Final Exam

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Problem 1

a) Let f be a continuous function defined on an open domain, then it is possible that the inverse image of an open set is not an open set.

   False. Otherwise f assumes same value twice, and f^{-1} DNE.

b) If f is a continuous function defined on $[0, 1]$, then it is possible that the image of f is unbounded.

   False. Continuous function on compact domain Theorem 4.2.3

c) If $f(x)$ is differentiable on open interval (a, b) then its derivative $f'(x)$ can not have any jump discontinuities on (a, b)

   True

d) $x - \sin(x) = o(x^2)$ as $x \to 0$

   True

e) If $f(x)$ is strictly increasing at x_0 and f is differentiable at x_0 then $f'(x_0) > 0$.

   False

Problem 2

Show that every infinite compact set has a limit-point. Is the same true for infinite closed set?

Problem 3

If f is a continuous function on $\mathbb{R}$, is it true that x is a limit-point of $x_1, x_2, ...$ implies $f(x)$ is a limit point of $f(x_1), f(x_2), ...$? Prove your conclusion.

Problem 4

Function $f(x)$ is defined by:

$$f(x) = \begin{cases} x^2 \cos(1/x^2), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Show that f is differentiable at all $x \in \mathbb{R}$. Is $f'(x)$ continuous at $x = 0$? Prove your statement.

Problem 5

Suppose $f(x)$ is continuously differentiable on an interval (a, b). Prove that on any closed subinterval $[c, d]$ of (a, b), the function is uniformly differentiable in the sense that given any $1/m$ there exists $1/n$ (independent of x_0, x) such that for all $x, x_0 \in [c, d]$ $|x - x_0| < 1/n$ we have

$$|f(x) - f(x_0) - f'(x_0)(x - x_0) \leq \frac{1}{m}|x - x_0|$$

Hint: Use mean value theorem on $f(x) - f(x_0)$ and the fact that $f'(x)$ is continuous function on the compact set $[c, d]$