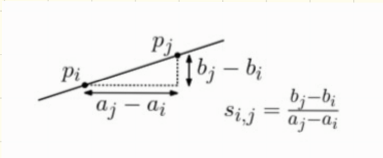
Warmup Problem

Given n points $P = \{p_1, p_2, \dots, p_n\} \subseteq \mathbb{R}^2$
$p_i = (a_i, b_i)$

simplifying assumption:
all a-coords
& all b-coords are distinct

for $p_i \neq p_j$
define $s_{i,j} = \dfrac{b_j - b_i}{a_j - a_i}$ ← i.e. the slope of the line through p_i and p_j



define
$S = \{ s_{i,j} \mid 1 \leq i < j \leq n \}$

How big is S?
$|S| = \binom{n}{2} = \frac{n(n-1)}{2} = O(n^2)$

Descriptive stats on lines throug pairs of points
- max/min: find max and min slope is S
- k^{th} smallest/largest: k^{th} smallest or largest elt in S
- average - avg slope
- (*) range counting - given $s^- \in \mathbb{R}$, $s^+ \in \mathbb{R}$ w/ $s^- \leq s^+$
  return # of elts in S w/ slope in $[s^-, s^+]$

Simple solution
- compute S — $O(n^2)$
- sort S — $O(n^2 \log n)$
- bin search s^- & s^+ — $O(\log n)$
- dist between entries — $O(1)$

⇒ $O(n^2 \lg n)$ preproc
⇒ $O(\lg n)$ query

Version 1:

Consider $s^- = 0$ & $s^+ = +\infty$

(num of non-neg sloped lines)

idea! compute # of neg sloped lines and compute $\binom{n}{2}$ − # of neg sloped lines

1. Sort points in increasing order of a-coordinate

relable

$P = \langle p_1, p_2, \ldots, p_n \rangle$

Let $B = \langle b_1, b_2, \ldots, b_n \rangle$ seq of b-coords

Obs for $1 \le i < j \le n$

$b_i > b_j$

$\iff s_{i,j} < 0$

why $a_i < a_j$

$\Rightarrow \text{sign}\left(\frac{b_j - b_i}{a_j - a_i}\right) = \text{sign}(b_j - b_i)$

$+$

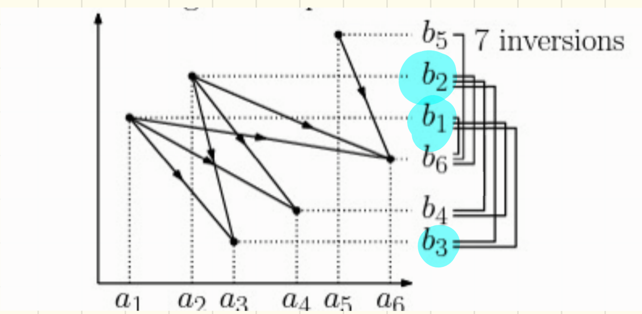
def For $1 \le i < j \le n$

the pair (i,j) is an inversion

if $b_i > b_j$

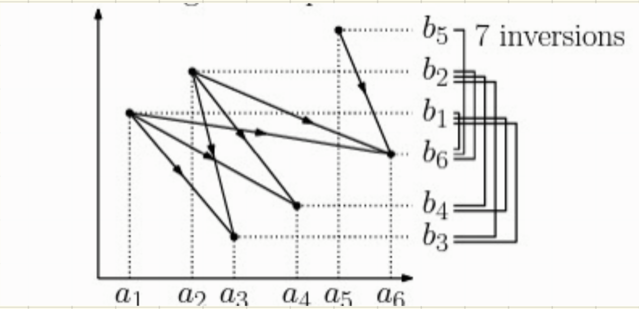
by obs counting # of neg slopes

is the same as counting inversions



D&C alg for counting inversions (based on merge sort)

idea:



b_3	b_4	b_6	b_1	b_2	b_5

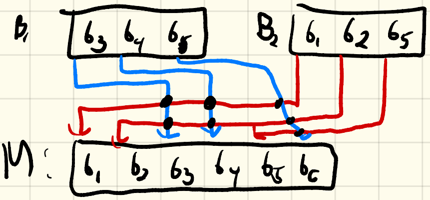
split into two lists

B_1: | b_3 b_4 b_5 | B_2: | b_1 b_2 b_5 |

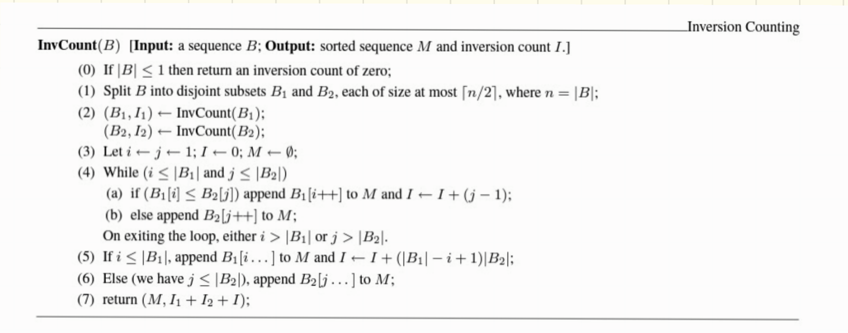
I_1 = compute # of inversions I_2 = compute # of inversions

I # of inversions from merging B_1 & B_2

$\Rightarrow$ total # of inversions are $I + I_1 + I_2$



$\Rightarrow$ # of items are 7



Inversion Counting

InvCount(B) [**Input:** a sequence B; **Output:** sorted sequence M and inversion count I.]

(0) If $|B| \leq 1$ then return an inversion count of zero;
(1) Split B into disjoint subsets B_1 and B_2, each of size at most $\lceil n/2 \rceil$, where $n = |B|$;
(2) $(B_1, I_1) \leftarrow$ InvCount(B_1);
$(B_2, I_2) \leftarrow$ InvCount(B_2);
(3) Let $i \leftarrow j \leftarrow 1$; $I \leftarrow 0$; $M \leftarrow \emptyset$;
(4) While ($i \leq |B_1|$ and $j \leq |B_2|$)
 (a) if ($B_1[i] \leq B_2[j]$) append $B_1[i{+}{+}]$ to M and $I \leftarrow I + (j-1)$;
 (b) else append $B_2[j{+}{+}]$ to M;
On exiting the loop, either $i > |B_1|$ or $j > |B_2|$.
(5) If $i \leq |B_1|$, append $B_1[i \ldots]$ to M and $I \leftarrow I + (|B_1| - i + 1)|B_2|$;
(6) Else (we have $j \leq |B_2|$), append $B_2[j \ldots]$ to M;
(7) return $(M, I_1 + I_2 + I)$;

Analysis: recurrence $T(n) = 2T(\frac{n}{2}) + n \Rightarrow O(n \log n)$ ← to get # of neg slopes

$\Rightarrow$ getting # of positive slopes is $O(n \lg n)$

Duality

$p: (a,b)$ a point
$p^* := y = ax - b$ a line

$\circlearrowleft$ duality

sim a line $\ell := y = ax - b \to \ell^* := (a,b)$

$(p^*)^* = p$
$(\ell^*)^* = \ell$

One prop
let $p_i = (a_i, b_i) \qquad p_i^* := y = a_i x - b_i$
$\quad p_j = (a_j, b_j) \qquad p_j^* := y = a_j x - b_j$

assumed $a_i \neq a_j \Rightarrow p_i^* \not\parallel p_j^*$

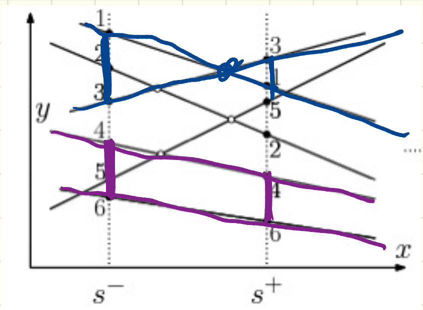
x-coord of $p_i^* \cap p_j^*$ is $a_i x - b_i = a_j x - b_j$

$$\Leftrightarrow x = \frac{b_j - b_i}{a_j - a_i} = s_{i,j}$$

slope of the line through pts p_i, p_j

Recast problem as:
count
~~find~~ all lines w/ slope $[s^-, s^+]$

$\rightsquigarrow$ Count all pairs of lines w/ an intersection in the vertical slab $[s^-, s^+]$



Observe that
2 lines intersect in slab
$\Leftrightarrow$ their order swaps

1. sort lines in decreasing order of the intsesetion w/ y-coord on left side of the slab $O(n \lg n)$
2. re-number $O(n \lg n)$
3. sort decresing on r.h.s of slab

4. count # of inversions $O(n \lg n)$

$\Rightarrow$ anser

Q

- min/max slope : $O(n \lg n)$

output size k

- report all lines in the range : $O(n \log n + k)$