# Exam 1

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**Problem 1**

True or False

(a) if $f(x)$ on a finite interval $[a,b]$ is Riemann integrable, then $f(x)$ can only have jump discontinuities.

False. - Hole

(b) If $|f(x)|$ is Riemann integrable on $[a,b]$ then $f(x)$ is also Riemann integrable on $[a,b]$.

False

(c) if $\sum_{k=1}^{\infty} a_k$ converges conditionally and $\sum_{k=1}^{\infty} b_k$ converges absolutely, then $\sum_{k=1}^{\infty} a_k \cdot b_k$ converges absolutely

True

(d) If $\sum_{k=1}^{\infty} a_k$ converges conditionally, then the sum of all positive terms of this series diverges

True

(e) Let $f_n$ be a sequence of $C^1$ functions defined on $(a,b)$ if $f_n'(x)$ converges uniformly to $g(x)$ on $(a,b)$. Then there exists a $C^1$ function $f$ on $(a,b)$ such that $f_n(x)$ converge uniformly to $f(x)$ and $f'(x) = g(x)$

True

**Problem 2**

Give the statement (using quantifiers) that a sequence of functions $f_n(x)$ on a common domain $\mathbb{D}$ does **NOT** converge uniformly to a function $f(x)$ on $\mathbb{D}$

a sequence of functions $f_n(x)$ on a common domain $\mathbb{D}$ does **NOT** converge uniformly to a function $f(x)$ on $\mathbb{D}$ if:

$$\exists 1/m \;\; st \;\; \forall N \;\; \exists x \in \mathbb{D}, \;\; \exists n \geq N \;\; st \;\; |f_n(x) - f(x)| \geq 1/m$$

**Problem 3**

Compute the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{4n^5 - 7n + 2}{3^n} x^n$$

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Here we have: $a_n = \frac{4n^5-7n+2}{3^n}$ We know the radius of convergence is given by: $\frac{1}{R} = \limsup_{n\to\infty} \sup \sqrt[n]{|a_n|}$

$$\begin{aligned}
\frac{1}{R} &= \limsup_{n\to\infty} \sqrt[n]{|a_n|} \\
&= \limsup_{n\to\infty} \sqrt[n]{\left|\frac{4n^5 - 7n + 2}{3^n}\right|} \\
&= \limsup_{n\to\infty} \left|\frac{\sqrt[n]{4n^5 - 7n + 2}}{\sqrt[n]{3^n}}\right| \\
&= \frac{1}{3}
\end{aligned}$$

(the numerator of the last step comes from the lemma that we used for 7.4.1 Or from the example we did showing radius of convergence of $a_n = p(n)/q(n)$ is 1)

So $R = 3$

## Problem 4

Prove the first part of the linearity of Riemann Integral. Namely, if both $f(x)$ and $g(x)$ are Riemann integrable on $[a, b]$, then $f + g$is Riemann integrable on $[a, b]$, and

$$\int_a^b (f+g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Here $(f + g)(x) = f(x) + g(x)$

**Hint:** Use part (e) of Theorem 6.2.1 and linear property of Cauchy sum

## Problem 5

Let $b_1, b_2, ...$ be a sequence of postiive numbers convergint monotonically to zero: $b_1 \geq b_2 \geq b_3 ...$ and $\lim_{n\to\infty} b_n = 0$. If $|a_n| \leq b_n - b_{n+1}$ for all $n$. Prove: $\sum_{n=1}^{\infty} a_n$ converges absolutely.

## Problem 6a

Let $f_n(x) \to f(x)$ uniformly on a finite interval $[a, b]$ and all $f_n(x)$ are Riemann integrable on $[a, b]$. Define $F_n(x) = \int_a^x f_n(t)dt$. Prove that $F_n \to F$ uniformly on $[a, b]$ for some $F(x)$, and give the expression of the limit function $F(x)$.

## Problem 6b

Is the same true on the whole line? Namely, let $f_n(x) \to f(x)$ uniformly on the entire real line $\mathbb{R}$ , and all $f_n(x)$ are Riemann integrable on any finite interval. Define $F_n(x) = \int_0^x f_n(t)dt$. Is it always true that $F_n(x) \to F(x)$ uniformly on $\mathbb{R}$ for some $F$? Prove it if your answer is Yes, or give a counter example if your answer is No.