Homework 6

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Problem 1 7.5.5 Problem 7

If f is C^1 on $[a, b]$ prove that there exists a cubic polynomial P such that $f - P$ and its first derivative vanish at the endpoints of the interval.

Hint: you can use the result of problem 1 without proving it

PROOF.

From problem 1 we know that there exists a polynomial P of degree $2n-1$ satisfying $P(x_k) = a_k$ and $P'(x_k) = b_k$ for $k = 1...n$. In our case we want a cubic polynomial, so $n = 2$. Thus we have $P(x_1) = a_1$, $P(x_2) = a_2$ and $P'(x_k) = b_k$, $P'(x_2) = b_2$. Let $x_1 = a$ and $x_2 = b$. We then let $a_1 = f(a)$ and $a_2 = f(b)$, and similarly $b_1 = f'(a), b_2 = f'(b)$.

Then by our construction of P we have that $f(a) - P(a) = f(a) - a_1 = 0$, and $f(b) - P(b) = f(b) - a_2 = 0$. Satisfying the first condition. Then similarly, $f'(a) - P'(a) = f'(a) - b_1 = 0$ and $f'(b) - P'(b) = f'(b) - b_2 = 0$. And P is of degree 3. Thus we have a cubic polynomial P such that $f - P$ and its first derivative vanish at the endpoints of the interval.

$\square$

Problem 2 7.5.5 Problem 9

If $f(c) = 0$ for some $c \in (a, b)$ prove that the polynomials approximating f on $[a, b]$ may be taken to vanish at c.

Hint: Here $f(x)$ is a continuous function on $[a, b]$. Assume $f_n(x)$ is the sequence of polynomials approximating $f(x)$ uniformly by WTA, consider $g_n(x) = f_n(x) - f_n(c)$

PROOF.

Let f_n be the sequence of polynomials approximating f by the Wierstrass Approximation Theorem By WAT, we know that $f_n \to f$ uniformly. Thus $\forall 1/m \quad \exists N \quad st \quad \forall n \geq N \quad \forall x \in [a, b] \quad |f_n(x) - f(x)| \leq 1/m$. So we know that at $x = c$ we have $\forall 1/m \quad \exists N \quad st \quad \forall n \geq N \quad |f_n(c)| \leq 1/m$ Which converges to 0. Thus $f_n \to 0$ at c. So the polynomials approximating f may be taken to vanish at c.

$\square$

Problem 3 7.5.5 Problem 14

(a) For $c_m = \int_{-1}^{1}(1-x^2)^m dx$ obtain the identity $c_m = c_{m-1} - (1/2m)c_m$ by integration by parts.

(b) Show that

$$c_m = 2\frac{2 \cdot 4 \cdot 6 \cdot \ldots \cdot 2m}{3 \cdot 5 \cdot 7 \cdot \ldots \cdot 2m+1} = \frac{2(2^m m!)^2}{(2m+1)!}$$

(a) We know the integration by parts formula $\int udv = uv - \int vdu$ from calculus Let $u = (1-x^2)^m$, and $v = x$. Then $dv = dx$ and $du = m \cdot (1-x^2)^{m-1}(-2x) = -2mx(1-x^2)^{m-1}$

$$\begin{aligned}
c_m &= \int_{-1}^{1}(1-x^2)^m dx \\
&= x(1-x^2)^m\Big|_{-1}^{1} - \int_{-1}^{1} x \cdot -2mx(1-x^2)^{m-1}dx \\
&= 0 - \int_{-1}^{1} -2m \cdot x^2(1-x^2)^{m-1}dx \\
&= \int_{-1}^{1} 2m \cdot (x^2 - 1 + 1)(1-x^2)^{m-1}dx \\
&= \int_{-1}^{1} 2m \cdot (-(1-x^2)+1)(1-x^2)^{m-1}dx \\
&= \int_{-1}^{1} 2m \cdot -(1-x^2)(1-x^2)^{m-1} + \int_{-1}^{1} 2m \cdot (1-x^2)^{m-1}dx \\
&= \int_{-1}^{1} 2m \cdot (1-x^2)^{m-1}dx - \int_{-1}^{1} 2m \cdot (1-x^2)^{m}dx \\
c_m &= 2m \cdot c_{m-1} - 2m \cdot c_m \\
c_m &= 2m \cdot (c_{m-1} - c_m) \\
1/2m \cdot c_m &= (c_{m-1} - c_m) \\
c_m &= c_{m-1} - 1/2m \cdot c_m
\end{aligned}$$

(b) PROOF. By induction

Base case: we show $c_1 = \frac{2(2^1 1!)^2}{(2+1)!} = 8/6 = 4/3$ We then verify that
$c_1 = 4/3$. $\int_{-1}^{1}(1-x^2)dx = \int_{-1}^{1} dx - \int_{-1}^{1} x^2 dx = 2 - 1/3x^3\big|_{-1}^{1} = 2 - 2/3 = 4/3$

Then the base case holds. So assume that $c_{m-1} = \frac{2(2^{m-1}(m-1)!)^2}{(2(m-1)+1)!}$

We know from above that $c_{m-1} = c_m(1 + 1/2m)$ So then

$$\begin{aligned}
c_m(1 + 1/2m) &= c_{m-1} \\
c_m(1 + 1/2m) &= \frac{2(2^{m-1}(m-1)!)^2}{(2m-1)!} \\
c_m &= \frac{2(2^{m-1}(m-1)!)^2}{(2m-1)!(1+1/2m)} \\
c_m &= \frac{2(2^{m-1}(m-1)!)^2}{(2m-1)! + (2m-1)!/2m} \\
c_m &= \frac{2(2m)(2^{m-1}(m-1)!)^2}{2m(2m-1)! + (2m-1)!} \\
c_m &= \frac{2(2m)(2m)(2^{m-1}(m-1)!)^2}{(2m-1)!(2m+1)(2m)} \\
c_m &= 2\frac{(2m)(2^{m-1}(m-1)!)(2m)(2^{m-1}(m-1)!)}{(2m+1)!} \\
c_m &= 2\frac{(2^m m!)(2^m m!)}{(2m+1)!} \\
c_m &= 2\frac{(2^m m!)^2}{(2m+1)!}
\end{aligned}$$

Then the inductive step holds. So the claim is true by mathematical induction.

□