1 Distributions

Uniform: $f(x) = \frac{1}{b-a}; a \leq x \leq b;$
$\bar{X} = \frac{a+b}{2}$; $Var(x) = \frac{(b-a)^2}{12}$
$mgf = \frac{e^{tb}-e^{ta}}{t(b-a)}$

Normal: $f(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp \frac{-1}{2\sigma^2}(x-\mu)^2$;
$\bar{X} = \mu$; $Var(x) = \sigma^2$;
$mgf = \exp \mu t + \frac{t^2\sigma^2}{2}$;

Exponential: $f(x) = \frac{1}{\beta} e^{\frac{-x}{\beta}}$;
$\bar{X} = \beta$; $Var(x) = \beta^2$;
$mgf = (1 = \beta t)^{-1}$;

Gamma: $f(x) = \frac{1}{\Gamma(\alpha)\beta^\alpha} x^{\alpha-1} e^{\frac{-x}{\beta}}$;
$\bar{X} = \alpha\beta$; $Var(x) = \alpha\beta^2$;
$mgf = (1 = \beta t)^{-\alpha}$; $\Gamma(\alpha) = \int_0^\infty y^{\alpha-1}e^{-y}dy$

Beta: $f(x) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$;
$\bar{X} = \frac{\alpha}{\alpha+\beta}$;
$Var(x) = \frac{\alpha\beta}{(\alpha+\beta)^2(\alpha+\beta+1)}$;

Binomial: $\binom{p(y)=n}{yp^y(1-p)^{n-y}}$
$\bar{X} = np, Var(x) = np(1-p)$

Geometric: $p(y) = p(1-p)^{y-1}$
$\bar{X} = 1/p, Var(x) = (1-p)/p^2$

Poisson: $p(y) = \frac{\lambda^y e^{-\lambda}}{y!}$
$\bar{X} = \lambda, Var(x) = \lambda$

2 Misc

Moment Generating Function:
$m_{X_1,\ldots X_n} = \mathbb{E}[e^{tX_1+\cdots+tX_n}] \overset{iid}{=} \prod_{i=1}^n \mathbb{E}[e^{tX_i}]$

Jacobian Method: $Y = g(X)$ with $g(X)$ monotone:
$f_Y(y) = f_X(g^{-1}) \left| \frac{d}{dy} g^{-1}(y) \right|$.
If $g(X)$ not monotone, then f_Y is sum of piecewise monotone $g_i(X)$.

3 Order Statistics

Maximum: $F_{X_{(n)}}(x) = F_X(x)^n$,
$f_{X_{(n)}} = n(F_X(x))^{n-1} f_X(x)$

Minimum: $F_{X_{(1)}}(x) = 1 - (1 - F_X(x))^n$,
$f_{X_{(1)}}(x) = n(1 - F_X(x))^{n-1} f_X(x)$

General:
$f_{X_{(j)}} = \frac{n!}{(j-1)!(n-j)!} F_X(x)^{j-1} f_X(x)(1 - F_X(x))^{n-j}$

4 Normal, Chi-Square Distributions

Mean: $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$.

Given $X \sim N(\mu, \sigma^2)$, $Z_i = \frac{X_i - \mu}{\sigma}$. Then:
$\sum_{i=1}^n Z_i^2 \sim \chi_n^2$

$\chi_\nu^2 = Gamma(\frac{\nu}{2}, \beta = 2)$, $\mathbb{E}[X] = \nu, Var(x) = 2\nu$.

Theorem 7.3: $\bar{x}, S^2$ are independent, and:

$$\frac{(n-1)S^2}{\sigma^2} \sim \chi_{n-1}^2 \qquad S^2 = \frac{1}{n-1}\sum_{i=1}^n (x_i - \bar{x})^2$$

Student T Distribution: if $Z \sim N(0,1)$ and $W \sim \chi_\nu^2$, and Z, W independent. Then RV: $T = \frac{Z}{\sqrt{W/\nu}}$ follows the student T distribution. Where:

$$f_T(x) = \frac{\Gamma(\frac{\nu+1}{2})}{\Gamma(\frac{\nu}{2})\sqrt{\pi\nu}} (1 + \frac{t^2}{\nu})^{-(\nu+1)/2}$$

Snedecor's F Distribution: if $U \sim \chi_u^2$ and $V \sim \chi_v^2$ and U, V independent. Then $X = \frac{U/u}{V/v} \sim F_{u,v}$. Where:

$$f_X(x; u, v) = \frac{\Gamma(\frac{u+v}{2})}{\Gamma(u/2)(v/2)} (\frac{u}{v}^{u/2}) x^{u/2-1} (1 + \frac{u}{v} * x)^{-(u+v)/2} \quad x > 0$$

5 Central Limit Theorem

Let $X_1, \ldots, X_n$ be iid RV with expected value μ and variance σ^2 and assume that mgf exists. The cdf of $\bar{X}$ converges to a normal distribution as $n \to \infty$. $Z_n = (\bar{X} - \mu)/(\sigma/\sqrt{n})$, then $\lim_{n\to\infty} P(Z_n \leq z) = P(Z \leq z)$ where $Z \sim N(0,1)$. I.e. the sampling distribution for the sample mean $\overset{a}{\sim} N(\mu, \sigma^2/n)$

6 Method of Moments

Goal is to match the pointwise moments to the distribution moments. Need one equation for each unknown. Set $\mathbb{E}[X] = \frac{1}{n}\sum x_i$, $\mathbb{E}[X^2] = \frac{1}{n}\sum x_i^2$, … for however many equations needed.

7 Maximum Likelihood Extimators

Compute likelihood function $\mathcal{L}(\theta|x)$ = joint distribution = $f(x|\theta)$. Find $\hat{\theta}_{MLE}$ that maximizes $\mathcal{L}(\theta|x)$. Solve by setting $\frac{\partial}{\partial\theta_i} L(\theta|x) = 0$. Can also compute log-likelihood which is easier as $l(\theta|x) = \ln(L(\theta|x))$. Don't forget! Verify that critical point is the maximum by verifying second derivative at critical point < 0.

8 Mean Squared Error and Bias

$Bias_\theta(\hat{\theta}) = \mathbb{E}[\hat{\theta}] - \theta$. An estimator is said to be unbiased if $\mathbb{E}[\hat{\theta}] = \theta$. The mean squared error is: $MSE_\theta(\hat{(\theta)}) = \mathbb{E}_\theta[(\hat{\theta} - \theta)^2] = Var_\theta(\hat{\theta}) + Bias_\theta(\hat{\theta})^2$. MSE combines accuracy and precision into one measure, if MSE increases the estimator is 'less good'.

Best unbiased estimator - Uniform Minimum Variance Unbiased Estimator (UMVUE) is $Bias_\theta(\hat{\theta}^*) = 0$ and $Var(\hat{\theta}^*) \leq Var(\hat{\theta}) \quad \hat{\theta} \in \{\hat{\theta} | Bias_\theta(\hat{\theta}) = 0\}$.

9 Sufficiency, Completeness, UMVUE

Definition 9.1 (UMVUE)**.** *An estimator $\hat{\theta}$ is UMVUE if: $Bias_\theta(\hat{\theta}) = 0$ and $Var(\hat{\theta}) \leq Var(\theta') \quad \forall \theta' \in \{\theta' | Bias_\theta(\theta') = 0\}$*

Definition 9.2 (Sufficient)**.** *Let $X_1, \ldots, X_n$ be a random sample from a probability distribution with parameter θ, The statistic, $T(X)$ is said to be sufficient for θ if the conditional distribution of $(X_1, \ldots, X_n)|T(X)$ does not depend on θ.*

Factorization Theorem: let $f(x;\theta)$ be joint pdf of $X_1, \ldots, X_n$. A statistic is sufficient for θ is sufficient iff $\exists g(T(x);\theta), h(x)$ such that: $f(x;\theta) = g(T(x);\theta) * h(x)$.

Definition 9.3 (Complete Statistic)**.** *A statistic is complete if $\mathbb{E}[g(T)] = 0 \quad \forall \theta$ implies that $P(g(T) = 0) = 1 \quad \forall \theta$*

Exponential Family Theorem: Let $X_1, \ldots, X_n$ be an iid RS from a pdf of the form

$$f(x;\theta) = h(x)c(\theta)\exp\left(\sum_{j=1}^{k} t_j(x)w_j(\theta)\right)$$

Then the set of statistics $T(X) = \sum_{i=1}^n t_1(X_i), \ldots, \sum_{i=1}^n t_k(X_i)$ is complete and sufficient for θ.