Homework 6

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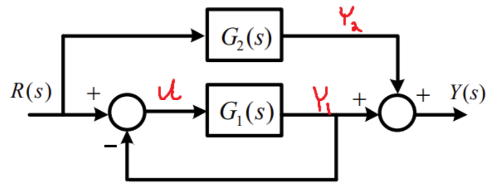
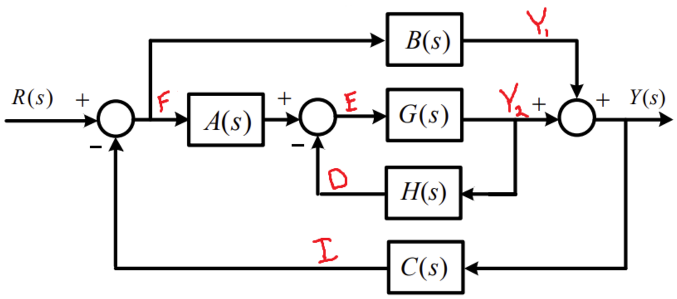
Problem 1

Statement:

Find the transfer function for the block diagrams:



(a)



(b)

Figure 1: Problem 1

Solution

a) $Y(s) = Y_1(s) + Y_2(s) = G_1(s)U(s) + G_2(s)R(s)$,
$U(s) = R(s) - G_1(s)U(s) \implies (1 + G_1(s))U(s) = R(s) \implies U(s) = \frac{R(s)}{1+G_1(s)}$
So, $Y(s) = \left[\frac{G_1(s)}{1+G_1(s)} + G_2(s)\right] R(s)$

b) So we are going to start in the inside and go outside.

$$Y_2(s) = G(s)E(s)$$
$$E(s) = A(s)F(s) - H(s)Y_2(s)$$
$$Y_2(s) = G(s)A(s)F(s) - G(s)H(s)Y_2(s)$$
$$Y_2(s) = \frac{G(s)A(s)F(s)}{1 + G(s)H(s)}$$

Now this gives us the middle component. Now we can use this to find the outer component. $Y_1(s) = B(s)F(s)$ is easy. And our last equation: $F(s) = R(s) - C(s)Y(s)$ will bring everything together. We know

$$
\begin{aligned}
Y(s) &= Y_1(s) + Y_2(s) \\
&= B(s)F(s) + \frac{G(s)A(s)F(s)}{1 + G(s)H(s)} \\
&= \left[B(s) + \frac{G(s)A(s)}{1 + G(s)H(s)}\right] F(s) \\
&= \left[B(s) + \frac{G(s)A(s)}{1 + G(s)H(s)}\right] [R(s) - C(s)Y(s)] \\
Y(s) &= \left[B(s) + \frac{G(s)A(s)}{1 + G(s)H(s)}\right] R(s) - \\
&\quad \left[B(s)C(s) + \frac{G(s)A(s)C(s)}{1 + G(s)H(s)}\right] Y(s) \\
Y(s)\left[1 + B(s)C(s) + \frac{G(s)A(s)C(s)}{1 + G(s)H(s)}\right] &= \left[B(s) + \frac{G(s)A(s)}{1 + G(s)H(s)}\right] R(s) \\
Y(s) &= \frac{\left[B(s) + \frac{G(s)A(s)}{1+G(s)H(s)}\right]}{\left[1 + B(s)C(s) + \frac{G(s)A(s)C(s)}{1+G(s)H(s)}\right]} R(s)
\end{aligned}
$$

Problem 2
Statement:
Determine the time constant of the system:

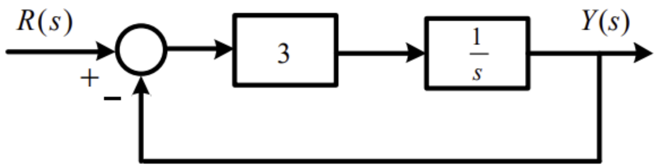


Figure 2: Problem 2

Solution

First we need to write the transfer function for the system. We can do this by using the block diagram. $Y(s) = \frac{3}{s}U(s)$, and $U(s) = R(s) - Y(s)$, so
$Y(s) = \frac{3}{s}\left[R(s) - Y(s)\right] \implies Y(s) = \frac{1}{1+3/s}R(s) = \frac{s}{s+3}R(s)$. We want the response to a unit step, so $R(s) = \frac{1}{s}$, so $Y(s) = \frac{1}{s+3}$, thus $y(t) = e^{-3t}$, so our time constant is $\frac{1}{3}$

Problem 3
Statement:
Specify the gain K of the proportional controller so that the output $y(t)$ has an overshoot of no more then 10% in response to a unit step

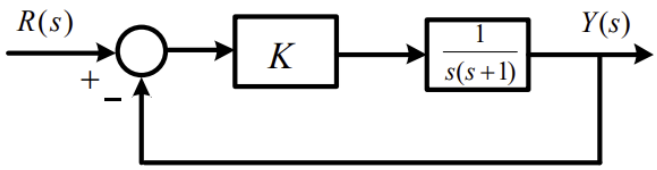


Figure 3: Problem 3

Solution

First, we go for the transfer function: $Y(s) = KG(s)U(s)$, and $U(s) = R(s) - Y(s)$, so $Y(s) = \frac{KG(s)}{1+G(s)}R(s)$.

$$\begin{aligned} Y(s) &= \frac{KG(s)}{1+G(s)}R(s) \\ &= \frac{K\frac{1}{s(s+1)}}{1+\frac{1}{s(s+1)}}R(s) \\ &= \frac{K\frac{1}{s(s+1)}}{\frac{s(s+1)+1}{s(s+1)}}R(s) \\ &= \frac{K}{s^2+s+1}R(s) \end{aligned}$$

We recognize the standard form of a second order system, so we can use the standard formula for the overshoot. $G(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$, so in our case $\omega_n = \sqrt{K}$, so $\xi = \frac{1}{2\sqrt{K}}$, then

$$\begin{aligned}
M_p &= \exp\left(\frac{-\pi\xi}{\sqrt{1-\xi^2}}\right)\\
&= \exp\left(\frac{-\pi\frac{1}{2\sqrt{K}}}{\sqrt{1-\frac{1}{4K}}}\right)\\
&= \exp\left(\frac{-\pi\frac{1}{2\sqrt{K}}}{\frac{\sqrt{4K-1}}{2\sqrt{K}}}\right)\\
&= \exp\left(\frac{-\pi}{\sqrt{4K-1}}\right)
\end{aligned}$$

We want $M_p \leq 0.1$, so we solve:

$$\begin{aligned}
0.1 &\leq \exp\left(\frac{-\pi}{\sqrt{4K-1}}\right)\\
\ln(0.1) &\leq \frac{-\pi}{\sqrt{4K-1}}\\
\ln(0.1)\sqrt{4K-1} &\leq -\pi\\
\sqrt{4K-1} &\geq -\frac{\pi}{\ln(0.1)}\\
4K-1 &\geq \frac{\pi^2}{\ln(0.1)^2}\\
K &\geq \frac{\pi^2}{4\ln(0.1)^2} + \frac{1}{4}\\
K &\geq 0.715381
\end{aligned}$$

So pick $K = 0.75$

Problem 4
Statement:

Consider the system

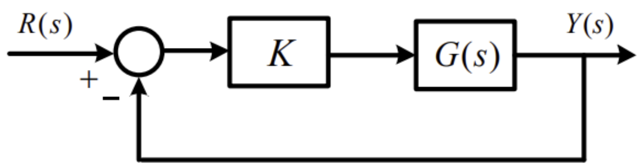


Figure 4: Problem 4

with

a) $KG(s) = \frac{4(s+2)}{s(s^3+2s^2+3s+4)}$

b) $KG(s) = \frac{2(s+4)}{s^2(s+1)}$

Use Routh's stability criterion to determine whether the each of the resulting closed-loop system will be asymptotically stable.

Solution

with

a) $KG(s) = \frac{4(s+2)}{s(s^3+2s^2+3s+4)}$ We multiply this out to get: $\frac{4s+8}{s^4+2s^3+3s^2+4s}$, and then we can write the Routh table:

$$\begin{array}{c|cc} s^4 & 1 & 3 \\ s^3 & 2 & 4 \\ s^2 & -\frac{4-6}{2} = 1 & 0 \\ s^1 & 4 & * \\ s^0 & 0 & * \end{array}$$

Which has no sign changes, so there are no poles with a positive real part. So the system is marginally stable.

b) $KG(s) = \frac{2(s+4)}{s^2(s+1)} = \frac{2s+8}{s^3+s^2}$. We can write the Routh table:

$$\begin{array}{c|cc} s^3 & 1 & 0 \\ s^2 & 1 & 0 \\ s^1 & 0 \to 2 & * \\ s^0 & \frac{0}{-2} = 0 & * \end{array}$$

for the s^1 row of zeros: $s^2 + 4 \to 2s + 0$. This also has no sign changes, but the last value is zero. So it is marginally stable.

Problem 5
Statement:
Using Routh's stability criterion to determine how many roots with positive real parts the following equations have:

a) $s^4 + 8s^3 + 32s^2 + 80s + 100 = 0$

b) $s^5 + 10s^4 + 30s^3 + 80s^2 + 344s + 480 = 0$

Solution

a) We write the Routh table:

$$\begin{array}{c|ccc}
s^4 & 1 & 32 & 100 \\
s^3 & 8 & 80 & 0 \\
s^2 & 22 & 100 & 0 \\
s^1 & \frac{960}{22} & * & * \\
s^0 & \frac{960*100}{22} * \frac{22}{960} = 100 & &
\end{array}$$

So no roots have positive real parts.

b) We write the Routh table:

$$\begin{array}{c|ccc}
s^5 & 1 & 30 & 344 \\
s^4 & 10 & 80 & 480 \\
s^3 & 22 & 296 & 0 \\
s^2 & -\frac{600}{11} & 480 & * \\
s^1 & -(22*480 + \frac{600}{11} * 296)\frac{-11}{600} = \frac{968}{5} + 296 & * & * \\
s^0 & 480 & * & *
\end{array}$$

There are two sign changes, so two roots with positive real parts.

Problem 6
Statement:

Consider the closed-loop system:

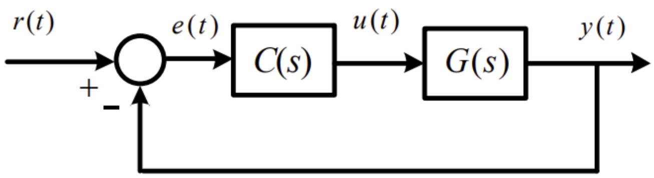


Figure 5: Problem 6

$G(s) = \frac{1}{s^2}$, and $C(s) = \frac{10(s+2)}{s+5}$.

Find the system type and determine the steady state tracking errors for:

a) $r(t) = 1(t)$

b) $r(t) = t1(t)$

c) $r(t) = 1/2t^2 1(t)$

Solution

So $L(s) = G(s)C(s) = \frac{10(s+2)}{s^2(s+5)}$. So it is a type 2 system with $L_0 = \frac{10(s+2)}{s+5}$, we evaluate this at $s = 0$ to get $L_0 = 4$. This is K_a. So the steady state error is:

a) $r(t) = 1(t) \implies 0$

b) $r(t) = t1(t) \implies 0$

c) $r(t) = 1/2t^2 1(t) \implies 1/4$

Problem 7
Statement:
Sketch the Nyquist plot for an open-loop system with transfer function:

a) $G(s) = \frac{1}{s^2}$

b) $G(s) = \frac{1}{s^2+4}$

Solution

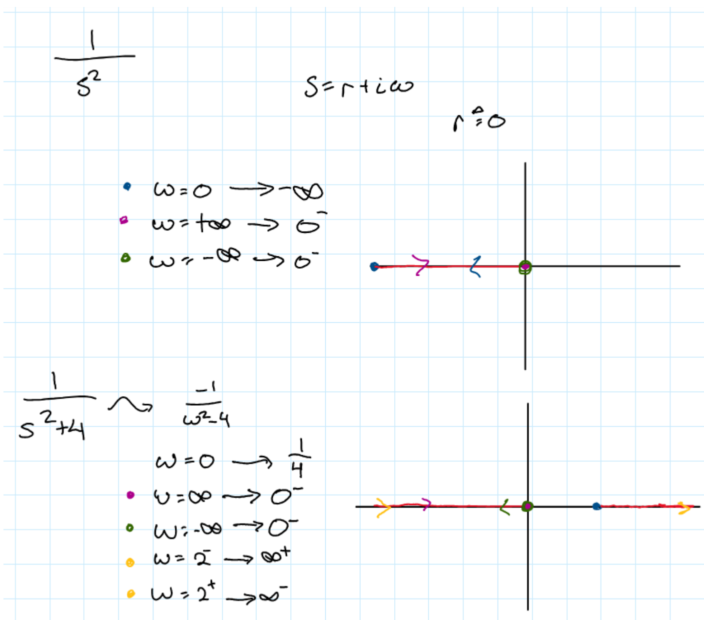


Figure 6

1. First we solve for $G(i\omega)$:

$$
\begin{aligned}
G(i\omega) &= \frac{1}{(i\omega)^2} \\
&= \frac{1}{-\omega^2} \\
&= -\frac{1}{\omega^2}
\end{aligned}
$$

So this is just the negative real line.

2. First we solve for $G(i\omega)$:

$$\begin{aligned} G(i\omega) &= \frac{1}{(i\omega)^2 + 4} \\ &= \frac{1}{-\omega^2 + 4} \\ &= -\frac{1}{\omega^2 - 4} \end{aligned}$$

So this starts at 1/4, goes to $+\infty$, then teleports to $-\infty$ and goes to zero before turning around.

Problem 8

Statement:

Consider the system with loop gain

$$L(s) = KG(s) = \frac{K(s+2)}{s+10}$$

Use Matlab command nyquist to plot nyquist plot for $G(s)$, and based on the nyquist plot, determine the range of K for which the closed-loop system is asymptotically stable

Solution

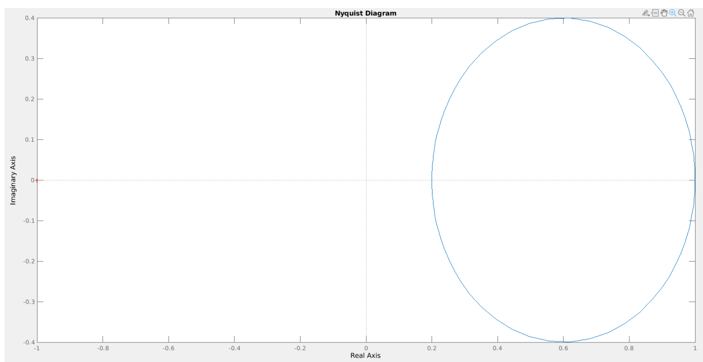


Figure 7

We also solve for

$$\begin{aligned} L(i\omega) &= \frac{K(i\omega+2)}{i\omega+10} \\ &= \frac{K(i\omega+2)(-i\omega+10)}{(i\omega+10)(-i\omega+10)} \\ &= \frac{K(\omega^2+8i\omega+20)}{\omega^2+100} \\ &= K\frac{\omega^2+20}{\omega^2+100} + iK\frac{8\omega}{\omega^2+100} \end{aligned}$$

So we can see that the real part is always positive, thus the system can never encircle -1, so the system is always asymptotically stable.

Problem 9
Statement:
Is the following system controllable? Observable?

$$\begin{aligned}\dot{x} &= \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} x + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \\ y &= \begin{pmatrix} 1 & 0 \end{pmatrix} x\end{aligned}$$

Solution

For controllability, we build the controllability matrix. $C = \begin{bmatrix} B & AB \end{bmatrix}$ $AB = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, so $C = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ which is full rank, so it is controllable.

For observability, we build the observability matrix. $O = \begin{bmatrix} C \\ CA \end{bmatrix}$ $CA = \begin{bmatrix} 0 & 1 \end{bmatrix}$, so $O = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ which is full rank, so it is observable.