STAT 422: Homework #7

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Problem 1

Statement

Section 8.9 in your text describes how to create a 100(1-α)% CI for σ^2.

a. Construct a 100(1-α)% CI for the sample standard deviation, σ.

b. Now, suppose that you are interested in the variability in credit card debt among MSU students, and in a RS of 40 students, you observe a sample standard devaition of $s = 200\$$. Construct a 95% CI for σ and *interpret the interval in the context of the problem.*

c. Will a 90% CI for σ be wider or smaller than a 95% CI for σ? Explain. Assume the same sample size and sample standard deviation as in part b.

Solution

a. First we need an unbiased estimator for σ. We know that S^2 is an unbiased estimator or σ^2, so its square root S as unbiased estimator of σ. We know from Theorem 7.3 that $\frac{(n-1)S^2}{\sigma^2} \sim \chi^2_{n-1}$, we take the square root of both sides to find that $\frac{\sqrt{n-1}S}{\sigma} \sim \chi_{n-1}$. We then need to follow the pivotal method to find two numbers χ_L, χ_U such that

$$P\left[\chi_L \leq \frac{\sqrt{n-1}S}{\sigma} \leq \chi_U\right] = 1-\alpha$$

There are many combinations for χ_L, χ_U, we choose the one that removes equal areas on left and right ie. $P(X \leq \chi_L) = P(X \geq \chi_U) = \alpha/2$ if $X \sim \chi_{n-1}$. This corresponds to picking χ_L to be the α-quantile of χ_{n-1} and χ_U to be the $(1-\alpha)$-quantile of χ_{n-1}.These selections for χ_L, χ_U are denoted $\chi_\alpha, \chi_{1-\alpha}$ respectively. Then we re-arrange the formula above to solve for σ.

$$P\left[\chi_\alpha \leq \frac{\sqrt{n-1}S}{\sigma} \leq \chi_{1-\alpha}\right] = 1-\alpha$$

$$P\left[\frac{\sqrt{n-1}S}{\chi_{1-\alpha}} \leq \sigma \leq \frac{\sqrt{n-1}S}{\chi_\alpha}\right] = 1-\alpha$$

Is a $100(1-\alpha)$% CI for σ.

```r
# Example of how to compute Chi_l and Chi_u
library(chi)  # for the chi distribution
n = 5  # for example
alpha = 0.05
result <- qchi(p=c(alpha, 1-alpha), df = n)
chi_l = result[1]
chi_u = result[2]
print(chi_l)
```

```
## [1] 1.070269
```

```r
print(chi_u)
```

```
## [1] 3.327236
```

b. So first we need to compute χ_L, χ_U with the following r script

```r
# Example of how to compute Chi_l and Chi_u
n = 40  # 40 students
alpha = 0.05
result <- qchi(p=c(alpha, 1-alpha), df = n)
chi_l = result[1]
chi_u = result[2]
print(chi_l)
```

```
## [1] 5.148719
```

```r
print(chi_u)
```

```
## [1] 7.46716
```

Then we compute the bounds:

```r
S = 200
lower_bound = (sqrt(n - 1) * S) / chi_u
upper_bound = (sqrt(n - 1) * S) / chi_l
print(paste0("The 95% confidence interval for sigma is ", lower_bound, " <= sigma <= ",
```

```
## [1] "The 95% confidence interval for sigma is 167.265679397627 <= sigma <= 242.584553
```

c. The interval for 90% will be more narrow than 95%. If we think about it, we cut off $\alpha/2$ probability on both ends. If α is bigger, then we cut out more of a region and thus the region we keep is smaller.

Problem 2

Statement

An experimenter has prepared a drug dosage level that she claims will induce sleep for 80% of people suffering from insomnia. After examining the dosage, we think that her claims regarding the effectiveness of the dosage are inflated. In an attempt to provide statistical evidence against her claim, we administer her prescribed dosage to 20 insomniacs, and we observe Y, the number for which the drug dose induces sleep. We wish to test the hypothesis $H_0 : p = 0.8$ versus the alternative $H_a : p > 0.8$ Assume that the rejection region $\{y \leq 12\}$ is used.

a. In terms of this problem, what is a Type I error?

b. Find the significance level,α of the test.

c. In terms of this problem, what is a Type II error?

d. Find the probability of a Type II error when $p = 0.6$

e. Find the probability of a Type II error when $p = 0.4$

f. Find the power for the following values of p

   - $p = 0.4$
   - $p = 0.5$
   - $p = 0.6$
   - $p = 0.7$

g. Graph the power function in R. Provide the graph as well as the code used to produce it.

h. What is the power when $p = 0.8$? Why is this the same value as your answer to part (b)

i. What would happen to the power at $p = 0.8$ if we change the rejection region to $\{y \leq 13\}$? What would happen to the power at $p = 0.8$ if we change the rejection region to $\{y \leq 11\}$? Briefly explain why each of these changes occurrs.

Solution

a. Type 1 error is when we reject H_0 if it is actually true. This is if more than 80% of tested insomniacs fall asleep when the drug is actually less than 80% effective.
b. This follows a binomial distribution. We reject the null if we observe ≥ 12 insomniacs falling asleep. We want to know the probability of $P(X \leq 12)$ if $X \sim Binomial(p = 0.8, n = 20)$ which is the probability that we reject the null hypothesis. This is computed iwth following R code:

```
alpha = sum(dbinom(0:12, 20, 0.8))
print(alpha)
```

```
## [1] 0.03214266
```

c. Type 2 error is when we fail to reject H_0 if it is actually false. This is if fewer than 80% of tested insomniacs fall asleep and the drug is actually 80% effective.
d. This is the probability that we observe more than 12 insomiacs falling asleep if $p = 0.6$. This is $P(X > 12) \quad X \sim Binomial(p = 0.6, n = 20)$. This is computed by

```
probability_type2 = sum(dbinom(13:20, 20, 0.6))
print(probability_type2)
```

```
## [1] 0.4158929
```

e. Similarly but $p = 0.4$

```
probability_type2 = sum(dbinom(13:20, 20, 0.4))
print(probability_type2)
```

```
## [1] 0.02102893
```

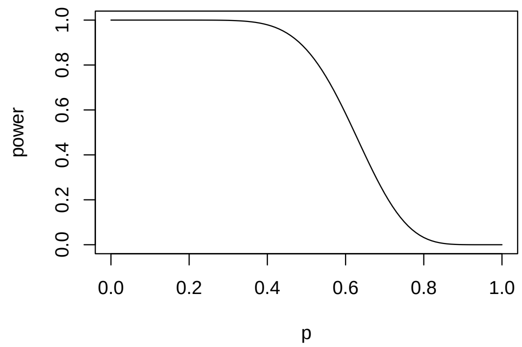
f. The power of the alternate hypothesis is 1 - Probability of type 2 error

```
probability_type2_4 = sum(dbinom(13:20, 20, 0.4))
probability_type2_5 = sum(dbinom(13:20, 20, 0.5))
probability_type2_6 = sum(dbinom(13:20, 20, 0.6))
probability_type2_7 = sum(dbinom(13:20, 20, 0.7))
cat("P = 0.4: " , 1 - probability_type2_4,
             "\nP = 0.5: " , 1 - probability_type2_5,
             "\nP = 0.6: " , 1 - probability_type2_6,
             "\nP = 0.7: " , 1 - probability_type2_7)
```

```
## P = 0.4:  0.9789711
## P = 0.5:  0.868412
## P = 0.6:  0.5841071
## P = 0.7:  0.2277282
```

g.

```
fun <- function(x){
  type2 = sum(dbinom(13:20, 20, x))
  return(1 - type2)
}
curve(Vectorize(fun)(x), from = 0, to=1, ylab='power', xlab='p')
```



h. This is the same as in (b) because when $p = 0.8$ the null hypothesis is true. So this becomes the probability we reject the null hypothesis given that it is true (type 1 error).

```
probability_type2_8 = sum(dbinom(13:20, 20, 0.8))
cat("Power, p=0.8 : ", 1 - probability_type2_8)
```

```
## Power, p=0.8 :  0.03214266
```

i. The power increases (α increases) if we change the rejection region. If the region is $y \leq 13$ this makes it easier to reject, thus making the confidence smaller. If the region is $y \leq 11$ it makes it harder to reject, so α decreases making it more likely that the test is correct.

```
probability_type2_13 = sum(dbinom(14:20, 20, 0.8))
probability_type2_11 = sum(dbinom(12:20, 20, 0.8))
cat("Y <= 13: " , 1 - probability_type2_13,
    "\nY <= 11: " , 1 - probability_type2_11
)
```

```
## Y <= 13:  0.08669251
## Y <= 11:  0.009981786
```

‘‘‘