Homework 10

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28 Oct 2020

Problem 1 5.3.4 Problem 1

Define:

$$x_+ = \begin{cases} x & if \;\; x \geq 0, \\ 0 & if \;\; x < 0 \end{cases}$$

Prove that $f(x) = x_+^k$ is continuously differentiable if $k \in \mathbb{Z}, \;\; k > 1$

First we show that: $\frac{d}{dx}x^k = k \cdot x^{k-1}$

PROOF. By induction:

We have the base case of $k = 2$. Then $f(x) = x^2 = x \cdot x$. We can use the product rule and theorem 5.3.1 to simplify this. Let $g(x) = h(x) = x$ so $f(x) = (g \cdot h)(x)$. Then $f' = f(x)g'(x) + f'(x)g(x) = (x)(1) + (1)(x) = 2x$. So the base case holds.

Now we assume that for $f(x) = x^m$ $f'(x) = mx^{m-1}$. Then we show that $f(x) = x^{m+1}$ $f'(x) = (m+1)x^m$. We can factor $f(x) = x^{m+1}$ into $g(x) = x$, $h(x) = x^m$. Then by the product rule and inductive assumption that $h'(x) = mx^{m-1}$: $f' = f(x)g'(x) + f'(x)g(x) = (x)(mx^{m-1}) + (1)(x^m) = mx^m + x^m = (m+1)x^m$.

Since the base case and inductive step hold, the claim is true by mathematical induction.

□

Then using the above relation it is easy to see that the derivative is continuous.

PROOF.

Clearly it is continuous for any $x \neq 0$. So we have to show that it is continuous at $x = 0$

We show that on both sides of zero the derivatives are equal. So on the left hand side $(x < 0)$ $f_-(x) = 0$ which is a constant function so $f'_-(x) = 0$. Then by the above theorem on the right hand side $(x \geq 0)$: $f_+(x) = x^k$ and $f'_+(x) = kx^{k-1}$. Since $k \geq 2$, $k - 1 > 0$. So at $x = 0$ $f'_+(0) = k \cdot 0^{k-1} = 0$.

So the derivative of both the left and right hand sides are equal. We can take the one sided limits of $\lim_{x\to 0^-} f'_-(x) = 0$ and $\lim_{x \to 0^+} f'_+(x) = 0$ so $\lim_{x\to 0} f(x)$ exists. So $f'(x)$ is continuous at 0

□

Problem 2 5.4.6 Problem 2

Suppose $f'(x_0) = 0, \quad f''(x_0) = 0, \quad \dots \quad , f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) > 0$ for a $f \in C^n$. Prove that f has a local minimum at x_0 if n is even and that x_0 is neither a local max or local min if n is odd.

Hint: Use Taylor's Theorem 5.4.5

PROOF.

Let T_n denote the n^{th} order Taylor approximation of f around x_0. $T_n = f(x_0) + f'(x_0)(x - x_0) + f''(x_0)(x - x_0)^2 + ... + f^{(n)}(x_0)(x - x_0)^n$. We use the fact that $f^{(k)} = 0$ for $k = 1, 2, ...n - 1$. Then we have $T_n(x) = f(x_0) + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$.

By Theorem 5.4.5, we know that $f(x) - T_n(x) = o(|x - x_0|^n)$, or equivalently $\lim_{x\to x_0} \frac{f(x)-T_n(x)}{|x-x_0|^n} = \lim_{x\to x_0} \frac{f(x)-f(x_0)-\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}{|x-x_0|^n} = 0$. Then we have:

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{|x - x_0|^n} = \lim_{x\to x_0} \frac{\frac{f^{(n)}(x_0)}{n!}(x - x_0)^n}{|x - x_0|^n}$$

If n is even, then $(x - x_0)^n \geq 0$ and $(x - x_0)^n = |x - x_0|^n$, so we get $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{|x-x_0|^n} = \frac{f^{(n)}(x_0)}{n!}$ which must be larger than zero since $f^{(n)} > 0$. So $f(x) - f(x_0) > 0$ thus x_0 is a local min.

If n is odd, then $(x - x_0)^n$ is not positive, but changes signs. On the left $(x < x_0)$ we have $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{|x-x_0|^n} = -\frac{f^{(n)}(x_0)}{n!}$ which is less than zero, so $f(x) - f(x_0) < 0$. But on the right $(x > x_0)$ we have $\lim_{x\to x_0} \frac{f(x)-f(x_0)}{|x-x_0|^n} = \frac{f^{(n)}(x_0)}{n!}$ which is greater than zero, so $f(x) - f(x_0) > 0$. So on the left of x_0 $f(x)$ is less than $f(x_0)$, but on the right f is larger than $f(x_0)$. So it neither a local min nor max.

$\square$