Exam 1

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Problem 1 True or false

1. The set of $\mathbf{Q}$ is countable?

   True

2. Let $A_1, A_2, A_3, ...$ be a sequence of countably many sets, and each $A_i, i = 1, 2, 3, ...$ is countable then their cartesian product $A_1 x \times A_2 \times A_3 \times ...$ i countable.

   False

3. A Cauchy sequence of positive rational numbers cannot be equivalent to a Cauchy sequence of negative rational numbers.

   False, $x_n = 1/n$ $y_n = -1/n$ both converge to 0 so have the same limit and are thus equivalent.

4. Let $x_1, x_2, ...$ be a convergent sequence of real numbers, then $limsup x_n = liminf x_n$.

   True

5. The union of any number of closed sets is a closed set.

   False

Problem 2 The statement "A real number x is a limit-point of a sequence of real numbers $x_1, x_2, ...$" can be written explicitly using quantifiers as follows: "For all $n \in \mathbf{N}$, for all $m \in \mathbf{N}$, there exists $j > m$ such that $|x - x_j| < 1/n$".

Now write the statement "A real number x is not a limit-point of a sequence of real numbers $x_1, x_2, ...$" explicitly using quantifiers

There exists an $n, m \in \mathbf{N}$ such that for all $j > m$ $|x - x_j| > 1/n$

Or equivalently

$$\exists m, n \in \mathbf{N} \; s.t. \forall j > m \; |x - x_j| > 1/n$$

Problem 3 Give an example of a set A that is not closed but such that every point of A is a limit point

We construct cantor set C. Then because C is the cantor set, it is closed and perfect. So it contains all of its limit points and all of its points are limit points. Then let $A = C \setminus \{0, 1\}$. Then every point in A lies within $(0, 1)$ so it is open. Since we only subtracted two points, every point remaining in A must also be a limit point of A (since C is perfect). But it does not contain all of its limit points (it is missing 0 and 1) so it is not closed. Thus A is not closed and every point in A is a limit point.

Problem 4 let $x_1, x_2, ...$ be a sequence of real numbers given by

$$x_n = \frac{\cos(\sqrt{n!}\pi)}{4^n}$$

and define $y_n = x_1 + x_2 + ... + x_n$ prove that the sequence $y_1, y_2, ...$ converges

PROOF.

We know that $y_n = \sum_{i=1}^{n} \frac{\cos(\sqrt{i!}\pi)}{4^i}$

We want to show that the sequence converges, so we show that it is Cauchy. So we must show that $\forall n \in \mathbf{N}\, \exists m \in \mathbf{N}\, \forall j, k \geq m\ |y_j - y_k| \leq 1/n$. We assume that $j \geq k$, then we have that:

$$|y_j - y_k| = \left|\sum_{i=1}^{j} \frac{\cos(\sqrt{i!}\pi)}{4^i} - \sum_{i=1}^{k} \frac{\cos(\sqrt{i!}\pi)}{4^i}\right| = \left|\sum_{i=k}^{j} \frac{\cos(\sqrt{i!}\pi)}{4^i}\right|$$

Then since $\cos(\sqrt{i!}\pi)$ is bounded $[-1, 1]$ $\left|\sum_{i=k}^{j} \frac{\cos(\sqrt{i!}\pi)}{4^i}\right| \leq \sum_{i=k}^{j} \frac{1}{4^i} < \sum_{i=k}^{\infty} \frac{1}{4^i}$. In order to simplify this further we provide two properties of geometric series $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ and $\sum_{n=0}^{x} r^n = \frac{1-r^x}{1-r}$ if $|r| < 1$. Using this we can then solve for an expression for $\sum_{i=k}^{\infty} \frac{1}{4^i}$

$$\sum_{i=0}^{\infty} \frac{1}{4^i} = \sum_{i=0}^{k-1} \frac{1}{4^i} + \sum_{i=k}^{\infty} \frac{1}{4^i}$$

$$\sum_{i=0}^{\infty} \frac{1}{4^i} - \sum_{i=0}^{k-1} \frac{1}{4^i} = \sum_{i=k}^{\infty} \frac{1}{4^i}$$

$$\frac{4}{3} - \left(\frac{1 - \frac{1}{4^{k-1}}}{1 - \frac{1}{4}}\right) = \sum_{i=k}^{\infty} \frac{1}{4^i}$$

$$\frac{1}{3} * \frac{1}{4^k} = \sum_{i=k}^{\infty} \frac{1}{4^i}$$

Now the smallest k can be is m, so $\sum_{i=k}^{\infty} \frac{1}{4^i} \leq \sum_{i=m}^{\infty} \frac{1}{4^i}$ so we must pick an m such that $\sum_{i=m}^{\infty} \frac{1}{4^i} = \frac{1}{3} * \frac{1}{4^m} \leq 1/n$. This is true if $\log_4(n/3) \leq m$ for simplicity, we set $m = n$ since $n > \log_4(n/3)\ \forall n$. So in conclusion:

$$|y_j - y_k| \leq \sum_{i=k}^{j} \frac{1}{4^i} < \sum_{i=k}^{\infty} \frac{1}{4^i} \leq \sum_{i=m}^{\infty} \frac{1}{4^i} = \frac{1}{3} * \frac{1}{4^m}$$

Now we have found an m such that $\forall n \in \mathbf{N}\ \forall j, k \geq m\ |y_j - y_k| \leq 1/n$. So the sequence is Cauchy. Then it must converge, since all Cauchy sequences are convergent in the reals.

□

Problem 5

prove

$$limsup\{x_n + y_n\} \leq limsup\{x_n\} + limsup\{y_n\}$$

if both $limsup\{x_n\}$ and $limsup\{y_n\}$ are finite, and give an example where the equality does not hold.

PROOF.

Fist define $z_n = \{x_n + y_n\}$. Let $X = limsup\{x_n\}$ and $Y = limsup\{y_n\}$ and $Z = limsup\{z_n\}$. By the definition of $limsup$ we have that there are infinitely many points in $\{z_n\}$ within a neighborhood of Z (by neighborhood we mean $|z_n - Z| \leq 1/n$). There must also be a convergent subsequence z' that converges to Z. Because there is a subsequence selection function $f : \mathbf{N} \to \mathbf{N}$: $z'_i = z_j = x_j + y_j$. Then $x_j \leq sup_{a \geq j}\{x_a\}$ and $y_j \leq sup_{a \geq j}\{x_a\}$. We know that the

$$\begin{aligned} Z = \lim_{\imath \to \infty} z'_i = \lim_{\jmath \to \infty} z_j = \lim_{\jmath \to \infty}(x_j + y_j) \\ \leq \lim_{\jmath \to \infty}(sup_{a \geq j}\{x_a\} + sup_{a \geq j}\{y_a\}) \qquad &= \lim_{\jmath \to \infty}(sup_{a \geq j}\{x_a\}) + \lim_{\jmath \to \infty}(sup_{a \geq j}\{y_a\}) \\ \leq limsup\{x_j\} + limsup\{y_j\} \end{aligned}$$

So we have $limsup\{x_n + y_n\} = limsup\{z_n\} = Z \leq limsup\{x_n\} + limsup\{y_n\}$ $\square$

The inequality is necessary, if we have $x_n = 0, 1, 0, 1, 0, 1, ...$ and $y_n = 2, -3, 2, -3, 2, -3, ...$ then $z_n = 2, -2, 2, -2, 2, -2, ...$ Clearly $limsup\{z_n\} = 2$ and $limsup\{x_n\} = 1$ and $limsup\{y_n\} = 2$. But $2 \leq 1 + 2$ so the inequality is necessary (ie equality does not hold).