Homework # 8

Elliott Pryor

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Problem 1

Statement

The length of time until the breakdown of an essential piece of equipment is important in the decision of the use of auxiliary equipment. Assume you are interested in the time to breakdown (in days) of a randomly chosen generator.

a. Define a random variable, Y that represents the quantity of interest.

b. What probability distribution(s) of those we have discussed, could you use to model the random variable of interest. Explain

c. Suppose that, historically the average time to breakdown of a randomly chosen generator is 10 days with a variance of 100 days. Specify a probability model for Y.

d. Using your model in (c), what is the probability a generator will break down in the next 21 days?

e. A company owns 7 such generators. Assuming the breakdown of any one generator is independent of breakdowns of the other generators, what is the probability that at least 6 of the 7 generators will operate for the next 21 days without a breakdown? Be sure to first define a RV, X for the quantity of interest, describe what X is in words, provide notation to communicate the assumed distribution of X, and write a probability statement using proper notation. Compute the probability by hand or using software. If you use software provide the code you used

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Solution

a. Let Y be the random variable representing the number of days a single generator will last without breaking down.

b. Of the probability distributions that we have discussed, the gamma distribution makes the most sense. This mostly depends on how we want to model the timeframe. If we consider continuous time, ie we are conserned with exactly when a generator will fail (say 1:30pm on the 3rd day) then we would use gamma. If we only care if a generator fails on a specific day (say on the 3rd day) we would use the discrete analog which is the geometric distribution. In all honesty, the phrasing of the question would imply that we are interested in the geometric distribution. It asks only about specific days. However, it is not possible to generate a geometric distribution with the parameters given in part (c). For a geometric distribution: $E(Y) = 10 = 1/p \implies p = 0.1$, but $Var(Y) = (1-p)/p^2 = 90$ so both can't be satisfied.

c. We know that the mean (expected value) of a geometric distribution is $E(Y) = 10 = \alpha\beta \implies \alpha = 10/\beta$. Then we can use this to solve for the variance which we know is $Var(Y) = 100 = \alpha\beta^2 = 10/\beta * \beta^2 = 10b \implies b = 10 \implies a = 1$. This is the special case of the gamma distribution called the exponential distribution. So $Y \sim Exp(10)$

d. We are asked to find the probability that it breaks down in the next 21 days, so within 21 days. We can use the cdf since we would still want to know the entire probability it fails before 21 days, so we solve

$$\int_0^{21} 1/\beta e^{-x/\beta} = \int_0^{21} 1/10 e^{-x/10} = -e^{-x/10}\Big|_0^{21} = 1 - exp(-2.1) = 0.877$$

So the probability that it fails within 21 days is 0.877. We solved the last expression using R

```
1- exp(-2.1)
```

```
## [1] 0.8775436
```

e. We let X be the number of generators that failed. We say $X \sim Binom(7, p)$ where p is the probability that a single generator fails. This was calculated in part (d) so $X \sim Binom(7, 0.877)$. We are looking at the probability that exactly one fails, so $P(X \leq 1) = F_Y(1)$. We solve it with the following R command

```
pbinom(q = 1, size = 7, prob = 1- exp(-2.1))
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## [1] 2.112656e-05
```

So We have that the probability that 6 remain functional after 21 days is very small. This makes sense since there is a high probability of a single generator failing within 21 days, so it is extremely likely that more than one would fail.

Problem 2

Statement

4.130

Prove that the variance of a beta distributed random variable with parameters α, β is:

$$\sigma^2 = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Solution

We know that the variance is $Var(Y) = E(Y^2) - E(Y)^2$. We know that $E(Y) = \frac{\alpha}{\alpha+\beta}$. We solve for $E(Y^2)$

$$\begin{aligned}
E(Y^2) &= \int_0^1 y^2 \frac{1}{\beta(\alpha,\beta)} y^{\alpha-1}(1-y)^{\beta-1} dy \\
&= \frac{1}{\beta(\alpha,\beta)} \int_0^1 y^{(\alpha+2)-1}(1-y)^{\beta-1} dy \\
&= \frac{1}{\beta(\alpha,\beta)} \int_0^1 \frac{\beta(\alpha+2,\beta)}{\beta(\alpha+2,\beta)} y^{(\alpha+2)-1}(1-y)^{\beta-1} dy \\
&= \frac{\beta(\alpha+2,\beta)}{\beta(\alpha,\beta)} \int_0^1 f_Z(z) dz \qquad Z \sim Beta(\alpha+1,\beta) \\
&= \frac{\Gamma(\alpha+2)\Gamma(\beta)}{\Gamma(\alpha+2+\beta)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \\
&= \frac{\alpha(\alpha+1)\Gamma(\alpha)\Gamma(\beta)}{(\alpha+\beta)(\alpha+\beta+1)\Gamma(\alpha+\beta)} \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \\
&= \frac{\alpha^2+\alpha}{(\alpha+\beta)(\alpha+\beta+1)}
\end{aligned}$$

Then we can plug this in. We get:

$$Var(Y) = \frac{\alpha^2 + \alpha}{(\alpha+\beta)(\alpha+\beta+1)} - \left(\frac{\alpha}{\alpha+\beta}\right)^2 = \frac{(\alpha+\beta)(\alpha^2+\alpha) - \alpha^2(\alpha+\beta+1)}{(\alpha+\beta)^2(\alpha+\beta+1)} = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$$

Problem 3

Statement

4.140

Identify the distributions of the random variables with the following moment generating functions:

a. $m(t) = (1-4t)^{-2}$

b. $m(t) = 1/(1-3.2t)$

c. $m(t) = e^{-5t+6t^2}$

Solution

Problem 4

Statement

Use the example code for plotting the Gamma pdf (posted on Brightspace) and do something similar for the Beta distribution. Make note of the parameterization R uses and describe how changing the values of α and β change the resulting Beta distributions

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Solution