

# Final Exam

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## Problem 1

- a) Let  $f$  be a continuous function defined on an open domain, then it is possible that the inverse image of an open set is not an open set.  
False. Otherwise  $f$  assumes same value twice, and  $f^{-1}$  DNE.
- b) If  $f$  is a continuous function defined on  $[0, 1]$ , then it is possible that the image of  $f$  is unbounded.  
False. Continuous function on compact domain Theorem 4.2.3
- c) If  $f(x)$  is differentiable on open interval  $(a, b)$  then its derivative  $f'(x)$  can not have any jump discontinuities on  $(a, b)$   
True
- d)  $x - \sin(x) = o(x^2)$  as  $x \rightarrow 0$   
True
- e) If  $f(x)$  is strictly increasing at  $x_0$  and  $f$  is differentiable at  $x_0$  then  $f'(x_0) > 0$ .  
False

**Problem 2**

Show that every infinite compact set has a limit-point. Is the same true for infinite closed set?

PROOF.

Suppose not, suppose that an infinite compact set  $A$  has no limit points. Then any point  $x$  is not a limit point, so  $\exists 1/n$  st.  $\forall y \in A, y \neq x \implies |y - x| \geq 1/n$ . Let  $a = \inf A$ ,  $b = \sup A$ . We know  $a, b$  must be finite since  $A$  is compact (theorem 3.3.1). Then, the points of  $A$  must be separated by at least  $1/n$ . For any  $x \in A$  we know that  $a \leq x \leq b$ . So there is at most a finite number of values in  $A$ , a contradiction. So there must be a limit point.  $\square$

The same is not true for an infinite closed set. It could contain no limit points and would thus also be closed.

**Problem 3**

If  $f$  is a continuous function on  $\mathbb{R}$ , is it true that  $x$  is a limit-point of  $x_1, x_2, \dots$  implies  $f(x)$  is a limit point of  $f(x_1), f(x_2), \dots$ ? Prove your conclusion.

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**Problem 4**

Function  $f(x)$  is defined by:

$$f(x) = \begin{cases} x^2 \cos(1/x^2), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Show that  $f$  is differentiable at all  $x \in \mathbb{R}$ . Is  $f'(x)$  continuous at  $x = 0$ ? Prove your statement.

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**Problem 5**

Suppose  $f(x)$  is continuously differentiable on an interval  $(a, b)$ . Prove that on any closed subinterval  $[c, d]$  of  $(a, b)$ , the function is uniformly differentiable in the sense that given any  $1/m$  there exists  $1/n$  (independent of  $x_0, x$ ) such that for all  $x, x_0 \in [c, d]$   $|x - x_0| < 1/n$  we have

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \frac{1}{m}|x - x_0|$$

*Hint: Use mean value theorem on  $f(x) - f(x_0)$  and the fact that  $f'(x)$  is continuous function on the compact set  $[c, d]$*