

Homework 4

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Problem 1 3.1.3 problem 1a)

Compute the sup, inf limsup, liminf and all the limit points of $x_n = 1 + (-1)^n/n$

$$x_n = 0, 3/2, 2/3, 5/4, 4/5, 5/6, 6/7, 9/8, \dots$$

Clearly the sup is $3/2$ and inf is 0 . We then show that the sequence is convergent to 1 . We need to show $\forall n \in \mathbf{N} \exists m \in \mathbf{N} \text{ s.t. } \forall j \geq m \ |x_j - 1| \leq 1/n$.

$$|x_j - 1| = |1 + (-1)^j/j - 1| = |(-1)^j/j| = 1/j$$

If we choose $m = n$ then $1/j \leq 1/n \ \forall j \geq n$ as required. Since it is a convergent sequence, by theorem 3.1.5 $\limsup = \liminf = 1$.

Problem 2 3.1.3 problem 2

1. If a bounded sequence is the sum of a monotone increasing and monotone decreasing sequence ($x_n = y_n + z_n$ where $\{y_n\}$ is monotone increasing and $\{z_n\}$ is monotone decreasing) does it follow that the sequence converges?
 2. What if $\{y_n\}$ and $\{z_n\}$ are bounded?
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1. No, the sequence could oscillate.

PROOF. By contradiction

Suppose that sequence $x_n = y_n + z_n$ where $\{y_n\}$ is monotone increasing and $\{z_n\}$ is monotone decreasing converges for any y_n, z_n , and x_n is bounded. We let $y_n = \begin{cases} n, & \text{if } n \text{ is even} \\ (n+1), & \text{if } n \text{ is odd} \end{cases}$

and $z_n = \begin{cases} -n, & \text{if } n \text{ is even} \\ -(n-1), & \text{if } n \text{ is odd} \end{cases}$

So $y_n = 2, 2, 4, 4, 6, \dots$ and $z_n = 0, -2, -2, -4, -4, \dots$. And then $x_n = (2+0), (2-2), (4-2), (4-4), (6-4), \dots = 2, 0, 2, 0, 2, \dots$. So clearly x_n is bounded and it does not converge since $|x_n - x_{n+1}| = 2 \forall n \in \mathbf{N}$

So $x_n = y_n + z_n$ does not converge for monotone increasing sequence y_n and monotone decreasing sequence z_n . A contradiction, so x_n does not converge for every y_n, z_n . \square

2. Yes x_n converges if y_n, z_n are bounded. Since y_n is bounded and monotone increasing it must have a finite limit equal to the *sup*, and since z_n is bounded and monotone decreasing it must have a finite limit equal to the *inf*. Thus $\lim_{k \rightarrow \infty} y_k = y$ and $\lim_{k \rightarrow \infty} z_k = z$. Then $\lim_{k \rightarrow \infty} y_k + z_k = y + z$. Since $y, z \in \mathbf{R}$ the sequence $x_n = y_n + z_n$ is convergent.

Problem 3 3.1.3 problem 4

Prove $\sup(A \cup B) \geq \sup(A)$ and $\sup(A \cap B) \leq \sup(A)$

Problem 4 3.1.3 problem 6

Is every subsequence of a subsequence of a sequence also a subsequence of the sequence?