Homework 3

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Problem 1 6.3.1 Problem 1

For which values of a and b does the improper integral

$$\int_0^{1/2} x^a |\log x|^b dx$$

exist

Hint: Consider $a > -1$, $a < -1$, $a = -1$ separately.

If $a > -1$ then $\exists \epsilon > 0$ such that $a - \epsilon > -1$ or $\epsilon = \frac{a+1}{2}$. Thus we have

$$\int_0^{1/2} x^a |\log x|^b dx = \int_0^{1/2} x^{a-\epsilon} x^{\epsilon} |\log x|^b dx$$

Since $\epsilon > 0$, fo any b, by L'Hopital's rule (you don't need to prove this), we have $\lim_{x \to 0^+} x^{\epsilon} |\log x|^b = 0$

So the convergence of the integral is determined by $x^{a-\epsilon}$

If $a < -1$, we use similar argument writing $x^a = x^{a+\epsilon} x^{-\epsilon}$, and use L'Hopital's rule to show $\lim_{x \to 0^+} x^{-\epsilon} |\log x|^b = \infty$

If $a = -1$ use substitutions $u = \log x$ to convert it to a form that the results are known.

Problem 2 7.2.4 Problem 1

Give an example of two convergent series $\sum_{k=1}^{\infty} x_k$ and $\sum_{k=1}^{\infty} y_k$ such that $\sum_{k=1}^{\infty} x_k y_k$ diverges. Can this happen if one of the series is absolutely convergent?

Hint: For the first part, consider the alternating series. For the second part, if one series is absolutely convergent, consider to use the Cauchy Criterion and the fact that every term in the other series is bounded.

Problem 3 7.2.4 Problem 2

State a contrapositive form of the comparison test that can be used to show divergence of a series

Hint You can assume the terms of both series are non-negative, and you don't need to prove the statement.

Problem 4 7.2.4 Problem 4

Prove the ratio test (Theorem 7.2.3a). What does this tell you if $lim_{n\to\infty}|x_{n+1}/n|$ exists?

Hint: Use comparison test with geometric series, and then use the Cauchy criterion.