Homework 8

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Problem 1 5.1.3 Problem 1

Show that $f(x) = O(|x - x_0|^2)$ as $x \to x_0$ implies $f(x) = o(|x - x_0|)$ as $x \to x_0$ but give an example to show the converse is not true.

PROOF.

We know that by the definition of big O $\exists 1/n, c \ \ |x - x_0| < 1/n \implies |f(x)| \leq c|g(x)$. So we have $\exists 1/n, c \ \ |x - x_0| < 1/n \implies |f(x)| \leq c|x - x_0|^2$. We want to show $\forall 1/m \ \ \exists 1/n \ \ st \ \ |x - x_0| < 1/n \implies |f(x)| < 1/m|x - x_0|$ or equivalently $\lim_{x \to x_0} \frac{f(x)}{|x - x_0|} = 0$. Then since $|f(x)| \leq c|x - x_0|^2$ we have $\frac{|f(x)|}{|x - x_0|} \leq c|x - x_0|$ within $x \in (x_0 - 1/n, x_0 + 1/n)$. Then we take the limit and non-strict inequality is preserved so $\lim_{x \to x_0} \frac{|f(x)|}{|x - x_0|} \leq \lim_{x \to x_0} c|x - x_0| = 0$. Since $\frac{|f(x)|}{|x - x_0|} > 0 \forall x$ then $\lim_{x \to x_0} \frac{|f(x)|}{|x - x_0|} = 0$ as required. □

For example: if we take the function $f(x) = |x - x_0|^{1.5}$ we have $\lim_{x \to x_0} \frac{|x - x_0|^{1.5}}{|x - x_0|} = \lim_{x \to x_0} \sqrt{|x - x_0|} = 0$ so $f \in o(|x - x_0|)$. Then we show that it is not $O(|x - x_0|^2)$ by showing $\frac{|x - x_0|^{1.5}}{|x - x_0|^2}$ is unbounded. We take $\lim_{x \to x_0} \frac{|x - x_0|^{1.5}}{|x - x_0|^2} = \frac{1}{\sqrt{|x - x_0|}} = +\infty$. So there is no constant c that could satisfy $|f(x)| \leq c|x - x_0|^2$

Problem 2 5.2.4 Problem 1 Let f and g be continuous functions on $[a,b]$ and differentiable at every point in the interior, with $g(a) \neq g(b)$. Prove that there exists a point in x_0 in (a,b) such that

$$\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(x_0)}{g'(x_0)}$$

This is also called second mean value theorem

PROOF.

We let $h(x) = (f(b)-f(a))g(x) - (g(b)-g(a))f(x)$. In order to apply mean value theorem we need to know $h(b) - h(a)$

$$\begin{aligned} h(b)-h(a) &= (f(b)-f(a))g(b) - (g(b)-g(a))f(b) - (f(b)-f(a))g(a) + (g(b)-g(a))f(a) \\ &= f(b)g(b) - f(a)g(b) - f(b)g(b) + f(b)g(a) - f(b)g(a) + f(a)g(a) + f(a)g(b) - f(a)g(a) \\ &= 0 \end{aligned}$$

So there is some $x_0 \in (a,b)$ such that $h'(x_0) = 0$. We compute $h'(x) = (f(b)-f(a))g'(x) - (g(b)-g(a))f'(x)$. So:

$$\begin{aligned} 0 &= (f(b)-f(a))g'(x_0) - (g(b)-g(a))f'(x_0) \\ (g(b)-g(a))f'(x_0) &= (f(b)-f(a))g'(x_0) \\ \frac{f'(x_0)}{g'(x_0)} &= \frac{(f(b)-f(a))}{(g(b)-g(a))} \end{aligned}$$

□

Problem 3 5.2.4 Problem 2

if f is a function satisfying $f(x) - f(y) \leq M|x-y|^\alpha$ for all x, y and some fixed M and $\alpha > 1$, prove that f is constant. *Hint: what is* f'. It is rumored that a graduate student once wrote a whole thesis on the class of functions satisfying this condition!

Problem 4 5.2.4 problem 3

Suppose f is defined on $[a, b]$ and g is defined on $[b, c]$ with $f(b) = g(b)$ then define:

$$h(x) = \begin{cases} f(x) & \text{if } a \leq x \leq b \\ g(x) & \text{if } b \leq x \leq c \end{cases}$$

give an example where f and g are differentiable but h is not. Give a definition of one-sided deriviatives $f'(b)$ $g'(b)$ and show that the equality of these is a necessary and sufficient condition for h to be differntiable. Given that $f, g$are differentiable.