Homework 9

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Problem 1

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by:

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & x,y \neq 0,0 \\ 0 & x,y = 0,0 \end{cases}$$

(a) Show that $\partial f/\partial x, \partial f/\partial y$ exist for all $(x,y) \in \mathbb{R}^2$

Hint: for $(x,y) \neq (0,0)$ calculate directly by formula. For $(x,y) = (0,0)$ calculate by its definition:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x\to 0} \frac{f(x,0) - f(0,0)}{x - 0}$$

(b) Show that both $\frac{\partial^2 f(0,0)}{\partial x \partial y}$ and $\frac{\partial^2 f(0,0)}{\partial y \partial x}$ exist, but $\frac{\partial^2 f(0,0)}{\partial x \partial y} \neq \frac{\partial^2 f(0,0)}{\partial y \partial x}$

Hint: note

$$\frac{\partial^2 f(0,0)}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \lim_{x\to 0} \frac{\frac{\partial f(x,0)}{\partial y} - \frac{\partial f(0,0)}{\partial y}}{x - 0}$$

Where $\frac{\partial f(x,0)}{\partial y}, \frac{\partial f(0,0)}{\partial y}$ are calculated in part (a)

Problem 2

For any $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ and any multi-index $\alpha = (\alpha_1, \dots, \alpha_n)$ prove that

$$|x^\alpha| \leq |x|^{|\alpha|}$$

where $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$, $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$