Homework 6

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Problem 1 7.5.5 Problem 7

If f is C^1 on $[a, b]$ prove that there exists a cubic polynomial P such that $f - P$ and its first derivative vanish at the endpoints of the interval.

Proof.

$\square$

Problem 2 7.5.5 Problem 9

If $f(c) = 0$ for some $c \in (a, b)$ prove that the polynomials approximating f on $[a, b]$ may be taken to vanish at c.

Hint: Here $f(x)$ is a continuous function on $[a, b]$. Assume $f_n(x)$ is the sequence of polynomials approximating $f(x)$ uniformly by WTA, consider $g_n(x) = f_n(x) - f_n(c)$

PROOF.

Let f_n be the sequence of polynomials approximating f by the Wierstrass Approximation Theorem By WAT, we know that $f_n \to f$ uniformly. Thus $\forall 1/m \quad \exists N \quad st \quad \forall n \geq N \quad \forall x \in [a, b] \quad |f_n(x) - f(x)| \leq 1/m$. So we know that at $x = c$ we have $\forall 1/m \quad \exists N \quad st \quad \forall n \geq N \ \ |f_n(c)| \leq 1/m$ Which converges to 0. Thus $f_n \to 0$ at c. So the polynomials approximating f may be taken to vanish at c.

$\square$

Problem 3 7.5.5 Problem 14

(a) For $c_m = \int_{-1}^{1}(1-x^2)^m dx$ obtain the identity $c_m = c_{m-1} - (1/2m)c_m$ by integration by parts.

(b) Show that

$$c_m = 2\frac{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2m}{3 \cdot 5 \cdot 7 \cdot \ldots \cdot 2m+1} = \frac{2(2^m m!)^2}{(2m+1)!}$$

(a) We know the integration by parts formula $\int u dv = uv - \int v du$ from calculus Let $u = (1-x^2)^m$, and $v = x$. Then $dv = dx$ and $du = m \cdot (1-x^2)^{m-1}(-2x) = -2mx(1-x^2)^{m-1}$

$$\begin{aligned} c_m &= \int_{-1}^{1}(1-x^2)^m dx \\ &= x(1-x^2)^m\big|_{-1}^{1} - \int_{-1}^{1} x \cdot -2mx(1-x^2)^{m-1} dx \\ &= 0 - \int_{-1}^{1} -2m \cdot x^2(1-x^2)^{m-1} dx \end{aligned}$$