Homework 7

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Problem 1 7.6.3 Problem 2

If $|f_n(x) - f_n(y)| \leq M|x-y|^\alpha$ for some fixed M and $\alpha > 0$ and all x, y in a compact interval. Show that $\{f_n\}$ is uniformly equicontinuous

Proof.

By definition of uniform equicontinuity we need to show: $\forall 1/m \ \ \exists 1/n \ \ st \ \ |x-y| < 1/n \implies |f_k(x) - f_k(y)| < 1/m \ \ \forall k$ We have from the definition of f_n that $|f_n(x) - f_n(y)| \leq M|x-y|^\alpha$, So $|f_k(x) - f_k(y)| \leq M|x-y|^\alpha \leq M(1/n)^\alpha$ which we need to be $< 1/m$. So we choose $1/n < \sqrt[\alpha]{\frac{1}{Mm}}$ since M, α are defined and $\alpha > 0$ we can compute this for any $1/m$ as required. $\square$

Problem 2 7.6.3 Problem 5

Give an example of a sequence that is uniformly equicontinuous but not uniformly bounded

$f_n = x + n$. Is uniformly equicontinuous since can choose $n = m$. But is not uniformly bounded since $f_m \to \infty$ as $n \to \infty$.

Problem 3 7.6.3 Problem 6

Prove that the family of all polynomials of degree $\leq N$ with coefficients in $[-1, 1]$ is uniformly bounded and uniformly equicontinuous on any compact interval.

Hint: Let the compact interval be $[a, b]$, and each polynomial in the family has the form

$p(x) = c_0 + c_1 x + c_2 x^2 + \cdots + c_N x^N \quad x \in [a, b], \quad c_i \in [-1, 1], \quad 0 \leq i \leq N$

To prove uniform equicontinuity, show that the derivatives of all polynomials in the family are uniformly bounded.

Proof.

Consider the derivatives of the family of functions: $p'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots + N c_N x^{N-1}$ Then let g be the polynomial with all coefficients $= 1$: $g(x) = 1 + x + x^2 + \cdots + x^N$, and $g'(x) = 1 + 2x + 3x^2 + \cdots + N x^{N-1}$

Then clearly outside of $(-1, 1)$ $p \leq g$ and $p' \leq g'$ for any p in the family. And inside $(-1, 1)$, $p'(x) \leq \sum_{i=0}^{n} i$ since for any term $i \cdot c_i x^{i-1} \leq i \cdot x^{i-1} \leq i$ inside $x \in (-1, 1)$. We call this summation $e = \sum_{i=0}^{n} i$. And let $f = \sup_{x \in [a,b]} g'(x)$ which we know exists since g' is continuous.

Then let $M = \max(e, f)$. So then clearly p' is uniformly bounded by M.

Then copied from the proof of Corollary 7.6.1: by MVT $\forall x, y \in [a, b] \quad \forall k \quad |f_k(x) - f_k(y)| = |f_k'(z)(x - y)| \leq |f_k'(z)||x - y| \leq M|x - y| < 1/m$ by choosing $1/n > \frac{1}{mM}$. Thus all p are equicontinuous $\square$

Problem 4 7.6.3 Problem 9

Give an example of a uniformly bounded and uniformly equicontinuous sequence of functions on the whole line that does not have any uniformly convergent subsequences.

Hint: Consider the following sequence of functions on $\mathbb{R}$

$$f_n(x) = \begin{cases} 0 & x \leq n-1 \\ x-(n-1) & n-1 < x \leq n \\ 1 & x > n \end{cases}$$