Exam 1

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Problem 1 True or false

1. The set of $\mathbf{Q}$ is countable?

   True

2. Let $A_1, A_2, A_3, ...$ be a sequence of countably many sets, and each $A_i, i = 1, 2, 3, ...$ is countable then their cartesian product $A_1 x \times A_2 \times A_3 \times ...$ i countable.

   False

3. A Cauchy sequence of positive rational numbers cannot be equivalent to a Cauchy sequence of negative rational numbers.

   False, $x_n = 1/n$ $y_n = -1/n$ both converge to 0 so have the same limit and are thus equivalent.

4. Let $x_1, x_2, ...$ be a convergent sequence of real numbers, then $limsup x_n = liminf x_n$.

   True

5. The union of any number of closed sets is a closed set.

   False

Problem 2 The statement "A real number x is a limit-point of a sequence of real numbers $x_1, x_2, ...$" can be written explicitly using quantifiers as follows: "For all $n \in \mathbf{N}$, for all $m \in \mathbf{N}$, there exists $j > m$ such that $|x - x_j| < 1/n$".

Now write the statement "A real number x is not a limit-point of a sequence of real numbers $x_1, x_2, ...$" explicitly using quantifiers

There exists an $n, m \in \mathbf{N}$ such that for all $j > m$ $|x - x_j| > 1/n$

Or equivalently

$$\exists m, n \in \mathbf{N}\ s.t. \forall j > m\ |x - x_j| > 1/n$$

Problem 3 Give an example of a set A that is not closed but such that every point of A is a limit point

We construct cantor set C. Then because C is the cantor set, it is closed and perfect. So it contains all of its limit points and all of its points are limit points. Then let $A = C \setminus \{0, 1\}$. Then every point in A lies within $(0, 1)$ so it is open. Since we only subtracted two points, every point remaining in A must also be a limit point of A (since C is perfect). But it does not contain all of its limit points (it is missing 0 and 1) so it is not closed. Thus A is not closed and every point in A is a limit point.

Problem 4 let $x_1, x_2, ...$ be a sequence of real numbers given by

$$x_n = \frac{\cos(\sqrt{n!}\pi)}{4^n}$$

and define $y_n = x_1 + x_2 + ... + x_n$ prove that the sequence $y_1, y_2, ...$ converges

PROOF.

We know that $y_n = \sum_{i=1}^{n} \frac{\cos(\sqrt{i!}\pi)}{4^i}$

We want to show that the sequence converges, so we show that it is Cauchy. So we must show that $\forall n \in \mathbf{N}\, \exists m \in \mathbf{N}\, \forall j, k \geq m\ |y_j - y_k| \leq 1/n$. We assume that $j \geq k$, then we have that:

$$|y_j - y_k| = \left| \sum_{i=1}^{j} \frac{\cos(\sqrt{i!}\pi)}{4^i} - \sum_{i=1}^{k} \frac{\cos(\sqrt{i!}\pi)}{4^i} \right| = \left| \sum_{i=k}^{j} \frac{\cos(\sqrt{i!}\pi)}{4^i} \right|$$

Then since $\cos(\sqrt{i!}\pi)$ is bounded $[-1, 1]$ $\left|\sum_{i=k}^{j} \frac{\cos(\sqrt{i!}\pi)}{4^i}\right| \leq \sum_{i=k}^{j} \frac{1}{4^i} < \sum_{i=k}^{\infty} \frac{1}{4^i}$. In order to simplify this further we provide two properties of geometric series $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ and $\sum_{n=0}^{x} r^n = \frac{1-r^x}{1-r}$ if $|r| < 1$. Using this we can then solve for an expression for $\sum_{i=k}^{\infty} \frac{1}{4^i}$

$$\sum_{i=0}^{\infty} \frac{1}{4^i} = \sum_{i=0}^{k-1} \frac{1}{4^i} + \sum_{i=k}^{\infty} \frac{1}{4^i}$$

$$\sum_{i=0}^{\infty} \frac{1}{4^i} - \sum_{i=0}^{k-1} \frac{1}{4^i} = \sum_{i=k}^{\infty} \frac{1}{4^i}$$

$$\frac{4}{3} - \left(\frac{1 - \frac{1}{4^{k-1}}}{1 - \frac{1}{4}} \right) = \sum_{i=k}^{\infty} \frac{1}{4^i}$$

$$\frac{1}{3} * \frac{1}{4^k} = \sum_{i=k}^{\infty} \frac{1}{4^i}$$

Now the smallest k can be is m, so $\sum_{i=k}^{\infty} \frac{1}{4^i} \leq \sum_{i=m}^{\infty} \frac{1}{4^i}$ so we must pick an m such that $\sum_{i=m}^{\infty} \frac{1}{4^i} = \frac{1}{3} * \frac{1}{4^m} \leq 1/n$. This is true if $\log_4(n/3) \leq m$ for simplicity, we set $m = n$ since $n > \log_4(n/3)\ \forall n$. So in conclusion:

$$|y_j - y_k| \leq \sum_{i=k}^{j} \frac{1}{4^i} < \sum_{i=k}^{\infty} \frac{1}{4^i} \leq \sum_{i=m}^{\infty} \frac{1}{4^i} = \frac{1}{3} * \frac{1}{4^m}$$

Now we have found an m such that $\forall n \in \mathbf{N}\ \forall j, k \geq m\ |y_j - y_k| \leq 1/n$. So the sequence is Cauchy. Then it must converge, since all Cauchy sequences are convergent in the reals.

$\square$

Problem 5

prove

$$limsup\{x_n + y_n\} \leq limsup\{x_n\} + limsup\{y_n\}$$

if both $limsup\{x_n\}$ and $limsup\{y_n\}$ are finite, and give an example where the equality does not hold.

PROOF.

Fist define $z_n = \{x_n + y_n\}$. Let $X = limsup\{x_n\}$ and $Y = limsup\{y_n\}$ and $Z = limsup\{z_n\}$. By the definition of $limsup$ we have that there are infinitely many points in $\{z_n\}$ within a neighborhood of Z (by neighborhood we mean $|z_n - Z| \leq 1/n$). There must also be a convergent subsequence z' that converges to Z. Because there is a subsequence selection function $f : \mathbf{N} \rightarrow \mathbf{N}$: $z'_i = z_j = x_j + y_j$. Then $x_j \leq sup_{a \geq j}\{x_a\}$ and $y_j \leq sup_{a \geq j}\{x_a\}$. We know that the

$$\begin{aligned} Z = \lim_{\imath\to\infty} z'_i = \lim_{\jmath\to\infty} z_j = \lim_{\jmath\to\infty}(x_j + y_j) \\ \leq \lim_{\jmath\to\infty}(sup_{a\geq j}\{x_a\} + sup_{a\geq j}\{y_a\}) \qquad &= \lim_{\jmath\to\infty}(sup_{a\geq j}\{x_a\}) + \lim_{\jmath\to\infty}(sup_{a\geq j}\{y_a\}) \\ \leq limsup\{x_j\} + limsup\{y_j\} \end{aligned}$$

So we have $limsup\{x_n + y_n\} = limsup\{z_n\} = Z \leq limsup\{x_n\} + limsup\{y_n\}$ $\square$

The inequality is necessary, if we have $x_n = 0, 1, 0, 1, 0, 1, ...$ and $y_n = 2, -3, 2, -3, 2, -3, ...$ then $z_n = 2, -2, 2, -2, 2, -2, ...$ Clearly $limsup\{z_n\} = 2$ and $limsup\{x_n\} = 1$ and $limsup\{y_n\} = 2$. But $2 \leq 1 + 2$ so the inequality is necessary (ie equality does not hold).

Problem 6 Find the sup, inf, limsup, liminf, and all limit-points of the sequence

$$x_n = (-1)^n + 1/n + 2cos(\frac{n\pi}{2}, \quad n = 1, 2, 3...$$

We start by listing a few elements to get an idea on the pattern

$x_n = 0, -1/2, -2/3, 13/4, -4/5, -5/6, -6/7, 25/8, ...$

Unless n mod $2 = 0$ the cos term is zero. So in these cases, the sequence is $-1^n + 1/n$. When n is divisible by 2 and not by 4, the cosine term is -2, but the $(-1)^n$ term is positive, so $1-2 = -1$. But when n is divisible by 4 both the $(-1)^n$ term and the cos term are positive. So we get $3+1/n$ on terms where n mod $4 = 0$ and $-1 + 1/n$ on all the other terms. These two subsequences we denote $y_n = 3 + 1/n$ and $z_n = -1 + 1/n$ where $x = z_1, z_2, z_3, y_4, z_5, z_6, z_7, y_8, z_9, ...$

In the first few terms we see that $sup = 13/4$, as this is the first term in the sequence of y_n which is monotone decreasing, so the first term is the largest. Then for the remainder we need to examine the limits of the sequences. For $limsup$ we examine the limit of y_n. $\lim_{n\to\infty} y_n = \lim_{n\to\infty} 3 + 1/n = 3$. So the $limsup = 3$. Since y_n is monotone decreasing and bounded, it converges to its one limit point. We show that it converges to 3 $\forall n \in \mathbf{N}\ \exists m\ s.t\ \forall j \geq m\ |y_j - 3| \leq 1/n$. We choose $m = n$ then $|y_j - 3| \leq |y_m - 3| = |3 + 1/n - 3| = 1/n \leq 1/n$ as required.

Then to compute the inf and $liminf$ we examine $\lim_{n\to\infty} z_n = \lim_{n\to\infty} -1 + 1/n = -1$. This sequence is also monotone decreasing. So by the same argument as above we can see that the limit point of the sequence is -1. This is the inf and $liminf$ of x_n.

$sup = 13/4 \qquad limsup = 3 \qquad inf = -1 \qquad liminf = -1.$