Homework 2

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Problem 1 Prove that between any two distinct rational numbers there are infinitely many other rational numbers.

PROOF. We first start by showing that between any two distinct rational numbers, there is at least one rational number. Let $p, q \in \mathbf{Q}$ s.t. $p \neq q$. By definition of $p = \frac{a}{b} = \frac{a*d}{bd}$, $q = \frac{c}{d} = \frac{c*b}{bd}$. Because $p \neq q$ $ad \neq cb$ so $|ad - cb| \geq 1$. Then we write $p = \frac{2a*d}{2bd}$ and $q = \frac{2c*b}{2bd}$. Now the difference $|2ad - 2cb| \geq 2$. So there must be an integer, x, in between $2ad$ and $2cb$. So we have a new rational number $r = \frac{x}{2bd}$ which lies between p and q. So we have shown that there is at least one rational number in between any two distinct rational numbers.

Next we show that there are infinitely many rational numbers in between two distinct rational numbers. Let $p, q \in \mathbf{Q}$ s.t. $p \neq q$. By above, there is a rational number r in between p and q. We can then always find a new rational number, r', in between p and the previous r value. So there are infinitely many rational numbers in between p and q.

$\square$

Problem 2 What kinds of real numbers are representable by Cauchy sequences of integers.

Only integers are representable by Cauchy sequences of integers.

Proof.

Let x_n be a Cauchy sequence of integers. Then we know that $\forall n \in \mathbf{N}\ \exists m \in \mathbf{N}\ s.t\ \forall j, k \geq m\ |x_j - x_k| \leq 1/n$. If $n \geq 2$ then x_j must equal x_k. Since $x_j = x_k \in \mathbf{Z}$ the sequence must converge to x_j which is an integer. $\square$

Problem 3 Prove that if a Cauchy sequence of rationals is modified by changing a finite number of terms, the result is an equivalent Cauchy sequence.

PROOF.

Let x_n be a Cauchy sequence of rationals that has limit of x. We know that $\forall n \in \mathbf{N}\ \exists m_n \in \mathbf{N}\ s.t\ \forall j,k \geq m_n\ |x_j - x_k| \leq 1/n$. Then x'_n is the resulting Cauchy sequence after changing a finite number of elements in x_n. Let a be the largest index of the modified elements. Because a finite number of elements were modified, a must be finite.

We show that x is equivalent to x'. So we are attempting to show

$\forall n \in \mathbf{N}\ \exists m \in \mathbf{N}\ s.t\ \forall k \geq m\ |x_k - x'_k| \leq 1/n.$

We know that $\forall k > a\ x_k = x'_k$. We select $m = \max(a+1, m_n)$. Then $\forall n \in \mathbf{N}\ \forall k \geq m\ |x_k - x'_k| = 0 \leq 1/n$ as required. So x and x' are equivalent.

$\square$

Problem 4 Can a Cauchy sequence of positive rational numbers be equivalent to a Cauchy sequence of negative rational numbers.

Yes.

PROOF.

We know that two Cauchy sequences are equivalent iff they have the same limit. So we show that there is a sequence of positive rational numbers and a sequence of negative rational numbers that have the same limit. We take a sequence of positive rational numbers that converges to zero. For example the sequence $x_i = 1/i^3$ converges to zero and is strictly positive.

In order to show that $x_i = 1/i^3$ converges to $x = 0$ we must show that $\forall n \in \mathbf{N}\ \exists m \in \mathbf{N}\ s.t\ \forall k \geq m\ |x_k - x| \leq 1/n$. If we choose $m = n$ then $|x_k - x| \leq |1/n^3 - x| = |1/n^3 - 0| = 1/n^3 \leq 1/n$ as required.

We can similarly find a Cauchy sequence of negative numbers that converge to zero. For example the sequence $x_j = -1/j^3$ converges to zero and is strictly negative.

In order to show that $x_j = -1/j^3$ converges to $x = 0$ we must show that $\forall n \in \mathbf{N}\ \exists m \in \mathbf{N}\ s.t\ \forall k \geq m\ |x_k - x| \leq 1/n$. If we choose $m = n$ then $|x_k - x| \leq |-1/n^3 - x| = |-1/n^3 - 0| = 1/n^3 \leq 1/n$ as required.

Since each sequence is convergent it must be Cauchy. Thus we have found two Cauchy sequences of rational numbers with the same limit so they must be equivalent.

$\square$

Problem 5 Show that if $x_1, x_2, ...$ is a Cauchy sequence of rational numbers there exists a positive integer N such that $x_j \leq N \; \forall j$.

PROOF.

By the definition of a Cauchy sequence we know that $\forall n \in \mathbf{N} \; \exists m \in \mathbf{N} \; s.t \; \forall j, k \geq m \; |x_j - x_k| \leq 1/n$. Then $|x_j - x_m| \leq 1/n$. We consider the case where $n = 1$ as it has the loosest bound on the sequence. Then by the Cauchy criterion every element x_j s.t. $j \geq m$ must be within 1 of x_m: $|x_j - x_m| \leq 1$. So any element that is larger than N must be in the first m elements. Since m is finite, we can find the maximum value x_p of the first $m - 1$ elements; $x_p = \max(x_1, x_2, ...x_{m-1})$. Then we set $N = \lceil \max(x_m + 1, x_p) \rceil$. Then $x_j \leq N \; \forall j$

$\square$