STAT 422: HW #6

Due: April 4, 2022, 11:59pm

Problem 1

Statement

Let $X_1, ..., X_n$ be a random sample from the distribution with the following pdf:

$$f(x;\theta) = \theta x^{\theta-1} I_{(0,1)}(x), \text{ where } \theta > 0$$

a. Find a complete sufficient statistic for θ

b. Using your answer in part (a), explain why $\prod_{i=1}^{n} X_i$ is also a sufficient statistic for θ.

Solution

a. Goal is to show that this is exponential family. So we want to write $f(x;\theta) = h(x)c(\theta)\exp\left(\sum_{j=1}^{k} t_j(x) w_j(\theta)\right)$.

$$\begin{aligned} f(x;\theta) &= \prod_{i=1}^{n} \theta x_i^{\theta-1} \\ &= \prod_{i=1}^{n} \theta \exp\left((\theta-1)\ln(x_i)\right) \\ &= \theta^n \exp\left(\sum_{i=1}^{n} \ln(x_i)(\theta-1)\right) \end{aligned}$$

where $h(x) = 1, c(\theta) = \theta^n, t(x) = \sum \ln(x_i), w(\theta) = (\theta - 1)$. Thus by the Exponential Family theorem: $\sum \ln(x_i)$ is a CSS.

b. We can see this via Factorization theorem in a (hand wavy way). We show sufficiency but not completeness. So for completeness, we need a one-to-one function that maps $\sum \ln(x_i)$ to $\prod x_i$. We essentially did this in part a, we just now have to walk backwards. e^x is one-to-one. So we take $\exp(\sum \ln(x_i)) = \prod \exp(\ln(x_i)) = \prod_{i=1}^{n} x_i$ As required :)

Problem 2

Statement

A real estate firm wants to estimate the rate of new houses sold in a week in Bozeman. Assume $X_1, ..., X_n$ is a random sample of weekly house sales in Bozeman, where each X_i is a Poisson random variable with mean μ. The observed number of new houses sold per week for 5 randomly chosen weeks, were 2, 3, 3, 4, and 6. Find the best unbiased estimator (i.e., the UMVUE) of μ and its estimate. Show all work, and carefully explain why it is the UMVUE.

Solution

First, what is the pdf of Poisson? $p(y) = \frac{\lambda^y e^{-y}}{y!}$. We want an UMVUE for λ. First we want to check if it is exponential family.

$$
\begin{aligned}
p(y;\lambda) &= \prod_{i=1}^{n} \frac{\lambda^{y_i} e^{-y_i}}{y_i!} \\
&= \prod_{i=1}^{n} \frac{1}{y_i!} \lambda^{y_i} e^{-y_i} \\
&= \prod_{i=1}^{n} \frac{1}{y_i!} \exp(y_i \ln(\lambda) - y_i) \\
&= \prod_{i=1}^{n} \frac{1}{y_i!} \exp((\lambda - 1) y_i) \\
&= \left(\prod_{i=1}^{n} \frac{1}{y_i!}\right) \left(\exp(\sum_{i=1}^{n} (\lambda - 1) y_i)\right)
\end{aligned}
$$

where $h(x) = \prod_{i=1}^{n} \frac{1}{y_i!}, c(\theta) = 1, t(x) = \sum y_i, w(\theta) = (\lambda - 1)$. Thus by the Exponential Family theorem: $\sum y_i$ is a CSS.

Next we need $g(\sum y_i)$ such that $\mathbb{E}[g(\sum y_i)] = \lambda$. Pick $g(\sum_{i=1}^{n} y_i) = 1/n \sum_{i=1}^{n} y_i$.

$$
\begin{aligned}
\mathbb{E}[g(\sum y_i)] &= \mathbb{E}[1/n \sum_{i=1}^{n} y_i] \\
&= 1/n \sum_{i=1}^{n} \mathbb{E}[y_i] \\
&= 1/n \sum_{i=1}^{n} \lambda \\
&= \lambda
\end{aligned}
$$

As required, where step three follows from the expected value of a poisson distribution.