Homework # 7

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Problem 1

Statement

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The length of time to failure (in hundreds of hours) for a transistor is a random variable Y with distribution function given by

$$F(y) = \begin{cases} 0, & y < 0 \\ 1 - e^{-y^2}, & y \geq 0 \end{cases}$$

a. Show that F(y) has the properties of a distribution function
b. Find the 0.30 -quartile $\phi_{0.30}$ of Y
c. Find $f(y)$
d. Find the probability that the transistor operates for at least 200 hours.
e. Find $P(Y > 100|Y \leq 200)$

Solution

a. In order to show that $F(y)$ has the properties of a distribution function it must have the following properties

$$\lim_{y \to -\infty} F_Y(y) = 0 \tag{1}$$
$$\lim_{y \to \infty} F_Y(y) = 1 \tag{2}$$
$$\text{if } y_1 \leq y_2 \qquad F(y_1) \leq F(y_2) \tag{3}$$
$$\lim_{y \to y_0^+} F_Y(y) = F_Y(y) \tag{4}$$

The first property is clearly true since $F(y) = 0 \quad \forall y < 0$. The second property we can see by taking the limit $\lim_{y\to\infty} 1 - e^{-y^2} = 1$ since $e^{-y^2} \to 0$. The third property is also true. This is clearly true for all $y < 0$ as it is constant. So we consider $y \geq 0$. If $y_1 \leq y_2$ then $y_1^2 \leq y_2^2$ for any $y \geq 0$. So $\frac{1}{e^{y_1^2}} \geq \frac{1}{e^{y_2^2}}$, and since we are subtracting a larger value in the y_1 case $1 - e^{-y_1^2} \leq 1 - e^{-y_2^2} \quad \forall y \geq 0$. The last property is also true. $1 - e^{-y^2}$ is continuous so this property holds for $y > 0$, so we just have to show that there is no jump discontinuity at $y = 0$. $1 - e^{-0^2} = 1 - 1 = 0$, so there is no discontinuity and the property holds. Thus it is a distribution function.

b. The $\phi_{0.30}$ quartile occurs when $F(y) = 0.30$. So solving we get

$$\begin{aligned} 1 - e^{-y^2} &= 0.30 \\ 0.70 &= e^{-y^2} \\ \ln(0.70) &= -y^2 \\ -\ln(0.70) &= y^2 \\ \sqrt{-\ln(0.70)} &= y \\ 0.597 &= y \end{aligned}$$

So the 0.30 quartile occurs at $y = 0.597$, so the likelihood of a value occurring between 0 and 0.597 is 0.30

c. We can find $f(y)$ which is the probability density function by differentiation.

$$f(y) = \begin{cases} 0, & y < 0 \\ 2ye^{-y^2}, & y \geq 0 \end{cases}$$

d. The probability that it works for at least 200 hours is the area under the curve of $f(y)$ from 2 to ∞ since the units of y are in hundreds of hours.

$$\int_2^{\infty} 2ye^{-2y^2} = F(\infty) - F(2) = 1 - 4e^{-8} = 0.9987$$

So it is very likely that the transistor will operate for over 200 hours.

e. Since the value $P(Y = y_0) = 0;\ \forall y_0$ we can exchange the strict inequality for the non-strict inequality. So, find $P(Y \geq 100 | Y \leq 200)$, so by the definition of conditional probability:

$$P(Y \geq 100 | Y \leq 200) = \frac{P(100 \leq Y \leq 200)}{P(Y \leq 200)} = \frac{\int_1^2 2ye^{-2y^2}}{\int_0^2 2ye^{-2y^2}} = \frac{F(2) - F(1)}{F(2) - F(0)} = \frac{0.981 - 0.632}{0.981 - 0} = 0.356$$

Problem 2

Statement

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In using the triangulation method to determine the range of an acoustic source, the test equipment must accurately measure the time at which the spherical wave front arrives at a recieving sensor. According to Perruzzi and Hilliard (1984), measurement errors in these times can be modeled as possesing a uniform distribuion from -0.05 to 0.05 μs (microseconds)

a. What is the probability that a particular arrival-time measurement will be accurate to within 0.01 μs?

b. Find the mean and variance of the measurement errors.

Solution

a. Since it is a uniform distribution, we know that the probability density in $[-0.05, 0.05]$ is 10. We are looking for the probability that the error is $|e| \leq 0.01$ which means $-0.01 \leq e \leq 0.01$. So if $Y \sim Unif(-0.05, 0.05)$ we are looking for $P(-0.01 \leq Y \leq 0.01) = \int_{-0.01}^{0.01} 10dy = 0.2$. So the probability that a particular arrival time is accurate to within 0.01 μs is 0.2.

b. The mean is the midpoint of the uniform distribution so $\frac{-0.05+0.05}{2} = 0\ \mu s$. We need to calculate a formula for $Var(Y)$. We start with $Var(Y) = E(Y^2) - E(Y)$. We then compute $E(Y^2)$

$$E(Y^2) = \int_{-\infty}^{\infty} y^2 f_Y(y)dy = \int_{\theta_1}^{\theta_2} y^2 \frac{1}{\theta_2 - \theta_1} dy = \frac{1}{\theta_2 - \theta_1} \left(1/3y^3 \Big|_{\theta_1}^{\theta_2} \right) = \frac{1}{3(\theta_2 - \theta_1)} \left(\theta_2^3 - \theta_1^3 \right)$$

Then $Var(Y) = \frac{1}{3(\theta_2-\theta_1)} \left(\theta_2^3 - \theta_1^3\right) - \left(\frac{\theta_2+\theta_1}{2}\right)^2 = \frac{(\theta_2-\theta_1)^2}{12}$. The last step in the simplification was given in class. So $Var(Y) = \frac{(\theta_2-\theta_1)^2}{12} = \frac{(0.05+0.05)^2}{12} = 0.0008333$

Problem 3

Statement

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A normally distributed random variable has density function

$$f_Y(y) = \frac{1}{\sigma\sqrt{2\pi}} \exp(-(y-\mu)^2/(2\sigma^2)), \qquad -\infty < y < \infty$$

Using the fundamental properties assosicated with any density function, argue that the parameter σ must be such that $\sigma > 0$

Solution

So any density function must (1) integrate to 1 over the reals. (2) $f_Y(y) \geq 0 \quad \forall y \in \mathbb{R}$.

If $\sigma = 0$ then $\frac{1}{\sigma\sqrt{2\pi}}$ is undefined, so $\sigma \neq 0$.

If $\sigma < 0$ then $\frac{1}{\sigma\sqrt{2\pi}} < 0$, and $e^a > 0 \quad \forall a \in \mathbb{R} \implies \frac{1}{\sigma\sqrt{2\pi}} \exp(-(y-\mu)^2/(2\sigma^2) < 0 \quad \forall y$, which violates (2).

So $\sigma > 0$

Problem 4

Statement

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Wires manufactured for use in a computer system are specified to have resistances between 0.12 and 0.14 ohms. The actual measured resistances of the wires produced by company A have a normal probability distribution with mean 0.13 ohms and standard deviation 0.005 ohm.

a. What is the probability that a randomly selected wire from company A's production will meet the specifications?

b. If four of these wires are used in each computer system and all are selected from company A, what is the probability that all four in a randomly selected system will meet the specifications?

Solution

a. So we are seeing if the probability that the actual resistance falls within ± 0.01 of the mean. This is two standard deviations. So by the empiricle rule, the probability that a wire is within 2 standard deviations is 95%. So there is a 95% probability the wire will meet the required specifications.

b. The probability that all four meet the specifications can be written as the probability that the first one meets specifications **and** the second one meets specifications **and** the third one meets specifications **and** the fourth one meets specifications. This logical and is multiplication. The probability that a single one meets specifications is $95\% = 0.95$. So the probability that all meet specifications $= 0.95 * 0.95 * 0.95 * 0.95 = 0.8145 = 81.45\%$. So there is a 91.45% probability that all meet the required specifications.