Homework 2

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Problem 1
Statement:
Write the state space model for the mass-spring-damper mechanical system shown in Fig. 1. The input of the system is the external force f and the output is the displacement of the mass y.

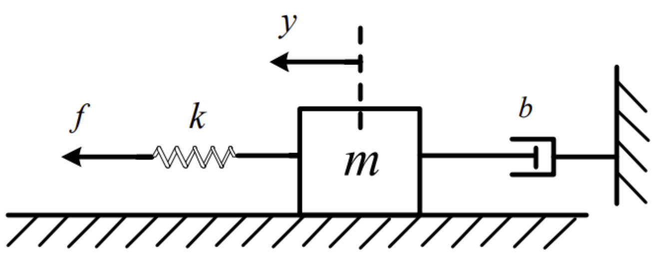


Figure 1: Problem 1 system

Solution

The spring is very confusing, as the displacement of the spring seems ill-defined. Since there is no defined left end of the spring. We can examine a free body diagram of just the spring, where f is applied on the left (our input), thus by Newton's first law, f must also be applied exiting the spring (on the right side). So essentially, f would be applied directly onto m, and the spring can be ignored.

The force of the pneumatic is $\dot{y} * b$. Then the total force on the mass is $F_m = m\ddot{y} = F - b\dot{y}$. Let $x_1 = \dot{y}$, $x_2 = y$, then the system can be expressed as:

$$\dot{x} = \begin{bmatrix} 0 & -\frac{b}{m} \\ 1 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} \frac{1}{m} \\ 0 \end{pmatrix} f \tag{1}$$

$$y = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \tag{2}$$

Problem 2

Statement:

Write the state model for the spring-damper mechanical system shown in Fig. 2. The input of the system is the external force f and the outputs are the displacements of the two masses y_1 and y_2

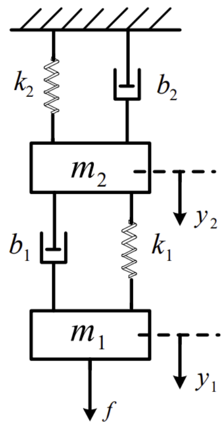


Figure 2: Problem 2 system

Solution

$$f_1 = f - b_1(\dot{y}_1 - \dot{y}_2) - k_1(y_1 - y_2)$$

(1) $$= f - b_1\dot{y}_1 + b_1\dot{y}_2 - k_1 y_1 + k_1 y_2$$

$$f_2 = b_1(\dot{y}_1 - \dot{y}_2) + k_1(y_1 - y_2) - k_2 y_2 - b_2\dot{y}_2$$

$$= b_1\dot{y}_1 - b_1\dot{y}_2 + k_1 y_1 - k_1 y_2 - k_2 y_2 - b_2\dot{y}_2$$

(2) $$= b_1\dot{y}_1 - (b_1+b_2)\dot{y}_2 + k_1 y_1 - (k_1+k_2) y_2$$

let $x_1 = \dot{y}_1$, $x_2 = \dot{y}_2$; $x_3 = y_1$, $x_4 = y_2$

from (1)

$$\dot{x}_1 = -\frac{b_1}{m_1}x_1 + \frac{b_1}{m_1}x_2 - \frac{k_1}{m_1}x_3 + \frac{k_1}{m_1}x_4 + \frac{1}{m}f$$

from (2) $$\dot{x}_2 = \frac{b_1}{m_2}x_1 - \frac{(b_1+b_2)}{m_2}x_2 + \frac{k_1}{m_2}x_3 - \frac{(k_1+k_2)}{m_2}x_4$$

$$\dot{x}_3 = x_1$$

$$\dot{x}_4 = x_2$$

Figure 3: Solution to problem 2

This results in the following matrix form:

$$\dot{x} = \begin{bmatrix} \frac{-b_1}{m_1} & \frac{b_1}{m_1} & \frac{-k_1}{m_1} & \frac{k_1}{m_1} \\ \frac{b_1}{m_2} & \frac{-(b_1+b_2)}{m_2} & \frac{k_1}{m_2} & \frac{-(k_1+k_2)}{m_2} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} + \begin{pmatrix} 1/m \\ 0 \\ 0 \\ 0 \end{pmatrix} f \tag{3}$$

$$y = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} \tag{4}$$

Problem 3
Statement:
Derive a state space model that describes the circuit shown in Fig. 4, with u as the input and y as the output.

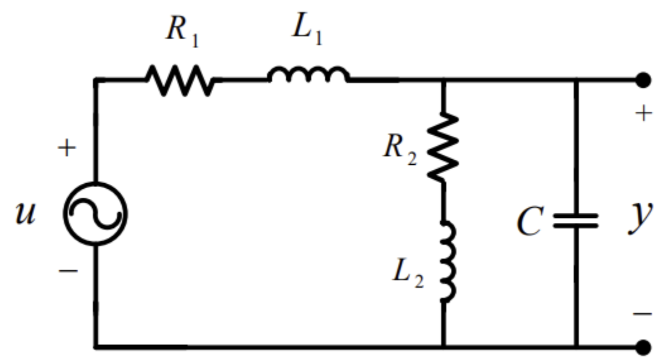
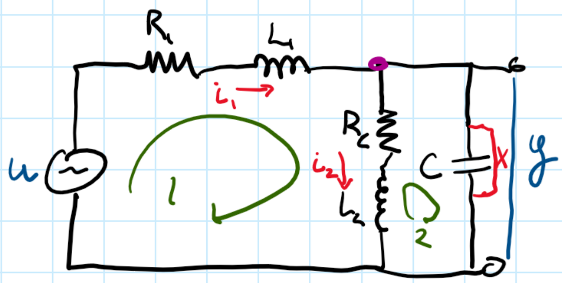


Figure 4: Problem 3

Solution

Solution in figure 5



inductor: $v = L\dot{i}$

capac $i = C\dot{v}$

1. $u - i_1 R_1 - L_1 \dot{i}_1 - i_2 R_2 - L_2 \dot{i}_2 = 0$
2. $i_2 R_2 + L_2 \dot{i}_2 - x = 0$

$i_3 = C\dot{x}$

KCL: $i_1 - i_2 - C\dot{x} = 0$

into 1 $\dot{i}_2 = \frac{1}{L_2}(x - i_2 R_2)$

$$u - i_1 R_1 - L_1 \dot{i}_1 - i_2 R_2 - (x - i_2 R_2) = 0$$

$$\dot{i}_1 = \frac{1}{L_1}(-R_1 i_1 - x + u)$$

$$\dot{x} = \frac{1}{C}(i_1 - i_2)$$

$$\begin{pmatrix} \dot{i}_1 \\ \dot{i}_2 \\ \dot{x} \end{pmatrix} = \begin{bmatrix} -R_1/L_1 & 0 & \frac{-1}{L_1} \\ 0 & -\frac{R_2}{L_2} & \frac{1}{L_2} \\ \frac{1}{C} & -\frac{1}{C} & 0 \end{bmatrix} \begin{pmatrix} i_1 \\ i_2 \\ x \end{pmatrix} + \begin{pmatrix} 1/L_1 \\ 0 \\ 0 \end{pmatrix} u$$

$$y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} i_1 \\ i_2 \\ x \end{pmatrix}$$

Figure 5: Solution to problem3

Problem 4
Statement:

Consider the op-amp circuit shown in Fig. 6. Show that the dynamics can be written in the state space form as

$$\begin{cases} \dot{x} = \begin{bmatrix} \frac{-1}{R_1C_1} - \frac{1}{R_aC_1} & 0 \\ \frac{-R_b}{R_a}\frac{1}{R_2C_2} & \frac{-1}{R_2C_2} \end{bmatrix} x + \begin{bmatrix} \frac{1}{R_1C_1} \\ 0 \end{bmatrix} u \\ y = \begin{bmatrix} 0 & 1 \end{bmatrix} x \end{cases}$$

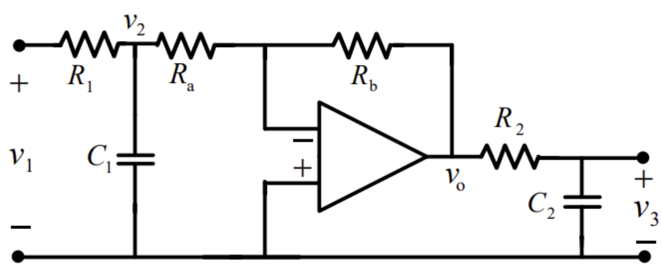
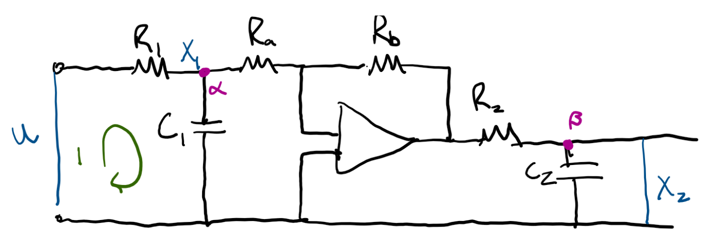


Figure 6: Problem 4

hint: let $u = v_1, y = v_3, x_1 = v_2, x_2 = v_3$

Solution



$x_1 = R_a i_a \qquad i_a = \frac{x_1}{R_a}$

$x_1 = -R_1 i_1 + u \qquad i_1 = -\frac{x_1}{R_1} + \frac{u}{R_1}$

$KCL\ \alpha = -\frac{x_1}{R_1} + \frac{u}{R_1} - \frac{x_1}{R_a} - C_1\dot{x}_1 = 0 \qquad \dot{x}_1 = \left(\frac{-1}{R_1} - \frac{1}{R_a}\right)x_1 + \frac{u}{R_1}$

$V_{out} = \frac{R_b}{R_f}x_1 \qquad x_2 = -R_2 i_2 + \frac{R_b}{R_a}x_1 \qquad i_2 = \frac{-1}{R_2}x_2 - \frac{R_b}{R_aR_2}x_1$

$i_2 - C_2\dot{x} = 0 \qquad \dot{x}_2 = \frac{-1}{R_2C_2}x_2 - \frac{R_b}{R_aR_2C_2}x_1$

Figure 7: Solution to problem 4

Problem 5
Statement:
A mathematical model that describes a wide variety of physical systems is the nth order differential equation

$$y^{(n)} = g(y, \dot{y}, \ldots, y^{(n-1)}, u)$$

where u and y are scalar variables. With u as input and y as output, derive a state space model for the system

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Solution

let $x_i = y^{(i-1)}$ for $i = 1 \ldots n+1$.
By definition, $\dot{x_i} = x_{i+1}$, then the special case of $\dot{x_{n+1}} = g(x_1, \ldots, x_{n+1}, u)$. And the output $y = x_1$

$$\begin{aligned}
\dot{x_1} &= x_2 \\
\dot{x_2} &= x_3 \\
&\vdots \\
\dot{x_{n+1}} &= g(x_1, x_2, \ldots, x_n, u) \\
\\
y &= x_1
\end{aligned}$$