

# Homework 1

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**Problem 1** Let  $P = \{p_1, \dots, p_n\}$  and  $P' = \{p'_1, \dots, p'_n\}$  be the vertex sets of two upper hulls in the plane. Each set is presented as a sequence of points sorted from left to right. Let  $p_i = (x_i, y_i)$  and  $p'_j = (x'_j, y'_j)$  denote the point coordinates. We assume that  $P$  lies entirely to the left of  $P'$ , meaning that there exists a value  $z$  such that for all  $i$  and  $j$ ,  $x_i < z < x'_j$ .

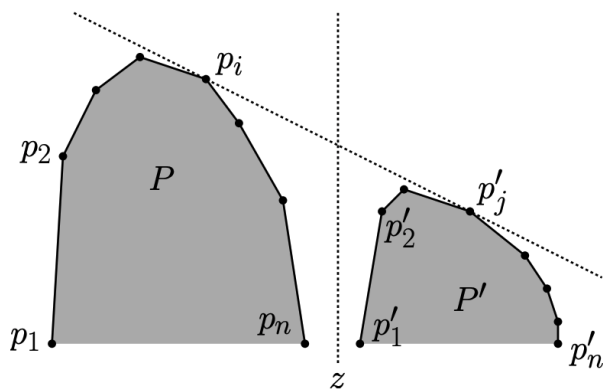


Figure 1: Problem 1: Computing the upper tangent of two hulls

Present an  $O(\log n)$ -time algorithm which, given  $P$  and  $P'$ , compute the two points  $p_i \in P$  and  $p'_j \in P'$  such that their common support line passes through these two points.

Briefly justify your algorithm's correctness and drive its running time. (**Hint:** The correctness proof involves a case analysis. Please be careful, a poorly drawn figure may lead to an incorrect hypothesis.)

Some sort of binary search.

Jointly search  $P$  and  $P'$  Pick  $a \in P$ ,  $b \in P'$  Want line with highest  $z$  intercept

Idea:  $O(n)$  run Graham's Scan on it. Already sorted so takes  $O(n)$  time.

Idea: Start at  $a$  compute point  $b \in P'$  tangent to  $a$  ( $O(\log n)$ ). Then reverse, find  $a$  tangent to  $b$ . Then forward, find  $b$  tangent to  $a$ . Then done????

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**Algorithm 1** Upper Tangent Function
 

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1: function PEAKFINDER( $a, P$ )
2:   Binary search to find maximum
3:    $low \leftarrow 0$ 
4:    $high \leftarrow |P|$ 
5:   while !found do
6:      $m \leftarrow \lceil (low + high)/2 \rceil$ 
7:      $s_1 \leftarrow$  slope of line between  $m - 1, m$ 
8:      $s_2 \leftarrow$  slope of line between  $m, m + 1$ 
9:     if  $s_1 > 0$  and  $s_2 < 0$  then
10:      found peak, slopes make a  $\wedge$  at point  $m$ 
11:      return  $m$ 
12:     else if  $s_1 > 0$  and  $s_2 > 0$  then
13:       Increasing on both sides, need to go more clockwise (right)
14:        $high \leftarrow m - 1$ 
15:     else if  $s_1 < 0$  and  $s_2 < 0$  then
16:       Decreasing on both sides, need to go more counter-clockwise (left)
17:        $low \leftarrow m + 1$ 
18:     end if
19:   end while
20: end function

1: function UPPERTANGENT( $P, P'$ )
2:   Run Tangent 3 times to find upper tangent.
3:    $a \leftarrow |P|/2$  // Random point in  $P$ 
4:    $b \leftarrow \text{Tangent}(a, P')$  // So we can 'see'  $p_i$  from  $b$ 
5:    $a \leftarrow \text{Tangent}(b, P)$  // Finds  $p_i$ 
6:    $b \leftarrow \text{Tangent}(a, P')$  // Finds  $p_j$ 
7:   return  $\overrightarrow{a, b}$ 
8: end function

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## Problem 2

Consider a set  $P = \{p_1, \dots, p_n\}$  of points in the plane, where  $p_i = (x_i, y_i)$ . A *Pareto set* for  $P$ , denoted  $\text{Pareto}(P)$ , (named after the Italian engineer and economist Vilfredo Pareto), is a subset of points  $p_i$  such that there is no  $p_j \in P (j \neq i)$  such that  $x_j \geq x_i$  and  $y_j \geq y_i$ . That is, each point of  $\text{Pareto}(P)$  has the property that there is no point of  $P$  that is both to the right and above it.

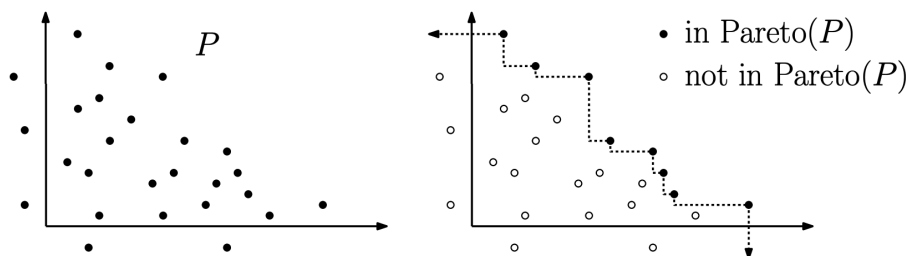


Figure 2: Problem 2: Pareto set

Pareto sets and convex hulls in the plane are similar in many respects. In this problem we will explore some of these connections.

1. (5 points) A point  $p$  lies on the convex hull of a set  $P$  if and only if there is a line passing through  $p$  such that all the points of  $P$  lie on one side of this line. Provide an analogous assertion for the points of  $\text{Pareto}(P)$  in terms of a different shape.
  2. (5 points) Devise an analogue of Graham's convex-hull algorithm for computing  $\text{Pareto}(P)$  in  $O(n \log n)$  time. Briefly justify your algorithm's correctness and derive its running time. (You do not need to explain the algorithm "from scratch", that is, you can explain with modifications would be made to Graham's algorithm.)
  3. (5 points) Devise an analogue of the Jarvis march algorithm for computing  $\text{Pareto}(P)$  in  $O(h \cdot n)$  time, where  $h$  is the cardinality of  $\text{Pareto}(P)$ . (As with the previous part, you can just explain the differences with Jarvis's algorithm.)
  4. (5 points) Devise an algorithm for computing  $\text{Pareto}(P)$  in  $O(n \log h)$  time, where  $h$  is the cardinality of  $\text{Pareto}(P)$ .
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1. The points on the  $\text{Pareto}(P)$  fall on a series of horizontal and vertical line segments. It follows a line in 'Manhattan' distance. In order to make the analogous assertion, we define corner  $C(p)$   $p \in P$  as the region  $(x, y) \in \mathbb{R}^2$   $x \leq p_x, y \leq p_y$ .  
Then a point  $p$  is on the Pareto of a set  $P$  if and only if there is no other corner  $C(p')$ ,  $p \neq p'$ , containing  $p$
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2. This is almost identical to Graham's Scan. We replace the Orient() function with a different comparison. We simply compare the  $y$  values of adjacent points. Since the points are in sorted,  $x$ , order if a point  $p_i$  has a larger  $y$ -coordinate than  $p_{i-1}$  the  $p_{i-1}$  would be in  $C(p_i)$  so is not on the Pareto. Since this comparison only needs the first point in the stack, we also adjust initialization of  $S$  to only push  $p_1$  onto the stack, and to not terminate the while loop unless there are 0 points in the stack.

This has the same running time as Graham's Scan. Since after sorting, the algorithm takes a linear pass through all of the points. It also only pops a point at most once from the stack. The only difference with this algorithm from Graham's Scan is the Orient. Since comparing  $y$ -coordinates is also a constant time lookup, our running time is  $O(n \log(n))$  due to the sorting in line 2.

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**Algorithm 2** Pareto Scan

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1: function PARETOSCAN( $P$ )
2:   sort  $P$  by increasing  $x$  value
3:   push  $p_1$  onto stack  $S$ 
4:   for  $i \leftarrow 2, \dots, n$  do
5:     while  $|S| \geq 1$  and  $p_{i,y} \geq S[top]_y$  do
6:       pop  $S$ 
7:     end while
8:     push  $p_i$  onto  $S$ 
9:   end for
10: end function

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**Correctness:** We show that algorithm 2 is correct by induction. A pareto front is at least one point, so initialization is trivially true.

We assume that the points on the stack are the pareto front for the first  $i - 1$  points in  $P$ . Then, we consider point  $p_i$ . We have two cases. If  $p_{i,y} < S[top]_y$  then, since points are in sorted  $x$  order,  $p$  is not in the corner of any of the points of  $S$ , and none of the points of  $S$  are in the corner of  $p$ . Thus,  $S \cup p$  is the pareto front for the first  $i$  points.

Second, if  $p_{i,y} \geq S[top]_y$ , then  $S[top]$  is in the corner of  $p$  (since in sorted  $x$  order). Then,  $S$  gets popped (all popped points meet the above condition) until  $|S| < 1$  or  $S[top]_y > p_{i,y}$ . Thus only points in the corner of  $p_i$  were popped, so are not on the pareto front. Now our stack either has size 1 from pushing  $p$  and is trivially a pareto front. Or it is a valid pareto front for the first  $i$  points since it now meets the first condition.

Thus at the end of the loop,  $S$  is a valid pareto front for the first  $n$  datapoints, and is thus a valid pareto.

3. Our modified Jarvis march algorithm would search for the point with the greatest  $y$  value. Then it would loop and search for the point with the next greatest  $y$ -coordinate to the right of the previous point ( $p_{i,x} > p_{i-1,x}$ ). This loop is repeated  $h - 1$  times (the first point was found in initialization step of finding point with largest  $y$ -coordinate).

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**Algorithm 3** Jarvis Stairs

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1: function JARVISSTAIRS( $P$ )
2:   find point  $p_1$  with largest  $y$ -coordinate
3:    $S \leftarrow p_1$ 
4:   for  $i \leftarrow 2, \dots, h$  do
5:     find point  $p_i$  with largest  $y$ -coordinate such that  $p_{i,x} > p_{i-1,x}$ 
6:   end for
7:   return  $S$ 
8: end function

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**Correctness:** The point with the maximum  $y$  value must be a part of the pareto. So at initialization we find the point. We then search for the next greatest point to the right of this. This point is also part of the pareto. We repeat this  $h$  times to find the  $h$  points with greatest  $y$  value to the right of the previous point.

4. We start by dividing  $P$  into  $n/h$  sets of size  $h$ . We run Pareto Scan (Algorithm 2) on each of the  $n/h$  sets. The runtime of this step is  $O(h \log(h))$  repeated  $n/h$  times:  $O(n \log(h))$ .

We can then merge these in  $O(n)$  time. We examine this in a 2D matrix. Each row is an individual pareto, and they are all at most  $h$  long. The algorithm explores this entire matrix, which is  $n/h \times h$ . So covering this entire matrix takes  $O(n/h \cdot h) = O(n)$  time.

We can also arrive at this an alternative way. Line 3 takes  $O(n \log(h))$ . Then the for loop is run  $h$  times. Line 7 and 8 both take  $O(n/h)$  time since they go through each mini-pareto once. Line 9 is constant time. So the for loop takes  $O(h \cdot n/h) = O(n)$ .

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**Algorithm 4** Chan Pareto

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1: function CHANPARETO( $P$ )
2:   divide  $P$  into  $P_1, P_2, \dots, P_{n/h}$  where  $|P_i| = h$ 
3:   Solve each  $P_i$  using Algorithm 2 and store paretos in  $S_i$ 
4:    $index \leftarrow$  list of  $n/h$  ones.
5:   initialize stack  $S$ 
6:   for  $1, \dots, h$  do
7:     find point  $p$  with largest  $y$  coordinate along the indicies ( $S_i[index[i]]$ )
8:     Go through paretos along indicies and increment index if point  $S_i[index[i]]$  is in corner of  $p$ 
9:     push  $p$  onto  $S$ 
10:  end for
11:  return  $S$ 
12: end function

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**Correctness:** We know that each mini-pareto is correct since Algorithm 2 is correct. So we need to show that the merge operation is correct. We know that points on the pareto are part of the mini paretos. By induction, we start with an empty set which is vacuously true. Then we assume that we have the first  $i - 1$  points of the pareto. And that  $index[i]$  points to the largest uncovered point in pareto  $S_i$ .

We show that this holds after the  $i^{th}$  iteration. To do this, we show that the point  $p$  with the largest  $y$  coordinate is part of the pareto. By our choice of  $p$ , it is certainly not covered by any of the points on  $S_i[index[i]]$ . Thus, we show that it is also not covered by any previous point, or future point. By the inductive assumption,  $p$  is not covered by any of the points in  $S$ . Then, suppose that  $p$  is covered by a future point  $p_j$ , that future point would also cover  $S_j[index[j]]$ . Then  $S_j$  would not be a valid pareto, a contradiction. So  $p$  belongs on the pareto.

### Problem 3

Assume you have an orientation test available which can determine in constant time whether three points make a left turn (i.e., the third point lies on the left of the oriented line described by the first two points) or a right turn. Now, let a point  $q$  and a convex polygon  $P = \{p_1, \dots, p_n\}$  in the plane be given, where the points of  $P$  are stored in an array in counter-clockwise order around  $P$  and  $q$  is outside of  $P$ . Give pseudo-code to determine the tangents from  $q$  to  $P$  in  $O(\log n)$  time.

We say  $\text{Orient}(a, b, c) > 0$  if it is oriented counter clockwise (makes left turn), and  $\text{Orient}(a, b, c) < 0$  if it is oriented clockwise (right turn)

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#### Algorithm 5 Tangent Function

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1: function TANGENT( $a, P$ )
2:   Binary search to find point of tangency
3:    $low \leftarrow 1$ 
4:    $high \leftarrow |P|$ 
5:    $p_1, p_2$ 
6:   while !found do
7:      $c \leftarrow \lceil (low + high)/2 \rceil$ 
8:      $b \leftarrow c - 1, d \leftarrow c + 1$    b is point counter-clockwise from a, d is point clockwise from a
9:     if  $\text{Orient}(a, c, b) = \text{Orient}(a, c, d)$  then
10:      found tangent
11:       $p_1 \leftarrow m$ 
12:      break while
13:     else if orientations at  $c, low, high$ , all match then
14:       this catches if both tangents are on one side of c,
15:       so we need to go around so we don't get stuck on low, or high
16:       move in opposite direction of  $\text{Orient}(a, c, low)$  and adjust  $low, high$  accordingly
17:       (same as lower rules)
18:     else if  $\text{Orient}(a, c, b) > 0$  and  $\text{Orient}(a, c, d) < 0$  then
19:       Need to move search point left (counter-clockwise)
20:        $high \leftarrow b$ 
21:     else if  $\text{Orient}(a, c, b) < 0$  and  $\text{Orient}(a, c, d) > 0$  then
22:       need to move search point right (clockwise)
23:        $low \leftarrow d$ 
24:     end if
25:   end while
26:   Repeat same while loop but flipping inequality signs to find other tangency point.
27:   return  $p_1, p_2$ 
28: end function

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This follows a binary search. Every operation within the while loop takes  $O(1)$  time. Then, the same as regular binary search, the algorithm jumps  $(high - low)/2$  every iteration, so it covers the entire list in  $O(\log(n))$  time.



**Correctness:** We use Figure 3 to justify the rules in the else if statements lines 18-23. This shows the orientation values when we need to move clockwise and counter-clockwise.

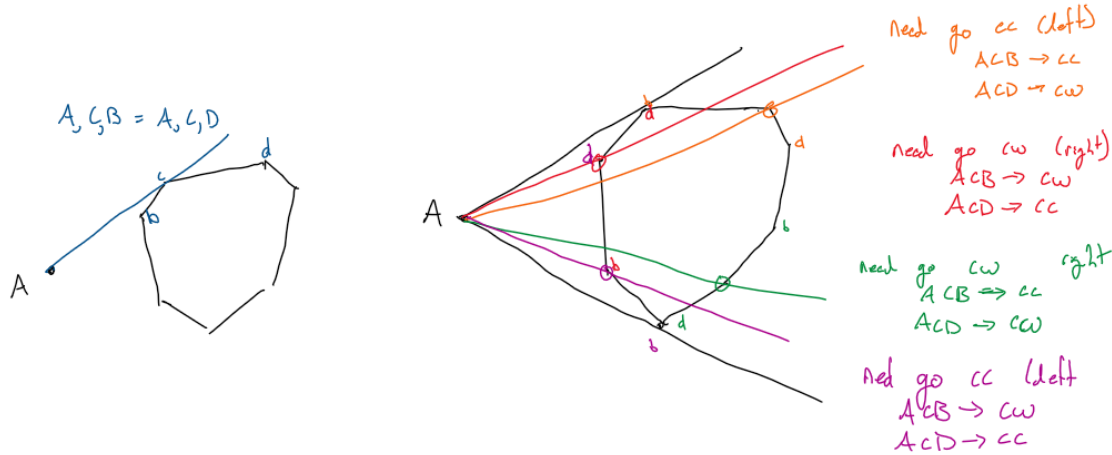


Figure 3: Orientation Rules Justification and example

The else if statement line 12-17 is used to catch the edge case where both tangents are on one side of our center line. This means that if we listen to the normal rules in the other else if statements, we may get caught and unable to pass our high or low marker. This case occurs when all 3 points are on one 'face', so we know that we will get trapped moving towards them. So we move in the opposite direction. This will eventually put c in between the two tangents and the algorithm will work normally.

This statement can be avoided if you can find better starting and ending points such that this does not occur. I do not know how to pick these points in  $O(\log(n))$  time, but supposing we had a method that would be better.

### Problem 4

Given a set  $S$  of  $n$  points in the plane, consider the subsets

$$\begin{aligned} S_1 &= S, \\ S_2 &= S_1 \setminus \{\text{set of vertices of } \text{conv}(S_1)\} \\ &\dots \\ S_i &= S_{i-1} \setminus \{\text{set of vertices of } \text{conv}(S_{i-1})\} \end{aligned}$$

until  $S_k$  has at most three elements. Give an  $O(n^2)$  time algorithm that computes all convex hull  $\text{conv}(S_1), \text{conv}(S_2), \dots, S_k$ . [Extra credit, provide an algorithm that is faster than  $O(n^2)$ ].

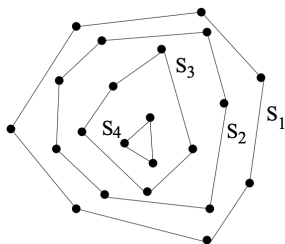


Figure 4: Problem 4: Onion peeling

We essentially do Graham's Scan, but instead of popping and removing them, we move the popped hull sections down one layer.

So for the runtime, the sort operation takes  $O(n \log(n))$ . Then at worst case, every single point gets sifted down every shell. There are at most  $O(n)$  shells. So it takes  $O(n^2)$  time.

**Correctness:** We prove this inductively on each shell. So considering just the outer shell, we have the exact same as Graham's Scan algorithm, so it is correct. The inner hulls are also correct since Graham's Scan is run on each of them.

Then for each of the lower shells, we take a convex shell segment and add it to the lower shell. We show that the shell segment gets added in its entirety.

Suppose not, suppose the entire shell segment  $S'$  should not be added. We say the first point in the shell segment is  $p$ . And the shell we consider is  $S$ . Suppose only the first portion of the shell segment can be added without breaking convexity. Then the remaining portion can be added without breaking convexity since the entire shell segment was convex.

Then suppose that some point that was popped, should be added partially through the segment. This is a contradiction since we are building the upper hull and the points are in sorted order. So any popped point has  $x < p_x$ . If we add the popped point back, the sorted  $x$  ordering is broken a contradiction.

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**Algorithm 6** Onion Problem
 

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1: function RECURR( $p', S$ )
2:   popped = []
3:   while  $|S| \leq 2$  and  $\text{Orient}(p', S[\text{top}], S[\text{top} - 1]) < 0$  do
4:     popped.append( $S.\text{pop}()$ )
5:   end while
6:   return popped
7: end function

1: function ONIONS( $P$ )
2:   sort  $P$  by increasing  $x$ 
3:   push  $p_1, p_2$  onto stack  $S_0$ 
4:   for  $i \leftarrow 3, \dots, n$  do
5:     Variable initialization to clean things up
6:      $S \leftarrow S_0$ 
7:      $\text{add\_to\_next} \leftarrow [p_i]$    mark  $p_i$  to be added to S
8:      $\text{popped} \leftarrow \text{recurr}(p_i, S)$    get points removed from S
9:     while  $\text{popped}$  is not empty do
10:       $S.\text{push}(\text{add\_to\_next})$    add the points to this shell
11:       $\text{add\_to\_next} \leftarrow \text{popped}$ 
12:       $p' \leftarrow \text{popped}[\text{top}]$ 
13:       $S \leftarrow$  next layer down stack
14:       $\text{popped} \leftarrow \text{recurr}(p', S)$ 
15:     end while
16:      $S.\text{push}(\text{add\_to\_next})$    Add to last layer
17:   end for
18:   repeat to make the lower hull.
19: end function

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