Homework 1

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Problem 1 6.2.4 Problem 6

Prove that if f is Riemann integrable on $[a, b]$ and $g(x) = f(x)$ for every x except for a finite number, then g is Riemann integrable.

Hint: Mimic the proof of Theorem 6.2.3

PROOF.

Let $a_1, a_2, ..., a_N$ denote the points where $g(a_i) \neq f(a_i)$. Given any $1/n$ surround each a_k by an interval I_k such that $|I_k| < 1/n$. Then, g is continuous and Riemann integrable on $[a, b]$ with $\cup_{k=1}^{N} I_k$ removed.

Then, we estimate the contribution to the oscillation from the I_k intervals. First, f must be bounded since it is Riemann-integrable. Then since g differs from f at a finite number of locations, it must also be bounded. Then let $M = \sup_{x \in [a,b]} g(x)$ and $M = \inf_{x \in [a,b]} g(x)$. Then the contribution of any of the intervals: I_k is at most $1/n \cdot (M - m)$. There are N such intervals, so the total contribution to oscillation is $N/n \cdot (M - m)$.

Then for the remaining intervals: $[a, b] \setminus \cup_{k=1}^{N} I_k,\ g = f$. Since f is Riemann-integrable, the oscillation on these intervals can be made sufficiently small by choosing the partition size sufficiently small. In other words, the total oscillation on a partition P' of $[a, b] \setminus \cup_{k=1}^{N} I_k$, $g = f$ can be, $\forall 1/n\ \ \exists 1/m\ \ st\ \ Osc(g, P') < 1/n$ for $|P'| < 1/m$ (part b of Theorem 6.2.1)

Then the total oscillation $Osc(g, P) < 1/n + N/n \cdot (M - m)$. Since N, M, m are constant, there exists a sequence of partitions such that $Osc(g, P_j) \to 0\ \ as\ \ j \to \infty$ by selecting n large enough.

$\square$

Problem 2 6.2.4 Problem 9, part b

If f is Riemann integrable on $[a, b]$, prove $F(x) = \int_a^x f(t)dt$ satisfies a Lipschitz condition

PROOF. We need to show that $|F(x) - F(x_0)| \leq M|x - x_0|$ for some M.

Without loss of generality assume $x > x_0$. Then by the definition of F. We have

$$|F(x) - F(x_0)| = \int_a^x f(t)dt - \int_a^{x_0} f(t)dt = \int_{x_0}^x f(t)dt$$

We say $M = \sup_{x \in [a,b]} f(x)$. Then

$$|F(x) - F(x_0)| = \int_{x_0}^x f(t)dt \leq M|x - x_0|$$

as required. $\square$

Problem 3 6.2.4 Problem 10

If f is Riemann integrable on $[a,b]$ and continuous at x_0, prove that $F(x) = \int_a^x f(t)dt$ is differentiable at x_0 and $F'(x_0) = f(x_0)$. Show that if f has a jump discontinuity at x_0, then F is not differentiable at x_0.

Hint: Refer to the proof of Theorem 6.1.2, note that $f(x)$ being continuous at x_0 can be written as $\forall 1/m, \quad \exists 1/n \quad st. \quad \forall x \in [a,b], \quad |x - x_0| < 1/n$, we have $|f(x) - f(x_0)| < 1/m$ or $f(x_0) - 1/m < f(x) < f(x_0) + 1/m$

Proof.

We show this in two parts. We first show that if $f(x)$ is continuous at x_0 then $F'(x_0)$ exists and equals $f(x_0)$. So we want to show $\lim_{h\to 0} \frac{F(x_0+h)-F(x_0)}{h} = f(x_0)$.

We have $\lim_{h\to 0} \frac{F(x_0+h)-F(x_0)}{h} = 1/h \int_{x_0}^{x_0+h} f(t)dt$. Let $M = \sup_{x\in[x_0,x_0+h]}$ and $m = \inf_{x\in[x_0,x_0+h]}$. Then certainly $m \leq 1/h \int_{x_0}^{x_0+h} f(t)dt \leq M$. Since $f(x)$ is continuous at x_0 we have
$\forall 1/q, \ \exists 1/n \quad s.t. \quad \forall x \in [a,b] \ |x - x_0| < 1/n$ we have $|f(x) - f(x_0)| < 1/q$, thus m, M must converge to $f(x_0)$. So we have by squeeze theorem
$\lim_{h\to 0} \frac{F(x_0+h)-F(x_0)}{h} = 1/h \int_{x_0}^{x_0+h} f(t)dt = f(x_0)$

The second part we show that $F'(x_0)$ DNE if $f(x_0)$ is discontinuous. We say that
$\lim_{x\to x_0^+} f(x) = q$ and $\lim_{x\to x_0^-} f(x) = r$ where $q \neq r$. We examine $\lim_{h\to 0} \frac{F(x_0+h)-F(x_0)}{h}$ as we approach from both sides. On the left side of zero $\lim_{h\to 0^-}$, $f(x) \to r$, and by the first part $F'(x) \to r$, but on the right side of zero: $\lim_{x\to x_0^+}$, $f(x) \to q$, and by the first part $F'(x) \to q$. Since the limit is not the same from both sides the limit DNE. $\square$