Homework 3

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Problem 1 2.2.4-3

If x is a real number, show that there exists a Cauchy sequence of rationals, $x_1, x_2, ...$ representing x such that $x_n < x$ for all n

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PROOF.

We first show that there is some rational number y s.t $x - 1/n \leq y \leq x$ for $x \in \mathbf{R}$ and $n \in \mathbf{N}$. Since the reals are closed under addition, $x - 1/n$ is a real number. By the density of rationals we can find rational numbers y_1 and y_2 such that $|x - y_1| \leq 1/4n$ and $|(x - 1/n) - y_2| \leq 1/4n$. We then let y be the midpoint of $[y_2, y_1]$. So in the worst case, where $y_1 = x - 1/4n$ and $y_2 = x - 5/4n$ then $y = \frac{(x-1/4n)-(x-5/4n)}{2} + (x - 5/4n) = 1/2n + x - 5/4n = x - 3/4n$. We then show that $y < x$ by examining the case where $y_1 = x + 1/4n$ and $y_2 = x - 3/4n$ so $y = \frac{x+1/4n-x+3/4n}{2} + x - 3/4n = x - 1/4n < x$. So $x - y < 1/n$.

From above, we can find some $y \in \mathbf{Q}$ s.t. $x - y < 1/n$. We then construct sequence of rationals $\{y_k\}$ that satisfy this relation. By the construction, $y_k < x \ \forall k$. Then $\forall n \in \mathbf{N} \ \exists m \in \mathbf{N} \ s.t. \ |x - y_k| \leq 1/n \ \forall k \geq m$. By our construction, if $m = n$ the previous statement is true. Therefore, $\{y_k\}$ converges to x. Since $\{y_k\}$ is convergent, then it must be Cauchy and it represents x since it has x as its limit. $\square$

Problem 2 2.2.4-7

Prove $|x-y| \geq |x| - |y|$ for any real numbers x and y.

Proof.

Let $\{x_k\}$ be a Cauchy sequence of rationals representing x and $\{y_k\}$ be a Cauchy sequence of rationals representing y. Then $\{x_k - y_k\}$ is a Cauchy sequence representing $x - y$. By the triangle inequality $|x_k - y_k| \geq |x_k| - |y_k|$. By definition $\lim_{k\to\infty} |x_k| = |x|$ and $\lim_{k\to\infty} |y_k| = |y|$. So $lim_{k\to\infty}|x_k - y_k| \geq \lim_{k\to\infty} |x_k| - \lim_{k\to\infty} |y_k| = |x| - |y|$. So we have $|x-y| \geq |x| - |y|$ for some $x, y \in \mathbf{R}$

$\square$

Problem 3 2.3.3-1

Write out a proof that $\lim_{k\to\infty}(x_k + y_k) = x + y$ if $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} y_k = y$ for sequences of real numbers.

PROOF.

We know that the sequence $\{x_k\}$ converges to x and $\{y_k\}$ converges to y. So $\forall n \in \mathbf{N}\ \exists m_1 \in \mathbf{N}\ s.t.\ \forall k \geq m_1\ |x_k - x| \leq 1/2n$ and $\forall n \in \mathbf{N}\ \exists m_2 \in \mathbf{N}\ s.t.\ \forall k \geq m_2\ |y_k - y| \leq 1/2n$. Since both $\{x_k\}$ and $\{y_k\}$ have limits, both must be Cauchy sequences.

So we want to show that $\{x_k + y_k\}$ converges to $x + y$. So we need to show $\forall n \in \mathbf{N}\ \exists m \in \mathbf{N}\ s.t.\ \forall k \geq m\ |(x + y) - (x_k - y_k)| \leq 1/n$. We choose $m = \max(m_1, m_2)$ then the following is true.

$$|(x + y) - (x_k + y_k)| = |(x - x_k) + (y - y_k)| \leq |x - x_k| + |y - y_k| \leq 1/2n + 1/2n = 1/n$$

So then by the definition of a limit $\lim_{k\to\infty}(x_k + y_k) = x + y$.

□

Problem 4 2.3.3-3

Let $x_1, x_2, ...$ be a sequence of real numbers such that $|x_n| \leq 1/2^n$, and set $y_n = x_1 + x_2 + ... + x_n$. Show that the sequence $y_1, y_2, ...$ converges.

PROOF.

We know that a sequence converges iff it is Cauchy. So we show that $y_1, y_2, ...$ is Cauchy. So we must show $\forall n \in \mathbf{N}\ \exists m \in \mathbf{N}\ s.t.\ \forall j, k \geq m\ |y_j - y_k| \leq 1/n$. Suppose $j \geq k$ then

$$|y_j - y_k| = \left| \sum_{i=1}^{j} (1/2)^i - \sum_{i=1}^{k} (1/2)^i \right| = \sum_{i=k}^{j} (1/2)^i$$

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Now we must find an m such that $\sum_{i=m}^{\infty} (1/2)^i \leq 1/n$. If this holds, then $\sum_{i=k}^{j} (1/2)^i \leq 1/n$ must also be true since $\sum_{i=k}^{j} (1/2)^i < \sum_{i=m}^{\infty} (1/2)^i$

Let $s = \sum_{i=1}^{n} (1/2)^i$. Then $2s = 1 + \sum_{i=1}^{n-1} (1/2)^i = 1 + s - 1/2^n$. So $s = 1 - 1/2^n$. Then as $\lim_{n \to \infty} s = 1$. So $\sum_{i=1}^{\infty} (1/2)^i = s = 1$.

Then $\sum_{i=m}^{\infty} (1/2)^i = (1/2)^m \sum_{i=1}^{\infty} (1/2)^i = (1/2)^m s = (1/2)^m$. So now we just choose an m such that $1/2^m \leq 1/n$. This holds if $m \geq \frac{\ln(1/n)}{\ln(1/2)}$. For simplicity we choose $m = n$ since $n > \frac{\ln(1/n)}{\ln(1/2)}$ $\forall n \in \mathbf{N}$.

So we have $\forall n \in \mathbf{N}\ \forall j, k \geq n\ |y_j - y_k| \leq 1/n$ as required.

$\square$