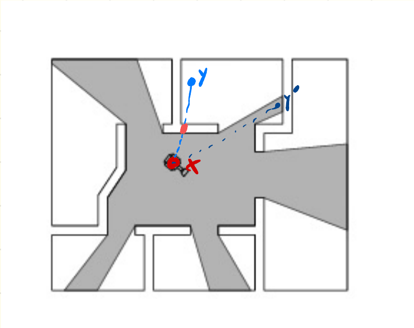
Guarding Δtions

2/11

Art gallary problem

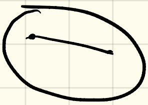
Gallery as a simple polygon P
points $x, y \in P$
are visible if the
open segment from x to y
is in the interior of P



Q: Given a polygon P find the minimal
\# of cameras for guarding P ... NP-hard :(

Q: Bound for any simple polygon w/ n verts

idea 1:



- any convex polygon can be guarded w/ 1 camera
- Δs are convex polygons

⇒ put 1 camera in each Δ

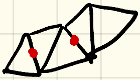
Q: how many Δs are in a simple polygon w/ n verts?
$\boxed{n-2}$

induction:

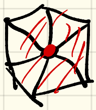
3 verts
$n-2 = 1$
Δ has 3 verts and 1 Δ $\quad 3-2=1$ ✓

assume poly w/ $n-1$ verts has $n-3$ Δs
since every simple polygon w/ 4 or more verts has an ear,
pluck ear ⇒ removes a vert ⇒ a polygon of $n-1$ verts
and gives 1 Δ $\quad$ ↳ $n-3$ Δs
$+1$ Δ ⇒ $n-2$ Δs ✓

idea 2: one per edge $\Rightarrow$ need about $n/2$ cameras (better!



idea 3: use verts



Let T_P be a Δtion of P

Goal: select a subset of verts from P

s.t. any Δ of T_P has one of these verts

Let black, white, gray be our 3 colors

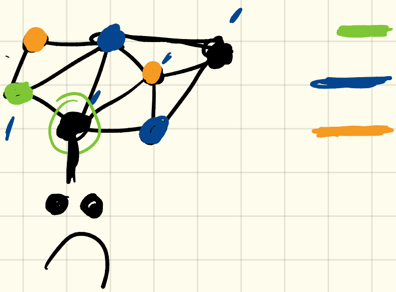
Can we 3 color our Δtion?

if we can 3 color our graph

$\Rightarrow$ pick set w/ min size

put a camera on each vertex

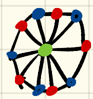
$\Rightarrow$ we have at most $n/3$ cameras



Does a 3 coloring of a Δ polygon always exist?

Side note for (planar graphs):

- does a 6 coloring always exist? Yes
- does a 5 coloring always exist? Yes
- does a 4 coloring always exist? Yes! Appel and Haken 1976



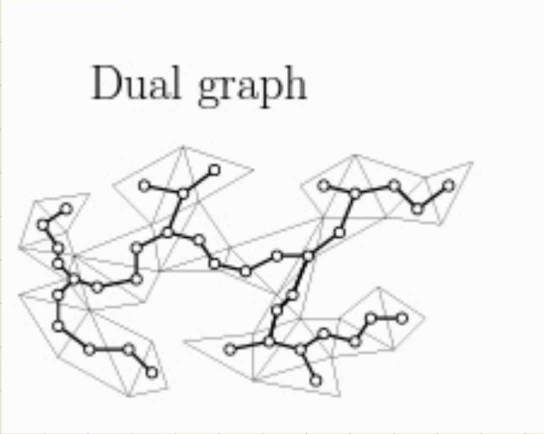
Back to 3 coloring:

consider dual graph

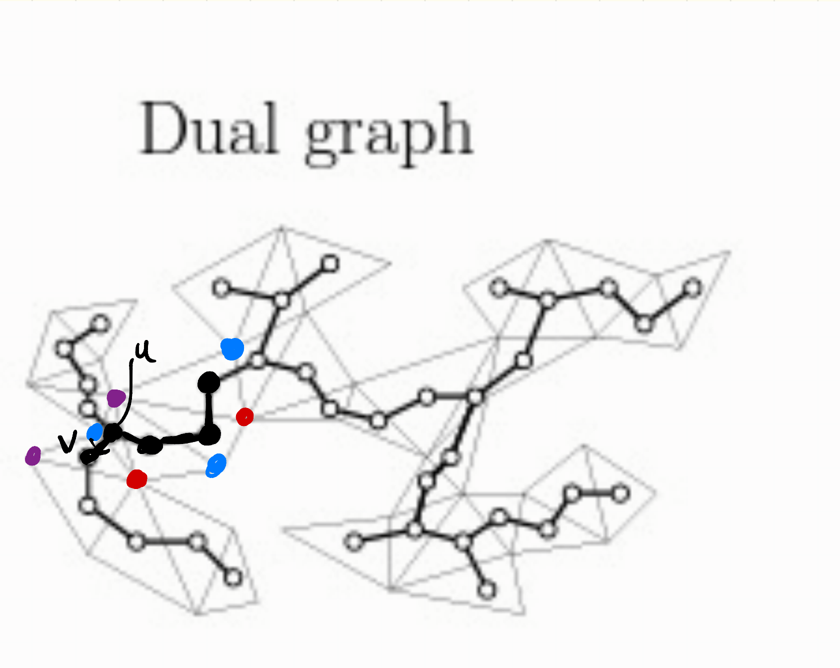
- $G(T_p)$
- node v in graph $t(v)$ is Δ in T_p
- remove a diag split the polygon into 2 polygons

$\Rightarrow$ remove an edge from $G(T_p) \Rightarrow$ splits graph into 2 parts

$\Rightarrow$ graph is a tree

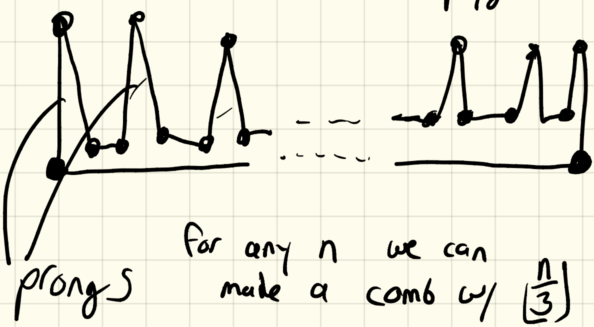


Tree $\Rightarrow$ color w/ DFS



Formally: Invariant: all verts of visited Δs
are alreay 3 colored
$\Rightarrow$ Start at any node of $G(T_p)$ and color vets w/ 3 colors
consider reacing node $v \in G(T_p)$ came from $u \in C(T_p)$
$\Rightarrow$ $t(v)$ and $t(u)$ share an edge
$t(u)$ was alreay 3-colored
2 verts are shared w/ $t(v)$
those vert already have colors
$\Rightarrow$ set the remaining vertex of $t(v)$ w 3rd color
$\Rightarrow$ since $G(T_p)$ is a tree
nodes adj to v are not visited
$\Rightarrow$ Δtion of a simple polygon is 3 colorable
$\Rightarrow$ $\lfloor \frac{n}{3} \rfloor$ cameas is enough to guard a simple polygon

Can we do better?
we cannot because there is a polygon that neede $\lfloor n/3 \rfloor$ cameas



For any n we can make a comb w/ $\lfloor \frac{n}{3} \rfloor$ prongs

$\Rightarrow$ can't find a stratagey
tha is always less than $\lfloor \frac{n}{3} \rfloor$ cameas

Theorem 3.2 (Art Gallery Theorem) *For a simple polygon with n vertices, $\lfloor n/3 \rfloor$ cameras are occasionally necessary and always sufficient to have every point in the polygon visible from at least one of the cameras.*