Homework 5

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Problem 1 3.2.3 Problem 1

Let A be an open set. Show that if a finite number of points are removed from A the remaining set is still open. Is the same true if a countable number of points are removed?

Proof.

Because A is an open set, we know that A can be represented as the union of disjoint open intervals $\cup^{\infty}(a_i, b_i)$. Furthermore, by the definition of an open set, we know that for any $x \in A$, x is in an open interval that is contained within A. Say that $x \in (a, b)$. Then $x \notin (a, x) \cup (x, b)$. Since every open interval representing A is disjoint, we replace the single interval (a, b) containing x with two more disjoint intervals (a, x) and (x, b). Then $A \setminus \{x\}$ can still be represented by a set of open intervals. So A is open $\square$

No, the same is not true if a countable number of points are removed

Problem 2 3.2.3 Problem 4

Let A be a set and x a number. Show that x is a limit point of A if and only if there exists a sequence $x_1, x_2, ...$ of distinct points in A that converges to x.

Proof.

We prove the forward direction $P \to Q$

We assume that x is a limit point of A and show that there is a sequence of distinct points $x_1, x_2, ...$ in A that converge to x. By the definition of a limit point we know that for any $n \in \mathbf{N}$ there exists a $y_n \in A$ s.t. $y_n \neq x$ and $|y_n - x| \leq 1/n$. Then by the axiom of Archimedes there must be infinitely many such y_n. Since if there was a finite number of y_n then there is some n_{max} and there is some other n satisfying $1/n < 1/n_{max}$.

We then construct the sequence $y_1, y_2, y_3, ...$ where y_n is the y_i satisfying $\max_i\{|y_i - x| < 1/n\}$. Or in other words, we select the point in A that is furthest from x while being in a neighborhood of $1/n$ from x. We choose subsequence y'_i such that $y'_i \neq y'_j$ if $i \neq j$. We can do this since there are infinitely many points within $1/n$ of x for any n so we can always find a unique one, and since each y_n it may only be repeated a finite number of times in a row in the original sequence. Thus, y'_n is a sequence of distinct points satsifying $|y'_n - x| < 1/n \; \forall n$. Thus the sequence y'_n converges to x. So if x is a limit point of A there exists a sequence of distinct points in A that converge to x.

We prove the reverse direction $P \leftarrow Q$

We assume that there is a sequence of points $x_1, x_2, ...$ of distinct points in A that converge to x. We know that since $x_1, x_2, ...$ converges to x then $\forall n \in \mathbf{N} \; \exists m \; s.t. \forall j \geq m \; |x_j - x| < 1/n$. Since the sequence is a sequence of distinct point we know that $x_j \neq x$ for any j. Then we have for any $n \in \mathbf{N}$ there exists a $x_j \in A$ s.t. $x_j \neq x$ and $|x_j - x| < 1/n$. Which is the definition of a limit point of the sequence, so x is a limit point of A

$\square$

Problem 3 3.2.3 Problem 5

Let A be a closed set, x a point in A, and B be the set A with x removed. Under what conditions is B closed.

A set is closed if it contains all of its limit points. So if B is closed x must not be a limit point of A. Since removing a single element from a set will not change its limit points.