Homework 7

Elliott Pryor

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Problem 1 7.6.3 Problem 2

If $|f_n(x) - f_n(y)| \leq M|x-y|^\alpha$ for some fixed M and $\alpha > 0$ and all x, y in a compact interval. Show that $\{f_n\}$ is uniformly equicontinuous

PROOF.

By definition of uniform equicontinuity we need to show: $\forall 1/m \;\; \exists 1/n \;\; st \;\; |x-y| < 1/n \implies |f_k(x) - f_k(y)| < 1/m \;\; \forall k$ We have from the definition of f_n that $|f_n(x) - f_n(y)| \leq M|x-y|^\alpha$, So $|f_k(x) - f_k(y)| \leq M|x-y|^\alpha \leq M(1/n)^\alpha$ which we need to be $< 1/m$. So we choose $1/n < \sqrt[\alpha]{\frac{1}{Mm}}$ since M, α are defined and $\alpha > 0$ we can compute this for any $1/m$ as required. $\square$

Problem 2 7.6.3 Problem 5

Give an example of a sequence that is uniformly equicontinuous but not uniformly bounded

Problem 3 7.6.3 Problem 6

Prove that the family of all polynomials of degree $\leq N$ with coefficients in $[-1, 1]$ is uniformly bounded and uniformly equicontinuous on any compact interval.

Hint: Let the compact interval be $[a, b]$, and each polynomial in the family has the form

$$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_N x^N \quad x \in [a, b], \quad c_i \in [-1, 1], \quad 0 \leq i \leq N$$

To prove uniform equicontinuity, show that the derivatives of all polynomials in the family are uniformly bounded.

Problem 4 7.6.3 Problem 9

Give an example of a uniformly bounded and uniformly equicontinuous sequence of functions on the whole line that does not have any uniformly convergent subsequences.

Hint: Consider the following sequence of functions on $\mathbb{R}$

$$f_n(x) = \begin{cases} 0 & x \leq n-1 \\ x-(n-1) & n-1 < x \leq n \\ 1 & x > n \end{cases}$$