Homework 8

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Problem 1 10.1.5 Problem 3

If f is differentiable at y, show that $d_u f(y)$ is linear in u, meaning $d_{(au+bv)}f(y) = ad_u(y) + bd_v f(y)$.

Hint: Apply Theorem 10.1.1

PROOF.

We know from theorem 10.1.1, since f is differentiable at y, then $d_{(au+bv)}f(y) = df(y)(au + bv) = a \cdot df(y)u + b \cdot df(y)v$ since matrix multiplication is distributive. We then recognize that $df(y)u$ has the form (from theorem 10.1.1) of $d_u(y)$, and similarly for $df(y)v$. So we have $d_{(au+bv)}f(y) = df(y)(au + bv) = a \cdot df(y)u + b \cdot df(y)v = ad_u(y) + bd_v f(y)$ $\square$

Problem 2 10.1.5 Problem 10

Let $g : [a, b] \to \mathbb{R}^n$ be differentiable. If $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable, what is the derivative $(d/dt)f(g(t))$

Hint: Use notation $g(t) = (g_1(t), \ldots, g_n(t)), t \in [a, b]$ and $f(z) = f(z_1, \ldots, z_n), z = (z_1, \ldots z_n) \in \mathbb{R}^n$ and Apply the chain rule.

Proof.

we know the Chain rule in general is:

$$\frac{\partial f}{\partial x_j} = \sum_{k=1}^{n} \frac{\partial f}{\partial z_k} \frac{\partial z_k}{\partial x_j}$$

where $z_k = g_k(x_1, \ldots x_n)$.

In our case, we are looking for $x_j = t$, and $g(t) = (g_1(t), \ldots, g_n(t)), t \in [a, b]$ So $z_k = g_k(t)$.

So we have

$$\frac{\partial f}{\partial t} = \sum_{k=1}^{n} \frac{\partial f}{\partial z_k} \frac{\partial g_k(t)}{\partial t}$$

$\frac{\partial g_k(t)}{\partial t}$ is just a number (scalar), so we can write this as two separate sums: $\sum_{k=1}^{n} \frac{\partial f}{\partial z_k} * \sum_{k=1}^{n} \frac{\partial g_k(t)}{\partial t}$. Then clearly this is just the sum of all partial derivatives of each function, which is the differential df, dg. We also note that this $df = \nabla f$ since $f : \mathbb{R}^n \to \mathbb{R}$. So we have:

$$\frac{\partial f}{\partial t} = df\, dg/dt = \nabla f * dg/dt$$

$\square$

Problem 3 10.1.5 Problem 13

Compute df of

a. $f : \mathbb{R}^2 \to \mathbb{R},\ \ f(x_1, x_2) = x_1 e^{x_2}$

b. $f : \mathbb{R}^3 \to \mathbb{R}^2,\ \ f(x_1, x_2, x_3) = (x_3, x_2)$

c. $f : \mathbb{R}^2 \to \mathbb{R}^3,\ \ f(x_1, x_2) = (x_1, x_2, x_1 \cdot x_2)$

We know that for the differential matrix df we know that $df_{k,j} = \frac{\partial f_k}{\partial x_j}$

a. $f : \mathbb{R}^2 \to \mathbb{R},\ \ f(x_1, x_2) = x_1 e^{x_2}$ There is just one, f_k and there are x_1, x_2. So we have that

$$df = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} e^{x_2} & x_1 e^{x_2} \end{bmatrix}$$

b. $f : \mathbb{R}^3 \to \mathbb{R}^2,\ \ f(x_1, x_2, x_3) = (x_3, x_2)$

$$df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \frac{\partial f_2}{\partial x_3} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

c. $f : \mathbb{R}^2 \to \mathbb{R}^3,\ \ f(x_1, x_2) = (x_1, x_2, x_1 \cdot x_2)$

$$df = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 1, 0 \\ 0, 1 \\ x_2, x_1 \end{bmatrix}$$

Problem 4 10.1.5 Problem 15

If $f : D \to \mathbb{R}$ is C^1 with $D \subseteq \mathbb{R}^n$ and D contains the line segment joining x and y, show that $f(y) = f(x) + \nabla f(z) \cdot (y - x)$ for some point z on the line segment. Explain why this is an n-dimensional analog of the mean value theorem

Hint: Define function $g : [0, 1] \to \mathbb{R}^n$ by $g(t) = x + t(y - x)$ and consider the composition function

$$h(t) = (f \circ g)(t) = f(g(t)) : \mathbb{R} \to \mathbb{R}$$

Apply Mean Value Theorem to $h(t)$ for $h(1) - h(0)$ and use the chain rule (formula derived in problem 10 above) to calculate h'

PROOF.

Let $g : [0, 1] \to \mathbb{R}^n$ by $g(t) = x + t(y - x)$, and let h be the composition function: $h(t) = (f \circ g)(t) = f(g(t)) : \mathbb{R} \to \mathbb{R}$. By our definition of g: $g(0) = x$, $g(1) = y$ and $g(t)$ is on the straight line-segment joining x to y. Thus the image of $g \in D$. Then $f \in C^1$ so h is differentiable.

From problem 10: we know that $d/dt(h) = \nabla f * dg/dt$, and we know that $dg/dt = (y - x)$ by definition of g. Then by Mean value Theorem we know that there exists a z such that: $(1 - 0) * h'(z) = h(1) - h(0) = f(g(1)) - f(g(0)) = f(y) - f(x)$ Since we know $h' = dh/dt = \nabla f(z) * (y - x)$ we have: $f(y) - f(x) = \nabla f(z)(y - x)$ we simply re-arrange this to get:

$$f(y) = f(x) + \nabla f(z)(y - x)$$

$\square$

We first note that: $\nabla f(z) = df(z)$ since $f : \mathbb{R}^n \to \mathbb{R}$. So our equation is $f(y) = f(x) + df(z)(y - x)$

We can re-arrange the classic MVT: $f'(z) = \frac{f(y) - f(x)}{y - x}$ to get: $f'(z)(y - x) + f(x) = f(y)$ Which matches our equation.

Essentially we have some point z on the line segment, such that the derivative at that point is the average change in the function's value ($f(y) - f(x) = \nabla f(z)(y - x)$) across the interval.