Homework 4

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Problem 1 7.3.4 Problem 2

Suppose $f_n \to f$ and all functions f_n satisfy the Lipschitz condition $|f_n(x) - f_n(y)| \leq M|x - y|$ for some constant M, independent of n. Prove that f also satisfies the same Lipschitz condition

PROOF. We know that by the definition of convergence that $\forall x_0$ the sequence of numbers $f_1(x_0), f_2(x_0), ...$ converges to $f(x_0)$. Then we have that $\forall 1/n \quad \exists m \quad st \quad \forall k > m \quad |f_k(x_0) - f(x_0)| < 1/n$.

We want to show that $|f(x) - f(y)| \leq M|x - y|$. So we expand,

$$\begin{aligned} |f(x) - f(y)| &= |f(x) - f_k(x) + f_k(x) - f(y) + f_k(y) - f_k(y)| \\ &\leq |f(x) - f_k(x)| + |f_k(x) - f_k(y)| + |f(y) - f_k(y)| \\ &\leq |f(x) - f_k(x)| + M|x - y| + |f(y) - f_k(y)| \end{aligned}$$

We can then set k such that $|f(x) - f_k(x)| \leq M|x - y|$, and same for y (we choose the maximum k). We know we can do this by the convergence of f_k.

Then we have $|f(x) - f(y)| \leq M|x - y| + M|x - y| + M|x - y| = 3M|x - y|$. Thus f satisfies the Lipschitz condition. $\square$

Problem 2 7.3.4 Problem 5

If $\lim_{n\to\infty} f_n = f$ and the functions f_n are all monotone increasing, must f be monotone increasing? What happens if f_n are all strictly increasing?

Yes, if f_n are all monotone increasing, then f is monotone increasing.

PROOF. By contradiction.

Suppose not, suppose $\exists y > x, \quad st \quad f(y) < f(x)$ (f decreases). Then the sequence of numbers $f_1(y), f_2(y), ...$ converges to $f(y)$ and $f_1(x), f_2(x), ...$ converges to $f(x)$. But since each $f_k(x)$ is monotone increasing. We have $f_1(x) \leq f_1(y), f_2(x) \leq f_2(y), ...$ Since non-strict inequality is preserved under the limit: we then have $f(x) \leq f(y)$ a contradiction. $\square$

If f_n are strictly increasing, then f is not necessarily strictly increasing. (Just monotone increasing).

Consider the common domain $\mathbb{D} = [2, 10]$. Let $f_n(x) = 1 - 1/nx$. Then clearly f_n are strictly increasing since if $x_1 > x_0$, then $1/x_1 < 1/x_0$. And f_n converge to $f(x) = 1$. Which is a constant function (by definition still monotone increasing). But is clearly not strictly increasing.

Problem 3 7.3.4 Problem 6

Give an example of a sequence of continuous functions converging pointwise to a function with a discontinuity of the second kind.

Hint: Consider the common domain $\mathbb{D} = [0, 1]$ and

$$f_n(x) = \begin{cases} nx & 0 \le x \le 1/n \\ 1 & 1/n \le x \le 1 \end{cases}$$

Find another function $g(x)$ which has a discontinuity of the second kind on $\mathbb{D}$ and define $g_n(x) = f_n(x) \cdot g(x)$ You need to prove that g_n are continuous on $\mathbb{D}$ and converges pointwise to a function with a discontinuity of the second kind.

So we pick $g(x)$ to be the zigzag function. $g(1) = 1, g(1/2) = 0, g(1/4) = 1, ...$ with linear line segments connecting the points. We know that $g(x)$ has a discontinuity of the second kind at $x = 0$.

Let

$$f_n(x) = \begin{cases} nx & 0 \le x \le 1/n \\ 1 & 1/n \le x \le 1 \end{cases}$$

Then $g_n(x) = f_n(x) \cdot g(x)$. We show that g_n is continuous.

Clearly, for $x > 1/n$ g_n is continuous since $f_n(x) = 1$, so $g_n(x) = g(x)$ which we know to be continuous. All the line segments are always continuous of g, so we just have to show that the discontinuity of second kind at $x = 0$ does not occur.

By the squeeze theorem, $g_n \le f_n$ for all $x < 1/n$ (this holds for larger than $1/n$ as well). Then $\lim_{x\to 0} f_n(x) = 0$. So then using squeeze theorem $\lim_{x\to 0} g_n(x) = 0$. Thus there is not a discontinuity of the second kind. There could be a jump discontinuity, so we check $g_n(0) = f_n(0) \cdot g(0) = 0 \cdot g(0) = 0$ therefore, g_n is continuous.

So next we show that g_n converges pointwise to g. By the construction of f_n, we have that for $x > 1/n,\ \ g_n(x) = g(x)$. So for any x, we choose n large enough such that $1/n < x$, thus $g_n(x) = g(x)$. So it converges pointwise to g. Since by choosing n large enough $g_n = g$.

Thus we found a sequence of continuous functions that converge pointwise to a function with a discontinuity of second kind.

Problem 4 7.3.4 Problem 7

If $|f_n(x)| \leq a_n$ for all x, and $\sum_{n=1}^{\infty} a_n$ converges, prove that $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly.

Hint: The series $\sum_{n=1}^{\infty} f_n(x)$ converges uniformly is the equivalent to that the sequence of partial sum functions $F_n(x) = \sum_{k=1}^{n} f_k(x)$ converges uniformly. Then prove F_n satisfies the Cauchy criterion for uniform convergence (Theorem 7.3.1).

PROOF. We define the function $F_n(x) = \sum_{k=1}^{n} f_k(x)$ to be the function of the partial sums. We show that this converges uniformly.

So, we show that F_n satisfies the Cauchy Criterion.

$\forall 1/m \;\; \exists N \;\; st \;\; \forall q, p \geq N \;\; \forall x \;\; |F_q(x) - F_p(x)| \leq 1/m.$

Without loss of generality, assume $q \geq p$. Then $|F_q(x) - F_p(x)| = \left|\sum_{i=p}^{q} f_i(x)\right| \leq \sum_{i=p}^{q} |f_i(x)| \leq \sum_{i=p}^{q} a_i$

Then, by the Cauchy Criterion for Convergence of Series (Theorem 7.2.1): $\forall 1/n \;\; \exists m \;\; st \;\; \forall q \geq p \geq m \;\; \left|\sum_{i=p}^{q} a_i\right| < 1/n$. We observe that $a_n \geq 0$, so we can drop the absolute values signs $\sum_{i=p}^{q} a_i < 1/n$.

Thus, we have that $\forall 1/n \;\; \exists m \;\; st \;\; \forall q \geq p \geq m \;\; |F_q(x) - F_p(x)| \leq \sum_{i=p}^{q} a_i < 1/n$

$\square$