Homework 10

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Problem 1 5.3.4 Problem 1

Define:

$$x_+ = \begin{cases} x & if \;\; x \geq 0, \\ 0 & if \;\; x < 0 \end{cases}$$

Prove that $f(x) = x_+^k$ is continuously differentiable if $k \in \mathbb{Z}, \;\; k > 1$

First we show that: $\frac{d}{dx}x^k = k \cdot x^{k-1}$

Proof. By induction:

We have the base case of $k = 2$. Then $f(x) = x^2 = x \cdot x$. We can use the product rule and theorem 5.3.1 to simplify this. Let $g(x) = h(x) = x$ so $f(x) = (g \cdot h)(x)$. Then $f' = f(x)g'(x) + f'(x)g(x) = (x)(1) + (1)(x) = 2x$. So the base case holds.

Now we assume that for $f(x) = x^m$ $f'(x) = mx^{m-1}$. Then we show that $f(x) = x^{m+1}$ $f'(x) = (m+1)x^m$. We can factor $f(x) = x^{m+1}$ into $g(x) = x$, $h(x) = x^m$. Then by the product rule and inductive assumption that $h'(x) = mx^{m-1}$: $f' = f(x)g'(x) + f'(x)g(x) = (x)(mx^{m-1}) + (1)(x^m) = mx^m + x^m = (m+1)x^m$.

Since the base case and inductive step hold, the claim is true by mathematical induction.

□

Then using the above relation it is easy to see that it is continuous.

Proof.

We show that on both sides of zero the derivatives are equal. So on the left hand side ($x < 0$) $f_-(x) = 0$ which is a constant function so $f'_-(x) = 0$. Then by the above theorem on the right hand side ($x \geq 0$): $f_+(x) = x^k$ and $f'_+(x) = kx^{k-1}$. Since $k \geq 2$, $k - 1 > 0$. So at $x = 0$ $f'_+(0) = k \cdot 0^{k-1} = 0$.

So the derivative of both the left and right hand sides are equal. We can take the one sided limits of $\lim_{x \to 0^-} f'_-(x) = 0$ and $\lim_{x \to 0^+} f'_+(x) = 0$ so $\lim_{x \to 0} f(x)$ exists. So $f'(x)$ is continuous at 0

□

Problem 2 5.4.6 Problem 2

Suppose $f'(x_0) = 0, \quad f''(x_0) = 0, \quad \dots \quad , f^{(n-1)}(x_0) = 0$ and $f^{(n)}(x_0) > 0$ for a $f \in C^n$. Prove that f has a local minimum at x_0 if n is even and that x_0 is neither a local max or local min if n is odd.

Hint: Use Taylor's Theorem 5.4.5