8 Nov. 2019

## Linear Prog.

min/max $\sum_{i=1}^{n} c_i x_i$

subject to $\sum_{i=1}^{n} a_{ij} x_i \le b_j$ "less than or ="

for $j = 1, \dots, m$

constraints

or

"greater or ="

$\sum_{i} a_{ij} x_i \ge b_j$

"equality"

or $\sum a_{ij} x_i = b_j$

## Std Form:

- max, not min
- $x_i \ge 0 \quad \forall i = 1, \dots, n$
- linear constraints are all $\le$

(*) e.g.,

max $x_1 + x_2$

s.t.

$4x_1 - x_2 \le 8$

$2x_1 + x_2 \le 10$

$5x_1 - 2x_2 \ge -2$

$x_1, x_2 \ge 0$

Std Form of (*)

max $x_1 + x_2$

subject to

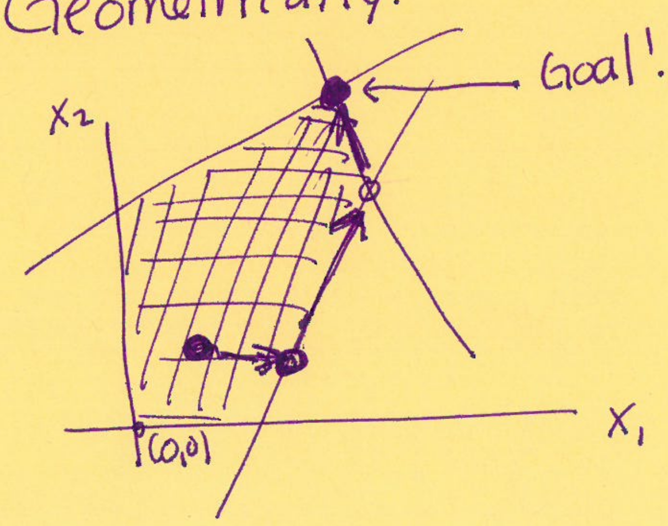
$4x_1 - x_2 \le 8$

$2x_1 + x_2 \le 10$

$-5x_1 + 2x_2 \le 2$

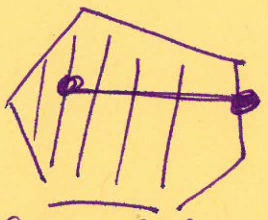
$x_1, x_2 \ge 0$

Geometrically:



Note: Since constraints are all linear, taking the intersection of all half-spaces is a convex subset of $\mathbb{R}^d$.

def: $X$ is convex if $\forall x, y \in X$, the line from $x$ to $y$ is inside $X$



convex

NOT convex

~~Slack Form~~:

constraint: $\underbrace{\sum_{i=1}^{n} a_{ij} x_i}$ $\leq b_j$

$$a_j = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix} \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

$a_j^T x \leq b_j$ ← one equation in Linear-Algebra notation

$$b = \begin{pmatrix} b_1 \\ \vdots \\ b_m \end{pmatrix} \quad A = \begin{pmatrix} a_{11} & & \\ & \ddots & \\ & & a_{mn} \end{pmatrix}$$

$$Ax \leq b$$

↑ all of our constraints!

constraint: $a_j^T x \le b_j$

slack $S = b_j - a_j^T x$

"how much wiggle room is left"

---

Slack Form, by example: (*) unknowns: $x_1, x_2$

max $x_1 + x_2$

subject to

$$x_3 = 8 - 4x_1 + x_2$$
$$x_4 = 10 - 2x_1 - x_2$$
$$x_5 = 2 + 5x_1 - 2x_2$$
$$x_1, x_2 \ge 0 \quad \} \text{non-basic variables}$$
$$x_3, x_4, x_5 \ge 0 \quad \} \text{basic variables}$$

In general:

n variables
m constraints

max $c^T x$

subject to

$$x_{n+j} = b_j - a_j^T x \quad \text{for } j = 1, \dots, m$$
$$x_i \ge 0 \quad \forall i$$

# Example:

$$\min \quad x_1 - x_2 + 2x_3$$

subject to

$$2x_1 + x_2 - x_3 \leq 8$$

$$-2x_1 - 3x_2 + x_3 \geq 2$$

$$x_1, x_2 \geq 0$$

$$x_3 \leq 0$$

---

## Standard Form:

$$\max \quad -x_1 + x_2 + 2x_3'$$

subject to:

$$2x_1 + x_2 + x_3' \leq 8$$

$$[2x_1 + 3x_2 + x_3' \leq -2]$$

$$x_1, x_2, x_3' \geq 0$$

note: Above, we made the substitution $x_3' = -x_3$. After solving, we can find $x_3 = -x_3'$.

## Slack Form:

$$\max \quad z = -x_1 + \underline{x_2} + \underline{2x_3'}$$

subject to:

$$x_4 = 8 - 2x_1 - x_2 - x_3'$$

$$x_5 = -2 - 2x_1 - 3x_2 - x_3'$$

$$x_1, x_2, x_3' \geq 0$$

$$x_4, x_5 \geq 0$$

---

Init solution:

- non-basic variables $= 0$

  $x_1 = 0, x_2 = 0, x_3' = 0$
- basic variables $x_{n+j} = b_j$

  $x_4 = 8, \ x_5 = -2$

NOT in the feasible space b/c 2nd equation Breaks!

Choose $x_2$ or $x_3'$ as pivot.

Question to think about:

How ~~can~~ can we formulate max flow as a linear program?

$$\max \sum_{v \in V} f_{sv} - \sum_{v \in V} f_{vs}$$

where $s \in V$ is the source

$f_{xy}$ = flow from $x$ to $y$ on edge $xy$
$= f(x,y)$

subject to...