

Homework 3

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Problem 1 2.2.4-3

If x is a real number, show that there exists a Cauchy sequence of rationals, x_1, x_2, \dots representing x such that $x_n < x$ for all n

PROOF.

We first show that there is some rational number y s.t $x - 1/n \leq y \leq x$ for $x \in \mathbf{R}$ and $n \in \mathbf{N}$. Since the reals are closed under addition, $x - 1/n$ is a real number. By the density of rationals we can find rational numbers y_1 and y_2 such that $|x - y_1| \leq 1/4n$ and $|(x - 1/n) - y_2| \leq 1/4n$. We then let y be the midpoint of $[y_2, y_1]$. So in the worst case, where $y_1 = x - 1/4n$ and $y_2 = x - 5/4n$ then $y = \frac{(x-1/4n)-(x-5/4n)}{2} + (x - 5/4n) = 1/2n + x - 5/4n = x - 3/4n$. We then show that $y < x$ by examining the case where $y_1 = x + 1/4n$ and $y_2 = x - 3/4n$ so $y = \frac{x+1/4n-x+3/4n}{2} + x - 3/4n = x - 1/4n < x$. So $x - y < 1/n$.

From above, we can find some $y \in \mathbf{Q}$ s.t. $x - y < 1/n$. We then construct sequence of rationals $\{y_k\}$ that satisfy this relation. By the construction, $y_k < x \forall k$. Then $\forall n \in \mathbf{N} \exists m \in \mathbf{N}$ s.t. $|x - y_k| \leq 1/n \forall k \geq m$. By our construction, if $m = n$ the previous statement is true. Therefore, $\{y_k\}$ converges to x . Since $\{y_k\}$ is convergent, then it must be Cauchy and it represents x since it has x as its limit. \square

Problem 2 2.2.4-7

Prove $|x - y| \geq |x| - |y|$ for any real numbers x and y .

PROOF.

Let $\{x_k\}$ be a Cauchy sequence of rationals representing x and $\{y_k\}$ be a Cauchy sequence of rationals representing y . Then $\{x_k - y_k\}$ is a Cauchy sequence representing $x - y$. By the triangle inequality $|x_k - y_k| \geq |x_k| - |y_k|$. By definition $\lim_{k \rightarrow \infty} |x_k| = |x|$ and $\lim_{k \rightarrow \infty} |y_k| = |y|$. So $\lim_{k \rightarrow \infty} |x_k - y_k| \geq |x| - |y|$. So we have $|x - y| \geq |x| - |y|$ for some $x, y \in \text{reals}$

\square

Problem 3 2.3.3-1

Write out a proof that $\lim_{k \rightarrow \infty} (x_k + y_k) = x + y$ if $\lim_{k \rightarrow \infty} x_k = x$ and $\lim_{k \rightarrow \infty} y_k = y$ for sequences of real numbers.

Problem 4 2.3.3-3

Let x_1, x_2, \dots be a sequence of real numbers such that $|x_n| \leq 1/2^n$, and set $y_n = x_1 + x_2 + \dots + x_n$. Show that the sequence y_1, y_2, \dots converges.
