Exam 1

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Problem 1

True or False

(a) if $f(x)$ on a finite interval $[a,b]$ is Riemann integrable, then $f(x)$ can only have jump discontinuities.

False. - Hole

(b) If $|f(x)|$ is Riemann integrable on $[a,b]$ then $f(x)$ is also Riemann integrable on $[a,b]$.

False

(c) if $\sum_{k=1}^{\infty} a_k$ converges conditionally and $\sum_{k=1}^{\infty} b_k$ converges absolutely, then $\sum_{k=1}^{\infty} a_k \cdot b_k$ converges absolutely

True

(d) If $\sum_{k=1}^{\infty} a_k$ converges conditionally, then the sum of all positive terms of this series diverges

True

(e) Let f_n be a sequence of C^1 functions defined on (a,b) if $f_n'(x)$ converges uniformly to $g(x)$ on (a,b). Then there exists a C^1 function f on (a,b) such that $f_n(x)$ converge uniformly to $f(x)$ and $f'(x) = g(x)$

True

Problem 2

Give the statement (using quantifiers) that a sequence of functions $f_n(x)$ on a common domain $\mathbb{D}$ does **NOT** converge uniformly to a function $f(x)$ on $\mathbb{D}$

a sequence of functions $f_n(x)$ on a common domain $\mathbb{D}$ does **NOT** converge uniformly to a function $f(x)$ on $\mathbb{D}$ if:

$$\exists 1/m \;\; st \;\; \forall N \;\; \exists x \in \mathbb{D}, \;\; \exists n \geq N \;\; st \;\; |f_n(x) - f(x)| \geq 1/m$$

Problem 3

Compute the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{4n^5 - 7n + 2}{3^n} x^n$$

Here we have: $a_n = \frac{4n^5-7n+2}{3^n}$ We know the radius of convergence is given by: $\frac{1}{R} = \limsup_{n\to\infty} \sup \sqrt[n]{|a_n|}$

$$\begin{aligned}
\frac{1}{R} &= \limsup_{n\to\infty} \sqrt[n]{|a_n|} \\
&= \limsup_{n\to\infty} \sqrt[n]{\left|\frac{4n^5 - 7n + 2}{3^n}\right|} \\
&= \limsup_{n\to\infty} \left|\frac{\sqrt[n]{4n^5 - 7n + 2}}{\sqrt[n]{3^n}}\right| \\
&= \frac{1}{3}
\end{aligned}$$

(the numerator of the last step comes from the lemma that we used for 7.4.1 Or from the example we did showing radius of convergence of $a_n = p(n)/q(n)$ is 1, since $4n^5 - 7n + 2$ is a polynomial)

So $R = 3$

Problem 4

Prove the first part of the linearity of Riemann Integral. Namely, if both $f(x)$ and $g(x)$ are Riemann integrable on $[a,b]$, then $f+g$ is Riemann integrable on $[a,b]$, and

$$\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Here $(f+g)(x) = f(x) + g(x)$

Hint: Use part (e) of Theorem 6.2.1 and linear property of Cauchy sum

PROOF. We take the Cauchy sum: $S(f+g,P)$ and show that it converges to $\int_a^b f(x)dx + \int_a^b g(x)dx$.

Since f, g are both bounded functions, then $f+g$ must also be bounded.

By linearity of Cauchy sums, we have $S(f+g,P) = S(f,P) + S(g,P)$. By Theorem 6.2.1 part e, we have $S(f,P) \to \int_a^b f(x)dx$ as the maximum interval length of P tends to zero.

So as $|P| \to 0$ we have that $S(f+g,P) = \int_a^b (f+g)(x)dx$. We use the linearity of S to find what the value of this is. $S(f+g,P) = S(f,P) + S(g,P) = \int_a^b f(x)dx + \int_a^b g(x)dx$ by Theorem 6.2.1 and f, g are Riemann Integrable.

$\square$

Problem 5

Let $b_1, b_2, ...$ be a sequence of positive numbers convergent monotonically to zero: $b_1 \geq b_2 \geq b_3 ...$ and $\lim_{n\to\infty} b_n = 0$. If $|a_n| \leq b_n - b_{n+1}$ for all n. Prove: $\sum_{n=1}^{\infty} a_n$ converges absolutely.

PROOF.

We show that $\sum_{n=1}^{\infty} a_n$ converges absolutely. So we show that: $\sum_{n=1}^{\infty} |a_n|$ converges by the Cauchy Criterion.

$\forall 1/n \;\; \exists m \;\; st \;\; \forall q \geq p \geq m \;\; \sum_{k=p}^{q} |a_k| < 1/n$. Since we know that a_n is a sequence of positive numbers converging monotonically to zero: $b_n - b_{n+1} \geq 0$, so we can drop the absolute value symbol since all terms are positive.

Then $\sum_{k=p}^{q} a_k = b_p - b_{p+1} + b_{p+1} - b_{p+2} + b_{p+2} + ... - b_q + b_q - b_{q+1} = b_p - b_{q+1}$ Since the sequence b_n converges to zero, we can choose m large enough that $b_m < 1/n$. Thus since $b_m \geq b_p \geq b_{q+1}$, we know that $b_p - b_{q+1} < 1/n$.

$\square$

Problem 6a

Let $f_n(x) \to f(x)$ uniformly on a finite interval $[a,b]$ and all $f_n(x)$ are Riemann integrable on $[a,b]$. Define $F_n(x) = \int_a^x f_n(t)dt$. Prove that $F_n \to F$ uniformly on $[a,b]$ for some $F(x)$, and give the expression of the limit function $F(x)$.

PROOF.

We want to show that: $\forall 1/m \;\; \exists N \;\; st \;\; \forall x \in [a,b] \;\; \forall n \geq N \;\; |F_n(x) - F(x)| < 1/m$

$|F_n(x) - F(x)| = \left|\int_a^x f_n(t)dt - \int_a^x f(x)dx\right|$ Which then by linearity of the integral we have $\left|\int_a^x f_n(t)dt - \int_a^x f(x)dx\right| = \left|\int_a^x (f_n(t) - f(t))dt\right| \leq \int_a^x |f_n(t) - f(t)|dt$

But since $f_n \to f$ uniformly: $\exists N \;\; st \;\; \forall x \in [a,b] \;\; \forall n \geq N \;\; |f_n(x) - f(x)| < \frac{1}{m(b-a)}$.

Then we have that $\int_a^x |f_n(t) - f(t)|dt \leq \int_a^x \frac{1}{m(b-a)}dt \leq \int_a^b \frac{1}{m(b-a)} = 1/m.$ $\square$

Problem 6b

Is the same true on the whole line? Namely, let $f_n(x) \to f(x)$ uniformly on the entire real line $\mathbb{R}$, and all $f_n(x)$ are Riemann integrable on any finite interval. Define $F_n(x) = \int_0^x f_n(t)dt$. Is it always true that $F_n(x) \to F(x)$ uniformly on $\mathbb{R}$ for some F? Prove it if your answer is Yes, or give a counter example if your answer is No.

No,

The problem comes in when we take $x \to \infty$. If we take the sequence of constant functions $f_n(x) = (10 - 1/n)$ they clearly converge uniformly to $f(x) = 10$. F_n also converges uniformly to F on any finite interval. But consider $\lim_{x\to\infty} |F_n(x) - F(x)|$. $|F_n(x) - F(x)| = 1/n \cdot x$. We cannot choose an N large enough such that for $n \geq N$, $x/n < 1/m \;\; \forall x$. Since we can always increase x slightly to make $x/n > 1/m$.