

Homework # 9

Elliott Pryor

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Problem 1

Statement

The random variables X and Y have the joint distribution $f_{X,Y}$ given by:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{y! \Gamma(\theta) \delta^\theta} x^{y+\theta-1} e^{-x(\frac{1}{\delta}+1)} & \text{if } y = 0, 1, 2, \dots \quad 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Where: $\delta, \theta > 0$

- Calculate the marginal pdf $f_X(x)$. Identify this distribution and its parameter(s).
 - Calculate the marginal pmf $f_Y(y)$
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Solution

- We have to sum over the support of Y so

$$\begin{aligned} f_X(x) &= \sum_{y=0}^{\infty} \frac{1}{y! \Gamma(\theta) \delta^\theta} x^{y+\theta-1} e^{-x(\frac{1}{\delta}+1)} \\ &= f_X(x) = \frac{e^{-x(\frac{1}{\delta})} x^{\theta-1}}{\Gamma(\theta) \delta^\theta} \sum_{y=0}^{\infty} \frac{1}{y!} e^{-x} x^y \implies Y \sim \text{Poisson}(\lambda = x) \\ &== f_X(x) = \frac{e^{-x(\frac{1}{\delta})} x^{\theta-1}}{\Gamma(\theta) \delta^\theta} I_{(0,\infty)}(x) \end{aligned}$$

$f_X(x)$ follows a gamma distribution with $\alpha = \theta$ and $\beta = \delta$

- We integrate out the support of X so: We have to sum over the support of Y so

$$\begin{aligned} f_Y(y) &= \int_0^{\infty} \frac{1}{y! \Gamma(\theta) \delta^\theta} x^{y+\theta-1} e^{-x(\frac{1}{\delta}+1)} dx \\ &= \frac{1}{y!} \int_0^{\infty} \frac{1}{\Gamma(\theta) \delta^\theta} x^y x^{\theta-1} e^{-x/\delta} e^{-x} dx \\ &= \frac{1}{y!} \left(\int_0^{\infty} x^y e^{-x} dx \right) \left(\int_0^{\infty} \frac{1}{\Gamma(\theta) \delta^\theta} x^{\theta-1} e^{-x/\delta} dx \right) \implies X \sim \text{Gamma}(\alpha = \theta, \beta = \delta) \\ &= \frac{1}{y!} \left(\int_0^{\infty} x^y e^{-x} dx \right) \\ &= \frac{1}{y!} \Gamma(y+1) \quad y \in \mathbb{N} \cup \{0\} \end{aligned}$$

I feel like this should simplify more, using the $\Gamma(y+1)$ and somehow using the beta distribution. But I can't figure out how since $\alpha = y$ $\beta = 1$ doesn't work

Problem 2

Statement

Find $P(X > \sqrt{Y})$ if X, Y are jointly distributed with pdf:

$$f_{X,Y}(x,y) = x + y \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1$$

Solution

We need a 2D integral. We also know that X, Y are continuous so we can substitute $P(X \geq \sqrt{Y})$

$$\begin{aligned} & \int_0^1 \int_{\sqrt{y}}^1 (x+y) dx dy \\ P(X \geq \sqrt{Y}) &= \int_0^1 \int_{\sqrt{y}}^1 (x+y) dx dy \\ &= \int_0^1 \left. \frac{1}{2}x^2 + xy \right|_{\sqrt{y}}^1 dy \\ &= \int_0^1 (1/2y + y^{3/2}) dy \\ &= \left. \frac{1}{4}y^2 + \frac{2}{5}y^{5/2} \right|_0^1 \\ &= 1/4 + 2/5 - 0 - 0 \\ &= 13/20 \end{aligned}$$

Problem 3

Statement

Find $P(X^2 < Y < X)$ if X, Y , are jointly distributed with pdf

$$f_{X,Y}(x,y) = 2x \quad 0 \leq x \leq 1, \quad 0 \leq y \leq 1$$

Solution

$$\begin{aligned} P(X \geq \sqrt{Y}) &= \int_0^1 \int_{x^2}^x (2x) dy dx \\ &= \int_0^1 \left. 2xy \right|_{x^2}^x dx \\ &= \int_0^1 (2x^2 - 2x^3) dx \\ &= \left. \frac{2}{3}x^3 - \frac{1}{2}x^4 \right|_0^1 \\ &= 1/6 \end{aligned}$$

Problem 4

Statement

A pdf is defined by

$$f_{X,Y}(x,y) = \begin{cases} C(x+2y) & \text{if } 0 < y < 1 \text{ } 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

- Find the value of C
 - Find the marginal pdf of X
 - Find the joint cdf of X and Y
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Solution

- C is the normalizing constant so it integrates to 1.

$$\begin{aligned} 1 &= C \int_0^1 \int_0^2 C(x+2y) dx dy \\ 1 &= C \int_0^1 1/2x^2 + 2xy \Big|_0^2 dy \\ 1 &= C \int_0^1 (2 + 4y) dy \\ 1 &= C 2y + 2y^2 \Big|_0^1 \\ 1 &= C(4) \\ 1/4 &= C \end{aligned}$$

- This was poor planning. I found marginal pdf of Y in part A.

$$\begin{aligned} f_X(x) &= C \int_0^1 C(x+2y) dy \\ f_X(x) &= C xy + y^2 \Big|_0^1 \\ f_X(x) &= 1/4(x+1)I_{(0,2)}(x) \end{aligned}$$

- This was pretty much done in part a, but needs to be done generically.

$$\begin{aligned} F_{X,Y}(x,y) &= 1/4 \int \int (x+2y) dx dy \\ F_{X,Y}(x,y) &= 1/4 \int 1/2x^2 + 2xy dy \\ F_{X,Y}(x,y) &= 1/4(1/2x^2y + y^2x) \end{aligned}$$

We also need the marginal for Y:

$$\begin{aligned} f_Y(y) &= C \int_0^2 C(x+2y) dx \\ f_Y(y) &= C 1/2x^2 + 2xy \Big|_0^2 \\ f_Y(y) &= 1/4(2 + 4y)I_{(0,1)}(y) \end{aligned}$$

$$F_{X,Y}(x,y) = \begin{cases} 1/4(1/2x^2y + y^2x) & 0 < y < 1, \ 0 < x < 2 \\ 0 & y \leq 0, \ x \leq 0 \\ 1 & y \geq 1, \ x \geq 2 \\ 1/4(x+1) & y \geq 1, \ 0 < x < 2 \\ 1/4(2+4y) & x \geq 2, \ 0 < y < 1 \end{cases}$$