

# Homework 8

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9 April 2021

**Problem 1** 10.1.5 Problem 3

If  $f$  is differentiable at  $y$ , show that  $d_u f(y)$  is linear in  $u$ , meaning  $d_{(au+bv)} f(y) = ad_u(y) + bd_v f(y)$ .

**Hint:** Apply Theorem 10.1.1

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PROOF.

We know from theorem 10.1.1, since  $f$  is differentiable at  $y$ , then  $d_{(au+bv)} f(y) = df(y)(au + bv) = a \cdot df(y)u + b \cdot df(y)v$  since matrix multiplication is distributive. We then recognize that  $df(y)u$  has the form (from theorem 10.1.1) of  $d_u(y)$ , and similarly for  $df(y)v$ . So we have  $d_{(au+bv)} f(y) = df(y)(au + bv) = a \cdot df(y)u + b \cdot df(y)v = ad_u(y) + bd_v f(y)$   $\square$

**Problem 2** 10.1.5 Problem 10

Let  $g : [a, b] \rightarrow \mathbb{R}^n$  be differentiable. If  $f : \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable, what is the derivative  $(d/dt)f(g(t))$

**Hint:** Use notation  $g(t) = (g_1(t), \dots, g_n(t))$ ,  $t \in [a, b]$  and  $f(z) = f(z_1, \dots, z_n)$ ,  $z = (z_1, \dots, z_n) \in \mathbb{R}^n$  and Apply the chain rule.

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PROOF.

we know the Chain rule in general is:

$$\frac{\partial f}{\partial x_j} = \sum_{k=1}^n \frac{\partial f}{\partial z_k} \frac{\partial z_k}{\partial x_j}$$

where  $z_k = g_k(x_1, \dots, x_n)$ .

In our case, we are looking for  $x_j = t$ , and  $g(t) = (g_1(t), \dots, g_n(t))$ ,  $t \in [a, b]$  So  $z_k = g_k(t)$ .

So we have

$$\frac{\partial f}{\partial t} = \sum_{k=1}^n \frac{\partial f}{\partial z_k} \frac{\partial g_k(t)}{\partial t}$$

$\frac{\partial g_k(t)}{\partial t}$  is just a number (scalar), so we can write this as two separate sums:  $\sum_{k=1}^n \frac{\partial f}{\partial z_k} * \sum_{k=1}^n \frac{\partial g_k(t)}{\partial t}$ . Then clearly this is just the sum of all partial derivatives of each function, which is the differential  $df, dg$ . We also note that this  $df = \nabla f$  since  $f : \mathbb{R}^n \rightarrow \mathbb{R}$ . So we have:

$$\frac{\partial f}{\partial t} = df \, dg/dt = \nabla f * dg/dt$$

□

**Problem 3** 10.1.5 Problem 13Compute  $df$  of

- a.  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $f(x_1, x_2) = x_1 e^{x_2}$
  - b.  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ ,  $f(x_1, x_2, x_3) = (x_3, x_2)$
  - c.  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ ,  $f(x_1, x_2) = (x_1, x_2, x_1 \cdot x_2)$
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We know that for the differential matrix  $df$  we know that  $df_{k,j} = \frac{\partial f_k}{\partial x_j}$ 

- a.  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$ ,  $f(x_1, x_2) = x_1 e^{x_2}$  There is just one,  $f_k$  and there are  $x_1, x_2$ . So we have that

$$df = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} e^{x_2} & x_1 e^{x_2} \end{bmatrix}$$

- b.  $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ ,  $f(x_1, x_2, x_3) = (x_3, x_2)$

$$df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

- c.  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^3$ ,  $f(x_1, x_2) = (x_1, x_2, x_1 \cdot x_2)$

$$df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1, 0 \\ 0, 1 \\ x_2, x_1 \end{bmatrix}$$

**Problem 4** 10.1.5 Problem 15

If  $f : D \rightarrow \mathbb{R}$  is  $C^1$  with  $D \subseteq \mathbb{R}^n$  and  $D$  contains the line segment joining  $x$  and  $y$ , show that  $f(y) = f(x) + \nabla f(z) \cdot (y - x)$  for some point  $z$  on the line segment. Explain why this is an  $n$ -dimensional analog of the mean value theorem

**Hint:** Define function  $g : [0, 1] \rightarrow \mathbb{R}^n$  by  $g(t) = x + t(y - x)$  and consider the composition function

$$h(t) = (f \circ g)(t) = f(g(t)) : \mathbb{R} \rightarrow \mathbb{R}$$

Apply Mean Value Theorem to  $h(t)$  for  $h(1) - h(0)$  and use the chain rule (formula derived in problem 10 above) to calculate  $h'$

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PROOF.

Let  $g : [0, 1] \rightarrow \mathbb{R}^n$  by  $g(t) = x + t(y - x)$ , and let  $h$  be the composition function:  $h(t) = (f \circ g)(t) = f(g(t)) : \mathbb{R} \rightarrow \mathbb{R}$ . By our definition of  $g$ :  $g(0) = x$ ,  $g(1) = y$  and  $g(t)$  is on the straight line-segment joining  $x$  to  $y$ . Thus the image of  $g \in D$ . Then  $f \in C^1$  so  $h$  is differentiable.

From problem 10: we know that  $d/dt(h) = \nabla f * dg/dt$ , and we know that  $dg/dt = (y - x)$  by definition of  $g$ . Then by Mean value Theorem we know that there exists a  $z$  such that:  $(1 - 0) * h'(z) = h(1) - h(0) = f(g(1)) - f(g(0)) = f(y) - f(x)$  Since we know  $h' = dh/dt = \nabla f(z) * (y - x)$  we have:  $f(y) - f(x) = \nabla f(z)(y - x)$  we simply re-arrange this to get:

$$f(y) = f(x) + \nabla f(z)(y - x)$$

□

We see that this is the mean value theorem by further re-arranging:  $\nabla f(z) = df(z)$  since  $f : \mathbb{R}^n \rightarrow \mathbb{R}$

$$df(z) = \frac{f(y) - f(x)}{y - x}$$

which exactly matches the mean value theorem.