Homework # 9

Elliott Pryor

10/28/2020

Problem 1

Statement

The random variables X and Y have the joint distribution $f_{X,Y}$ given by:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{y!\,\Gamma(\theta)\delta^\theta} x^{y+\theta-1} e^{-x\left(\frac{1}{\delta}+1\right)} & \text{if } y = 0, 1, 2, ... \;\; 0 < x < \infty \\ 0 & \text{otherwise} \end{cases}$$

Where: $\delta, \theta > 0$

a. Calculate the marginal pdf $f_X(x)$. Identify this distribution and its parameter(s).

b. Calculate the marginal pmf $f_Y(y)$

Solution

a. We have to sum over the support of Y so

$$\begin{aligned} f_X(x) &= \sum_{y=0}^{\infty} \frac{1}{y!\,\Gamma(\theta)\delta^\theta} x^{y+\theta-1} e^{-x\left(\frac{1}{\delta}+1\right)} \\ &= f_X(x) = \frac{e^{-x\left(\frac{1}{\delta}\right)} x^{\theta-1}}{\Gamma(\theta)\delta^\theta} \sum_{y=0}^{\infty} \frac{1}{y!} e^{-x} x^y \qquad \implies Y \sim Poisson(\lambda = x) \\ &== f_X(x) = \frac{e^{-x\left(\frac{1}{\delta}\right)} x^{\theta-1}}{\Gamma(\theta)\delta^\theta} \end{aligned}$$

$f_X(x)$ follows a gamma distribution with $\alpha = \theta$ and $\beta = \delta$

b. We integrate out the support of X so: We have to sum over the support of Y so

$$\begin{aligned} f_Y(y) &= \int_0^\infty \frac{1}{y!\,\Gamma(\theta)\delta^\theta} x^{y+\theta-1} e^{-x\left(\frac{1}{\delta}+1\right)} dx \\ &= \frac{1}{y!} \int_0^\infty \frac{1}{\Gamma(\theta)\delta^\theta} x^y x^{\theta-1} e^{-x/\delta} e^{-x} dx \\ &= \frac{1}{y!} \left(\int_0^\infty x^y e^{-x} dx \right) \left(\int_0^\infty \frac{1}{\Gamma(\theta)\delta^\theta} x^{\theta-1} e^{-x/\delta} dx \right) \qquad \implies X \sim Gamma(\alpha = \theta, \beta = \delta) \\ &= \frac{1}{y!} \left(\int_0^\infty x^y e^{-x} dx \right) \\ &= \frac{1}{y!} \Gamma(y+1) \end{aligned}$$

I feel like this should simplify more, using the $\Gamma(y+1)$ *and somehow using the beta distribution. But I can't figure out how*

Problem 2

Statement

Find $P(X > \sqrt{Y}$ if X, Y are jointly distributed with pdf:

$$f_{X,Y}(x,y) = x + y \quad 0 \leq x \leq 1, \;\; 0 \leq y \leq 1$$

Solution

We need a 2D integral. We also know that X, Y are continuous so we can substitute $P(X \geq \sqrt{Y})$

$$\int_0^1 \int_{\sqrt{y}}^1 (x+y)dxdy$$

$$\begin{aligned}
P(X \geq \sqrt{Y}) &= \int_0^1 \int_{\sqrt{y}}^1 (x+y)dxdy \\
&= \int_0^1 \left. 1/2x^2 + xy \right|_{\sqrt{y}}^1 dy \\
&= \int_0^1 (1/2y + y^{3/2})dy \\
&= \left. 1/4y^4 + 2/5y^{5/2} \right|_0^1 \\
&= 1/4 + 2/5 - 0 - 0 \\
&= 13/20
\end{aligned}$$

Problem 3

Statement

Find $P(X^2 < Y < X)$ if X, Y, are jointly distributed with pdf

$$f_{X,Y}(x,y) = 2x \quad 0 \leq x \leq 1, \;\; 0 \leq y \leq 1$$

Solution

$$\begin{aligned}
P(X \geq \sqrt{Y}) &= \int_0^1 \int_{x^2}^x (2x)dydx \\
&= \int_0^1 \left. 2xy \right|_{x^2}^x dx \\
&= \int_0^1 (2x^2 - 2x^3)dx \\
&= \left. 2/3x^3 - 1/2x^4 \right|_0^1 \\
&= 1/6
\end{aligned}$$

Problem 4

Statement

A pdf is defined by

$$f_{X,Y}(x,y) = \begin{cases} C(x+2y) & \text{if } 0 < y < 1 \ \ 0 < x < 2 \\ 0 & \text{otherwise} \end{cases}$$

a. Find the value of C

b. Find the marginal pdf of X

c. Find the joint cdf of X and Y

Solution

a. C is the normalizing constant so it integrates to 1.

$$\begin{aligned} 1 &= C \int_0^1 \int_0^2 C(x+2y)dxdy \\ 1 &= C \int_0^1 1/2x^2 + 2xy\Big|_0^2 dy \\ 1 &= C \int_0^1 (2+4y)dy \\ 1 &= C\ 2y + 2y^2\Big|_0^1 \\ 1 &= C(4) \\ 1/4 &= C \end{aligned}$$

b. This was poor planning. I found marginal pdf of Y in part A.

$$\begin{aligned} f_X(x) &= C \int_0^1 C(x+2y)dy \\ f_X(x) &= C \\ Evalxy + y^2 01 & \\ f_X(x) &= 1/4(x+1) \end{aligned}$$

c. This was pretty much done in part a, but needs to be done generically.

$$\begin{aligned} F_{X,Y}(x,y) &= 1/4 \int \int (x+2y)dxdy \\ F_{X,Y}(x,y) &= 1/4 \int 1/2x^2 + 2xydy \\ F_{X,Y}(x,y) &= 1/4(1/2x^2y + y^2x) \end{aligned}$$