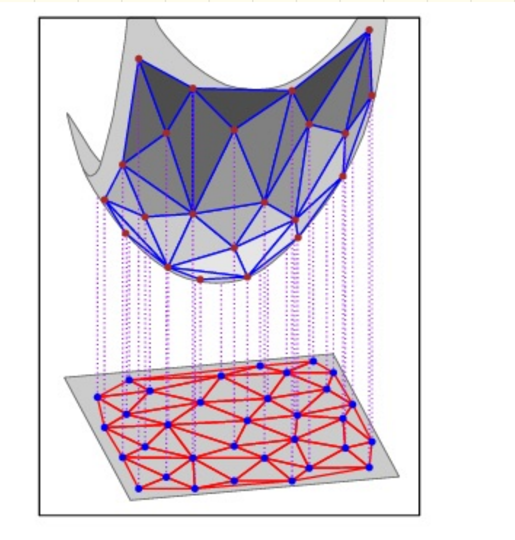
Hulls, Envelopes, DT & Vor

We will show
- DT of points in dim d by computing convex hull in dim $d+1$
- Vor of points in dim d by computing upper envelope of hyperplanes in dim $d+1$

DT & CH
- define paraboloid Ψ (psi) $z = x^2 + y^2$
- map each point $p = (p_x, p_y)$ w/ vertical projection to a point on Ψ
  $p^\uparrow = (p_x, p_y, p_x^2 + p_y^2) \in \mathbb{R}^3$

Given a pointset $P \subset \mathbb{R}^2$
let $P^\uparrow$ be the projection of each point in P to Ψ

Claim: $Del(P)$ is the projection of lower hull of $P^\uparrow$



if $p, q, r \in P$
w/ lifted points $p^\uparrow, q^\uparrow, r^\uparrow$ of Ψ

then $\Delta p^\uparrow q^\uparrow r^\uparrow$ is a face of lower hull of $P^\uparrow$
$\Leftrightarrow$ Δpqr is Δ of $Del(P)$

Pf sketch

Recall: Del condition: $p, q, r \in P$ form Δ in $Del(P)$

$\Leftrightarrow$ No other point in P can be in circumcircle of pqr

Recall: CH condition: $p^\uparrow q^\uparrow r^\uparrow \in P^\uparrow$ form a face of $CH(P^\uparrow)$

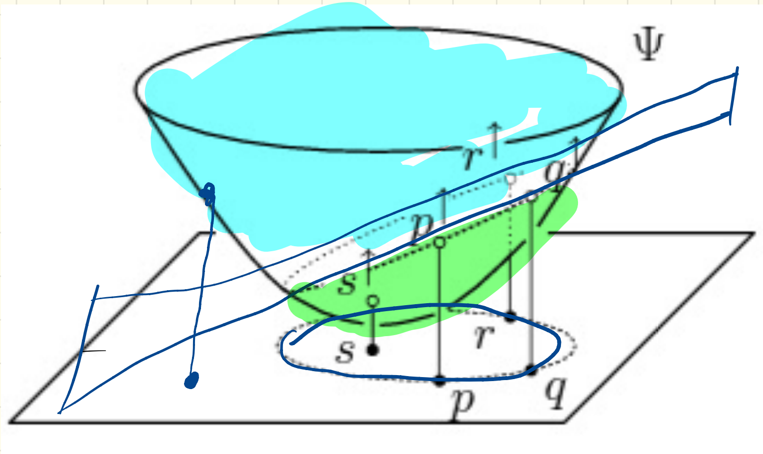
$\Leftrightarrow$ No other points of $P^\uparrow$ below the plane through $p^\uparrow q^\uparrow r^\uparrow$

Lemma: Consider distinct points $p, q, r, s \in \mathbb{R}^2$

Let $p^\uparrow q^\uparrow r^\uparrow s^\uparrow$ be projections to Ψ

s lies w/in circumcircle of pqr

$\Leftrightarrow$ $s^\uparrow$ lies beneath the plane through $p^\uparrow q^\uparrow r^\uparrow$



Consider $p, q, r \in P$
lift to $p^\uparrow q^\uparrow r^\uparrow$ of Ψ
define a plane that intersects Ψ
$\Rightarrow$ project the intersection curve is is the circle through pqr
$\Rightarrow$ s is in circumcircle
$\Leftrightarrow$ $s^\uparrow$ is below the plane through $p^\uparrow q^\uparrow r^\uparrow$

Thm: Given a pointset in the plane (no 4 co-circular)
& $p, q, r \in P$
$\Delta pqr \in Del(P)$

$\Leftrightarrow$ $p^\uparrow q^\uparrow s^\uparrow$ is a face of the lower hull

Sketch:
$\Delta pqr \in Del(P)$

$\Leftrightarrow$ no $s \in P$ in circumcirle of pqr

Lem $\Rightarrow$ any $s^\uparrow \in P^\uparrow$ on or above the plane through $p^\uparrow q^\uparrow r^\uparrow$

$\Rightarrow$ $p^\uparrow q^\uparrow r^\uparrow$ is a face of the lower hull

Voroni diagram & Upper envelopes

Del $\xleftrightarrow{\text{related to}}$ lower hulls $\xleftrightarrow{\text{related to}}$ upper envelopes $\xleftrightarrow{\text{related to}}$ Voroi diagrams

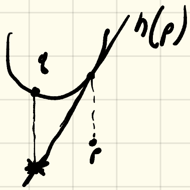
related to

For a point $p \in \mathbb{R}^2$ $p=(a,b)$

Tangent plane to Ψ at $p^{\uparrow}$ is

$z = 2ax + 2by - (a^2+b^2)$ ← lets define $h(p)$ to be plane

Consider $q = (q_x, q_y) \in \mathbb{R}^2$



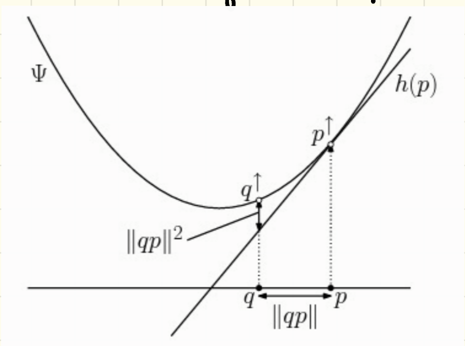
and vert dist from $q^{\uparrow}$ to $h(p)$

vert dist between $q^{\uparrow}$ to $h(p)$ is

$$q_z - (2aq_x + 2bq_y - a^2+b^2) = q_x^2 + q_y^2 - (2aq_x + 2bq_y - (a^2+b^2))$$

$$= (q_x^2 - 2aq_x + a^2) + (q_y^2 - 2bq_y + b^2) = \delta_E(q,p)^2$$

vert dist from $q^{\uparrow}$ to $h(p)$ is squared Euclided dist from p to q

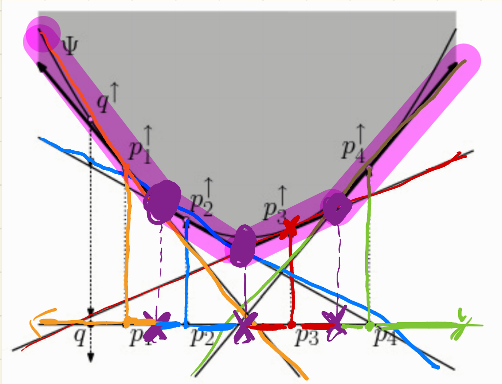


Lemma: Given a set of points $P = \{p_1, \ldots, p_n\}$ in the plane
Let $H(P) = \{ h(p) : p \in P \}$

for $q \in \mathbb{R}^2$
a ray from q to $q^\uparrow$ intersects planes of $H(P)$
in the same order as the dist of points from P to q

Let $U(P)$ be the upper envelope of $H(P)$
- $U(P)$ is an unbounded convex polytope

project $U(P)$ the plane
(it covers the plane)
- label each point of $H(P)$ w/ the face of the face of polytope



Th Given a set of Points P in the plane
$U(P)$ is upper envelope of the tangent planes through $p^\uparrow$ for $p \in P$

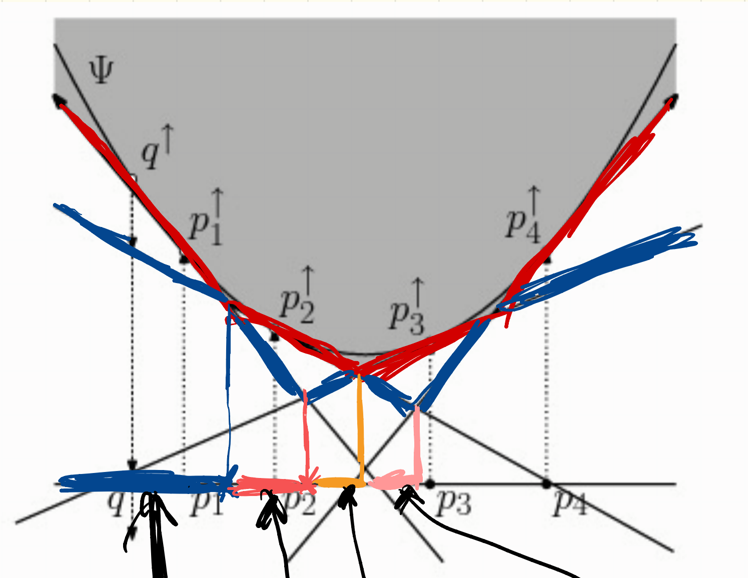
Vor(P) is the same as the vertical projection onto (x,y)-plane of the boundary of $U(P)$

Higher order Voronoi

order k-voroi

- each point in a cell has the same set of k-closest sites

find k-order voroin w/ UE trans

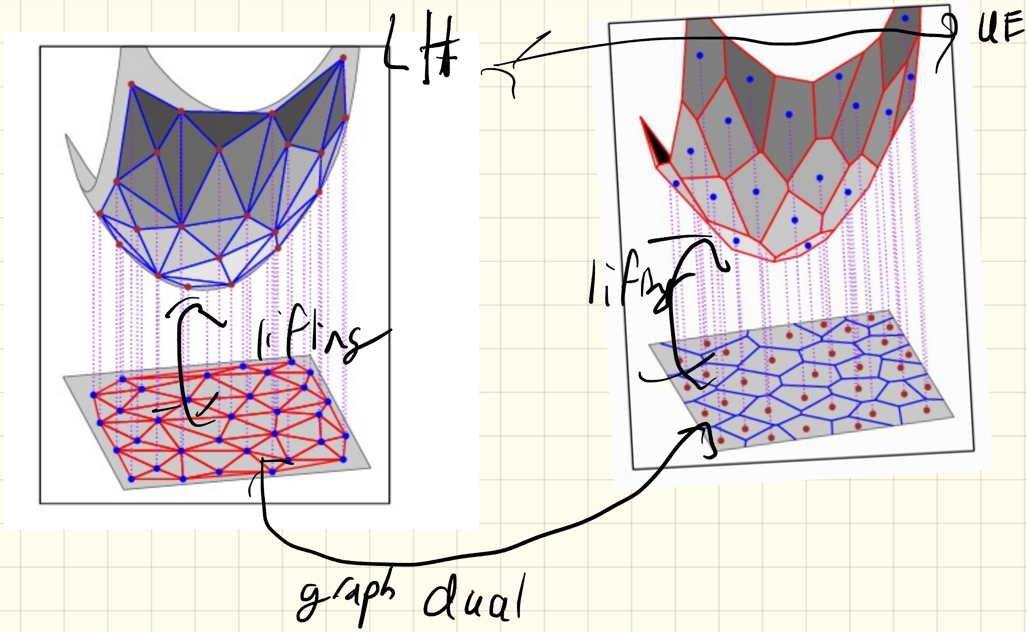


p_2 is closest
p_3 is 2^{nd} closest

p_1 is closest
p_2 is 2^{nd} closest

p_3 closest
p_4 2^{nd} closest

p_2 is closest
p_1 is 2^{nd} closest


LH
UE
lifting
lifting
graph dual