Homework 4

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Problem 1 3.1.3 problem 1a)

Compute the sup, inf limsup, liminf and all the limit points of $x_n = 1 + (-1)^n/n$

$x_n = 0, 3/2, 2/3, 5/4, 4/5, 5/6, 6/7, 9/8, ...$

Clearly the sup is 3/2 and inf is 0. We then show that the sequence is convergent to 1. We need to show $\forall n \in \mathbf{N}\ \exists m \in \mathbf{N}\ s.t. \forall j \geq m\ |x_j - 1| \leq 1/n$.

$$|x_j - 1| = |1 + (-1)^j/j - 1| = |(-1)^j/j| = 1/j$$

If we choose $m = n$ then $1/j \leq 1/n\ \forall j \geq n$ as required. Since it is a convergent sequence, by theorem 3.1.5 $limsup = liminf = 1$.

Problem 2 3.1.3 problem 2

1. If a bounded sequence is the sum of a monotone increasing and monotone decreasing sequence ($x_n = y_n + z_n$ where $\{y_n\}$ is monotone increasing and $\{z_n\}$ is monotone decreasing) does it follow that the sequence converges?
2. What if $\{y_n\}$ and $\{z_n\}$ are bounded?

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1. No, the sequence could oscillate.

   PROOF. By contradiction

   Suppose that sequence $x_n = y_n + z_n$ where $\{y_n\}$ is monotone increasing and $\{z_n\}$ is monotone decreasing converges for any y_n, z_n, and x_n is bounded. We let $y_n = \begin{cases} n, & \text{if n is even} \\ (n+1), & \text{if n is odd} \end{cases}$ and $z_n = \begin{cases} -n, & \text{if n is even} \\ -(n-1), & \text{if n is odd} \end{cases}$

   So $y_n = 2, 2, 4, 4, 6, ...$ and $z_n = 0, -2, -2, -4, -4, ...$ And then $x_n = (2+0), (2-2), (4-2), (4-4), (6-4), ... = 2, 0, 2, 0, 2, ...$ So clearly x_n is bounded and it does not converge since $|x_n - x_{n+1}| = 2 \; \forall n \in \mathbf{N}$

   So $x_n = y_n + z_n$ does not converge for monotone increasing sequence y_n and monotone decreasing sequence z_n. A contradiction, so x_n does not converge for every y_n, z_n. $\square$

2. Yes x_n converges if y_n, z_n are bounded. Since y_n is bounded and monotone increasing it must have a finite limit equal to the sup, and since z_n is bounded and monotone decreasing it must have a finite limit equal to the inf. Thus $\lim_{k\to\infty} y_k = y$ and $\lim_{k\to\infty} z_n = z$. Then $\lim_{k\to\infty} y_k + z_k = y + z$. Since $y, z \in \mathbf{R}$ the sequence $x_n = y_n + z_n$ is convergent.

Problem 3 3.1.3 problem 4

Prove $sup(A \cup B) \geq sup(A)$ and $sup(A \cap B) \leq sup(A)$

PROOF.

First we show $sup(A \cup B) \geq sup(A)$ by contradiction. We suppose that $sup(A \cup B) < sup(A)$. By definition $sup(A \cup B) \geq x \, \forall x \in A \cup B$. Then since every element in A is also in $A \cup B$ it is true that $sup(A \cup B) \geq x \, \forall x \in A$. Since $sup(A) \geq x \, \forall x \in A$ and $sup(A \cup B) < sup(A)$ then $sup(A)$ is not a least upper bound, a contradiction so $sup(A \cup B) \geq sup(A)$

Next we show $sup(A \cap B) \leq sup(A)$ by contradiction. We suppose that $sup(A \cap B) > sup(A)$. Since $sup(A \cap B)$ is the least upper bound, $\exists x \in A \cap B \; s.t|x - sup(A \cap B)| \leq 1/n \, \forall n$. Clearly everything in $A \cap B$ is also in A. Since $sup(A \cap B) > sup(A)$ then $\exists x \in A \cap B > sup(A)$ which is a contradiction since $A \cap B \subseteq A$. So $sup(A \cap B) \leq sup(A)$

$\square$

Problem 4 3.1.3 problem 6

Is every subsequence of a subsequence of a sequence also a subsequence of the sequence?

Yes.

PROOF.

Let x_n be some sequence, and x'_n be a subsequence. We show that x''_n is also a subsequence of x_n. First clearly every element in x'_n is in x_n since x'_n is a subsequence, then it follows that every element of x''_n is an element of x_n by the same reasoning. We need to show that there is a strictly increasing subsequence selection function f. There is a subsequence selection function g that selects elements from x to create x', and another subsequence selection function h that selects elements from x' to create x''. The subsequence selection function $f = h(g(n))$. $f : \mathbf{N} \to \mathbf{N}$ since $g : \mathbf{N} \to \mathbf{N}$ and $h : \mathbf{N} \to \mathbf{N}$. We show that equation 1 is strictly increasing

$$h(g(n+1)) > h(g(n)) \tag{1}$$

$g(n+1) > g(n)$ by definition. Let $a = g(n)$ then $g(n+1) \geq a+1$, so in the worst case we have $g(n+1) = a+1$. So we substitute this into equation 1. $h(a+1) > h(a)$. This is true because h is a subsequence selection function so is strictly increasing. Thus f which is the composition of h and g ($f = h(g(n))$) must be strictly increasing. So f is a subsequence selection function, and x''_n must be a subsequence of x_n as required.

$\square$