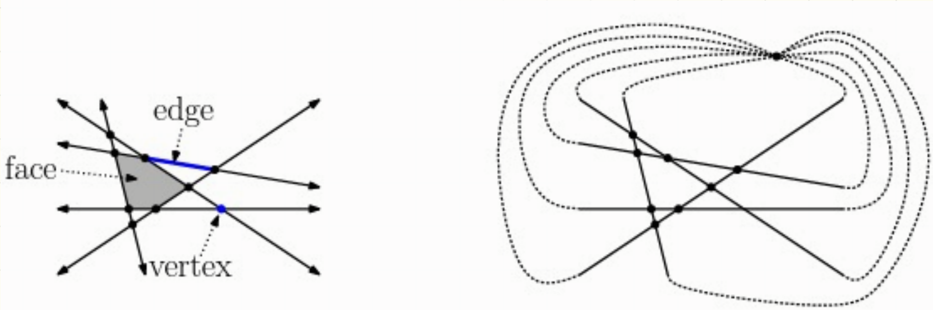
Arrangements

Given a set of n lines in the plane L
divide plane into cellular complex call arrangement $A(L)$



Combinatorial props:
Simple: no 3 lines intersecting at the same point
non-deg: simple & no parallel lines

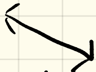
What is the combinatorial complexity?

Lemma: Let $A(L)$ be arrangement of n lines in plane:
(i) # of vert (not counting inf vertex) in $A(L)$ is $\binom{n}{2}$
(ii) # of edges in $A(L)$ is n^2
(iii) # of faces in $A(L)$ $\binom{n}{2}+n+1$

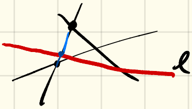
Pf (i) n non-parallel lines ⇒ each line intersects all other lines
(by assumption arrangement is "simple") ⇒ 2 lines per intersection ⇒ $\binom{n}{2}$

(ii) induction

$\underline{n=1}$

1 line, 1 unbounded edge 

assume $(n-1)$ lines has $(n-1)^2$ edges



add line ℓ

$(n-1)$ lines intersect ℓ

chop edges in n new edges

$(n-1)$ edges that split into 2 edges

$$(n-1)^2+n+n-1 = n^2-2n+1+2n-1 = n^2 \checkmark$$

(iii) $v-e+f=2$

$v=\binom{n}{2}+1 \leftarrow$ inf vertex

$e=n^2$

... algebra

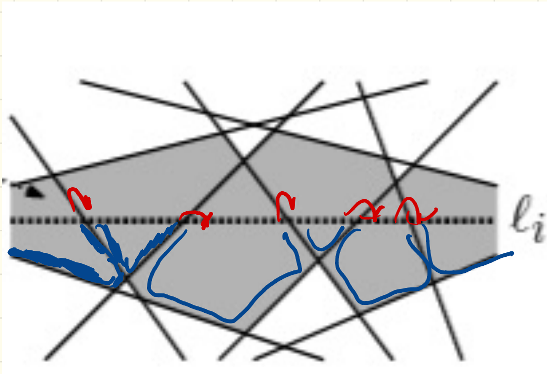
Incremental construction

$L = \{\ell_1, \dots, \ell_n\}$ set of lines

* want to insert ℓ_i so that it takes $O(i)$ time to insert $\Rightarrow O(n^2)$

Suppose $\{\ell_1, \dots, \ell_{i-1}\}$ are inserted

Consider ℓ_i



start at $x = \infty$

(use fact that lines are sorted by slope at $x = \infty$)

- find "entry" $O(i)$
- find $i-1$ edges intersected by ℓ via "walking"

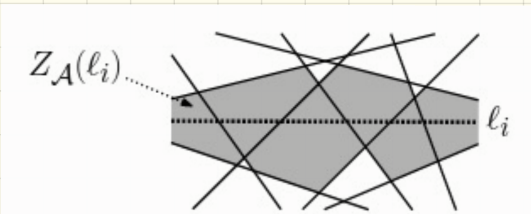
Analysis of walk:

- intersect $i-1$ lines
- each face is $O(i)$

$\Rightarrow O(i^2)$ to intersect i^{th} line

$\Rightarrow$ (Problem) $O(n^3)$

Claim: traversal only takes $O(i)$ time



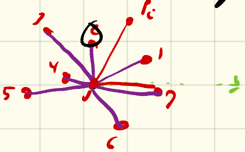
$Z_A(\ell)$ is the zone of a line in an arrangement the set of faces intersected by ℓ

Zone theorem: Given $A(L)$ of n lines in the plane
# of edges in all cells of $Z_A(\ell)$ is at most $6n$

Pf induction proof

Incremental construction
$L = \{\ell_1, \ldots, \ell_n\}$ set of lines
* want to insert ℓ_i so that it takes $O(i)$ time to insert $\Rightarrow O(n^2)$
$\Rightarrow$
Inc alg for computing arrangement of n lines is $O(n^2)$

Example of arrangement in action

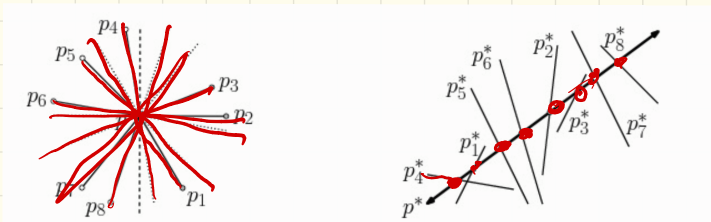


Fact 1: radially sort all point around a point $O(n\lg n)$

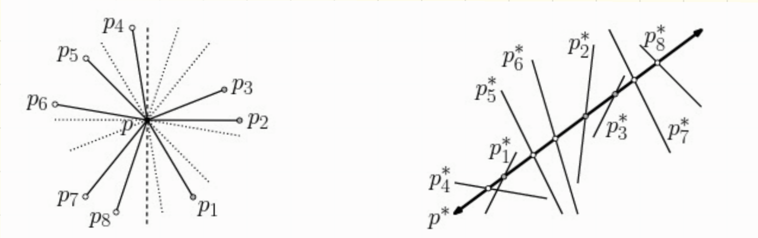
Radially sorting: give a pointset P in the plane
for each p_i radially sort $P \setminus \{p_i\}$ around p_i

Ul: for each point $p_i \in P$
radially sor $P \setminus \{p_i\}$ around p_i
$\Rightarrow$ time $O(n^2 \lg n)$

Recall $p = (a, b) \rightarrow p^* := y = ax - b$ consider p^* as a line and intersections of dual lines along p^*



- each vertex along p^* is a line ℓ in primal
- order of intersection along p^* lines order by slope
- 2 option for pt in radial ordering $p_i.x \lesseqgtr p.x$



$\Rightarrow$ we get all radial orderings along all points in $O(n^2)$ time