# Homework 1

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**Problem 1** 6.1.5 Problem 3

Derive the integration of the derivative theorem from the differentiation of the integral theorem. Can you prove the converse implication?

You **don't** need to prove the converse case. Here, assume the differentiation of the integral theorem is true, namely,

$$\frac{d}{dx}\int_a^x g(t)dt = g(x)$$

for any continuous function $g(x)$ on $[a, b]$ and $a \leq x \leq b$, you need to use this to prove:

$$\int_a^b f'(x)dx = f(b) - f(a)$$

for any $C^1$ function $f$ on $[a, b]$.

**Hint:** Consider function

$$G(x) = \int_a^x f'(t)dt, \quad a \leq x \leq b$$

and see how $G(x)$ and $f(x)$ are related.

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Proof. We want to show

$$\int_a^b f'(x)dx = f(b) - f(a)$$

From theorem 6.1.2, we know that:

$$F(x) = \int_a^x f(t)dt, \quad F'(x) = f(x)$$

.

We consider the function: $G(x) = \int_a^x f'(t)dt$. Then from theorem 6.1.2, we know that $G'(x) = f'(x)$, so $G(x) = f(x) + c$ where $c \in \mathbb{R}$

We evaluate $G(b) - G(a)$ using this information.

$$G(b) - G(a) = (f(b) + c) - (f(a) + c) = f(b) - f(a)$$

We also know from the definition of $G(x)$ that:

$$G(b) - G(a) = \int_a^b f'(t)dt - \int_a^a f'(t)dt = \int_a^b f'(t)dt$$

since $\int_i^j h(x)dx \leq (j - a) \sup_{x \in [i,j]}(h(x))$, so $\int_a^a f'(t)dt = 0$.

Then we know that $\int_a^b f'(t)dt = G(b) - G(a) = f(b) - f(a)$

$\square$

**Problem 2** 6.1.5 Problem 4

Prove the integral mean value theorem: if $f$ is continuous on $[a, b]$ then there exists $y$ in $(a, b)$ such that $\int_a^b f(x)dx = (b-a)f(y)$

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PROOF. Given a function $f$ is continuous on $[a, b]$, we need to find a $y$ such that $\int_a^b f(x)dx = (b-a)f(y)$. Consider $M = \sup_{x\in[a,b]} f(x)$ and $m = \inf_{x\in[a,b]} f(x)$. Then clearly $m \leq f(x) \leq M \quad \forall x \in [a, b]$. So consider $\int_a^b f(x)dx \leq \int_a^b M dx = M \int_a^b dx = M(b-a)$, and $\int_a^b f(x)dx \geq \int_a^b m dx = m \int_a^b dx = m(b-a)$. Thus we have $m(b-a) \leq \int_a^b f(x)dx \leq M(b-a)$.

We know that $\int_a^b f(x)dx = (b-a)K$ for some $K$. So then, from above we have $m(b-a) \leq K(b-a) \leq M(b-a) \to m \leq K \leq M$. So we are looking for $f(y) = K$. Since $f$ is a continuous function, by the Inermediate Value Theorem (theorem 4.2.2), there exists some $y \in [a, b]$ such that $f(y) = K$.

$\square$

**Problem 3** 6.1.5 Problem 8

Let $f$ be a $C^1$ function on the line, and let $g(x) = \int_0^1 f(xy)y^2 dy$. Prove that $g$ is a $C^1$ function and establish a formula for $g'(x)$ in terms of $f$

**Hint:** Use theorem 6.1.7

PROOF. First we say that $h(x,y) = f(xy)y^2$. Then we can use Theorem 6.1.7.

$$g(x) = \int_{a(x)}^{b(x)} h(x,y)dy$$

and

$$g'(x) = h(x,b(x))b'(x) - h(x,a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{\partial h}{\partial x}(x,y)dy$$

In this case: $a(x) = 0, \quad a'(x) = 0$ and $b(x) = 1, \quad b'(x) = 0$. We then evaluate $\frac{\partial h}{\partial x}(x,y) = yf'(xy)y^2 = f'(xy)y^3$ We plug this into our expression of $g'$ resulting in

$$g'(x) = \int_0^1 y^3 \cdot f'(xy)dy$$

Since $f$ is a $C^1$ function on the line, $f'(xy)$ exists and is continous everywhere. Thus the derivative of $g$, exists and is continuous everywhere. Therefore $g \in C^1$ $\square$

**Problem 4** 6.1.5 Problem 10

For a continuous, positive function $w(x)$ on $[a, b]$, define the weighted average operator $A_w$ to be:

$$A_w(f) = \frac{\int_a^b f(x)w(x)dx}{\int_a^b w(x)dx}$$

for continuous functions $f$. Prove that $A_w$ is linear and lies between the maximum and mininum values of $f$.

**Hint:** $A_w$ is linear if

$$A_w(c_1 f + c_2 g) = c_1 A_w(f) + c_2 A_w(g)$$

for any constants $c_1, c_2$ and continuous functions $f, g$ on $[a, b]$

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We will prove this in two parts: PROOF. We start by showing that $A_w$ is linear. So we need to show that $A_w(c_1 f + c_2 g) = c_1 A_w(f) + c_2 A_w(g)$

$$\begin{aligned}
A_w(c_1 f + c_2 g) &= \frac{\int_a^b (c_1 f(x) + c_2 g(x))w(x)dx}{\int_a^b w(x)dx} \\
&= \frac{\int_a^b c_1 f(x)w(x) + c_2 g(x)w(x)dx}{\int_a^b w(x)dx} \\
&= \frac{\int_a^b c_1 f(x)w(x)dx + \int_a^b c_2 g(x)w(x)dx}{\int_a^b w(x)dx} \\
&= \frac{\int_a^b c_1 f(x)w(x)dx}{\int_a^b w(x)dx} + \frac{\int_a^b c_2 g(x)w(x)dx}{\int_a^b w(x)dx} \\
&= c_1 \frac{\int_a^b f(x)w(x)dx}{\int_a^b w(x)dx} + c_2 \frac{\int_a^b g(x)w(x)dx}{\int_a^b w(x)dx} \\
&= c_1 A_w(f(x)) + c_2 A_w(g(x))
\end{aligned}$$

$\square$

PROOF. We next show that $A_w$ must be between the minimum and maximum values of $f$. Let $m = \inf_{x\in[a,b]} f(x)$, and $M = \sup_{x\in[a,b]} f(x)$. Then:

$$\frac{\int_a^b m w(x)dx}{\int_a^b w(x)dx} \leq \frac{\int_a^b f(x)w(x)dx}{\int_a^b w(x)dx} \leq \frac{\int_a^b M w(x)dx}{\int_a^b w(x)dx}$$

$$m \leq \frac{\int_a^b f(x)w(x)dx}{\int_a^b w(x)dx} \leq M$$

$$m \leq A_w(f) \leq M$$

$\square$