Homework 5

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Problem 1
Statement:
Find the laplace transform of:

$$f(t) = \begin{cases} 0, & t < 0 \\ te^{-3t}, & t \geq 0 \end{cases}$$

Solution

This is frequency shift $(e^{-\alpha t} f(t))$, which is $F(s+\alpha)$, so $f(t) = t, \implies F(s) = 1/s^2$. So our answer is $\mathcal{L}[f(t)] = \frac{1}{(s+3)^2}$

Problem 2
Statement:
Find the laplace transform of:

$$f(t) = \begin{cases} 0, & t < 0 \\ \sin(\omega t + \theta), & t \geq 0 \end{cases}$$

Solution

We know from trig rules that $\sin(a+b) = \sin(a)\cos(b) + \cos(a)\sin(b)$. So we re-write $f(t) = \sin(\omega t)\cos(\theta) + \cos(\omega t)\sin(\theta)$ Then we know the laplace:

$$F(s) = \frac{\cos(\theta)\omega}{s^2+\omega^2} + \frac{\sin(\theta)s}{s^2+w^2} = \frac{\cos(\theta)\omega + \sin(\theta)s}{s^2+\omega^2}$$

Problem 3
Statement:
Given:

$$F(s) = \frac{2(s+2)}{s(s+1)(s+3)}$$

use the Initial Value theorem to determine $f(0)$

Solution

Initial value theorem is $f(0) = \lim_{s\to\infty} sF(s)$, so we get $\lim_{s\to\infty} \frac{2s(s+2)}{s(s+1)(s+3)} = 0$ since it is quadratic on top and cubic on bottom.

Problem 4
Statement:
Given

$$F(s) = \frac{5(s+2)}{s(s+1)(s+3)}$$

use the Final Value theorem to determine $f(\infty)$

Solution

Final value theorem is $f(\infty) = \lim_{s\to 0} sF(s)$, so we get
$\lim_{s\to 0} \frac{5s(s+2)}{s(s+1)(s+3)} = \lim_{s\to 0} \frac{5s+10}{s^2+4s+3} = 10/3$

Problem 5
Statement:
Find the inverse laplace transform of:

$$F(s) = \frac{s+3}{(s+1)(s+2)}$$

Solution

We do partial fractions expansion: $F(s) = \frac{N(s)}{D(s)}$. We know that $N(s) = (s+3)$ and $D(s) = (s+1)(s+2)$. We expand: $F(s) = \frac{a_1}{s+1} + \frac{a_2}{s+2}$. We find $a_1 = \frac{s+3}{(s+2)}\Big|_{s=-1} = 2$, and $a_2 = \frac{s+3}{(s+1)}\Big|_{s=-2} = -1$. So $F(s) = \frac{2}{s+1} - \frac{1}{s+2}$.

Then inverse laplace: $f(t)$ is the sum of the inverse laplace of each term. So, $f(t) = 2e^{-t} - e^{-2t}$

Problem 6
Statement:
Find the inverse laplace transform of:

$$F(s) = \frac{1}{s(s^2+2s+2)}$$

Solution

So we have a repeated pole problem. $F(s) = \frac{1}{s(s^2+2s+2)} = \frac{a_1}{s} + \frac{a_2}{(s+1)^2} + \frac{a_3}{s+1}$ $a_1 = \frac{1}{s^2+2s+2}\Big|_{s=0} = \frac{1}{2}$, $a_2 = \frac{1}{s}\Big|_{s=-1} = -1$, and the tricky one: $a_3 = \frac{d}{ds}\frac{1}{s}\Big|_{s=-1} = \frac{-1}{s^2}\Big|_{s=-1} = -1$
So: $F(s) = \frac{1}{2s} + \frac{-1}{(s+1)^2} + \frac{-1}{s+1}$

Then $f(t) = 1/2 - te^{-t} - e^{-t}$

Problem 7
Statement:
A LTI system is described by:

$$\dot{x} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} x + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$
$$y = \begin{pmatrix} 1 & 2 \end{pmatrix} x$$

Find the transfer function of the system.

Solution

$G(s) = C(sI - A)^{-1}B + D$. So we first need $(sI-A)^{-1} = \begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}^{-1} = \frac{1}{(s-1)^2} \begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix}$

Now just matrix multiply:

$$\begin{aligned} G(s) &= \frac{1}{(s-1)^2} \begin{pmatrix} 1 & 2 \end{pmatrix} \begin{bmatrix} s-1 & 0 \\ -1 & s-1 \end{bmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{1}{(s-1)^2} \begin{pmatrix} s-3 & 2s-2 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ &= \frac{s-3}{(s-1)^2} \end{aligned}$$

Problem 8
Statement:
Find a state space realization of the transfer function:

$$G(s) = \frac{s^3}{s^3 + 3s^2 + 2s + 1}$$

Solution

First we need to long divide: $1 + \frac{-3s^2 - 2s - 1}{s^3 + 3s^2 + 2s + 1}$. Then we have controller canonical form: $b_2 = -3, b_1 = -2, b_0 = -1$, and $a_2 = 3, a_1 = 2, a_0 = 1$. So we have:

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix} x + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u \tag{1}$$

$$y = \begin{pmatrix} -1 & -2 & -3 \end{pmatrix} x + u \tag{2}$$

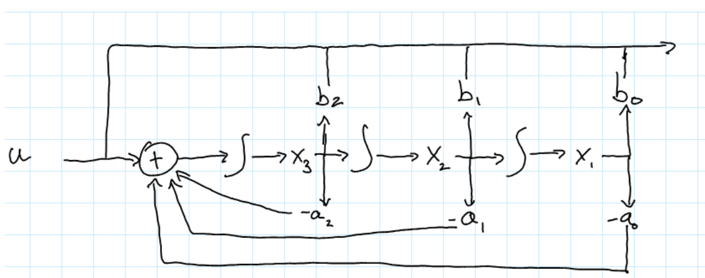

Figure 1: Realization of problem 8