

## Homework 4

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### Problem 1 True or false

1. The set of  $\mathbf{Q}$  is countable?

True

2. Let  $A_1, A_2, A_3, \dots$  be a sequence of countably many sets, and each  $A_i, i = 1, 2, 3, \dots$  is countable then their cartesian product  $A_1 \times A_2 \times A_3 \times \dots$  is countable.

False

3. A Cauchy sequence of positive rational numbers cannot be equivalent to a Cauchy sequence of negative rational numbers.

False,  $x_n = 1/n$   $y_n = -1/n$  both converge to 0 so have the same limit and are thus equivalent.

4. Let  $x_1, x_2, \dots$  be a convergent sequence of real numbers, then  $\limsup x_n = \liminf x_n$ .

True

5. The union of any number of closed sets is a closed set.

False

**Problem 2** The statement “A real number  $x$  is a limit-point of a sequence of real numbers  $x_1, x_2, \dots$ ” can be written explicitly using quantifiers as follows: “For all  $n \in \mathbf{N}$ , for all  $m \in \mathbf{N}$ , there exists  $j > m$  such that  $|x - x_j| < 1/n$ ”.

Now write the statement “A real number  $x$  is not a limit-point of a sequence of real numbers  $x_1, x_2, \dots$ ” explicitly using quantifiers

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There exists an  $n, m \in \mathbf{N}$  such that for all  $j > m$   $|x - x_j| > 1/n$

Or equivalently

$$\exists m, n \in \mathbf{N} \text{ s.t. } \forall j > m \ |x - x_j| > 1/n$$

**Problem 3** Give an example of a set  $A$  that is not closed but such that every point of  $A$  is a limit point

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We construct cantor set  $C$ . Then because  $C$  is the cantor set, it is closed and perfect. So it contains all of its limit points and all of its points are limit points. Then let  $A = C \setminus \{0, 1\}$ . Then every point in  $A$  lies within  $(0, 1)$  so it is open. Since we only subtracted two points, every point remaining in  $A$  must also be a limit point of  $A$  (since  $C$  is perfect). But it does not contain all of its limit points (it is missing 0 and 1) so it is not closed. Thus  $A$  is not closed and every point in  $A$  is a limit point.

**Problem 4** let  $x_1, x_2, \dots$  be a sequence of real numbers given by

$$x_n = \frac{\cos(\sqrt{n!}\pi)}{4^n}$$

and define  $y_n = x_1 + x_2 + \dots + x_n$  prove that the sequence  $y_1, y_2, \dots$  converges

PROOF.

We know that  $y_n = \sum_{i=1}^n \frac{\cos(\sqrt{i!}\pi)}{4^i}$

We want to show that the sequence converges, so we show that it is Cauchy. So we must show that  $\forall n \in \mathbf{N} \exists m \in \mathbf{N} \forall j, k \geq m |y_j - y_k| \leq 1/n$ . We assume that  $j \geq k$ , then we have that:

$$|y_j - y_k| = \left| \sum_{i=1}^j \frac{\cos(\sqrt{i!}\pi)}{4^i} - \sum_{i=1}^k \frac{\cos(\sqrt{i!}\pi)}{4^i} \right| = \left| \sum_{i=k}^j \frac{\cos(\sqrt{i!}\pi)}{4^i} \right|$$

Then since  $\cos(\sqrt{i!}\pi)$  is bounded  $[-1, 1]$   $\left| \sum_{i=k}^j \frac{\cos(\sqrt{i!}\pi)}{4^i} \right| \leq \sum_{i=k}^j \frac{1}{4^i} < \sum_{i=k}^{\infty} \frac{1}{4^i}$ . In order to simplify this further we provide two properties of geometric series  $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$  and  $\sum_{n=0}^x r^n = \frac{1-r^{x+1}}{1-r}$  if  $|r| < 1$ . Using this we can then solve for an expression for  $\sum_{i=k}^{\infty} \frac{1}{4^i}$

$$\begin{aligned} \sum_{i=0}^{\infty} \frac{1}{4^i} &= \sum_{i=0}^{k-1} \frac{1}{4^i} + \sum_{i=k}^{\infty} \frac{1}{4^i} \\ \sum_{i=0}^{\infty} \frac{1}{4^i} - \sum_{i=0}^{k-1} \frac{1}{4^i} &= \sum_{i=k}^{\infty} \frac{1}{4^i} \\ \frac{4}{3} - \left( \frac{1 - \frac{1}{4^{k-1}}}{1 - \frac{1}{4}} \right) &= \sum_{i=k}^{\infty} \frac{1}{4^i} \\ \frac{1}{3} * \frac{1}{4^k} &= \sum_{i=k}^{\infty} \frac{1}{4^i} \end{aligned}$$

Now the smallest  $k$  can be is  $m$ , so  $\sum_{i=k}^{\infty} \frac{1}{4^i} \leq \sum_{i=m}^{\infty} \frac{1}{4^i}$  so we must pick an  $m$  such that  $\sum_{i=m}^{\infty} \frac{1}{4^i} = \frac{1}{3} * \frac{1}{4^m} \leq 1/n$ . This is true if  $\log_4(n/3) \leq m$  for simplicity, we set  $m = n$  since  $n > \log_4(n/3) \forall n$ . So in conclusion:

$$|y_j - y_k| \leq \sum_{i=k}^j \frac{1}{4^i} < \sum_{i=k}^{\infty} \frac{1}{4^i} \leq \sum_{i=m}^{\infty} \frac{1}{4^i} = \frac{1}{3} * \frac{1}{4^m}$$

Now we have found an  $m$  such that  $\forall n \in \mathbf{N} \forall j, k \geq m |y_j - y_k| \leq 1/n$ . So the sequence is Cauchy. Then it must converge, since all Cauchy sequences are convergent in the reals.

□

**Problem 5**

prove

$$\limsup\{x_n + y_n\} \leq \limsup\{x_n\} + \limsup\{y_n\}$$

if both  $\limsup\{x_n\}$  and  $\limsup\{y_n\}$  are finite, and give an example where the equality does not hold.

PROOF.

First define  $z_n = \{x_n + y_n\}$ . Let  $X = \limsup\{x_n\}$  and  $Y = \limsup\{y_n\}$  and  $Z = \limsup\{z_n\}$ . By the definition of  $\limsup$  we have that there are infinitely many points in  $\{z_n\}$  within a neighborhood of  $Z$  (by neighborhood we mean  $|z_n - Z| \leq 1/n$ ). There must also be a convergent subsequence  $z'$  that converges to  $Z$ . Because there is a subsequence selection function  $f : \mathbb{N} \rightarrow \mathbb{N}$ :  $z'_i = z_j = x_j + y_j$ . Then  $x_j \leq \sup_{a \geq j}\{x_a\}$  and  $y_j \leq \sup_{a \geq j}\{y_a\}$ . We know that the

$$\begin{aligned} Z &= \lim_{i \rightarrow \infty} z'_i = \lim_{j \rightarrow \infty} z_j = \lim_{j \rightarrow \infty} (x_j + y_j) \\ &\leq \lim_{j \rightarrow \infty} (\sup_{a \geq j}\{x_a\} + \sup_{a \geq j}\{y_a\}) = \lim_{j \rightarrow \infty} (\sup_{a \geq j}\{x_a\}) + \lim_{j \rightarrow \infty} (\sup_{a \geq j}\{y_a\}) \\ &\leq \limsup\{x_j\} + \limsup\{y_j\} \end{aligned}$$

So we have  $\limsup\{x_n + y_n\} = \limsup\{z_n\} = Z \leq \limsup\{x_n\} + \limsup\{y_n\}$  □

The inequality is necessary, if we have  $x_n = 0, 1, 0, 1, 0, 1, \dots$  and  $y_n = 2, -3, 2, -3, 2, -3, \dots$  then  $z_n = 2, -2, 2, -2, 2, -2, \dots$ . Clearly  $\limsup\{z_n\} = 2$  and  $\limsup\{x_n\} = 1$  and  $\limsup\{y_n\} = 2$ . But  $2 \leq 1 + 2$  so the inequality is necessary (ie equality does not hold).