Homework 8

Elliott Pryor

28 Oct 2020

Problem 1 5.1.3 Problem 1

Show that $f(x) = O(|x-x_0|^2)$ as $x \to x_0$ implies $f(x) = o(|x-x_0|)$ as $x \to x_0$ but give an example to show the converse is not true.

PROOF.

We know that by the definition of big O $\exists 1/n, c \ \ |x-x_0| < 1/n \implies |f(x)| \leq c|g(x)$. So we have $\exists 1/n, c \ \ |x-x_0| < 1/n \implies |f(x)| \leq c|x-x_0|^2$. We want to show $\forall 1/m \ \ \exists 1/n \ \ st \ \ |x-x_0| < 1/n \implies |f(x)| < 1/m|x-x_0|$ or equivalently $\lim_{x\to x_0} \frac{f(x)}{|x-x_0|} = 0$. Then since $|f(x)| \leq c|x-x_0|^2$ we have $\frac{|f(x)|}{|x-x_0|} \leq c|x-x_0|$ within $x \in (x_0 - 1/n, x_0 + 1/n)$. Then we take the limit and non-strict inequality is preserved so $\lim_{x\to x_0} \frac{|f(x)|}{|x-x_0|} \leq \lim_{x\to x_0} c|x-x_0| = 0$. Since $\frac{|f(x)|}{|x-x_0|} > 0 \forall x$ then $\lim_{x\to x_0} \frac{|f(x)|}{|x-x_0|} = 0$ as required. □

For example: if we take the function $f(x) = |x-x_0|^{1.5}$ we have $\lim_{x\to x_0} \frac{|x-x_0|^{1.5}}{|x-x_0|} = \lim_{x\to x_0} \sqrt{|x-x_0|} = 0$ so $f \in o(|x-x_0|)$. Then we show that it is not $O(|x-x_0|^2)$ by showing $\frac{|x-x_0|^{1.5}}{|x-x_0|^2}$ is unbounded. We take $\lim_{x\to x_0} \frac{|x-x_0|^{1.5}}{|x-x_0|^2} = \frac{1}{\sqrt{|x-x_0|}} = +\infty$. So there is no constant c that could satisfy $|f(x)| \leq c|x-x_0|^2$

Problem 2 5.2.4 Problem 1 Let f and g be continuous functions on $[a, b]$ and differentiable at every point in the interior, with $g(a) \neq g(b)$. Prove that there exists a point in x_0 in (a, b) such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(x_0)}{g'(x_0)}$$

This is also called second mean value theorem

Proof.

We let $h(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x)$. In order to apply mean value theorem we need to know $h(b) - h(a)$

$$\begin{aligned} h(b) - h(a) &= (f(b) - f(a))g(b) - (g(b) - g(a))f(b) - (f(b) - f(a))g(a) + (g(b) - g(a))f(a) \\ &= f(b)g(b) - f(a)g(b) - f(b)g(b) + f(b)g(a) - f(b)g(a) + f(a)g(a) + f(a)g(b) - f(a)g(a) \\ &= 0 \end{aligned}$$

So there is some $x_0 \in (a, b)$ such that $h'(x_0) = 0$. We compute $h'(x) = (f(b) - f(a))g'(x) - (g(b) - g(a))f'(x)$. So:

$$\begin{aligned} 0 &= (f(b) - f(a))g'(x_0) - (g(b) - g(a))f'(x_0) \\ (g(b) - g(a))f'(x_0) &= (f(b) - f(a))g'(x_0) \\ \frac{f'(x_0)}{g'(x_0)} &= \frac{(f(b) - f(a))}{(g(b) - g(a))} \end{aligned}$$

$\square$

Problem 3 5.2.4 Problem 2

if f is a function satisfying $f(x) - f(y) \leq M|x-y|^\alpha$ for all x, y and some fixed M and $\alpha > 1$, prove that f is constant. *Hint: what is* f'. It is rumored that a graduate student once wrote a whole thesis on the class of functions satisfying this condition!

We re-write this as $\frac{f(x)-f(y)}{|x-y|} \leq M|x-y|^{\alpha-1}$. We know that $\alpha > 1$ so $\alpha - 1 > 0$. We then examine the limit as $x \to y$.

$$\lim_{x \to y} \frac{f(x) - f(y)}{|x-y|} \leq \lim_{x \to y} M|x-y|^{\alpha-1} = 0$$

We note that this is the definition of the derivative of f at y. We have $f'(y) = 0$ at an arbitrary y in the domain, so this could be repeated at every point in the domain and we have $f'(y) = 0 \ \ \forall y$. Then the derivative is zero at every point in the domain, so by theorem 5.2.2 f is constant.

Problem 4 5.2.4 problem 3

Is the converse of the mean value theorem true, in the sense that if f is continuous on $[a, b]$ and differentiable on (a, b), a given point x_0 in (a, b) there must exist points $x_1, x_2 \in (a, b)$ such that:

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(x_0)$$

No

Take $f(x) = x^3$ (or any odd polynomial with repeated root at 0). Choose $x_0 = 0$ then $f'(x_0) = 0$. For any $x_2 > x_1$ in (a, b) we have $\frac{f(x_2)-f(x_1)}{x_2-x_1} > 0$ since f is monotone increasing. So you cannot find a pair of points x_1, x_2 such that $\frac{f(x_2)-f(x_1)}{x_2-x_1} = f'(x_0)$. So we found a counterexample showing the converse of the mean value theorem cannot be true.