Homework 6

Elliott Pryor

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Problem 1 Pg 106 Problem 1

Show that compact sets are closed under arbitrary intersections and finite unions. (Hint: You need to show the intersection of finite or infinte compact sets is compact and the union of finitely many compact sets is a compact set.

PROOF.

We first show that the intersection of finite or infinite compact sets is compact. Let A be the union of any number of compact sets A_i. If $A = \emptyset$ then it is trivially compact. So we examine the case where $A \neq \emptyset$. We know by Theorem 3.2.3 that the intersection of any number of closed sets is a closed set. Since a compact set is closed we know that the intersection of any number of these is at least closed. We show that A must be bounded by contradiction. We assume A is unbounded, so we take a sequence of points $x_1, x_2, ...$ in A that is unbounded. Then by the construction of A the sequence $x_1, x_2, ...$ must be in each A_i. But A_i is compact so it is closed and bounded, so cannot contain an unbounded sequence. A contradiction. So the intersection of any number of compact sets is closed and bounded, so by Theorem 3.3.1 it is compact.

Next we show that the union of a finite number of compact sets is compact. Let $A \cup_{i=1}^{n} A_i$ where A_i is compact. We do this in much the same way as above. We know that the union of finitely many closed sets is closed. So the union of a finite number of compact sets is at least closed. Then we show that A must be bounded. We assume not, we assume A is unbounded. Then there is a sequence of points $x_1, x_2, ...$ in A that is unbounded. Then $\lim = -\infty$ or $\sup = \infty$. Thus there must be infinitely many terms such that $x_j < -n$ or $x_j > n$. Since A is the union of a finite number of sets, by the pigeon hole principle one set A_i must contain infinity many of these. Then A_i is not bounded, a contradiction since A_i is compact. So A is bounded. Thus the union of a finite number of compact sets is compact.

$\square$

Problem 2 Pg 107 Problem 4

If $A \subseteq B_1 \cup B_2$ where B_1 and B_2 are disjoint open sets and A is compact, show that $A \cap B_1$ is compact.

Is the same true if B_1 and B_2 not disjoint?

PROOF.

So we start with $A \cap B_1$ must be bounded since A is bounded. We then show that A is closed. We consider some sequence of points $x_1, x_2, ...$ in $A \cap B_1$. Since $x_1, x_2, ...$ is also in A it must have some finite limit point $x \in A$. By the construction of A $x \in B_1$ or $x \in B_2$. We show that $x \notin B_2$ by contradiction. Suppose $x \in B_2$. By the definition of a limit point in a set, x is a limit point of B_2 if for every neighborhood of x there exists a point in B_2 not equal to x. So any neighborhood of $x \in B_2$ contains infinitely many points. But each $x_1, x_2, ...$ is in B_1 which is disjoint from B_2. So B_2 cannot contain infinitely many points of $x_1, x_2, ...$ A contradiction, so $x \in A \cap B_1$ so $A \cap B_1$ is closed and bounded. Thus it is compact.

□

No the same is not true if B_1 and B_2 overlap. B_2 could contain a limit point of a sequence $x_1, x_2, ...$ in B_1 whose limit point is not in B_1. For example $B_1 = (0, 1)$ and $B_2 = (0.75, 2)$. Then if $A = [0.5, 1.5]$ A is certainly compact. But $A \cap B_1 = [0.5, 1)$ which is not closed, thus not compact.

Problem 3 Pg 107 Problem 8

If A is compact, show that $\sup A$ and $\inf A$ belong to A.

Give an example of a non-compact set A such that both $\sup A$ and $\inf A$ belong to A.

PROOF.

Given a compact set A we show that $\sup A \in A$ and $\inf A \in A$ by contradiction. Suppose $\sup A \notin A$ and $\inf A \notin A$. Then there must be a sequence $x_1, x_2, ...$ in A whose limit point is $\sup A$ and a sequence $y_1, y_2, ...$ in A whose limit point is $\inf A$. If this were not the case, then either $\sup A$ and $\inf A$ are singular points in A, thus a contradiction. Or $\sup A$ is not the least upper bound and $\inf A$ is not the greatest lower bound, a contradiction of the definition of sup and inf. Then A does not contain the limit points of any sequence of points in A, a contradiction since A is compact. $\square$

If $A = [0, 1) \cup (1, 2]$ then the $\inf A = 0$ and $\sup A = 2$ both of which are in A, but it is not compact since it does not contain the point 1. Ie. there is a sequence $x_n = 1/n + 1$ in A whose limit point is clearly 1, but $1 \notin A$ so A is not compact.