Final Exam

Elliott Pryor
-02508408

20 Nov 2020

Problem 1

a) Let f be a continuous function defined on an open domain, then it is possible that the inverse image of an open set is not an open set.

   False. Otherwise f assumes same value twice, and f^{-1} DNE.

b) If f is a continuous function defined on $[0, 1]$, then it is possible that the image of f is unbounded.

   False. Continuous function on compact domain Theorem 4.2.3

c) If $f(x)$ is differentiable on open interval (a, b) then its derivative $f'(x)$ can not have any jump discontinuities on (a, b)

   True

d) $x - \sin(x) = o(x^2)$ as $x \to 0$

   True

e) If $f(x)$ is strictly increasing at x_0 and f is differentiable at x_0 then $f'(x_0) > 0$.

   False

Problem 2

Show that every infinite compact set has a limit-point. Is the same true for infinite closed set?

Proof.

Suppose not, suppose that an infinite compact set A has no limit points. Then any point x is not a limit point, so $\exists 1/n \quad st. \quad \forall y \in A, \quad y \neq x \quad |y-x| \geq 1/n$. Let $a = infA, \quad b = supA$. We know a, b must be finite since A is compact and thus bounded (theorem 3.3.1). Then, the points of A must be separated by at least $1/n$. For any $x \in A$ we know that $a \leq x \leq b$. So there is at most a finite number of values in A, a contradiction. So there must be a limit point. □

The same is not true for an infinite closed set. It could contain no limit points and would thus also be closed.

Problem 3

If f is a continuous function on $\mathbb{R}$, is it true that x is a limit-point of $x_1, x_2, ...$ implies $f(x)$ is a limit point of $f(x_1), f(x_2), ...$? Prove your conclusion.

Yes

PROOF.

By theorem 4.1.2, a function f on domain $\mathbb{D}$ is continuous iff for every sequence of points $x_1, x_2, ...$ that has a limit in $\mathbb{D}$ the sequence $f(x_1), f(x_2), ...$ is convergent. Since x is a real number then $x \in \mathbb{D}$, and since f is continuous we have that $f(x_1), f(x_2), ...$ is convergent to $f(x)$ and thus $f(x)$ is a limit point.

□

Problem 4

Function $f(x)$ is defined by:

$$f(x) = \begin{cases} x^2 \cos(1/x^2), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

Show that f is differentiable at all $x \in \mathbb{R}$. Is $f'(x)$ continuous at $x = 0$? Prove your statement.

PROOF.

So first we show that f' exists at $x \neq 0$. Let $g(x) = x^2$ and $H(x) = \cos(1/x^2)$By product rule, $f' = g'(x)H(x) + g(x)H'(x)$. We use the chain rule to find $H'(x) = -\sin(1/x^2)(-2/x^3)$. We combine this to get: $f'(x) = 2x\cos(1/x^2) + 2/x\sin(1/x^2)$. This is well defined $\forall x \neq 0$.

Then at $x = 0$ we must show that the derivative exists. We show that $\lim_{x\to 0} \frac{f(x)-f(0)}{x-0}$ exists.

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{x^2\cos(1/x^2)}{x} = \lim_{x\to 0} x\cos(1/x) = 0$$

So f is differentiable everywhere. But $\lim_{x\to 0} f'(x) = \lim_{x\to 0} 2x\cos(1/x^2) + 2/x\sin(1/x^2) = DNE$ so it is not continuous at $x = 0$

$\square$

Problem 5

Suppose $f(x)$ is continuously differentiable on an interval (a, b). Prove that on any closed subinterval $[c, d]$ of (a, b), the function is uniformly differentiable in the sense that given any $1/m$ there exists $1/n$ (independent of x_0, x) such that for all $x, x_0 \in [c, d]$ $|x - x_0| < 1/n$ we have

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| \leq \frac{1}{m}|x - x_0|$$

Hint: Use mean value theorem on $f(x) - f(x_0)$ and the fact that $f'(x)$ is continuous function on the compact set $[c, d]$

Proof.

By the mean value theorem $\exists x_1 \in (x_0, x)$ $f'(x_1) = \frac{f(x)-f(x_0)}{x-x_0}$. We can arrange this to get $|f(x) - f(x_0) - f'(x_1)(x - x_0)| = 0$. We then add $|f'(x_1)(x - x_0) - f'(x_0)(x - x_0)|$ resulting in: $|f(x) - f(x_0) - f'(x_0)(x - x_0)| = |f'(x_1)(x - x_0) - f'(x_0)(x - x_0)|$. Since $f'(x)$ is continuous on a compact domain, by theorem 4.2.5, f' is uniformly continuous. So independent of x, x_0 we have $\forall 1/m \ \ \exists 1/n \ \ st. \ \forall x_1, x_0 \ \ |x - x_0| < 1/n \implies |f'(x_1) - f'(x_0)| < 1/m$. If we multiply this last by $|x - x_0|$ we get $|f'(x_1)(x - x_0) - f'(x_0)(x - x_0)| < 1/m|x - x_0|$.

Thus in conclusion we have, independent of x, x_0, $\forall 1/m \ \ \exists 1/n \ \ st. \ \forall x_1, x_0 \ \ |x - x_0| < 1/n$

$$|f(x) - f(x_0) - f'(x_0)(x - x_0)| = |f'(x_1)(x - x_0) - f'(x_0)(x - x_0)| \leq 1/m|x - x_0|$$

$\square$