

# Homework 1

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**Problem 1****Statement:**

Consider the problem of imitation learning within a discrete MDP with horizon  $T$  and an expert policy  $\pi^*$ . We gather expert demonstrations from  $\pi^*$  and fit an imitation policy  $\pi_\theta$  to these trajectories so that

$$\mathbb{E}_{p_{\pi^*}(s)} \pi_\theta(a \neq \pi^*(s)) = \frac{1}{T} \sum_{t=1}^T \mathbb{E}_{p_{\pi^*}(s_t)} \pi_\theta(a \neq \pi^*(s_t)) \leq \epsilon \quad (1)$$

i.e., the expected likelihood that the learned policy  $\pi_\theta$  disagrees with the expert  $\pi^*$  within the training distribution  $p_{\pi^*}$  of states drawn from random expert trajectories is at most  $\epsilon$ . For convenience, the notation  $p_\pi(s_t)$  indicates the state distribution under  $\pi$  at time step  $t$  while  $p(s)$  indicates the state marginal of  $\pi$  across time steps, unless indicated otherwise

1. Show that  $\sum_{s_t} |p_{\pi_\theta}(s_t) - p_{\pi^*}(s_t)| \leq 2T\epsilon$
2. Consider the expected return of the learned policy  $\pi_\theta$  for a state-dependent reward function  $r(s_t)$ . where we assume the reward is bounded  $|r(s_t)| \leq R_{max}$ :

$$J(\pi) = \sum_{t=1}^T \mathbb{E}_{p_\pi(s_t)} r(s_t)$$

- (a) Show that  $|J(\pi_\theta) - J(\pi^*)| \in \mathcal{O}(T\epsilon)$  when the reward only depends on the last state. I.e.,  $r(s_t) = 0 \ \forall t < T$
- (b) Show that  $J(\pi^*) - J(\pi_\theta) \in \mathcal{O}(T^2\epsilon)$

**Solution**

1. So we want to show that  $\sum_{s_t} |p_{\pi_\theta}(s_t) - p_{\pi^*}(s_t)| \leq 2T\epsilon$ . We know that  $\mathbb{E}_{p_{\pi^*}(s)} \pi_\theta(a \neq \pi^*(s)) = \frac{1}{T} \sum_{t=1}^T \mathbb{E}_{p_{\pi^*}(s_t)} \pi_\theta(a \neq \pi^*(s_t)) \leq \epsilon$ . This means that the expected value of the probability that  $\pi_\theta$  disagrees with  $\pi^*$  is at most  $\epsilon$ .

We note that if there is no mistake, then there is no difference in the state distribution. In the worst case, after a mistake the state distribution is completely different, so  $|p_{\pi_\theta}(s_t) - p_{\pi^*}(s_t)| = 2 \ \forall t > t_{mistake}$ . Thus the cost after the mistake would be:  $2(T - t_{mistake})$ . We partition the space by the time of the mistake:  $\epsilon * cost_1, (1 - \epsilon)\epsilon * cost_2, (1 - \epsilon)^2\epsilon * cost_3, \dots, (1 - \epsilon)^T * 0$ . Where cost is the total difference in the state distribution after the mistake. So all the possible costs that we can incur are:  $\epsilon * 2T, (1 - \epsilon)\epsilon * 2(T - 1), (1 - \epsilon)^2\epsilon * 2(T - 2), \dots, (1 - \epsilon)^T * 0$ . So the worst case is we make a mistake on the very first sample. Thus  $\sum_{s_t} |p_{\pi_\theta}(s_t) - p_{\pi^*}(s_t)| \leq 2T\epsilon$

2. (a) We want to show that  $|J(\pi_\theta) - J(\pi^*)| \in \mathcal{O}(T\epsilon)$  when the reward only depends on the last state. This bound is somewhat misleading as it appears to grow linearly, but in fact is constant. This is because the reward only depends on the last state, so  $|J(\pi_\theta) - J(\pi^*)| = |\mathbb{E}_{p_{\pi^*}(s_T)}(r(s_T)) - \mathbb{E}_{p_{\pi_\theta}(s_T)}(r(s_T))|$ . We know that  $r(s)$  is a bounded function, so this is clearly  $\leq R_{max}$  which is constant.

We can also derive the original (linear) bound fairly easily. We partition it into the case of no mistakes and the case of mistakes. No mistake occurs with probability  $(1 - \epsilon)^T$  which by Bernoulli's inequality is  $\geq 1 - T\epsilon$ . This means the probability of a mistake is  $\leq 1 - (1 - T\epsilon) = T\epsilon$ . We can assume worst case cost if a mistake occurs. Thus the expected value of the cost  $\leq (1 - T\epsilon) * 0 + (T\epsilon) * R_{max} \in \mathcal{T}\epsilon$

- (b) Given the previous result, we consider the generic case where the reward depends on all states, as a sum of cases where the reward only depends on the last state. Let  $j(T)$  denote the cost discovered in the previous section  $j(t) = t\epsilon * R_{max}$ . Then the generic case:

$$\begin{aligned}
 |J(\pi^*) - J(\pi_\theta)| &\leq j(1) + j(2) + \dots + j(T) \\
 &= \epsilon * R_{max} * (1 + 2 + \dots + T) \\
 &= \epsilon * R_{max} * \frac{T(T+1)}{2} \\
 &= \epsilon * R_{max} * \left( \frac{T^2}{2} + \frac{T}{2} \right) \\
 &\in \mathcal{O}(T^2\epsilon)
 \end{aligned}$$