Homework 5

Elliott Pryor

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Problem 1 7.4.5 Problem 2

If f is analytic in a neighborhood of x_0 and $f(x_0) = 0$, show that $f(x)/(x - x_0)$ is analytic in the same neighborhood.

Hint: Write $f(x)$ as a power series expanded at x_0 and pay attention to what is a_0

PROOF.

Since f is analytic around x_0 we can write it as the power series: $f(x) = \sum n = 0^{\infty} a_n(x-x_0)^n = a_0 + a_1(x - x_0) + a_2(x - x_0)^2$. Then $f(x_0) = a_0 = 0$ by definition of f.

So then we can write it as $\sum n = 1^{\infty} a_n(x - x_0)^n = a_1(x - x_0) + a_2(x - x_0)^2$. We can divide by $x - x_0$: $\sum n = 1^{\infty} a_n(x - x_0)^{n-1}$. We then let $n' = n - 1$ so we have
$\frac{f(x)}{x-x_0} = \sum_{n'=0}^{\infty} a_{n'+1}(x - x_0)^{n'}$ which is a power series expansion about x_0.

Then in order to show that they are analytic we just need to show that they have the same radius of convergence. Then by analytic continuation they are analytic on the same neighborhood $(x_0 - R, x_0 + R)$. We know that $\limsup_{n\to\infty} |a_n|^{1/n} = 1/R$ by definition. We need to show that $\limsup_{n\to\infty} |a_{n+1}|^{1/n} = 1/R$

$$\begin{aligned}
&\limsup_{n\to\infty} |a_{n+1}|^{1/n} \\
&\limsup_{n\to\infty} |a_{n+1}|^{\frac{1}{n+1}\cdot\frac{n+1}{n}} \\
&\limsup_{n\to\infty} \left(|a_{n+1}|^{\frac{1}{n+1}}\right)^{\frac{n+1}{n}} \\
&\limsup_{n\to\infty} (1/R)^{\frac{n+1}{n}} \\
&1/R
\end{aligned}$$

where the last step follows from $\lim_{n\to\infty} \frac{n+1}{n} = 1$ $\square$

Problem 2 7.4.5 Problem 6

Prove that if $f(x)$ is analytic on (a, b), then $F(x) = \int_c^x f(t)dt$ is also analytic on (a, b), where c is any point in (a, b)

Hint: You need to prove that for any $x_0 \in (a, b)$, $F(x)$ has a power series expansion about x_0 for x close to x_0. Now pick any fixed $x_0 \in (a, b)$ and write $F(x)$ as:

$$F(x) = \int_c^x f(t)dt = \int_c^{x_0} f(t)dt + \int_{x_0}^x f(t)dt = C_0 + \int_{x_0}^x f(t)dt$$

Proof.

We need to show that $F(x)$ has a power series expansion about any point $x_0 \in (a, b)$. So we pick any fixed $x_0 \in (a, b)$. Then: $F(x) = \int_c^x f(t)dt = \int_c^{x_0} f(t)dt + \int_{x_0}^x f(t)dt = C_0 + \int_{x_0}^x f(t)dt$ Then we can write $f(t) = \sum_{n=0}^{\infty} a_n(x - x_0)^n$ since it is analytic on (a, b) and $x_0 \in (a, b)$, so f has a power series expansion about x_0.

So we have:

$$\begin{aligned}
F(x) &= C_0 + \int_{x_0}^x f(t)dt \\
&= C_0 + \int_{x_0}^x \sum_{n=0}^{\infty} a_n(t - x_0)^n dt \\
&= C_0 + \sum_{n=0}^{\infty} a_n \int_{x_0}^x (t - x_0)^n dt \qquad \text{By linearity of integral} \\
&= C_0 + \sum_{n=0}^{\infty} a_n \left(1/(n+1)(t - x_0)^{n+1}\right|_{x_0}^x \\
&= C_0 + \sum_{n=0}^{\infty} a_n \left(1/(n+1)(x - x_0)^{n+1} - 1/(n+1)(x_0 - x_0)^{n+1}\right) \\
&= C_0 + \sum_{n=0}^{\infty} a_n/(n+1)(x - x_0)^{n+1}
\end{aligned}$$

Let $b_0 = C_0$ and $b_n = \frac{a_{n-1}}{n+1}$ for $n \geq 1$. Then we have: $F(x) = \sum_{n=0}^{\infty} b_n(x - x_0)^{n+1}$.

$\square$

Problem 3 7.4.5 Problem 7 a.

Compute the power-series expansion of $f(x) = \frac{x^2}{1-x^2}$ about $x = 0$

Let $u = x^2$, then $f(u) = \frac{u}{1-u} = u\left(\frac{1}{1-u}\right)$. We know that $g(u) = \frac{1}{1-u}$ has power series expansion: $\sum_{n=0}^{\infty} u^n$ for $|u| < 1$. And $f(u) = u \cdot h(u) = u \cdot \sum_{n=0}^{\infty} u^n = \sum_{n=0}^{\infty} u^{n+1}$. Then we can substitute back in $u = x^2$: $f(x) = \sum_{n=0}^{\infty} x^{2n+2}$. This works for $|u| = x^2 < 1$.

Problem 4

Compute the radius of convergence of the following power series:

a. $\sum(n^4/n!) \cdot x^n$

b. $\sum \sqrt{n}x^n$

c. $(n^2 2^n)x^n$

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a. $1/R = \limsup_{n\to\infty} |n^4/n!|^{1/n} = \limsup_{n\to\infty} \frac{(n^4)^{1/n}}{(n!)^{1/n}} = 0$. Since we know that $(n^4)^{1/n} \to 1$, and $(n!)^{1/n} \to \infty$. So $R = \infty$

b. $1/R = \limsup_{n\to\infty} |\sqrt{n}|^{1/n} = \limsup_{n\to\infty}(n^{1/2})^1/n = \limsup_{n\to\infty}(n^{1/n})^{1/2} = 1^{1/2} = 1$. So $R = 1$.

c. $1/R = \limsup_{n\to\infty}(n^2 2^n)^{1/n} = \limsup_{n\to\infty}(n^2)^{1/n}(2^n)^{1/n} == 1 \cdot 2$. So $R = 1/2$