Homework 9

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Problem 1

Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by:

$$f(x,y) = \begin{cases} \frac{xy(x^2-y^2)}{x^2+y^2} & x,y \neq 0,0 \\ 0 & x,y = 0,0 \end{cases}$$

(a) Show that $\partial f/\partial x, \partial f/\partial y$ exist for all $(x,y) \in \mathbb{R}^2$

Hint: for $(x,y) \neq (0,0)$ calculate directly by formula. For $(x,y) = (0,0)$ calculate by its definition:

$$\frac{\partial f}{\partial x}(0,0) = \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x - 0}$$

(b) Show that both $\frac{\partial^2 f(0,0)}{\partial x \partial y}$ and $\frac{\partial^2 f(0,0)}{\partial y \partial x}$ exist, but $\frac{\partial^2 f(0,0)}{\partial x \partial y} \neq \frac{\partial^2 f(0,0)}{\partial y \partial x}$

Hint: note

$$\frac{\partial^2 f(0,0)}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \lim_{x \to 0} \frac{\frac{\partial f(x,0)}{\partial y} - \frac{\partial f(0,0)}{\partial y}}{x - 0}$$

Where $\frac{\partial f(x,0)}{\partial y}, \frac{\partial f(0,0)}{\partial y}$ are calculated in part (a)

(a) Proof. We start by considering $x, y \neq 0$. Then we directly compute $\partial f/\partial x$ We have $\frac{xy(x^2-y^2)}{x^2+y^2} = \frac{yx^3-xy^3}{x^2+y^2}$

We use the quotient rule to get: $\partial f/\partial x = \frac{(3yx^2-y^3)(x^2+y^2)-(yx^3-xy^3)2x}{(x^2+y^2)^2} = \frac{(3yx^2-y^3)(x^2+y^2)-2yx^4+2x^2y^3}{(x^2+y^2)^2}$

Similarly, we solve: $\partial f/\partial y = \frac{(x^3-3xy^2)(x^2+y^2)-(yx^3-xy^3)2y}{(x^2+y^2)^2} = \frac{(x^3-3xy^2)(x^2+y^2)-2y^2x^3+2xy^4}{(x^2+y^2)^2}$

Then at $x, y = 0$ we use the definition. So for $y = 0$

$$\begin{aligned}
\partial f/\partial x &= \lim_{x \to 0} \frac{f(x,0) - f(0,0)}{x - 0} \\
&= \lim_{x \to 0} \frac{\frac{0 \cdot x^3 - x \cdot 0^3}{x^2+0^2}}{x} \\
&= \lim_{x \to 0} 0/x^2 \cdot 1/x \\
&= \lim_{x \to 0} 0/x^3 \\
&= \lim_{x \to 0} 0/3! = 0
\end{aligned}$$

Where the last step follows from applying L'hopitals rule 3 times.

We repeat the same process for y

$$\begin{aligned}
\partial f/\partial y &= \lim_{y\to 0} \frac{f(0,y)-f(0,0)}{y-0} \\
&= \lim_{y\to 0} \frac{\frac{y\cdot 0^3 - 0\cdot y^3}{0^2+y^2}}{y} \\
&= \lim_{y\to 0} 0/y^2 \cdot 1/y \\
&= \lim_{y\to 0} 0/y^3 \\
&= \lim_{y\to 0} 0/3! = 0
\end{aligned}$$

Thus the partial derivatives exist everywhere.

$\square$

(b) We compute the second partial derivatives: Proof.

$$\begin{aligned}
\frac{\partial^2 f(0,0)}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) &= \lim_{x\to 0} \frac{\frac{\partial f(x,0)}{\partial y} - \frac{\partial f(0,0)}{\partial y}}{x-0} \\
&= \lim_{x\to 0} \frac{\frac{(x^3-3x\cdot 0^2)(x^2+0^2)+2\cdot 0^2 x^3 + 2x\cdot 0^4}{(x^2+0^2)^2} - 0}{x} \\
&= \lim_{x\to 0} \frac{\frac{x^5}{x^4} - 0}{x} \\
&= 1
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^2 f(0,0)}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) &= \lim_{y\to 0} \frac{\frac{\partial f(0,y)}{\partial x} - \frac{\partial f(0,0)}{\partial x}}{y-0} \\
&= \lim_{y\to 0} \frac{\frac{(3y\cdot 0^2-y^3)(0^2+y^2)+2y\cdot 0^4 + 2\cdot 0^2 y^3}{(0^2+y^2)^2} - 0}{y} \\
&= \lim_{y\to 0} \frac{\frac{-y^5}{y^4} - 0}{y} \\
&= -1
\end{aligned}$$

$\square$

Problem 2

For any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ and any multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ prove that

$$|x^\alpha| \leq |x|^{|\alpha|}$$

where $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$, $|x| = \sqrt{x_1^2 + \cdots + x_n^2}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$

Proof.

We know that $|x^\alpha| = |x_1^{\alpha_1} \ldots x_n^{\alpha_n}| = |\prod_{i=1}^n x_i^{\alpha_i}|$ Which: $|\prod_{i=1}^n x_i^{\alpha_i}| \leq |\prod_{i=1}^n |x|^{\alpha_i}|$ since $x_i \leq |x|$.

Which if we write this out is $|x|^{\alpha_1} \cdot |x|^{\alpha_2} \cdot \ldots |x|^{\alpha_n} = |x|^{\alpha_1 + \cdots + \alpha_n} = |x|^{|\alpha|}$

$\square$