Homework 4

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Problem 1 7.3.4 Problem 2

Suppose $f_n \to f$ and all functions f_n satisfy the Lipschitz condition $|f_n(x) - f_n(y)| \leq M|x - y|$ for some constant M, independent of n. Prove that f also satisfies the same Lipschitz condition

Problem 2 7.3.4 Problem 5

If $\lim_{n\to\infty} f_n = f$ and the functions f_n are all monotone increasing, must f be monotone increasing? What happens if f_n are all strictly increasing?

Problem 3 7.3.4 Problem 6

Give an example of a sequence of continuous functions converging pointwise to a function with a discontinuity of the second kind.

Hint: Consider the common domain $\mathbb{D} = [0, 1]$ and

$$f_n(x) = \begin{cases} nx & 0 \leq x \leq 1/n \\ 1 & 1/n \leq x \leq 1 \end{cases}$$

Find another function $g(x)$ which has a discontinuity of the second kind on $\mathbb{D}$ and define $g_n(x) = f_n(x) \cdot g(x)$ You need to prove that g_n are continuous on $\mathbb{D}$ and converges pointwise to a function with a discontinuity of the second kind.

Problem 4 7.3.4 Problem 7

If $|f_n(x)| \leq a_n$ for all x, and $\sum_{n=1}^{\infty} a_n$ converges, prove that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly.

Hint: The series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly is the equivalent to that the sequence of partial sum functions $F_n(x) = \sum_{k=1}^{n} f_k(x)$ converges uniformly. Then prove F_n satisfies the Cauchy criterion for uniform convergence (Theorem 7.3.1).