Homework 3

Elliott Pryor

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Problem 1
Statement:
Show that by definition that the following vectors are linearly independent:

$$x_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, x_2 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, x_3 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Solution

Suppose not, suppose they are linearly dependent. Then $\exists \alpha_1, \alpha_2, \alpha_3$ such that $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$. This becomes the set of equations

$$\begin{aligned} \alpha_1 + \alpha_2 + \alpha_3 &= 0 && (1) \\ \alpha_2 + \alpha_3 &= 0 && (2) \\ \alpha_3 &= 0 && (3) \\ & && (4) \end{aligned}$$

Clearly $\alpha_3 = 0$, then we substitute into (2) to find $\alpha_2 = 0$, and then α_1 must also equal 0. This is a contradiction, thus they must be independent.

Problem 2
Statement:

Show that by definition that the following vectors are linearly dependent:

$$x_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, x_2 = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}, x_3 = \begin{pmatrix} 0 \\ 2 \\ 5 \end{pmatrix}$$

Solution

By definition: $\exists \alpha_1, \alpha_2, \alpha_3$ such that $\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3 = 0$. This becomes the set of equations

$$\alpha_1 - \alpha_2 = 0 \tag{5}$$
$$\alpha_1 + \alpha_3 = 0 \tag{6}$$
$$3\alpha_1 + 2\alpha_2 + 5\alpha_3 = 0 \tag{7}$$

Let $\alpha_1 = \alpha_2 = 1$ and $\alpha_3 = -1$. This satisfies equations 5-7, and the α_i are not all zero, so they are linearly dependent by definition.

Problem 3
Statement:
Determine the rank, nullity, and null space of

$$A = \begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 2 & 3 & 4 \\ 2 & 0 & 2 & 0 \end{bmatrix}$$

Solution

This has rank 2, we can see that $c_3 = c_1 + c_2$ and $c_4 = 2 * c_2$, so there are only 2 independent columns. Thus the nullity is also 2 (since 4 - 2 is 2).

Row reduction of A results in

$$A' = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

So, we have $x_1 + x_3 = 0$ and $x_2 + x_3 + 2x_4 = 0$. We get $x_3 \begin{pmatrix} -1 \\ 0 \\ 1 \\ -1/2 \end{pmatrix}$ and $x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1/2 \end{pmatrix}$

Problem 4
Statement:
Find all solutions to the following equation

$$\begin{bmatrix} 0 & 1 & 1 & 2 \\ 1 & 2 & 3 & 4 \\ 2 & 0 & 2 & 0 \end{bmatrix} x = \begin{pmatrix} -4 \\ -8 \\ 0 \end{pmatrix}$$

Solution

So we know from the previous problem what the null space is. This is helpful, we just need to find a nominal solution, then we can add the null vectors to get all of them.

By inspection, this is $-4c_2$, so $x_0 = \begin{pmatrix} 0 \\ -4 \\ 0 \\ 0 \end{pmatrix}$, then we add the null vectors to get a general solution of the form:

$$\begin{pmatrix} 0 \\ -4 \\ 0 \\ 0 \end{pmatrix} + \alpha_1 \begin{pmatrix} -1 \\ 0 \\ 1 \\ -1/2 \end{pmatrix} + \alpha_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1/2 \end{pmatrix} \forall \alpha_1, \alpha_2 \in \mathbb{R}$$

Problem 5
Statement:

Compute the determinate of the following matrices:

$$A_1 = \begin{bmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ -1 & -1 & 1 & 1 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 4 & 3 \end{bmatrix}$$

Solution

Why do we need to do so much busywork linear algebra homework. I appreciate the review is good for some, but this feels like wasting my time. This is not Math 101. I will step through A_2 and give you the matlab code for A_1.

Going down the first column we get:

$$\begin{aligned} |A_2| &= \begin{vmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 4 & 3 \end{vmatrix} \\ &= 1 \begin{vmatrix} 1 & 2 \\ 4 & 3 \end{vmatrix} + 0 \begin{vmatrix} 0 & 1 \\ 4 & 3 \end{vmatrix} + 0 \begin{vmatrix} 0 & 1 \\ 1 & 2 \end{vmatrix} \\ &= 1((1 * 3) - (2 * 4)) \\ &= -5 \end{aligned}$$

$$\begin{aligned} |A_2| &= \begin{vmatrix} 1 & 2 & 3 & 4 \\ -1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 2 \\ -1 & -1 & 1 & 1 \end{vmatrix} \\ &= 1(1) \begin{vmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ -1 & 1 & 1 \end{vmatrix} + -1(-1) \begin{vmatrix} 2 & 3 & 4 \\ 1 & 1 & 2 \\ -1 & 1 & 1 \end{vmatrix} + 1(1) \begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ -1 & 1 & 1 \end{vmatrix} + -1(-1) \begin{vmatrix} 2 & 3 & 4 \\ 1 & 2 & 3 \\ 1 & 1 & 2 \end{vmatrix} \\ &= -1 - 3 - 2 + 1 \\ &= -5 \end{aligned}$$

Problem 6
Statement:

Find the inverse of the following matrices:

$$A_1 = \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 3 & 4 \\ 3 & -1 & 6 \\ -1 & 5 & 1 \end{bmatrix}$$

Solution

We use the little trick for inverse of 2x2. $A_1^{-1} = \frac{1}{-5} \begin{bmatrix} -2 & -1 \\ -1 & 2 \end{bmatrix}$.

Then use the augmented matrix and row reduce for A_2.

$$\begin{bmatrix} 1 & 3 & 4 & 1 & 0 & 0 \\ 3 & -1 & 6 & 0 & 1 & 0 \\ -1 & 5 & 1 & 0 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & -10 & -6 & -3 & 1 & 0 \\ 0 & 8 & 5 & 1 & 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 3 & 4 & 1 & 0 & 0 \\ 0 & -10 & -6 & -3 & 1 & 0 \\ 0 & 0 & 1/5 & 7/5 & 4/5 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & 31/2 & -17/2 & -11 \\ 0 & 1 & 0 & 9/2 & -5/2 & -3 \\ 0 & 0 & 1 & -7 & 4 & 5 \end{bmatrix}$$

$$A_2^{-1} = \begin{bmatrix} 31/2 & -17/2 & -11 \\ 9/2 & -5/2 & -3 \\ -7 & 4 & 5 \end{bmatrix}$$

Problem 7
Statement:
Find the eigenvalues and eigenvectors of the following matrices:

$$A_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Solution

Eigenvalues of A_1 are trivial since it is diagonal, we have $\lambda_1 = \lambda_2 = 1, \lambda_3 = 2$. We solve $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} v_{1,2} = 0$ for eigenvectors with $\lambda = 1$, we see that $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, $v_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ satisfy this. Then we solve $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} v_3 = 0$ for the last vector, $v_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ satisfies this.

Once again, the eigenvalues are trivial since it is triangular $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 3$ We solve: $\begin{bmatrix} 0 & 4 & 10 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} v_1 = 0$, $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ satisfies this. We solve, $\begin{bmatrix} -1 & 4 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} v_2 = 0$ We get $v_2 = \begin{pmatrix} 4 \\ 1 \\ 0 \end{pmatrix}$ satisfies this.

We solve, $\begin{bmatrix} -2 & 4 & 10 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} v_3 = 0$ We get $v_3 = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}$ satisfies this.

Problem 8
Statement:
Find similarity transformations that transform the following matrices into Jordan form:

$$A_1 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -4 & -3 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix},\ A_3 = \begin{bmatrix} 0 & 4 & 3 \\ 0 & 20 & 16 \\ 0 & -25 & -20 \end{bmatrix}$$

Solution

1. First we need to find the generalized eigenvectors. We find the characteristic polynomial: $-\lambda(3\lambda+\lambda^2+4)+-2(1) = -\lambda^3-3\lambda^2-4\lambda-2$. The roots of which are: $\lambda_1 = -1+i, \lambda_2 = -1-i, \lambda_3 = -1$. We get $v_1 = \begin{pmatrix} i/2 \\ -1/2-i/2 \\ 1 \end{pmatrix}, v_2 = \begin{pmatrix} -i/2 \\ -1/2+i/2 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$ as the corresponding eigenvectors.
   Let $Q = \begin{bmatrix} i/2 & -i/2 & 1 \\ -1/2-i/2 & -1/2+i/2 & -1 \\ 1 & -1 & 1 \end{bmatrix}$. Then $J = Q^{-1}AQ$
2. First we need the eigenvectors. We find the characteristic polynomial: $(1-\lambda)((1-\lambda)(2-\lambda)) = -\lambda^3+4\lambda^2-5\lambda+2$. It has roots $\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 2$. We find the corresponding eigenvectors: $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$. Then, we let $Q = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, and thus $J = Q^{-1}AQ$
3. We repeat the same process: The characteristic polynomial is: $-\lambda((20-\lambda)(-20-\lambda)+400) = \lambda(\lambda^2-400+400) = \lambda^3$. Thus they are all repeated with $\lambda_i = 0$. Unfortunately, all the eigenvectors are not unique so we need to consider the generalized form. The first one, v_1 is in the nullspace of A, and is $v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$. Then we solve $Av_2 = v_1$, we see $v_2 = \begin{pmatrix} 0 \\ 4 \\ -5 \end{pmatrix}$. Then we solve $Av_3 = v_2$, we see that $v_3 = \begin{pmatrix} 0 \\ -3 \\ 4 \end{pmatrix}$. Then let $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & -3 \\ 0 & -5 & 4 \end{bmatrix}$, then $J = Q^{-1}AQ$

Problem 9
Statement:
Let

$$A = \begin{bmatrix} \sigma & \omega \\ -\omega & \sigma \end{bmatrix}$$

Which has eigenvalues $\sigma \pm i\omega$. Find e^{At}

Solution

So we know $\Delta(\lambda) = (\lambda - \sigma + i\omega)(\lambda - \sigma + i\omega)$, then $h(\lambda) = \beta_0 + \beta_1\lambda$, where such that $f(\sigma + i\omega) = h(\sigma + i\omega)$, and $f(\sigma - i\omega) = h(\sigma - i\omega)$ We want

$$\begin{aligned} e^{t(\sigma+i\omega)} &= e^{\sigma t}(\cos(\omega t) + i\sin(\omega t)) = \beta_0 + \beta_1(\sigma + i\omega) \\ e^{t(\sigma-i\omega)} &= e^{\sigma t}(\cos(\omega t) - i\sin(\omega t)) = \beta_0 + \beta_1(\sigma - i\omega) \end{aligned}$$

We find $\beta_0 = e^{t(\sigma+i\omega)} - \beta_1(\sigma + i\omega)$, we substitute this into the second equation:

$$\begin{aligned} e^{t(\sigma-i\omega)} &= e^{t(\sigma+i\omega)} - \beta_1(\sigma + i\omega) + \beta_1(\sigma - i\omega) \\ e^{t(\sigma-i\omega)} - e^{t(\sigma+i\omega)} &= \beta_1[(\sigma - i\omega) - (\sigma + i\omega)] \\ (e^{\sigma t}(\cos(\omega t) - i\sin(\omega t))) - e^{\sigma t}(\cos(\omega t) + i\sin(\omega t)) &= \beta_1(-2i\omega) \\ -2ie^{\sigma t}\sin(\omega t) &= \beta_1(-2i\omega) \\ \frac{e^{\sigma t}}{\omega}\sin(\omega t) &= \beta_1 \end{aligned}$$

Then from here

$$\begin{aligned} \beta_0 &= e^{t(\sigma+i\omega)} - (\frac{e^{\sigma t}}{\omega}\sin(\omega t))(\sigma + i\omega) \\ &= e^{\sigma t}(\cos(\omega t) + i\sin(\omega t)) - (\frac{e^{\sigma t}}{\omega}\sin(\omega t))(\sigma + i\omega) \\ &= e^{\sigma t}\cos(\omega t) + ie^{\sigma t}\sin(\omega t) - (\frac{e^{\sigma t}}{\omega}\sin(\omega t)\sigma + e^{\sigma t}\sin(\omega t)i) \\ \beta_0 &= e^{\sigma t}\cos(\omega t) - \frac{\sigma e^{\sigma t}}{\omega}\sin(\omega t) \end{aligned}$$

Now we can get the matrix exponential:

$$\begin{aligned} e^{At} &= \beta_0 I + \beta_1 A \\ &= \begin{bmatrix} e^{\sigma t}\cos(\omega t) - \frac{\sigma e^{\sigma t}}{\omega}\sin(\omega t) & 0 \\ 0 & e^{\sigma t}\cos(\omega t) - \frac{\sigma e^{\sigma t}}{\omega}\sin(\omega t) \end{bmatrix} + \begin{bmatrix} \sigma\frac{e^{\sigma t}}{\omega}\sin(\omega t) & \omega\frac{e^{\sigma t}}{\omega}\sin(\omega t) \\ -\omega\frac{e^{\sigma t}}{\omega}\sin(\omega t) & \sigma\frac{e^{\sigma t}}{\omega}\sin(\omega t) \end{bmatrix} \\ &= \begin{bmatrix} e^{\sigma t}\cos(\omega t) & e^{\sigma t}\sin(\omega t) \\ -e^{\sigma t}\sin(\omega t) & e^{\sigma t}\cos(\omega t) \end{bmatrix} \end{aligned}$$

Problem 10
Statement:
Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$. Compute A^{10}, A^{103}, and e^{At}

Solution

We unfortunately can't diagonalize this, so lets start multiplying and try to find a pattern.

$A^2 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $A^3 = \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $A^4 = \begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$,
extrapolating from here we get $A^n = \begin{bmatrix} 1 & 1 & n-1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$.

So $A^{10} = \begin{bmatrix} 1 & 1 & 9 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, $A^{103} = \begin{bmatrix} 1 & 1 & 102 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$

Since we have a generic form of the power of A, we can use the definition of the matrix exponential:

$$\begin{aligned}
e^{At} &= \sum_{k=0}^{\infty} \frac{1}{k!} t^k A^k \\
&= \sum_{k=0}^{\infty} \frac{1}{k!} t^k \begin{bmatrix} 1 & 1 & k-1 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \sum_{k=0}^{\infty} \frac{1}{k!} t^k & \sum_{k=0}^{\infty} \frac{1}{k!} t^k & \sum_{k=0}^{\infty} \frac{1}{k!} t^k (k-1) \\ 0 & 0 & \sum_{k=0}^{\infty} \frac{1}{k!} t^k \\ 0 & 0 & \sum_{k=0}^{\infty} \frac{1}{k!} t^k \end{bmatrix} \\
&= \begin{bmatrix} e^t & e^t & \sum_{k=0}^{\infty} \frac{1}{k!} t^k 1 - \sum_{k=0}^{\infty} \frac{1}{k!} t^k k \\ 0 & 0 & e^t \\ 0 & 0 & e^t \end{bmatrix} \\
&= \begin{bmatrix} e^t & e^t & e^t - te^t \\ 0 & 0 & e^t \\ 0 & 0 & e^t \end{bmatrix}
\end{aligned}$$

Problem 11
Statement:
Determine if the following matrices are positive definite or positive semidefinite

$$A_1 = \begin{bmatrix} 2 & 3 & 2 \\ 3 & 1 & 0 \\ 2 & 0 & 2 \end{bmatrix},\ A_2 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix},\ A_3 = \begin{bmatrix} a_1a_1 & a_1a_2 & a_1a_3 \\ a_2a_1 & a_2a_2 & a_2a_3 \\ a_3a_1 & a_3a_2 & a_3a_3 \end{bmatrix}$$

Solution

A_1 is neither, Consider the leading minor $\begin{bmatrix} 2 & 3 \\ 3 & 1 \end{bmatrix}$ has determinate -7, which is negative. So it is not positive definite nor positive semi-definite by the 3rd equivalency statement in the book.

A_2, lets find the eigenvalues. We will do it down the middle row. $(1)(-\lambda)\begin{vmatrix} -\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = -\lambda(\lambda^2 - 2\lambda - 1)$, which has roots $0, 1 \pm \sqrt{2}$. This has negative roots, so it is not positive definite nor positive semidefinite.

A_3, is actually a proof that I remember having fun with in Undergrad. First, note that this is an outer-product of $B = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$ (ie. $A = BB^T$). We use the definition of definiteness, we want to show $\forall x \in \mathbb{R}^3\ x^TAx \geq 0$. $x^TAx = x^TBB^Tx = (Bx)^TBx = \|Bx\|_2^2$ which is greater than or equal to zero for all x by definition of norm. So it is semidefinite!!

Problem 12
Statement:
Compute the singular values of:

$$A_1 = \begin{bmatrix} -1 & 0 & 1 \\ 2 & -1 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} -1 & 2 \\ 2 & 4 \end{bmatrix}$$

Solution

1. First we compute the singular values, which are $\sqrt{\lambda(AA^T)}$. $AA^T = \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix}$, which has characteristic polynomial $-\lambda^2 + 7\lambda - 6$, with roots 6, 1. So our singular values are: $\sqrt{6}, 1$: $\Sigma = \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$ We find the eigenvectors that correspond to these eigenvalues, $v_1 = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, v_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$. We should normalize, and put them as columns of U: $U = \begin{bmatrix} -1/\sqrt{5} & 2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$ Last, we compute V from the eigenvectors of A^TA. $A^TA = \begin{bmatrix} 5 & -2 & -1 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$, which we know has eigenvalues 6,1,0. Then I used matlab to solve for the eigenvectors: $\bar{V} = \begin{bmatrix} -5 & 0 & 1 \\ 2 & 1 & 2 \\ 1 & -2 & 1 \end{bmatrix}$, we normalize each column to get: $V = \begin{bmatrix} -5/\sqrt{30} & 0 & 1/\sqrt{6} \\ 2/\sqrt{30} & 1/\sqrt{5} & 2/\sqrt{6} \\ 1/\sqrt{30} & -2/\sqrt{5} & 1/\sqrt{6} \end{bmatrix}$

2. First, note $A_2 = \begin{bmatrix} -1 & 2 \\ 2 & 4 \end{bmatrix}$ is diagonal, so the eigenvectors are the same ($A^TA = AA^T$, so $A^TAu = AA^Tv = \lambda u = \lambda v$). $AA^T = \begin{bmatrix} 5 & 6 \\ 6 & 20 \end{bmatrix}$, which has characteristic polynomial $\lambda^2 - 25\lambda + 64$. Which has eigenvalues: $\frac{25 \pm \sqrt{625 - 4*64}}{2} = \frac{25 \pm 3\sqrt{41}}{2}$. So the singular values are the square root of this: $\Sigma \approx \begin{bmatrix} 4.7016 & 0 \\ 0 & 1.7016 \end{bmatrix}$. Perhaps a better way to do this would be the absolute value of eigenvalues of A: $= \left| \frac{3 \pm \sqrt{(41)}}{2} \right|$. Now we need to find the eigenvectors:

$AA^T - \lambda_1 I = \begin{bmatrix} -1 - \frac{25+3\sqrt{41}}{2} & 2 \\ 2 & 4 - \frac{25+3\sqrt{41}}{2} \end{bmatrix} v = 0$. We simplify:

$$\begin{bmatrix} 5 - \frac{25+3\sqrt{41}}{2} & 6 \\ 6 & 20 - \frac{25+3\sqrt{41}}{2} \end{bmatrix} v = 0$$

$$\begin{bmatrix} -\frac{15+3\sqrt{41}}{2} & 6 \\ 6 & -\frac{-15+3\sqrt{41}}{2} \end{bmatrix} v = 0$$

pick $v_2 = 1$, then $v_1 = 6/\frac{15+3\sqrt{41}}{2} = \frac{12}{15+3\sqrt{41}} = \frac{-5+\sqrt{41}}{4}$. So $v_1 = \begin{pmatrix} \frac{-5+\sqrt{41}}{4} \\ 1 \end{pmatrix}$. Similarly, we just flip the sign of the radical and get: $v_2 = \begin{pmatrix} \frac{-5-\sqrt{41}}{4} \\ 1 \end{pmatrix}$. We have to normalize, where we divide v_1 by $\sqrt{1 + \frac{-5+\sqrt{41}}{16}}$, and v_2 by $\sqrt{1 + \frac{-5-\sqrt{41}}{16}}$. This is ugly, so we just write the decimal form: So then $U = \begin{bmatrix} 0.331 & -0.9436 \\ 0.9436 & 0.331 \end{bmatrix}$. We know $A^T u_i = \sigma_i * v_i$, since $A = A^T$ we would think that $v_i = u_i$, but we can have a sign error due to a negative eigenvalue of A. Since λ_2 is negative, we need to flip the sign: $v_2 = -u_2$, thus: $V = \begin{bmatrix} 0.331 & 0.9436 \\ 0.9436 & -0.331 \end{bmatrix}$