# Homework 7

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**Problem 1** 7.6.3 Problem 2

If $|f_n(x) - f_n(y)| \leq M|x-y|^\alpha$ for some fixed $M$ and $\alpha > 0$ and all $x, y$ in a compact interval. Show that $\{f_n\}$ is uniformly equicontinuous

Proof.

By definition of uniform equicontinuity we need to show: $\forall 1/m \;\; \exists 1/n \;\; st \;\; |x-y| < 1/n \implies |f_k(x) - f_k(y)| < 1/m \;\; \forall k$ We have from the definition of $f_n$ that $|f_n(x) - f_n(y)| \leq M|x-y|^\alpha$, So $|f_k(x) - f_k(y)| \leq M|x-y|^\alpha \leq M(1/n)^\alpha$ which we need to be $< 1/m$. So we choose $1/n < \sqrt[\alpha]{\frac{1}{Mm}}$ since $M, \alpha$ are defined and $\alpha > 0$ we can compute this for any $1/m$ as required. $\square$

**Problem 2** 7.6.3 Problem 5

Give an example of a sequence that is uniformly equicontinuous but not uniformly bounded

$f_n = x + n$. Is uniformly equicontinuous since can choose $n = m$. But is not uniformly bounded since $f_m \to \infty$ as $n \to \infty$.

**Problem 3** 7.6.3 Problem 6

Prove that the family of all polynomials of degree $\leq N$ with coefficients in $[-1, 1]$ is uniformly bounded and uniformly equicontinuous on any compact interval.

**Hint:** Let the compact interval be $[a, b]$, and each polynomial in the family has the form

$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_N x^N \;\; x \in [a, b], \;\; c_i \in [-1, 1], \;\; 0 \leq i \leq N$

To prove uniform equicontinuity, show that the derivatives of all polynomials in the family are uniformly bounded.

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Proof.

Consider the derivatives of the family of functions: $p'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots + Nc_N x^{N-1}$ Then let $g$ be the polynomial with all coefficients $= 1$: $g(x) = 1 + x + x^2 + \cdots + x^N$, and $g'(x) = 1 + 2x + 3x^2 + \cdots + Nx^{N-1}$

Then clearly outside of $(-1, 1)$ $p \leq g$ and $p' \leq g'$ for any $p$ in the family. And inside $(-1, 1)$, $p'(x) \leq \sum_{i=0}^{n} i$ since for any term $i \cdot c_i x^{i-1} \leq i \cdot x^{i-1} \leq i$ inside $x \in (-1, 1)$. We call this summation $e = \sum_{i=0}^{n} i$. And let $f = \sup_{x \in [a,b]} g'(x)$ which we know exists since $g'$ is continuous.

Then let $M = \max(e, f)$. So then clearly $p'$ is uniformly bounded by $M$.

Then copied from the proof of Corollary 7.6.1: by MVT $\forall x, y \in [a, b] \;\; \forall k \;\; |f_k(x) - f_k(y)| = |f_k'(z)(x - y)| \leq |f_k'(z)||x - y| \leq M|x - y| < 1/m$ by choosing $1/n > \frac{1}{mM}$. Thus all $p$ are equicontinuous $\square$

**Problem 4** 7.6.3 Problem 9

Give an example of a uniformly bounded and uniformly equicontinuous sequence of functions on the whole line that does not have any uniformly convergent subsequences.

**Hint:** Consider the following sequence of functions on $\mathbb{R}$

$$f_n(x) = \begin{cases} 0 & x \leq n-1 \\ x-(n-1) & n-1 < x \leq n \\ 1 & x > n \end{cases}$$

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Proof.

Let $\{f_n\}$ be:

$$f_n(x) = \begin{cases} 0 & x \leq n-1 \\ x-(n-1) & n-1 < x \leq n \\ 1 & x > n \end{cases}$$

Clearly $f_n$ are uniformly bounded by $M = 1$ for all $x \in \mathbb{R}$. Then we show that it is equicontinuous: $\forall 1/m \quad \exists 1/n \quad st \quad |x-y| < 1/n \implies |f_k(x) - f_k(y)| < 1/m \quad \forall k$ If $x, y \notin (n-1, n]$ then $|f_k(x) - f_k(y)| = 0$. So then we only need to consider the middle portion. This is just a line of slope 1 with x-intercept at $n-1$, so it slants from 0, to 1 over interval $[n-1, n]$. If $x, y \in (n-1, n]$ then choose $1/n = 1/m$ since: $|f_k(x) - f_k(y)| = |x-(n-1)-y+(n-1)| = |x-y|$ if only $x$ or $y$ are in $(n-1, n]$ then also choose $1/n = 1/m$ since $(x, y)$ spans the 'corner' of the function, so $|f_k(x) - f_k(y)| \leq |x-y|$.

Consider any subsequence of $f_n$. Then suppose it meets the uniform convergence criteria. So then the Cauchy criteria is met: $\forall 1/m \quad \exists N \quad st \quad \forall j, k \geq N \quad \forall x \in \mathbb{R} |f_j(x) - f_k(x)| < 1/m$ Consider $k \geq j+4$ and $x = j+2$. Then $f_j(j+2) = 1$ and $f_k(j+2) = 0$ Thus $f_j(x) - f_k(x) = 1 > 1/m$ for $m > 1$. We can always do this for any $j, k$. So it violates the Cauchy criterion for any subsequence, thus there are no uniformly convergent subsequences.

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