Homework 7

Elliott Pryor

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Problem 1 4.1.5 Problem 2

Let A be the set defined by the equations $f_1(x) = 0$, $f_2(x) = 0$, ... $f_n(x) = 0$. Where $f_1, ...f_n$ are continuous functions defined on the whole line. Show that A is closed. Must A be compact? *Hint:* you can use result of 4.1.5 Problem 1 without proof

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So if $\mathcal{D}_i$ is the set of x satisfying $f_i(x) = 0$ then $A = \cup_{i=1}^{n} \mathcal{D}_i$. We know that $\mathcal{D}_i = f^{-1}(0)$. From 4.1.5 Problem 1 we know that a function f is continuous iff the inverse image of every closed set is a closed set. Since f_i is continuous, $\mathcal{D}_i$ must be closed. Then A is the finite union of a finite number of closed sets which is closed by Theorem 3.2.3. So then A must be closed.

Problem 2 4.1.5 Problem 4

Give a definition of $\lim_{x\to\infty} f(x) = y$. Show that this is true iff for every sequence $x_1, x_2, \ldots$ of points in the domain of f such that $\lim_{n\to\infty} x_n = \infty$ we have $\lim_{n\to\infty} f(x_n) = y$. *Hint:* For the proof of the 2nd part of the problem, refer to the proof of Theorem 4.1.1.

We define $\lim_{x\to\infty} f(x) = y$ as if $\forall 1/m$ there exists an n such that $\forall x > n$ $|f(x) - y| < 1/m$.

Proof.

We first prove the forward direction. $P \to Q$

Given that $\lim_{x\to\infty} f(x) = y$ exists. We take any sequence of points $x_1, x_2, \ldots$ such that $\lim_{n\to\infty} x_n = \infty$. We show that this implies $\lim_{n\to\infty} f(x_n) = y$. By our definition, we know that $\forall 1/m$ there exists an n such that $\forall x > n$ $|f(x) - y| < 1/m$. We also know that $\lim_{n\to\infty} x_n = \infty$. Which means that there are infinitely many terms of x_n satisfying $x_j > n$. So $\lim_{j\to\infty} x_j = \infty \implies \exists k\ s.t.\ \forall j \geq k\ x_j > n \implies |f(x_j) - y| < 1/m \implies \lim_{j\to\infty} f(x_j) = y$

We now prove the reverse direction.

Given that for every sequence $x_1, x_2, \ldots$ of points in the domain of f such that $\lim_{n\to\infty} x_n = \infty$ we have $\lim_{n\to\infty} f(x_n) = y$. We show that $lim_{x\to\infty} = y$. We need to show $\forall 1/m$ there exists an n such that $\forall x > n$ $|f(x) - y| < 1/m$. Suppose not, suppose that $\exists 1/m$ st. $\forall n$ $\exists n < z_n \in \mathbb{D}$ st. $|f(z_n) - y| \geq 1/m$. We can construct a sequence of the points z_n. By definition, $lim_{n\to\infty} = \infty$. But this sequence of points does not converge to y. A contradiction. So for every sequence $x_1, x_2, \ldots$ of points in the domain of f such that $\lim_{n\to\infty} x_n = \infty$ we have $\lim_{n\to\infty} f(x_n) = y$ $\implies \lim_{x\to\infty} f(x) = y$.

$\square$

Problem 3 4.1.5 Problem 7

Give an example of a continuous function with domain $\mathbb{R}$ such that the inverse image of a compact set is not compact.

Let $f = \sin(x)$. Then $\sin(x) \in [-1, 1]$ is compact and $x \in \mathbb{R}$, but $\arcsin(y) \in (-\infty, \infty)$ if $y \in [-1, 1]$, which is not compact.

Problem 4 4.1.5 Problem 10

Show that a function that satisfies a Lipschitz condition is uniformly continuous.

Proof.

A function satisfies the Lipschitz condition if $\exists m > 0$ st. $|f(x) - f(x_0)| \leq m|x - x_0| \;\; \forall x, x_0 \in \mathcal{D}$. Then we need to show that $\forall 1/k \;\; \exists 1/n \;\; s.t. \;\; |f(x) - f(x_0)| < 1/k \;\; \forall x, x_0 \in \mathcal{D}$ satisfying $|x - x_0| < 1/n$. We choose $n = km$ then $|x - x_0| < 1/km \implies |f(x) - f(x_0)| < m * 1/km = 1/k$. Thus we have $\forall 1/k \;\; \exists 1/n \;\; s.t. \;\; |f(x) - f(x_0)| < 1/k \;\; \forall x, x_0 \in \mathcal{D}$ satisfying $|x - x_0| < 1/n$

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