1 Dynamics

Generally: $\dot{x} = f(x,u), y = h(x,u)$, $x \in \mathbb{R}^n, u \in \mathbb{R}^m$.
Jacobian Linearization: given $u = u_0, x = x_0$ is a stationary point ($f(x_0, u_0) = 0$), then $\dot{x} \approx J_{f,x}x + J_{f,u}u, y \approx J_{h,x}x + J_{h,u}u$

2 Linear Algebra

Linear dependence $\exists \alpha_1, \dots \alpha_m$ not all zero, $\alpha_1 x_1 + \dots + \alpha_m x_m = 0$.
Simlarity: B is similar to A if $\exists Q, st. B = QAQ^{-1}$. Columns of Q are eigenvectors, B is diagonal of eigenvalues.
Generalized eigenvectors and Jordan: if not n linearly independent eigenvectors; solve $Av_1 = v_2$, until chain ends.
$rank(A) = \#$ of independent columns = size of largest-square submatrix with nonzero det. $rank(A) + nullity(A) = \#$ of columns.
Positive definite if any of

1. Every eigenvalue of M is positive (non-negative)
2. All leading principal minors of M are positive (All principal minors are non-negative)
3. There exists a non-singular matrix (matrix) $N \in \mathbb{R}^{m \times n}$ such that $M = N^T N$

Singular Value Decomposition (SVD) is a matrix factorization technique that decomposes a matrix into three matrices: $A = U \Sigma V^T$ where A is an $m \times n$ matrix, U is an $m \times m$ orthogonal matrix, Σ is an $m \times n$ diagonal matrix with non-negative real numbers on the diagonal (known as singular values), and V is an $n \times n$ orthogonal matrix. $A^T A$ has eigenvalues σ_i^2 where σ_i are singular values of A. $A^T A$ has eigenvectors v_i where v_i are right singular vectors of A. AA^T has eigenvalues σ_i^2 where σ_i are singular values of A. AA^T has eigenvectors u_i where u_i are left singular vectors of A. $A = \sum_{i=1}^r \sigma_i u_i v_i^T$ where $r = rank(A)$. $AV = U\Sigma \implies Av_1 = \sigma_1 u_1$

Cayley-Hamilton: let $\Delta(\lambda) = det(A - \lambda I)$, then $\Delta(A) = 0$.
For any $f(A)$: $f(A) = \beta_0 I + \dots + \beta_{n-1} A^{n-1}$. If f polynomial, let $h(\lambda) = \beta_0 + \beta_1 \lambda + \dots + \beta_{n-1}\lambda^{n-1}$, solved from $f^{(l)}(\lambda_i) = h^{(l)}(\lambda_i)$.

Matrix exponential: $e^{At} = \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k$
$e^{A(t_1+t_2)} = e^{At_1}e^{At_2}$, $(e^{At})^{-1} = e^{-At}$, $\frac{d}{dt}e^{At} = e^{At}A = Ae^{At}$, $e^{At} = \mathcal{L}^{-1}((sI - A)^{-1})$

Compute A^{100} with $A = \begin{pmatrix} 0 & 1 \\ -1 & -2 \end{pmatrix}$

$\Rightarrow f(\lambda) = \lambda^{100}$

$\Delta(\lambda) = \lambda^2 + 2\lambda + 1 = (\lambda+1)^2 \Rightarrow \lambda_1 = -1,\ n_1 = 2$

Let $h(\lambda) = \beta_0 + \beta_1 \lambda$

$$\Rightarrow \begin{cases} f(-1) = h(-1): & (-1)^{100} = \beta_0 - \beta_1 \\ f'(-1) = h'(-1): & 100 \times (-1)^{99} = \beta_1 \end{cases} \Rightarrow \begin{cases} 1 = \beta_0 - \beta_1 \\ -100 = \beta_1 \end{cases}$$

$$\Rightarrow \begin{cases} \beta_0 = -99 \\ \beta_1 = -100 \end{cases} \Rightarrow h(\lambda) = -99 - 100\lambda$$

$$\Rightarrow A^{100} = h(A) = -99I - 100A = \begin{pmatrix} -199 & -100 \\ 100 & 101 \end{pmatrix}$$

3 Other

Capacitor - $i = C\frac{dv}{dt}$, Inductor: $v = L\frac{di}{dt}$, Friction $F = -kv$

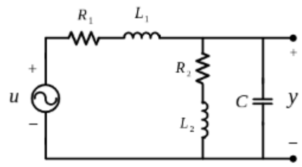

Figure: Problem 2.6

Let x_1 be the current through L_1 (from left to right), x_2 be the current through L_2 (from top down), and x_3 be the voltage across capacitor C (the same as y). By the Kirchhoff current and voltage laws, we have

$$\begin{cases} L_1\dot{x}_1 + R_1x_1 + x_3 = u, \\ L_2\dot{x}_2 + R_2x_2 = x_3, \\ C\dot{x}_3 + x_2 = x_1, \\ y = x_3, \end{cases}$$

from which we solve for $\dot{x}_1, \dot{x}_2, \dot{x}_3$ and obtain the state space model,

$$\begin{cases} \dot{x}_1 = -\frac{R_1}{L_1}x_1 - \frac{1}{L_1}x_3 + \frac{1}{L_1}u, \\ \dot{x}_2 = -\frac{R_2}{L_2}x_2 + \frac{1}{L_2}x_3, \\ \dot{x}_3 = \frac{1}{C}x_1 - \frac{1}{C}x_2, \\ y = x_3, \end{cases}$$

which can be written in the compact form

$$\begin{cases} \dot{x} = Ax + Bu, \\ y = Cx, \end{cases}$$

with

$$A = \begin{bmatrix} -\frac{R_1}{L_1} & 0 & -\frac{1}{L_1} \\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix}, B = \begin{bmatrix} \frac{1}{L_1} \\ 0 \\ 0 \end{bmatrix}, C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}.$$