STAT 422: Homework #4

Due: March 02, 2022 (midnight)

Problem 1

Statement

Assume $X_1, X_2 \dots X_n$ are independent, normal random variables with $E(X_i) = \mu_i$ and $Var(X_i) = \sigma_i^2$, $i = 1, 2, \dots, n$. Recall that the mgf of the $N(\mu, \sigma^2)$ distribution is $e^{\mu t + \frac{t^2 \sigma^2}{2}}$.

a. Is $X_1, X_2 \dots X_n$ a random sample, why or why not?

b. Find the distribution (including parameters) of $Y = \sum_{i=1}^{n} X_i$ using mgfs. Explain, in words, what this result indicates and how you think it is useful.

c. THM 6.3 in your book is a generalization of the result you found in part (b). Explain, in words, how this builds upon the result you found in (b).

d. Give the distribution (including parameters) of each of the following random variables. Justify/explain each step in your path to determining each constructed RVs corresponding distribution. *You may use the theorems you proved in parts a-c and THM 6.3.*

- $3X_1 + 5X_2$
- $Z_1^2 + Z_2^2$, where $Z_i = \dfrac{X_i - \mu_i}{\sigma_i}$ for $i = 1, 2, ..., n$
- $\dfrac{Z_2 - Z_1}{\sqrt{2}}$, where $Z_i = \dfrac{X_i - \mu_i}{\sigma_i}$ for $i = 1, 2, ..., n$
- $\dfrac{(Z_1 + Z_2)^2}{(Z_3 - Z_4)^2}$, where $Z_i = \dfrac{X_i - \mu_i}{\sigma_i}$ for $i = 1, 2, ..., n$

Solution

a. Yes, they are sampled independently and are random variables (given in the statement)

b. Proof:

$$\begin{aligned} m_Y = \mathbb{E}\left[e^{tY}\right] &= \mathbb{E}\left[e^{t\sum_{i=1}^n X_i}\right] \\ &= \mathbb{E}\left[\prod_{i=1}^n e^{tX_i}\right] \stackrel{iid}{=} \prod_{i=1}^n \mathbb{E}\left[e^{tX_i}\right] \\ &= \prod_{i=1}^n e^{\mu_i t + \frac{t^2\sigma_i^2}{2}} = \prod_{i=1}^n e^{\mu_i t} e^{\frac{t^2\sigma_i^2}{2}} \\ &= e^{\sum_{i=1}^n \mu_i t} e^{\sum_{i=1}^n \frac{t^2\sigma_i^2}{2}} \\ &= e^{\sum_{i=1}^n \mu_i t + \frac{t^2\sigma_i^2}{2}} \end{aligned}$$

This is the mgf of another normal distribution, with $\mu = \sum_{i=1}^n \mu_i$, $sigma^2 = \sum_{i=1}^n \sigma_i^2$. So we observe that means and standard deviations add together when summing normal distributions.

c. This result in THM 6.3 allows for scalar multiples of normal distributions to be added together. This allows us to multiply by a constant and show that the result is still normal. We can also subtract distributions if the constant < 0. Our proof is just the special case where all the constants $= 1$.

d.
- $3X_1 + 5X_2 \sim N(3\mu_1 + 5\mu_2, 9\sigma_1^2 + 25\sigma_2^2$. This is by theorem 6.3 with $a_1 = 3$ and $a_2 = 5$.
- $Z_1^2 + Z_2^2 \sim \chi_2^2$ We know that $Z_i \sim N(0,1)$ for $i = 1, 2$ so both Z_1, Z_2 are drawn from the same distribution $N(0,1)$. Then we can apply theorem 7.2 with \$n = 2 \$.
- $\frac{Z_2 - Z_1}{\sqrt{2}} \sim N(0,1)$. This is a funny trick of math. We have $Z_i \sim N(0,1)$. So this is from Theorem 6.3 with $a_1 = \frac{-1}{\sqrt{2}}$ and $a_2 = \frac{1}{\sqrt{2}}$, thus $\mu = a_1 * 0 + a_2 * 0 = 0$ and $\sigma^2 = a_1^2 + a_2^2 = 1/2 + 1/2 = 1$ so we get back to the same distribution :-)
- $\frac{(Z_1+Z_2)^2}{(Z_3-Z_4)^2} \sim F_{1,1}$, so first we note that $Z_1 + Z_2 \sim N(0,2)$, and $Z_3 - Z_4 \sim N(0,2)$ by theorem 6.3. So by theorem 7.2 $(Z_1 + Z_2)^2 = 2 * (A/\sqrt{2})^2 \sim \chi_1^2$ where $Z_1 + Z_2 = A \sim N(0,2)$, so $A/\sqrt{2} \sim N(0,1)$. Similarly $(Z_3 + Z_4)^2 = 2 * (B/\sqrt{2})^2 \sim \chi_1^2$ where $Z_3 - Z_4 = B \sim N(0,2)$, so $B/\sqrt{2} \sim N(0,1)$. So we have $\frac{2(A/\sqrt{2})^2}{2(B/\sqrt{2})^2} = \frac{(A/\sqrt{2})^2}{(B/\sqrt{2})^2}$. Since $(A/\sqrt{2})^2 \sim \chi_1^2$ and $(B/\sqrt{2})^2 \sim \chi_1^2$ we can use Def 7.3 to get that $\frac{(A/\sqrt{2})^2}{(B/\sqrt{2})^2} \sim F_{1,1}$ (Snedecor's F distribution with u=v=1 degrees of freedom)

Problem 2

Statement

Let X_i, $i = 1, 2, 3$ be independent with $N(i, i^2)$ distributions. That is, $X_1 \sim N(1, 1^2)$, $X_2 \sim N(2, 2^2)$, and $X_3 \sim N(3, 3^2)$. For each of the following situations, use any subset of the X_is to construct a random variable with the stated distribution. Justify why each constructed random variable follows the corresponding distribution of interest.

a. Standard normal distribution
b. Chi-squared distribution with 3 degrees of freedom.
c. t distribution with 2 degrees of freedom
d. F distribution with 1 numerator and 2 denominator degrees of freedom

Solution

a. $\frac{1}{\sqrt{8}}X_2 - \frac{2}{\sqrt{8}}X_1 \sim N(0,1)$ we have $\mu = \frac{2}{\sqrt{8}} - \frac{2}{\sqrt{8}} = 0$, and $\sigma^2 = \frac{4}{8} + \frac{4}{8} = 1$.

b. $\sum_{i=1}^{3}\left(\frac{1}{\sqrt{8}}X_2 - \frac{2}{\sqrt{8}}X_1\right)^2 = 3 * \left(\frac{1}{\sqrt{8}}X_2 - \frac{2}{\sqrt{8}}X_1\right)^2 \sim \chi^2_3$. From a we have that $\frac{1}{\sqrt{8}}X_2 - \frac{2}{\sqrt{8}}X_1 \sim N(0,1)$. Then we have a sum of 3 squared standard normal distributions, which by theorem 7.2 is χ^2_3.

c. From b we can easily get $2 * \left(\frac{1}{\sqrt{8}}X_2 - \frac{2}{\sqrt{8}}X_1\right)^2 \sim \chi^2_2$. Now it would be nice to re-use our solution from for our N(0,1), but we can't do that because we would have just $\frac{1}{\frac{1}{\sqrt{8}}X_2 - \frac{2}{\sqrt{8}}X_1}$, so we have to make a new one. $\frac{1}{\sqrt{18}}X_3 - \frac{3}{\sqrt{18}}X_1 \sim N(0,1)$, we have $\mu = \frac{3}{\sqrt{18}} - \frac{3}{\sqrt{18}} = 0$ and $\sigma^2 = \frac{9}{18} + \frac{9}{18} = 1$. So now we can put this together to get: (note that $\sqrt{8} * \sqrt{2} = \sqrt{16} = 4$ for dividing by degrees of freedom)

$$\frac{\frac{1}{\sqrt{18}}X_3 - \frac{3}{\sqrt{18}}X_1}{2 * \sqrt{2} * \left(\frac{1}{4}X_2 - \frac{2}{4}X_1\right)^2} \sim t_2$$

d. Thankfully, this one is pretty easy to construct. We can build a χ^2_2 distribution pretty easily from the new $N(0,1)$ we just defined above the same way as we did in b. So we have: $1 * \left(\frac{1}{\sqrt{18}}X_3 - \frac{3}{\sqrt{18}}\right)^2 \sim \chi^2_1$ and from c: $2 * \left(\frac{1}{\sqrt{8}}X_2 - \frac{2}{\sqrt{8}}X_1\right)^2 \sim \chi^2_2$. Thus putting it together

$$\frac{\left(\frac{1}{\sqrt{18}}X_3 - \frac{3}{\sqrt{18}}X_1\right)^2}{2 * \sqrt{2} * \left(\frac{1}{4}X_2 - \frac{2}{4}X_1\right)^2} \sim F_{1,2}$$