

# Homework 9

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**Problem 1**

Let  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined by:

$$f(x, y) = \begin{cases} \frac{xy(x^2 - y^2)}{x^2 + y^2} & x, y \neq 0, 0 \\ 0 & x, y = 0, 0 \end{cases}$$

- (a) Show that  $\partial f / \partial x, \partial f / \partial y$  exist for all  $(x, y) \in \mathbb{R}^2$

**Hint:** for  $(x, y) \neq (0, 0)$  calculate directly by formula. For  $(x, y) = (0, 0)$  calculate by its definition:

$$\frac{\partial f}{\partial x}(0, 0) = \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x - 0}$$

- (b) Show that both  $\frac{\partial^2 f(0,0)}{\partial x \partial y}$  and  $\frac{\partial^2 f(0,0)}{\partial y \partial x}$  exist, but  $\frac{\partial^2 f(0,0)}{\partial x \partial y} \neq \frac{\partial^2 f(0,0)}{\partial y \partial x}$

**Hint:** note

$$\frac{\partial^2 f(0, 0)}{\partial x \partial y} = \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \lim_{x \rightarrow 0} \frac{\frac{\partial f(x, 0)}{\partial y} - \frac{\partial f(0, 0)}{\partial y}}{x - 0}$$

Where  $\frac{\partial f(x, 0)}{\partial y}, \frac{\partial f(0, 0)}{\partial y}$  are calculated in part (a)

- (a) **PROOF.** We start by considering  $x, y \neq 0$ . Then we directly compute  $\partial f / \partial x$  We have  $\frac{xy(x^2 - y^2)}{x^2 + y^2} = \frac{yx^3 - xy^3}{x^2 + y^2}$

We use the quotient rule to get:  $\partial f / \partial x = \frac{(3yx^2 - y^3)(x^2 + y^2) - (yx^3 - xy^3)2x}{(x^2 + y^2)^2} = \frac{(3yx^2 - y^3)(x^2 + y^2) - 2yx^4 - 2x^2y^3}{(x^2 + y^2)^2}$

Similarly, we solve:  $\partial f / \partial y = \frac{(x^3 - 3xy^2)(x^2 + y^2) - (yx^3 - xy^3)2y}{(x^2 + y^2)^2} = \frac{(x^3 - 3xy^2)(x^2 + y^2) - 2y^2x^3 - 2xy^4}{(x^2 + y^2)^2}$

Then at  $x, y = 0$  we use the definition. So for  $y = 0$

$$\begin{aligned} \partial f / \partial x &= \lim_{x \rightarrow 0} \frac{f(x, 0) - f(0, 0)}{x - 0} \\ &= \lim_{x \rightarrow 0} \frac{0 \cdot x^3 - x \cdot 0^3}{x^2 + 0^2} \\ &= \lim_{x \rightarrow 0} \frac{x}{x} \\ &= \lim_{x \rightarrow 0} 0/x^2 \cdot 1/x \\ &= \lim_{x \rightarrow 0} 0/x^3 \\ &= \lim_{x \rightarrow 0} 0/1 = 0 \end{aligned}$$

Where the last step follows from applying L'hopitals rule 3 times.

We repeat the same process for  $y$

$$\begin{aligned}
 \partial f / \partial y &= \lim_{y \rightarrow 0} \frac{f(0, y) - f(0, 0)}{y - 0} \\
 &= \lim_{y \rightarrow 0} \frac{\frac{y \cdot 0^3 - 0 \cdot y^3}{0^2 + y^2}}{y} \\
 &= \lim_{y \rightarrow 0} 0 / y^2 \cdot 1 / y \\
 &= \lim_{y \rightarrow 0} 0 / y^3 \\
 &= \lim_{y \rightarrow 0} 0 / 1 = 0
 \end{aligned}$$

Thus the partial derivatives exist everywhere.

□

(b) We compute the second partial derivatives: PROOF.

$$\begin{aligned}
 \frac{\partial^2 f(0, 0)}{\partial x \partial y} &= \frac{\partial}{\partial x} \left( \frac{\partial f}{\partial y} \right) = \lim_{x \rightarrow 0} \frac{\frac{\partial f(x, 0)}{\partial y} - \frac{\partial f(0, 0)}{\partial y}}{x - 0} \\
 &= \lim_{x \rightarrow 0} \frac{\frac{(x^3 - 3x \cdot 0^2)(x^2 + 0^2) - 2 \cdot 0^2 x^3 - 2x \cdot 0^4}{(x^2 + 0^2)^2} - 0}{x} \\
 &= \lim_{x \rightarrow 0} \frac{\frac{x^5}{x^4} - 0}{x} \\
 &= 1
 \end{aligned}$$

$$\begin{aligned}
 \frac{\partial^2 f(0, 0)}{\partial y \partial x} &= \frac{\partial}{\partial y} \left( \frac{\partial f}{\partial x} \right) = \lim_{y \rightarrow 0} \frac{\frac{\partial f(0, y)}{\partial x} - \frac{\partial f(0, 0)}{\partial x}}{y - 0} \\
 &= \lim_{y \rightarrow 0} \frac{\frac{(3y \cdot 0^2 - y^3)(0^2 + y^2) - 2y \cdot 0^4 - 2 \cdot 0^2 y^3}{(0^2 + y^2)^2} - 0}{y} \\
 &= \lim_{y \rightarrow 0} \frac{\frac{-y^5}{y^4} - 0}{y} \\
 &= -1
 \end{aligned}$$

□

**Problem 2**

For any  $x = (x_1, x_2, \dots, x_n) \in \mathbb{R}^n$  and any multi-index  $\alpha = (\alpha_1, \dots, \alpha_n)$  prove that

$$|x^\alpha| \leq |x|^{|\alpha|}$$

where  $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$ ,  $|x| = \sqrt{x_1^2 + \dots + x_n^2}$ ,  $|\alpha| = \alpha_1 + \dots + \alpha_n$

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