

Homework 4

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Problem 1 True or false

1. The set of \mathbf{Q} is countable?

True

2. Let A_1, A_2, A_3, \dots be a sequence of countably many sets, and each $A_i, i = 1, 2, 3, \dots$ is countable then their cartesian product $A_1 \times A_2 \times A_3 \times \dots$ is countable.

False

3. A Cauchy sequence of positive rational numbers cannot be equivalent to a Cauchy sequence of negative rational numbers.

False, $x_n = 1/n$ $y_n = -1/n$ both converge to 0 so have the same limit and are thus equivalent.

4. Let x_1, x_2, \dots be a convergent sequence of real numbers, then $\limsup x_n = \liminf x_n$.

True

5. The union of any number of closed sets is a closed set.

False

Problem 2 The statement “A real number x is a limit-point of a sequence of real numbers x_1, x_2, \dots ” can be written explicitly using quantifiers as follows: “For all $n \in \mathbf{N}$, for all $m \in \mathbf{N}$, there exists $j > m$ such that $|x - x_j| < 1/n$ ”.

Now write the statement “A real number x is not a limit-point of a sequence of real numbers x_1, x_2, \dots ” explicitly using quantifiers

There exists an $n, m \in \mathbf{N}$ such that for all $j > m$ $|x - x_j| > 1/n$

Or equivalently

$$\exists m, n \in \mathbf{N} \text{ s.t. } \forall j > m \ |x - x_j| > 1/n$$

Problem 3 Give an example of a set A that is not closed but such that every point of A is a limit point

We construct cantor set C . Then because C is the cantor set, it is closed and perfect. So it contains all of its limit points and all of its points are limit points. Then let $A = C \setminus \{0, 1\}$. Then every point in A lies within $(0, 1)$ so it is open. Since we only subtracted two points, every point remaining in A must also be a limit point of A (since C is perfect). But it does not contain all of its limit points (it is missing 0 and 1) so it is not closed. Thus A is not closed and every point in A is a limit point.

Problem 4 let x_1, x_2, \dots be a sequence of real numbers given by

$$x_n = \frac{\cos(\sqrt{n!}\pi)}{4^n}$$

and define $y_n = x_1 + x_2 + \dots + x_n$ prove that the sequence y_1, y_2, \dots converges

PROOF.

We know that $y_n = \sum_{i=1}^n \frac{\cos(\sqrt{i!}\pi)}{4^i}$

We want to show that the sequence converges, so we show that it is Cauchy. So we must show that $\forall n \in \mathbf{N} \exists m \in \mathbf{N} \forall j, k \geq m |y_j - y_k| \leq 1/n$. We assume that $j \geq k$, then we have that:

$$|y_j - y_k| = \left| \sum_{i=1}^j \frac{\cos(\sqrt{i!}\pi)}{4^i} - \sum_{i=1}^k \frac{\cos(\sqrt{i!}\pi)}{4^i} \right| = \left| \sum_{i=k}^j \frac{\cos(\sqrt{i!}\pi)}{4^i} \right|$$

Then since $\cos(\sqrt{i!}\pi)$ is bounded $[-1, 1]$ $\left| \sum_{i=k}^j \frac{\cos(\sqrt{i!}\pi)}{4^i} \right| \leq \sum_{i=k}^j \frac{1}{4^i} < \sum_{i=k}^{\infty} \frac{1}{4^i}$. In order to simplify this further we provide two properties of geometric series $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ and $\sum_{n=0}^x r^n = \frac{1-r^{x+1}}{1-r}$ if $|r| < 1$. Using this we can then solve for an expression for $\sum_{i=k}^{\infty} \frac{1}{4^i}$

$$\begin{aligned} \sum_{i=0}^{\infty} \frac{1}{4^i} &= \sum_{i=0}^{k-1} \frac{1}{4^i} + \sum_{i=k}^{\infty} \frac{1}{4^i} \\ \sum_{i=0}^{\infty} \frac{1}{4^i} - \sum_{i=0}^{k-1} \frac{1}{4^i} &= \sum_{i=k}^{\infty} \frac{1}{4^i} \\ \frac{4}{3} - \left(\frac{1 - \frac{1}{4^{k-1}}}{1 - \frac{1}{4}} \right) &= \sum_{i=k}^{\infty} \frac{1}{4^i} \\ \frac{1}{3} * \frac{1}{4^k} &= \sum_{i=k}^{\infty} \frac{1}{4^i} \end{aligned}$$

Now the smallest k can be is m , so $\sum_{i=k}^{\infty} \frac{1}{4^i} \leq \sum_{i=m}^{\infty} \frac{1}{4^i}$ so we must pick an m such that $\sum_{i=m}^{\infty} \frac{1}{4^i} = \frac{1}{3} * \frac{1}{4^m} \leq 1/n$. This is true if $\log_4(n/3) \leq m$ for simplicity, we set $m = n$ since $n > \log_4(n/3) \forall n$. So in conclusion:

$$|y_j - y_k| \leq \sum_{i=k}^j \frac{1}{4^i} < \sum_{i=k}^{\infty} \frac{1}{4^i} \leq \sum_{i=m}^{\infty} \frac{1}{4^i} = \frac{1}{3} * \frac{1}{4^m}$$

Now we have found an m such that $\forall n \in \mathbf{N} \forall j, k \geq m |y_j - y_k| \leq 1/n$. So the sequence is Cauchy. Then it must converge, since all Cauchy sequences are convergent in the reals.

□

Problem 5

prove

$$\limsup\{x_n + y_n\} \leq \limsup\{x_n\} + \limsup\{y_n\}$$

if both $\limsup\{x_n\}$ and $\limsup\{y_n\}$ are finite, and give an example where the equality does not hold.