Homework 8

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Problem 1 4.2.4 Problem 1

If f is monotone increasing on an interval and has a jump discontinuity at x_0 in the interior of the domain, show that the jump is bounded above by $f(x_2) - f(x_1)$ for any two points x_1, x_2 in the domain surrounding x_0: $x_1 < x_0 < x_2$

Proof.

Suppose not, suppose that the jump is larger than $f(x_2) - f(x_1)$ for some x_2, x_1 in the domain surrounding x_0. Let j denote the jump, $f(x_2) - f(x_1) < j$. Then we evaluate the value of the function, since it is monotone increasing the smallest the gap between x_1, x_2 could be is j (ie the function is constant on either side of the jump discontinuity). So $f(x_2) - f(x_1) \geq j$, a contradiction since $f(x_2) - f(x_1) < j$.

□

Problem 2 4.2.4 Problem 3

If the domain of a continuous function is an interval, show that the image is an interval. Give examples where the image is an open interval. *Hint: Consider the interval with end points inf{f(D)} and sup{f(D)} where f is the function and D is the domain and use the intermediate Value theorem*

PROOF.

Let $a = f^-1(\inf\{f(D)\})$ and $b = f^-1(sup\{f(D)\})$. If a, b both exist and are finite then we can construct a closed interval $[a, b]$ which by the intermediate value theorem there must exist some $x' \in [a, b]$ st $f(x') \in [f(a), f(b)]$ which is a closed interval.

Then if a or b does not exist it must asymptotically approach $\inf\{f(D)\}$ or $sup\{f(D)\}$. If $\inf\{f(D)\}$ or $sup\{f(D)\}$ are finite, then we can construct closed interval $[a', b']$ where $a' = f^-1(\inf\{f(D)\} + 1/n)$ $b' = f^-1(\sup\{f(D)\} - 1/n)$ for some $n \in \mathbb{N}$ which is contained in D. $\inf\{f(D)\}$ or $sup\{f(D)\}$ are infinite then we can similarly construct a closed interval $[a', b']$ in D where $a' = f^-1(-n)$, $b' = f^-1(n)$. Then by intermediate value theorem there is a closed interval $[f(a'), f(b')]$ contained in the image of f. So the image $f(D) = \cup_{i=1}^{\infty}[f(a'), f(b')]$ which is an open interval.

Note we assumed $a \leq b$, these can be swapped if $b < a$ and the argument still holds.

$\square$

For example, the function $f(x) = \arctan(x)$ is defined on $\mathbb{R}$ and its image is $(-1, 1)$. Or $f(x) = x$ has domain $\mathbb{R}$ and image $\mathbb{R}$.

Problem 3 4.2.4 Problem 9

If f and g are uniformly continuous, show that $f+g$ is uniformly continuous

PROOF.

Let f, g be continuous functions defined on a domain D. By the definition of uniform continuity $\forall 1/m \;\; \exists 1/n_1$ st. $\forall x, x_0 \in D \;\; |x-x_0| < 1/n_1 \implies |f(x)-f(x_0)| < 1/2m$ and for g $\forall 1/m \;\; \exists 1/n_2$ st. $\forall x, x_0 \in D \;\; |x-x_0| < 1/n_2 \implies |g(x)-g(x_0)| < 1/2m$.

Then we want to show $\forall 1/m \;\; \exists 1/n$ st. $\forall x, x_0 \in D \;\; |x-x_0| < 1/n \implies |f(x)+g(x)-f(x_0)-g(x_0)| < 1/m$. We select $n = \min(1/n_1, 1/n_2)$ then: $|f(x)+g(x)-f(x_0)-g(x_0)| = |f(x)-f(x_0)+g(x)-g(x_0)| \leq |f(x)-f(x_0)| + |g(x)-g(x_0)| < 1/2m + 1/2m = 1/m$ $\square$

Problem 4 4.2.4 problem 11

If f is a continuous function on a compact set, show that either f has a zero or f is bounded away from zero ($|f(x)| > 1/n$ for all x in domain and some $1/n$).

Proof.

By theorem 4.2.4 we know that the image of $f(D)$ is compact. Then if $0 \in f(D)$ we are done it has a zero. Then if $0 \notin f(D)$ we show it must be bounded away from 0. Since $f(D)$ is compact it contains all its limit points, and $0 \notin f(D)$ so 0 is not a limit point of $f(D)$. Then by the definition: for any sequence of numbers x_j in $f(D)$ $\exists n \in \mathbb{N}$ st. $\exists m \in naturals$ st. $\forall j \geq m \ \ |x_j - 0| \geq 1/n$. Since $x_j \in f(D)$, $\exists x \in D \ \ st. \ \ f(x) = x_j$. So we have for any $x \in D$ $\exists n \in \mathbb{N}$ st. $|f(x) - 0| \geq 1/n$ and is thus bounded away from 0.

$\square$