

# Homework 1

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**Problem 1** 6.1.5 Problem 3

Derive the integration of the derivative theorem from the differentiation of the integral theorem. Can you prove the converse implication?

You **don't** need to prove the converse case. Here, assume the differentiation of the integral theorem is true, namely,

$$\frac{d}{dx} \int_a^x g(t)dt = g(x)$$

for any continuous function  $g(x)$  on  $[a, b]$  and  $a \leq x \leq b$ , you need to use this to prove:

$$\int_a^b f'(x)dx = f(b) - f(a)$$

for any  $C^1$  function  $f$  on  $[a, b]$ .

**Hint:** Consider function

$$G(x) = \int_a^x f'(t)dt, \quad a \leq x \leq b$$

and see how  $G(x)$  and  $f(x)$  are related.

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PROOF. We want to show

$$\int_a^b f'(x)dx = f(b) - f(a)$$

From theorem 6.1.2, we know that:

$$F(x) = \int_a^x f(t)dt, \quad F'(x) = f(x)$$

.

We consider the function:  $G(x) = \int_a^x f'(t)dt$ . Then from theorem 6.1.2, we know that  $G'(x) = f'(x)$ , so  $G(x) = f(x) + c$  where  $c \in \mathbb{R}$

We evaluate  $G(b) - G(a)$  using this information.

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$$G(b) - G(a) = (f(b) + c) - (f(a) + c) = f(b) - f(a)$$

We also know from the definition of  $G(x)$  that:

$$G(b) - G(a) = \int_a^b f'(t)dt - \int_a^a f'(t)dt = \int_a^b f'(t)dt$$

since  $\int_i^j h(x)dx \leq (j - a) \sup_{x \in [i,j]}(h(x))$ , so  $\int_a^a f'(t)dt = 0$ .

Then we know that  $\int_a^b f'(t)dt = G(b) - G(a) = f(b) - f(a)$

□

**Problem 2** 6.1.5 Problem 4

Prove the integral mean value theorem: if  $f$  is continuous on  $[a, b]$  then there exists  $y$  in  $(a, b)$  such that  $\int_a^b f(x)dx = (b - a)f(y)$

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PROOF. Given a function  $f$  is continuous on  $[a, b]$ , we need to find a  $y$  such that  $\int_a^b f(x)dx = (b - a)f(y)$ . Consider  $M = \sup_{x \in [a, b]} f(x)$  and  $m = \inf_{x \in [a, b]} f(x)$ . Then clearly  $m \leq f(x) \leq M \quad \forall x \in [a, b]$ . So consider  $\int_a^b f(x)dx \leq \int_a^b Mdx = M \int_a^b dx = M(b - a)$ , and  $\int_a^b f(x)dx \geq \int_a^b mdx = m \int_a^b dx = m(b - a)$ . Thus we have  $m(b - a) \leq \int_a^b f(x)dx \leq M(b - a)$ .

We know that  $\int_a^b f(x)dx = (b - a)K$  for some  $K$ . So then, from above we have  $m(b - a) \leq K(b - a) \leq M(b - a) \rightarrow m \leq K \leq M$ . So we are looking for  $f(y) = K$ . Since  $f$  is a continuous function, by the Intermediate Value Theorem (theorem 4.2.2), there exists some  $y \in [a, b]$  such that  $f(y) = K$ .

□

**Problem 3** 6.1.5 Problem 8

Let  $f$  be a  $C^1$  function on the line, and let  $g(x) = \int_0^1 f(xy)y^2 dy$ . Prove that  $g$  is a  $C^1$  function and establish a formula for  $g'(x)$  in terms of  $f$

**Hint:** Use theorem 6.1.7

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**Problem 4** 6.1.5 Problem 10

For a continuous, positive function  $w(x)$  on  $[a, b]$ , define the weighted average operator  $A_w$  to be:

$$A_w(f) = \frac{\int_a^b f(x)w(x)dx}{\int_a^b w(x)dx}$$

for continuous functions  $f$ . Prove that  $A_w$  is linear and lies between the maximum and minimum values of  $f$ .

**Hint:**  $A_w$  is linear if

$$A_w(c_1f + c_2g) = c_1A_w(f) + c_2A_w(g)$$

for any constants  $c_1, c_2$  and continuous functions  $f, g$  on  $[a, b]$