1 Dynamics

Generally: $\dot{x} = f(x,u), y = h(x,u)$, $x \in \mathbb{R}^n, u \in \mathbb{R}^m$.
Jacobian Linearization: given $u = u_0, x = x_0$ is a stationary point ($f(x_0,u_0) = 0$), then $\dot{x} \approx J_{f,x}x + J_{f,u}u, y \approx J_{h,x}x + J_{h,u}u$

2 Linear Algebra

Linear dependence $\exists \alpha_1, \dots \alpha_m$ not all zero, $\alpha_1 x_1 + \dots + \alpha_m x_m = 0$.
Simlarity: B is similar to A if $\exists Q, st. B = QAQ^{-1}$. Columns of Q are eigenvectors, B is diagonal of eigenvalues.
Generalized eigenvectors and Jordan: if not n linearly independent eigenvectors; solve $Av_1 = v_2$, until chain ends.
$rank(A) = \#$ of independent columns = size of largest-square submatrix with nonzero det. $rank(A) + nullity(A) = \#$ of columns.
Positive definite if any of

1. Every eigenvalue of M is positive (non-negative)
2. All leading principal minors of M are positive (All principal minors are non-negative)
3. There exists a non-singular matrix (matrix) $N \in \mathbb{R}^{m \times n}$ such that $M = N^T N$

Singular Value Decomposition (SVD) is a matrix factorization technique that decomposes a matrix into three matrices: $A = U\Sigma V^T$ where A is an $m \times n$ matrix, U is an $m \times m$ orthogonal matrix, Σ is an $m \times n$ diagonal matrix with non-negative real numbers on the diagonal (known as singular values), and V is an $n \times n$ orthogonal matrix. $A^T A$ has eigenvalues σ_i^2 where σ_i are singular values of A. $A^T A$ has eigenvectors v_i where v_i are right singular vectors of A. AA^T has eigenvalues σ_i^2 where σ_i are singular values of A. AA^T has eigenvectors u_i where u_i are left singular vectors of A. $A = \sum_{i=1}^r \sigma_i u_i v_i^T$ where $r = rank(A)$. $AV = U\Sigma \implies Av_1 = \sigma_1 u_1$

Cayley-Hamilton: let $\Delta(\lambda) = det(A - \lambda I)$, then $\Delta(A) = 0$.
For any $f(A)$: $f(A) = \beta_0 I + \dots + \beta_{n-1}A^{n-1}$. If f polynomial, let $h(\lambda) = \beta_0 + \beta_1\lambda + \dots + \beta_{n-1}\lambda^{n-1}$, solved from $f^{(l)}(\lambda_i) = h^{(l)}(\lambda_i)$.

Matrix exponential: $e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k$
$e^{A(t_1+t_2)} = e^{At_1}e^{At_2}$, $(e^{At})^{-1} = e^{-At}$, $\frac{d}{dt}e^{At} = e^{At}A = Ae^{At}$, $e^{At} = \mathcal{L}^{-1}((sI - A)^{-1})$

Compute A^{100} with $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$

$\Rightarrow f(\lambda) = \lambda^{100}$

$\Delta(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2 \Rightarrow \lambda_1 = -1,\ n_1 = 2$

Let $h(\lambda) = \beta_0 + \beta_1\lambda$

$$\Rightarrow \begin{cases} f(-1) = h(-1): & (-1)^{100} = \beta_0 - \beta_1 \\ f'(-1) = h'(-1): & 100 \times (-1)^{99} = \beta_1 \end{cases} \Rightarrow \begin{cases} 1 = \beta_0 - \beta_1 \\ -100 = \beta_1 \end{cases}$$

$$\Rightarrow \begin{cases} \beta_0 = -99 \\ \beta_1 = -100 \end{cases} \Rightarrow h(\lambda) = -99 - 100\lambda$$

$$\Rightarrow A^{100} = h(A) = -99I - 100A = \begin{pmatrix} -199 & -100 \\ 100 & 101 \end{pmatrix}$$

3 Dynamic Behavior

Lipschitz condition: $||F(x) - F(y)|| \leq L||x - y||$, $x, y \in \{\mathbb{R}^n : ||x - x_0|| \leq r\}, L, r > 0$ Globally Lipschitz implies $||F(x)|| \leq L||x||$

Poincare-Bendixon Criterion: Let M be a closed and bounded set such that it contains no equilibrium points, or the Jacobion has eigenvalues with positive real parts. Every trajectory starting in M stays in M, then it contains a periodic orbit (show $\dot{boundary} \leq 0$).

Lyaponov Stability: $V : D \to \mathbb{R}$ such that $V(x) \succ$

0, $x \in D \setminus \{0\}$, and $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \preceq 0$(stable) $\prec 0$(asymtotically). Require $\lim_{||x|| \to \infty} V(x) = \infty$ for global.

LaSalle's theorem: Let M be a closed and bounded set, and let $V : M \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \preceq 0$ for all $x \in M$. Let $S = \{x \in M : \dot{V}(x) = 0\}$, then if no solution can stay identically in S (other than trivial soln), the equilibrium is asymptotically stable.

Given $\begin{cases} x(t_0) \\ u(t), t \geq t_0 \end{cases} \to y(t)$, superposition: $\begin{cases} \alpha x_1(t_0) + \beta x_2(t_0) \\ \alpha u_1(t) + \beta u_2(t), t \geq t_0 \end{cases} \to \alpha y_1(t) + \beta y_2$ Discritization $x(k+1) \approx e^{AT} x(k) + (\int_{\sigma=0}^{T} e^{A\sigma} d\sigma) B u(k)$. If A is non-singular: $B_d = A^{-1}(A_d - I)B$.

Globally asymptotically stable iff A has all eigenvalues with negative real part. BIBO stable same condition, can also linearize and same condition.

Realization (controller): $G(x) = c + \frac{b_{n-1}s^{n-1} + \ldots b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}$, $\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \ldots & -a_{n-1} \end{bmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ c \end{pmatrix}$, $y = \begin{pmatrix} b_0 & b_1 & \ldots & b_{n-1} \end{pmatrix} x$

4 Laplace Transform

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t) e^{-st} dt$$

$Y(z) = Z(y(k)) = \sum_{k=0}^{\infty} y(k) z^{-k}$

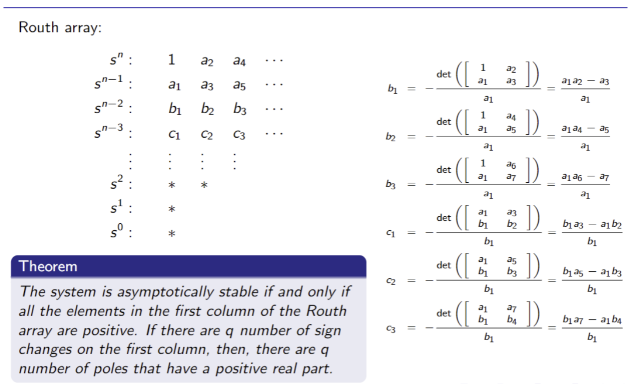
Unit step: $\mathcal{L}[1] = 1/s$
Unit ramp: $\mathcal{L}[t] = 1/s^2$
Power function: $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$
Exponential: $\mathcal{L}[e^{-\alpha t}] = \frac{1}{s+\alpha}$
Sine: $\mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$
Cosine: $\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2}$

Linearity: $\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 \mathcal{L}[f_1(t)] + a_2 \mathcal{L}[f_2(t)]$
Differentiation: $\mathcal{L}[\frac{d}{dt} f(t)] = sF(s) - f(0)$, or in general $\mathcal{L}[\frac{d^n}{dt^n} f(t)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - s f^{(n-2)}(0) - f^{(n-1)}(0)$
Integration: $\mathcal{L}(\int f(t) dt) = \frac{F(s)}{s} + \frac{\int f(t)dt\big|_{t=0}}{s}$
Time shift: $\mathcal{L}[f(t-\alpha)] = e^{-\alpha s} F(s)$
Frequency shift: $\mathcal{L}[e^{-\alpha t} f(t)] = F(s+\alpha)$
Time scale: $\mathcal{L}[f(t/\alpha)] = \alpha F(\alpha s)$
Multiplication by time: $\mathcal{L}[t f(t)] = -\frac{d}{ds} F(s)$

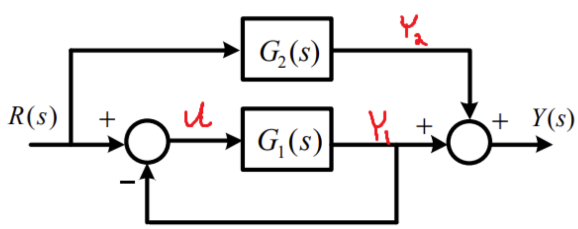
5 Frequency Domain

First order: $G(s) = \frac{\sigma}{s+\sigma} = \frac{1}{\frac{s}{\sigma}+1}$, $1/\sigma$: time constant
Second order: $G(s) = \frac{\omega_n^2}{s^2 + 2\sigma\epsilon\omega_n s + \omega_n^2}$, ϵ : damping ratio, $\omega_d = \omega_n \sqrt{1-\epsilon^2}$: damped frequency, ω_n: natural frequency, $t_r \approx \frac{1.8}{\omega_n}$: rise time, $t_s \approx \frac{4.6}{\omega_d}$: settling time, $M_p = e^{-\pi\epsilon/\sqrt{1-\epsilon^2}}$: overshoot, $t_p = \pi/\omega_d$: peak time

$T(s) = \frac{b(s)}{a(s)} = \frac{b_0 s^m + \cdots + b_m}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n}$, system is asymptotically stable iff $a(s)$ has all roots in LHP. The Routh array is. If zero in row, but remaining elements are non-zero. Replace with $\epsilon > 0$ and limit $\epsilon \to 0$. If entire row s^i is zero, and s^{i+1} has coeff $\alpha_1, \alpha_2, \ldots$, define aux $a^i = \alpha_1 s^{i=1} + \alpha_2 s^{i-1} + \alpha_3 s^{i-3} + \ldots$, take its derivative and use coefficients to fill in row.



$Y(s) = Y_1(s) + Y_2(s) = G_1(s)U(s) + G_2(s)R(s)$, $U(s) = R(s) - G_1(s)U(s) \implies (1 + G_1(s))U(s) = R(s) \implies U(s) = \frac{R(s)}{1+G_1(s)}$
So, $Y(s) = \left[\frac{G_1(s)}{1+G_1(s)} + G_2(s)\right] R(s)$



Let $L(s) = \frac{L_0(s)}{s^n}$ then it is type n system. Type 0 tracks $r(t) = 1(t) \to \frac{1}{1+K_p}$, Type 1 tracks $r(t) = t1(t) \to \frac{1}{K_v}$, and Type 2 tracks $r(t) = \frac{t^2}{2}1(t) \to \frac{1}{K_a}$.

PID is $u(t) = k_p(e + \frac{1}{T_1}\int_0^t e(\tau)d\tau + T_D\frac{de}{dt}) \to C(s) = U(s)/E(s) = k_p + \frac{k_p}{T_I s} + k_pT_Ds$.

Nyquist: Contour of $L(s)$ as s traverses the RHP. Circle -1 is bad. gain margin g_m the smalest factor L can be increased before circling -1. Phase margin ϕ_m the largest phase shift L can have before circling -1. Stability margin s_m, shortest distance from the Nyquist plot to -1.

Bode: plot $\lg\omega$ vs $20\lg|G(j\omega)|$. ω_{pc} is where phase cross 180, ω_{gc} is where gain cross 0. $g_m = 1/|G(j\omega_{gc})|$, $\phi_m = 180 + \angle G(j\omega_{pc})$.

Lead compensator: $C(s) = \frac{Ts+1}{\alpha Ts+1}$, $\phi_{max} = \arcsin\frac{1-\alpha}{1+\alpha}$, $\omega_{max} = \frac{1}{T\sqrt{\alpha}}$. Choose $\omega_{max} = \omega_{gc}$, and α so $\phi_{max} \le 60°$

6 Time Domain

controllability: (A, B) is controllable if $C = \begin{bmatrix} B & AB & \dots & A^{n-1}B \end{bmatrix}$ has full row rank.
Observability: (A, C) is observable if $O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ has full column rank.

Pole placement: Compute C, and desired $\bar{C}$ (controller realization, negative of desired pole in last row). $T = \bar{C}C^{-1}$. Compute desired characteristic polynomial: $\Delta(s) = s^n + \bar{a_n}s^{n-1} + \cdots + \bar{a_1}$, then $\bar{F} = \begin{bmatrix} \bar{a_1} - a_1 & \dots & \bar{a_n} - a_n \end{bmatrix}$ (from A), and finally $F = \bar{F}T$

LQR: $J = \int_0^\infty (x^TQx + u^TRu)dt$, $Q \succ 0, R \succ 0$. $u = -R^{-1}B^TPx$, P comes from Riccati: $A^TP + PA + Q - PBR^{-1}B^TP = 0$

7 Other

Capacitor - $i = C\frac{dv}{dt}$ (state=Voltage), Inductor: $v = L\frac{di}{dt}$ (state=Current), Friction $F = -kv$

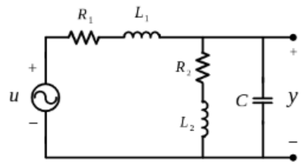


Figure: Problem 2.6

Let x_1 be the current through L_1 (from left to right), x_2 be the current through L_2 (from top down), and x_3 be the voltage across capacitor C (the same as y). By the Kirchhoff current and voltage laws, we have

$$\begin{cases} L_1\dot{x}_1 + R_1x_1 + x_3 = u, \\ L_2\dot{x}_2 + R_2x_2 = x_3, \\ C\dot{x}_3 + x_2 = x_1, \\ y = x_3, \end{cases}$$

from which we solve for $\dot{x}_1, \dot{x}_2, \dot{x}_3$ and obtain the state space model,

$$\begin{cases} \dot{x}_1 = -\frac{R_1}{L_1}x_1 - \frac{1}{L_1}x_3 + \frac{1}{L_1}u, \\ \dot{x}_2 = -\frac{R_2}{L_2}x_2 + \frac{1}{L_2}x_3, \\ \dot{x}_3 = \frac{1}{C}x_1 - \frac{1}{C}x_2, \\ y = x_3, \end{cases}$$

which can be written in the compact form

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx, \end{cases}$$

with

$$A = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix}, B = \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$