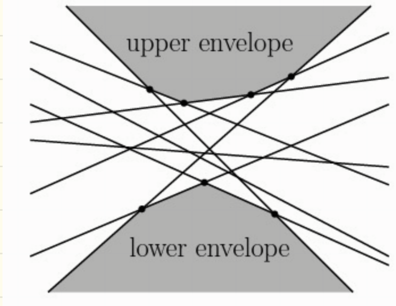
lower bound $\Omega(n \lg n)$ - sorting

Higher dim: n halfspaces in $\mathbb{R}^d$

boundary $\Theta(n^{\lfloor \frac{d}{2} \rfloor})$ today we stay in 2d

See book or DM lecture notes for D&C also

Lower envelopes (and duality)



lower envelope:

given lines $L = \{\ell_1, \dots, \ell_n\}$

$\ell_i : y = a_i x - b_i$

Consider lower halfplanes $y \leq a_i x - b_i$

lower envelope is an intersection of halfplanes

we can compute lower envelope (bdd) by computing a upper hull

" " upper envelope (bdd) by computing a lower hull

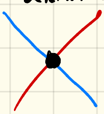
Duality for lines, points, incedences

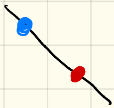
line $y = ax - b \rightarrow$ rep as 2 coeff (a, b)

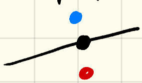
eg $\ell : y = 2x + 1 \rightarrow$ point $\ell^* : (2, -1)$

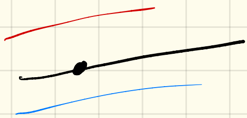
Relationship

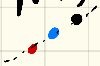
Primal	dual
2 lines determine a point	2 points determine a line
A point may lie above/on/below a line	A line may pass above/below or through a point
3 points collinear	3 lines through same point

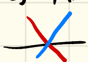












Point-line duality map:

$$l: y = l_a x - l_b \longrightarrow l^* = (l_a, l_b)$$
$$p = (p_x, p_y) \longrightarrow p^*: b = p_x a - p_y$$

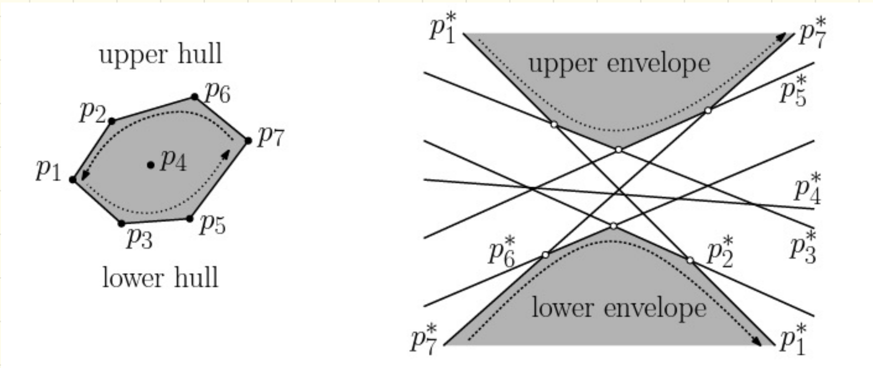
Props:

self inverse: $p^{**} = p$

order preserving: Point p above/on/below l in primal $\iff$ p^* is below/on/above l^* in the dual

intersection preserving: l_1, l_2 intersect a p in primal $\iff$ dual line p^* passes through poins l_1^* & l_2^*

Colinear/coincidence: 3 points collinear in primal $\iff$ dual lines intersect at common point



Convex hulls & envelopes

Lemma: Let P be a set of points in the plane

CCW of points in the upper hull of P

is the same seq of lines in the lower envelope of dual P^*

Pf Upper hull (assume no 3 are collinear)

consider p_i and p_j consecutive on upper hull
def $l_{ij} := \overleftrightarrow{p_i p_j}$

p_i, p_j on upper hull $\Rightarrow$ all points of $P \setminus \{p_i, p_j\}$ below l_{ij}

Consider dual lines p_i^* p_j^*

incidence prop $\Rightarrow$ $l_{ij}^* = p_i^* \cap p_j^*$
order preserving prop $\Rightarrow$ dual lines of P^* are above point l_{ij}^*
$\Rightarrow$ l_{ij}^* is on the lower envelope of P^*

How is order preserved?
observe: CCW order of pts on upper hull
slope of edges between points increase monotonically
slope of line in the primal $\rightarrow$ a-coord in dual
$\Rightarrow$ CCW in the upper hull $\rightarrow$ left/right in the lower envelope