Homework 4

Elliott Pryor

25 Sept 2020

Problem 1 3.1.3 problem 1a)

Compute the sup, inf limsup, liminf and all the limit points of $x_n = 1/n + (-1)^n$

$x_n = 0, 3/2, -2/3, 5/4, -4/5, 7/6, -6/7, 9/8, ...$

Clearly the sup is 3/2.

If we take the $\lim_{n\to\infty} x_n = \lim_{n\to\infty} 1/n + (-1)^n$ the $1/n$ term goes to 0 and the $(-1)^n$ term is always either ± 1. So $liminf = -1$ and $limsup = 1$.

Then to compute the inf we also look at the limit as $n \to \infty$ -1 is the inf of x_n. -1 is an upper bound since $-1 + 1/n > -1\ \forall n$. There cannot be a greater lower bound $y > -1$, by the Axiom of Archimedes $y - (-1) \geq 1/a$ for some $a \in \mathbf{N}$. We choose $n = a + 1$ if a is even and $n = a + 2$ if a is odd, in both cases n is odd so $(-1)^n = -1$. So from Axiom of Archimedes $y - (-1) > 1/n \leftrightarrow y > 1/n - 1 = x_n$, so $y > x_n$ and thus is not an lower bound. So -1 is the greatest lower bound.

Problem 2 3.1.3 problem 2

1. If a bounded sequence is the sum of a monotone increasing and monotone decreasing sequence ($x_n = y_n + z_n$ where $\{y_n\}$ is monotone increasing and $\{z_n\}$ is monotone decreasing) does it follow that the sequence converges?

2. What if $\{y_n\}$ and $\{z_n\}$ are bounded?

---

1. No, the sequence could oscillate.

   PROOF. By contradiction

   Suppose that sequence $x_n = y_n + z_n$ where $\{y_n\}$ is monotone increasing and $\{z_n\}$ is monotone decreasing converges for any y_n, z_n, and x_n is bounded. We let $y_n = \begin{cases} n, & \text{if n is even} \\ (n+1), & \text{if n is odd} \end{cases}$ and $z_n = \begin{cases} -n, & \text{if n is even} \\ -(n-1), & \text{if n is odd} \end{cases}$

   So $y_n = 2, 2, 4, 4, 6, ...$ and $z_n = 0, -2, -2, -4, -4, ...$ And then $x_n = (2+0), (2-2), (4-2), (4-4), (6-4), ... = 2, 0, 2, 0, 2, ...$ So clearly x_n is bounded and it does not converge since $|x_n - x_{n+1}| = 2 \; \forall n \in \mathbf{N}$

   So $x_n = y_n + z_n$ does not converge for monotone increasing sequence y_n and monotone decreasing sequence z_n. A contradiction, so x_n does not converge for every y_n, z_n. $\square$

2. Yes x_n converges if y_n, z_n are bounded. Since y_n is bounded and monotone increasing it must have a finite limit equal to the *sup*, and since z_n is bounded and monotone decreasing it must have a finite limit equal to the *inf*. Thus $\lim_{k\to\infty} y_k = y$ and $\lim_{k\to\infty} z_n = z$. Then $\lim_{k\to\infty} y_k + z_k = y + z$. Since $y, z \in \mathbf{R}$ the sequence $x_n = y_n + z_n$ is convergent.

Problem 3 3.1.3 problem 4

Prove $sup(A \cup B) \geq sup(A)$ and $sup(A \cap B) \leq sup(A)$

PROOF.

First we show $sup(A \cup B) \geq sup(A)$ by contradiction. We suppose that $sup(A \cup B) < sup(A)$. By definition $sup(A \cup B) \geq x \, \forall x \in A \cup B$. Then since every element in A is also in $A \cup B$ it is true that $sup(A \cup B) \geq x \, \forall x \in A$. Since $sup(A) \geq x \, \forall x \in A$ and $sup(A \cup B) < sup(A)$ then $sup(A)$ is not a least upper bound, a contradiction so $sup(A \cup B) \geq sup(A)$

Next we show $sup(A \cap B) \leq sup(A)$ by contradiction. We suppose that $sup(A \cap B) > sup(A)$. Since $sup(A \cap B)$ is the least upper bound, $\exists x \in A \cap B \; s.t|x - sup(A \cap B)| \leq 1/n \; \forall n$. Clearly everything in $A \cap B$ is also in A. Since $sup(A \cap B) > sup(A)$ then $\exists x \in A \cap B > sup(A)$ which is a contradiction since $A \cap B \subseteq A$. So $sup(A \cap B) \leq sup(A)$

$\square$

Problem 4 3.1.3 problem 6

Is every subsequence of a subsequence of a sequence also a subsequence of the sequence?

Yes.

PROOF.

Let x_n be some sequence, and x'_n be a subsequence. We show that x''_n is also a subsequence of x_n. First clearly every element in x'_n is in x_n since x'_n is a subsequence, then it follows that every element of x''_n is an element of x_n by the same reasoning. We need to show that there is a strictly increasing subsequence selection function f. There is a subsequence selection function g that selects elements from x to create x', and another subsequence selection function h that selects elements from x' to create x''. The subsequence selection function $f = h(g(n))$. $f : \mathbf{N} \to \mathbf{N}$ since $g : \mathbf{N} \to \mathbf{N}$ and $h : \mathbf{N} \to \mathbf{N}$. We show that equation 1 is strictly increasing

$$h(g(n+1)) > h(g(n)) \tag{1}$$

$g(n+1) > g(n)$ by definition. Let $a = g(n)$ then $g(n+1) \geq a+1$, so in the worst case we have $g(n+1) = a+1$. So we substitute this into equation 1. $h(a+1) > h(a)$. This is true because h is a subsequence selection function so is strictly increasing. Thus f which is the composition of h and g ($f = h(g(n))$) must be strictly increasing. So f is a subsequence selection function, and x''_n must be a subsequence of x_n as required.

$\square$

Problem 5 Can there exist a sequence whose set of limit points is exactly $1, 1/2, 1/3, ...$? (Hint: what is the liminf of the sequence?)

No

PROOF.

By theorem 3.1.4 the $liminf$ of a sequence is a limit point of the sequence. By the same theorem we have that the $liminf = inf\{\text{limit points}\}$. The $inf\{1, 1/2, 1/3, ...\} = inf\{1/n \mid n \in \mathbf{N}\} = 0$. So 0 must be a limit point of the sequence. A contradiction since $0 \neq 1/n\ \forall n \in \mathbf{N}$, so there cannot exist a sequence whose set of limit points is exactly $1, 1/2, 1/3, ...$ $\square$