

Exam 1

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Problem 1

True or False

- (a) if $f(x)$ on a finite interval $[a, b]$ is Riemann integrable, then $f(x)$ can only have jump discontinuities
- (b) If $|f(x)|$ is Riemann integrable on $[a, b]$ then $f(x)$ is also Riemann integrable on $[a, b]$
- (c) if $\sum_{k=1}^{\infty} a_k$ converges conditionally and $\sum_{k=1}^{\infty} b_k$ converges absolutely, then $\sum_{k=1}^{\infty} a_k \cdot b_k$ converges absolutely
- (d) If $\sum_{k=1}^{\infty} a_k$ converges conditionally, then the sum of all positive terms of this series diverges
- (e) Let f_n be a sequence of C^1 functions defined on (a, b) if $f'_n(x)$ converges uniformly to $g(x)$ on (a, b) . Then there exists a C^1 function f on (a, b) such that $f_n(x)$ converge uniformly to $f(x)$ and $f'(x) = g(x)$

Problem 2

Give the statement (using quantifiers) that a sequence of functions $f_n(x)$ on a common domain \mathbb{D} does **NOT** converge uniformly to a function $f(x)$ on \mathbb{D}

Problem 3

Compute the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{4n^5 - 7n + 2}{3^n} x^n$$

Problem 4

Prove the first part of the linearity of Riemann Integral. Namely, if both $f(x)$ and $g(x)$ are Riemann integrable on $[a, b]$, then $f + g$ is Riemann integrable on $[a, b]$, and

$$\int_a^b (f + g)(x)dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Here $(f + g)(x) = f(x) + g(x)$

Hint: Use part (e) of Theorem 6.2.1 and linear property of Cauchy sum

Problem 5

Let b_1, b_2, \dots be a sequence of positive numbers converging monotonically to zero: $b_1 \geq b_2 \geq b_3 \dots$ and $\lim_{n \rightarrow \infty} b_n = 0$. If $|a_n| \leq b_n - b_{n+1}$ for all n . Prove: $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Problem 6a

Let $f_n(x) \rightarrow f(x)$ uniformly on a finite interval $[a, b]$ and all $f_n(x)$ are Riemann integrable on $[a, b]$. Define $F_n(x) = \int_a^x f_n(t)dt$. Prove that $F_n \rightarrow F$ uniformly on $[a, b]$ for some $F(x)$, and give the expression of the limit function $F(x)$.

Problem 6b

Is the same true on the whole line? Namely, let $f_n(x) \rightarrow f(x)$ uniformly on the entire real line \mathbb{R} , and all $f_n(x)$ are Riemann integrable on any finite interval. Define $F_n(x) = \int_0^x f_n(t)dt$. Is it always true that $F_n(x) \rightarrow F(x)$ uniformly on \mathbb{R} for some F ? Prove it if your answer is Yes, or give a counter example if your answer is No.
