Homework 4

Elliott Pryor

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Problem 1
Statement:
Show that the following two systems both have a finite escape time for any value of α and any value of the initial condition specified.

(a) $\dot{x} = x^\alpha,\ \alpha > 1, x(0) > 0$

(b) $\dot{x} = -x^\alpha,\ \alpha > 1 \text{ is even}, x(0) < 0$

Solution

Problem 2
Statement:
Show that if $f_1 : \mathbb{R} \to \mathbb{R}$ and $f_2 : \mathbb{R} \to \mathbb{R}$ are locally Lipschitz, then $f_1 + f_2$, $f_1 f_2$, $f_2 \circ f_1$ are all locally Lipschitz.

Solution

Problem 3
Statement:
For each of the functions $f(x) : \mathbb{R} \to \mathbb{R}$ given below, determine whether it is locally Lipschitz or globally Lipschitz. If it is locally Lipschitz, identify the domain on which it is locally Lipschitz.

1. $f(x) = x^2$
2. $f(x) = |x|$
3. $f(x) = x + sgn(x)$
4. $f(x) = \sin(x)$
5. $f(x) = sat(x) = sgn(x)min(1, |x|)$

Solution

Problem 4
Statement:
use the Poincare-Bendixon's criterion to show that the system has a periodic orbit; $\dot{x_1} = x_2$, $\dot{x_2} = -x_1 + x_2(2 - 3x_1^2 - 2x_2^2)$

Solution

Problem 5
Statement:
Consider the following nonlinear system:

$$\dot{x_1} = x_1(x_1^2 + x_2^2)(A - \sqrt{x_1^2 + x_2^2}) - x_2$$
$$\dot{x_2} = x_1 + x_2(x_1^2 + x_2^2)(A - \sqrt{x_1^2 + x_2^2})$$

Where A is a positive constant. Show that the system has a stable limit cycle.

Solution

Problem 6
Statement:

Euler equations for a rotating rigid statecraft are given by

$$
\begin{aligned}
J_1\dot{\omega}_1 &= (J_2 - J_3)(\omega_2\omega_3) + u_1 \\
J_2\dot{\omega}_2 &= (J_3 - J_1)\omega_3\omega_1 + u_2 \\
J_3\dot{\omega}_3 &= (J_1 - J_2)\omega_1\omega_2 + u_3
\end{aligned}
$$

Where $\omega_1, \omega_2, \omega_3$ are the components of the angular velocity vector ω along the principal axes, $u_1, u_2 u_3$ are the torque inputs applied about the principal axes, and J_1, J_2, J_3 are the principal moments of inertia.

(a) Show that with $u_1, u_2, u_3 = 0$, the origin $\omega = 0$ is stable. Is it asymtotically stable?

(b) Suppose the torque inputs apply the feedback control $u_i = -k_i\omega_i$ where k_1, k_2, k_3 are positive constants. Show that the origin $\omega = 0$ is globally asymptotically stable.

Problem 7
Statement:
Consider a linear system with input u and output y. Three experiments are performed on the system using inputs $u_1(t), u_2(t), u_3(t)$ for $t \geq 0$. In each case, the initial state $x(0)$ at time $t = 0$ is the same. The corresponding outputs are denoted by $y_1(t), y_2(t), y_3(t)$, respectively. Which of the following statements are correct if $x(0) \neq 0$ Which are correct if $x(0) = 0$?

1. If $u_3 = u_1 + u_2$ then $y_3 = y_1 + y_2$
2. If $u_3 = 0.5(u_1 + u_2)$, then $y_3 = 0.5(y_1 + y_2)$
3. If $u_3 = u_1 - u_2$ then $y_3 = y_1 - y_2$

Solution

Problem 8
Statement:

An oscillation can be generated by:

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$$

Show that its solution is

$$x(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} x(0)$$

Solution

Problem 9
Statement:
Find the unit-step response of the linear system.

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$
$$y = \begin{pmatrix} 2 & 3 \end{pmatrix} x$$

Solution

Problem 10
Statement:
Consider the system:

$$\dot{x_1} = -x_1$$
$$\dot{x_2} = (x_1x_2 - 1)x_2^3 + (x_1x_2 - 1 - x_1^2)x_2$$

(a) Show that the system has a unique equilibrium point

(b) Using linearization, show the equilibrium point is asymptotically stable

(c) Show the equilibrium point is not globally asymptotically stable

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Solution