

# Homework 4

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**Problem 1** 7.3.4 Problem 2

Suppose  $f_n \rightarrow f$  and all functions  $f_n$  satisfy the Lipschitz condition  $|f_n(x) - f_n(y)| \leq M|x - y|$  for some constant  $M$ , independent of  $n$ . Prove that  $f$  also satisfies the same Lipschitz condition

PROOF. We know that by the definition of convergence that  $\forall x_0$  the sequence of numbers  $f_1(x_0), f_2(x_0), \dots$  converges to  $f(x_0)$ . Then we have that  $\forall 1/n \exists m$  st  $\forall k > m \quad |f_k(x_0) - f(x_0)| < 1/n$ .

We want to show that  $|f(x) - f(y)| \leq M|x - y|$ . So we expand,

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_k(x) + f_k(x) - f(y) + f_k(y) - f_k(y)| \\ &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f(y) - f_k(y)| \\ &\leq |f(x) - f_k(x)| + M|x - y| + |f(y) - f_k(y)| \end{aligned}$$

We can then set  $k$  such that  $|f(x) - f_k(x)| \leq M|x - y|$ , and same for  $y$  (we choose the maximum  $k$ ). We know we can do this by the convergence of  $f_k$ .

Then we have  $|f(x) - f(y)| \leq M|x - y| + M|x - y| + M|x - y| = 3M|x - y|$ . Thus  $f$  satisfies the Lipschitz condition.  $\square$

**Problem 2** 7.3.4 Problem 5

If  $\lim_{n \rightarrow \infty} f_n = f$  and the functions  $f_n$  are all monotone increasing, must  $f$  be monotone increasing? What happens if  $f_n$  are all strictly increasing?

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Yes, if  $f_n$  are all monotone increasing, then  $f$  is monotone increasing.

PROOF. By contradiction.

Suppose not, suppose  $\exists y > x$ , st  $f(y) < f(x)$  ( $f$  decreases). Then the sequence of numbers  $f_1(y), f_2(y), \dots$  converges to  $f(y)$  and  $f_1(x), f_2(x), \dots$  converges to  $f(x)$ . But since each  $f_k(x)$  is monotone increasing. We have  $f_1(x) \leq f_1(y), f_2(x) \leq f_2(y), \dots$ . Since non-strict inequality is preserved under the limit: we then have  $f(x) \leq f(y)$  a contradiction.  $\square$

If  $f_n$  are strictly increasing, then  $f$  is not necessarily strictly increasing. (Just monotone increasing).

Consider the common domain  $\mathbb{D} = [2, 10]$ . Let  $f_n(x) = 1 - 1/nx$ . Then clearly  $f_n$  are strictly increasing since if  $x_1 > x_0$ , then  $1/x_1 < 1/x_0$ . And  $f_n$  converge to  $f(x) = 1$ . Which is a constant function (by definition still monotone increasing). But is clearly not strictly increasing.

**Problem 3** 7.3.4 Problem 6

Give an example of a sequence of continuous functions converging pointwise to a function with a discontinuity of the second kind.

**Hint:** Consider the common domain  $\mathbb{D} = [0, 1]$  and

$$f_n(x) = \begin{cases} nx & 0 \leq x \leq 1/n \\ 1 & 1/n \leq x \leq 1 \end{cases}$$

Find another function  $g(x)$  which has a discontinuity of the second kind on  $\mathbb{D}$  and define  $g_n(x) = f_n(x) \cdot g(x)$ . You need to prove that  $g_n$  are continuous on  $\mathbb{D}$  and converges pointwise to a function with a discontinuity of the second kind.

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**Problem 4** 7.3.4 Problem 7

If  $|f_n(x)| \leq a_n$  for all  $x$ , and  $\sum_{n=1}^{\infty} a_n$  converges, prove that  $\sum_{n=1}^{\infty} f_n(x)$  converges uniformly.

**Hint:** The series  $\sum_{n=1}^{\infty} f_n(x)$  converges uniformly is the equivalent to that the sequence of partial sum functions  $F_n(x) = \sum_{k=1}^n f_k(x)$  converges uniformly. Then prove  $F_n$  satisfies the Cauchy criterion for uniform convergence (Theorem 7.3.1).

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