1 Dynamic Behavior

Lipschitz condition: $||F(x) - F(y)|| \leq L||x - y||$, $x, y \in \{\mathbb{R}^n : ||x - x_0|| \leq r\}, L, r > 0$ Globally Lipschitz implies $||F(x)|| \leq L||x||$

Can identify stable limit cycle with polar coordinate. $\dot{\theta}$ will be strictly positive, while r has zeros.

Poincare-Bendixon Criterion: Let M be a closed and bounded set such that it contains no equilibrium points, or the Jacobion has eigenvalues with positive real parts. Every trajectory starting in M stays in M, then it contains a periodic orbit. Show $\dot{boundary} = \frac{\partial V}{\partial x_1} f_1 + \frac{\partial V}{\partial x_2} f_2 \leq 0$.

Lyaponov Stability: $V : D \to \mathbb{R}$ such that $V(x) \succ 0$, $x \in D \setminus \{0\}$, and $\dot{V}(x) = \frac{\partial V}{\partial x} f(x) \preceq 0$(stable) $\prec 0$(asymtotically). Require $\lim_{||x|| \to \infty} V(x) = \infty$ for global. Normally choose $V = x_1^2 + x_2^2 + \dots$

LaSalle's theorem: Let M be a closed and bounded set, and let $V : M \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \preceq 0$ for all $x \in M$. Let $S = \{x \in M : \dot{V}(x) = 0\}$, then if no solution can stay identically in S (other than trivial soln), the equilibrium is asymptotically stable.

Given $\begin{cases} x(t_0) \\ u(t), t \geq t_0 \end{cases} \to y(t)$, superposition: $\begin{cases} \alpha x_1(t_0) + \beta x_2(t_0) \\ \alpha u_1(t) + \beta u_2(t), t \geq t_0 \end{cases} \to \alpha y_1(t) + \beta y_2$ Discritization $x(k+1) \approx e^{AT} x(k) + (\int_{\sigma=0}^{T} e^{A\sigma} d\sigma) B u(k)$. If A is non-singular: $B_d = A^{-1}(A_d - I)B$.

Globally asymptotically stable iff A has all eigenvalues with negative real part. BIBO stable same condition, can also linearize and same condition.

Realization (controller): $G(x) = c + \frac{b_{n-1}s^{n-1} + \dots b_0}{s^n + a_{n-1}s^{n-1} + \dots + a_0}$,

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix} + \begin{pmatrix} 0 \\ 0 \\ \vdots \\ c \end{pmatrix},$$

$y = \begin{pmatrix} b_0 & b_1 & \dots & b_{n-1} \end{pmatrix} x$

2 Laplace Transform

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t) e^{-st} dt$$

$Y(z) = Z(y(k)) = \sum_{k=0}^{\infty} y(k) z^{-k}$

Unit step: $\mathcal{L}[1] = 1/s$
Unit ramp: $\mathcal{L}[t] = 1/s^2$
Power function: $\mathcal{L}[t^n] = \frac{n!}{s^{n+1}}$
Exponential: $\mathcal{L}[e^{-\alpha t}] = \frac{1}{s+\alpha}$
Sine: $\mathcal{L}[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$
Cosine: $\mathcal{L}[\cos(\omega t)] = \frac{s}{s^2+\omega^2}$

Linearity: $\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 \mathcal{L}[f_1(t)] + a_2 \mathcal{L}[f_2(t)]$
Differentiation: $\mathcal{L}[\frac{d}{dt} f(t)] = sF(s) - f(0)$, or in general $\mathcal{L}[\frac{d^n}{dt^n} f(t)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \dots - s f^{(n-2)}(0) - f^{(n-1)}(0)$
Integration: $\mathcal{L}(\int f(t) dt) = \frac{F(s)}{s} + \frac{\int f(t)dt \big|_{t=0}}{s}$
Time shift: $\mathcal{L}[f(t - \alpha)] = e^{-\alpha s} F(s)$
Frequency shift: $\mathcal{L}[e^{-\alpha t} f(t)] = F(s + \alpha)$
Time scale: $\mathcal{L}[f(t/\alpha)] = \alpha F(\alpha s)$
Multiplication by time: $\mathcal{L}[t f(t)] = -\frac{d}{ds} F(s)$
Initial value: $f(0) = \lim_{s \to \infty} sF(s)$
Final Value: $f(\infty) = \lim_{s \to 0} sF(s)$

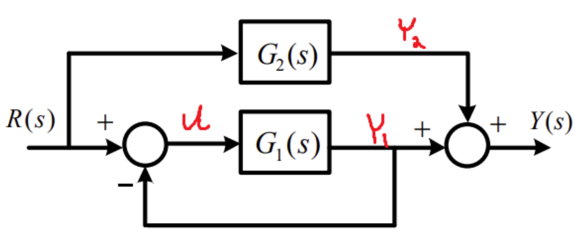
3 Frequency Domain

First order: $G(s) = \frac{\sigma}{s+\sigma} = \frac{1}{\frac{s}{\sigma}+1}$, $1/\sigma$: time constant
Second order: $G(s) = \frac{\omega_n^2}{s^2 + 2\sigma\epsilon\omega_n s + \omega_n^2}$, ϵ : damping ratio, $\omega_d = \omega_n \sqrt{1 - \epsilon^2}$: damped frequency,

ω_n: natural frequency, $t_r \approx \frac{1.8}{\omega_n}$: rise time, $t_s \approx \frac{4.6}{\omega_d}$: settling time, $M_p = e^{-\pi\epsilon/\sqrt{1-\epsilon^2}}$: overshoot, $t_p = \pi/\omega_d$: peak time

$T(s) = \frac{b(s)}{a(s)} = \frac{b_0 s^m + \cdots + b_m}{s^n + a_1 s^{n-1} + a_2 s^{n-2} + \cdots + a_n}$, system is asymptotically stable iff $a(s)$ has all roots in LHP. The Routh array is used for determining stability. First two rows are every other term of $a(s)$. Row 1 (s^n) is: $1, a_2, a_4, \ldots$ and row 2 (s^{n-1}) is: $a_1, a_3, a_5, \ldots$. Then the other rows are filled in by: $s_j^i = \frac{-1}{s_1^{i+1}} \begin{vmatrix} s_1^{i+2} & s_{j+1}^{i+2} \\ s_1^{i+1} & s_{j+1}^{i+1} \end{vmatrix}$ If zero in row, but remaining elements are non-zero. Replace with $\epsilon > 0$ and limit $\epsilon \to 0$. If entire row s^i is zero, and s^{i+1} has coeff $\alpha_1, \alpha_2, \ldots$, define aux $a^i = \alpha_1 s^{i=1} + \alpha_2 s^{i-1} + \alpha_3 s^{i-3} + \ldots$, take its derivative and use coefficients to fill in row.

$Y(s) = Y_1(s) + Y_2(s) = G_1(s)U(s) + G_2(s)R(s)$,
$U(s) = R(s) - G_1(s)U(s) \implies (1 + G_1(s))U(s) = R(s) \implies U(s) = \frac{R(s)}{1+G_1(s)}$
So, $Y(s) = \left[\frac{G_1(s)}{1+G_1(s)} + G_2(s)\right] R(s)$



Tracking: Let $L(s) = \frac{L_0(s)}{s^n}$ then it is type n system. Type 0 tracks $r(t) = 1(t) \to \frac{1}{1+K_p}$, Type 1 tracks $r(t) = t1(t) \to \frac{1}{K_v}$, and Type 2 tracks $r(t) = \frac{t^2}{2}1(t) \to \frac{1}{K_a}$. $K = L_0(0)$

PID is $u(t) = k_p(e + \frac{1}{T_1}\int_0^t e(\tau)d\tau + T_D\frac{de}{dt}) \to C(s) = U(s)/E(s) = k_p + \frac{k_p}{T_I s} + k_p T_D s$.

Nyquist: Contour of $L(s)$ as s traverses the RHP. Circle -1 is bad. gain margin g_m the smalest factor L can be increased before circling -1. Phase margin ϕ_m the largest phase shift L can have before circling -1. Stability margin s_m, shortest distance from the Nyquist plot to -1.

Bode: plot $\lg\omega$ vs $20\lg|G(j\omega)|$. ω_{pc} is where phase cross 180, ω_{gc} is where gain cross 0. $g_m = 1/|G(j\omega_{gc})|$, $\phi_m = 180 + \angle G(j\omega_{pc})$.

Lead compensator: $C(s) = \frac{Ts+1}{\alpha Ts+1}$, $\phi_{max} = \arcsin\frac{1-\alpha}{1+\alpha}$, $\omega_{max} = \frac{1}{T\sqrt{\alpha}}$. Choose $\omega_{max} = \omega_{gc}$, and α so $\phi_{max} \leq 60°$

4 Time Domain

controllability: (A, B) is controllable if $C = \begin{bmatrix} B & AB & \ldots & A^{n-1}B \end{bmatrix}$ has full row rank.
Observability: (A, C) is observable if $O = \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix}$ has full column rank.

Pole placement: Compute C, and desired $\bar{C}$ (controller realization, negative of desired pole in last row). $T = \bar{C}C^{-1}$. Compute desired characteristic polynomial: $\Delta(s) = s^n + \bar{a_n}s^{n-1} + \cdots + \bar{a_1}$, then $\bar{F} = \begin{bmatrix} \bar{a_1} - a_1 & \ldots & \bar{a_n} - a_n \end{bmatrix}$ (from A), and finally $F = \bar{F}T$

LQR: $J = \int_0^\infty (x^T Q x + u^T R u)dt$, $Q \succ 0, R \succ 0$. $u = -R^{-1}B^T P x$, P comes from Riccati: $A^T P + PA + Q - PBR^{-1}B^T P = 0$

5 Other

Capacitor - $i = C\frac{dv}{dt}$ (state=Voltage), Inductor: $v = L\frac{di}{dt}$ (state=Current), Friction $F = -kv$