

# Homework 5

Elliott Pryor

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**Problem 1** 7.4.5 Problem 2

If  $f$  is analytic in a neighborhood of  $x_0$  and  $f(x_0) = 0$ , show that  $f(x)/(x - x_0)$  is analytic in the same neighborhood.

**Hint:** Write  $f(x)$  as a power series expanded at  $x_0$  and pay attention to what is  $a_0$

PROOF.

Since  $f$  is analytic around  $x_0$  we can write it as the power series:  $f(x) = \sum_{n=0}^{\infty} a_n(x - x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2$ . Then  $f(x_0) = a_0 = 0$  by definition of  $f$ .

So then we can write it as  $\sum_{n=1}^{\infty} a_n(x - x_0)^n = a_1(x - x_0) + a_2(x - x_0)^2$ . We can divide by  $x - x_0$ :  $\sum_{n=1}^{\infty} a_n(x - x_0)^{n-1}$ . We then let  $n' = n - 1$  so we have  $\frac{f(x)}{x - x_0} = \sum_{n'=0}^{\infty} a_{n'+1}(x - x_0)^{n'}$  which is a power series expansion about  $x_0$ .

Then in order to show that they are analytic we just need to show that they have the same radius of convergence. Then by analytic continuation they are analytic on the same neighborhood  $(x_0 - R, x_0 + R)$ . We know that  $\limsup_{n \rightarrow \infty} |a_n|^{1/n} = 1/R$  by definition. We need to show that  $\limsup_{n \rightarrow \infty} |a_{n+1}|^{1/n} = 1/R$

$$\begin{aligned} & \limsup_{n \rightarrow \infty} |a_{n+1}|^{1/n} \\ & \limsup_{n \rightarrow \infty} |a_{n+1}|^{\frac{1}{n+1} \cdot \frac{n+1}{n}} \\ & \limsup_{n \rightarrow \infty} \left( |a_{n+1}|^{\frac{1}{n+1}} \right)^{\frac{n+1}{n}} \\ & \limsup_{n \rightarrow \infty} (1/R)^{\frac{n+1}{n}} \\ & 1/R \end{aligned}$$

where the last step follows from  $\lim_{n \rightarrow \infty} \frac{n+1}{n} = 1$

□

**Problem 2** 7.4.5 Problem 6

Prove that if  $f(x)$  is analytic on  $(a, b)$ , then  $F(x) = \int_c^x f(t)dt$  is also analytic on  $(a, b)$ , where  $c$  is any point in  $(a, b)$

**Hint:** You need to prove that for any  $x_0 \in (a, b)$ ,  $F(x)$  has a power series expansion about  $x_0$  for  $x$  close to  $x_0$ . Now pick any fixed  $x_0 \in (a, b)$  and write  $F(x)$  as:

$$F(x) = \int_c^x f(t)dt = \int_c^{x_0} f(t)dt + \int_{x_0}^x f(t)dt = C_0 + \int_{x_0}^x f(t)dt$$

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**Problem 3** 7.4.5 Problem 7 a.

Compute the power-series expansion of  $f(x) = \frac{x^2}{1-x^2}$  about  $x = 0$

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**Problem 4**

Compute the radius of convergence of the following power series:

a.  $\sum (n^4/n!) \cdot x^n$

b.  $\sum \sqrt{n}x^n$

c.  $(n^2 2^n)x^n$

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