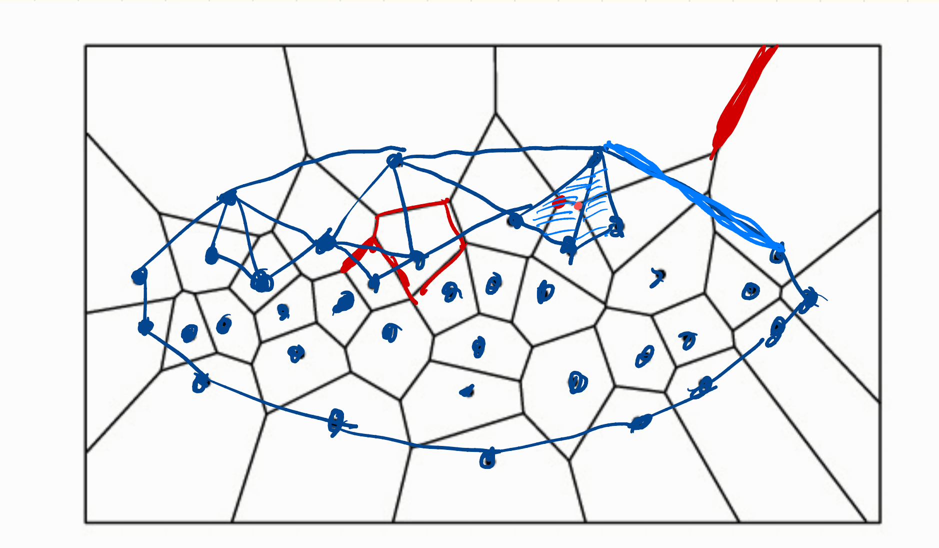
Delaunay △tion (Props)

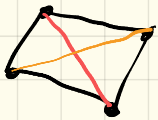
Delaunay as dual of Voronoi:

- for each face of the voronoi diagram create a vertex (located at input site)
- for each edge of the voronoi between sites p_i, p_j add edge connecting the "dual verts"
- for each vertex of the voronoi add a face in the dual



(Gp assumption no more than 3 sites are cocircular)
⇒ Voronoi verts are deg 3
⇒ any face in the dual is a triangle

Result is the Delaunay △tion



Props:

- convex hull: boundary of exterior faces of Voroin diagram
  is convex hull of pointset
- Circumcircle prop: Circumcircle of any Δ in the Delaunay Δtion is
  empty of sites from P

  Pf

  circle's center is a voronoi vertex

  $\Rightarrow$ 3 sites defining the vertex

  & 3 sites are nearest neighbor
- Empty circle prop:

  2 sites p_i, p_j connected by an edge in Delaunay

  $\Leftrightarrow$ empty circle through p_i and p_j

  pf (sketch)

  p_i, p_j are in Delaunay

  $\Rightarrow$ cells are neighbors in $Vor(P)$

  $\Rightarrow$ any point on the edge in Vor
  is the center of a circle passing through p_i and p_j
  and empty of points in P

- Closest pair

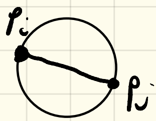
  suppose p_i and p_j are the closest pair in P

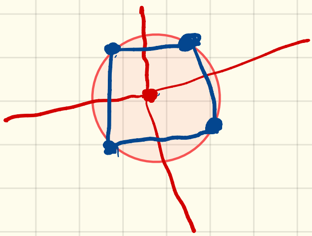
  $\Rightarrow$ circle w/ p_i and p_j w/ $p_i p_j$ as diameter

  $\Rightarrow$ circle cannot contain any other sites from P

  $\Rightarrow$ center of the circle is on a voronoi edge

  $\Rightarrow$ p_i and p_j are neighbors in Delaunay





Size

Given n points in $\mathbb{R}^2$ P, h verts on the convex hull

Del (P) has:

- n verts
- $2n-2-h$ Δs
- $3n-3-h$ edges

use eulers formula $v-e+f=2$ (for plar graph)

Higher dim:

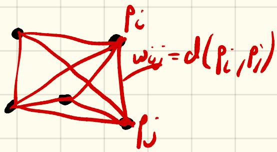
3d tetrahedra size can be from $O(n)$ to $O(n^2)$

in dim d # of simp can $O(n^{\lceil \frac{d}{2} \rceil})$

Min Spanning tree (MST)

Given n points in the plane

Euclidean graph is a graph w/ V (vertex set) are our points
E (edge set) are all $\binom{n}{2}$ pairs of points
w/ weight as the Eucideen dist between p_i p_j (for edge (p_i, p_j))



MST: is a set of $n-1$ edges connecting verts w/ min total weight

Version 1: Kruskal's algo

n^2 edges

$\Rightarrow O(n^2 \lg n)$ time (for edge sort)

In the plane we can do better

- Compute Delanay (via Fortune and take dual) - $O(n \lg n)$ time
- Run Krurka's algo on edges of Delauney - $O(n \lg n)$

Theorem MST $\subseteq$ DT (true in any dim)

Pf Let T MST of P

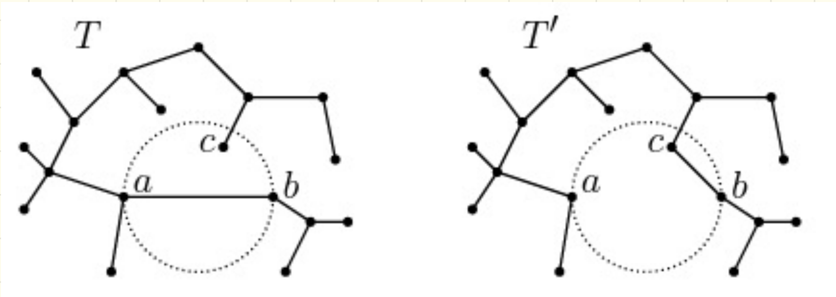
let $w(T)$ be the weight of T

pf by contradiction

Suppose an edge ab that is in T but not DT

$\Rightarrow$ no empty circle through ab

$\Rightarrow$ point c in the diametric circle of ab



- remove $\overline{ab}$ from T split tree into 2 trees
- wolg assume that c is in the tree w/ a
- remove $\overline{ab}$ from MST and add $\overline{bc}$
- result is a spanning tree T' w/ weight

$$w(T') = w(T) + \boxed{\delta(b,c) - \delta(a,b)} < w(T)$$

$\Rightarrow \not\subseteq$ T was MST

because c was in diametric circle of ab $\delta(b,c) < \delta(a,b)$

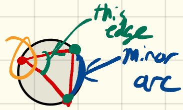
Max angle & flipping

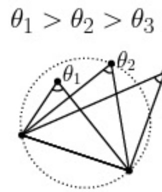
Delaunay maximizes the min angle over all Δtions

Def angle seq $(\alpha_1, \ldots, \alpha_m)$
as increasing seq of angles in a Δtion

Th among all Δtions of a planar point set
Del Δtion has the lexographically largest $\angle$ seq
$\Rightarrow$ max min $\angle$

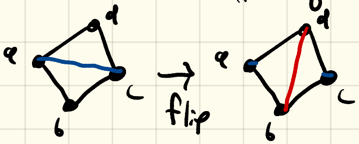
Some geometric facts
Consider circumcircle of 3 points
for an angle of Δ
consider angle subtended by point
w/ other two verts w/ minor arc



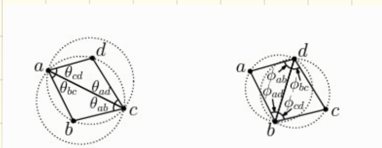


(a)

Pf Δ fails the empty circle prop
we can perform an edge flip
and increase lex seq or $\angle$s



Suppose Δ pair violates the empty circle prop
$\Rightarrow$ for Δabc $\exists$ d in the circumcirle of Δabc
$\Rightarrow$ b in the circumcircle of Δcda
If we flip the resulting Δ are Δabd and Δbcd
and new Δs are empty



Observe

$\phi_{ab} > \theta_{ab}$ $\qquad$ $\phi_{cd} > \theta_{cd}$

$\phi_{bc} > \theta_{bc}$ $\qquad$ $\phi_{ad} > \theta_{ad}$

Only a finite # of Δtions
process terminate w/ lex max Δtion
& Δtion satisfys empty circle prop $\Rightarrow$ Del Δtion