1 Nov 2019

flow network $G = (V, E, c)$
← capacity fcn
$c : E \to \mathbb{R}$

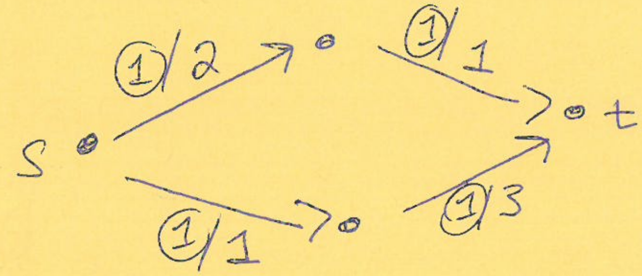
$f : E \to \mathbb{R}$ is a flow if:

① $0 \le f(e) \le c(e) \quad \forall e \in E$

② conservation of flow $\forall v \in V$ s.t. $v \neq s, v \neq t$

$$\sum_{u \in V} f(u,v) = \sum_{w \in V} f(v,w)$$

e.g.)



x = the flow
y ← the capacity

A cut (S,T) is a partition of vertices

$$V = \underset{\ni s}{S} \sqcup T_{\ni t}$$

TWO equiv problems (proof later today)

Max Flow ≡ min cut

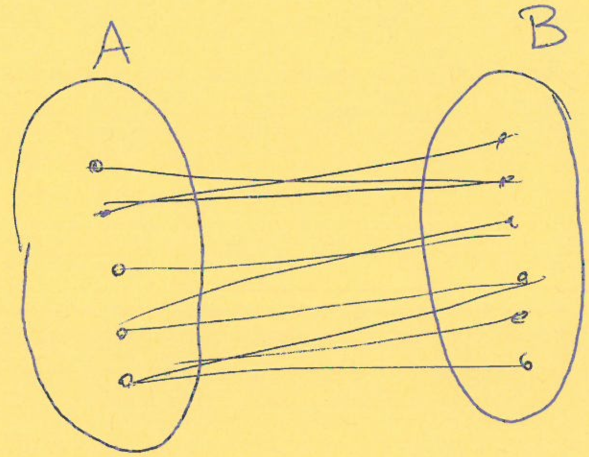
max value of flow

$$|f| = \sum_{v \in V} f(s,v)$$

$$c(S,T) = \sum_{v \in V} \sum_{u \in V} c(u,v)$$

Bipartite Graph Matching

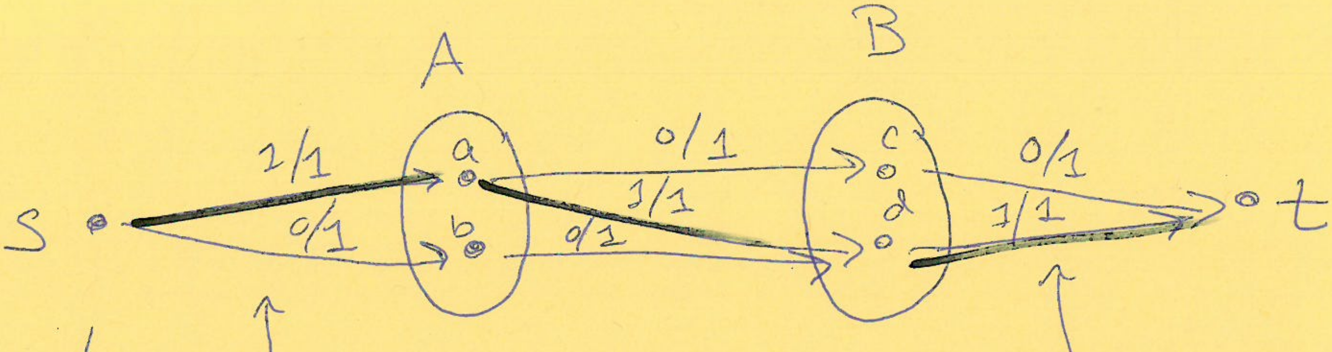
$G=(V,E)$ is a bipartite graph
if $V = A \sqcup B$ s.t. $E \subseteq A \times B$



eg., A = kids
B = presents
edge = kid likes the present

goal: match presents to kids in order to
maximize # of happy kids.
no vertex is in more than
one edge of the matching $M \subseteq E$

TO Formulate as Max Flow, we are missing:
1. capacities on edges (+ direction)
2. need to ensure 1 present is not split
or given twice
3. unclear what s/t are.

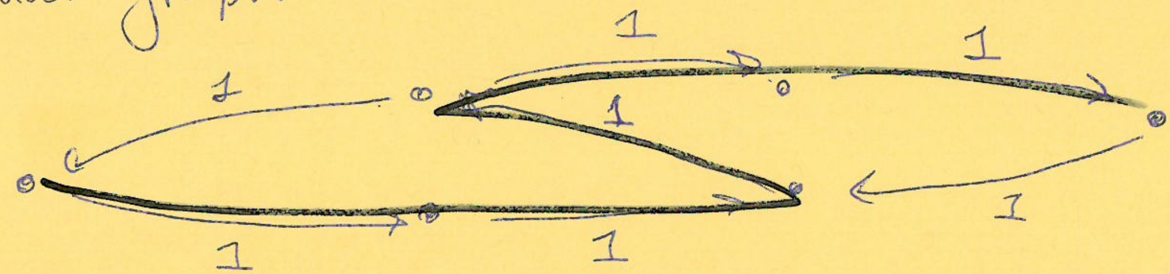


How many toys the kid in A can get

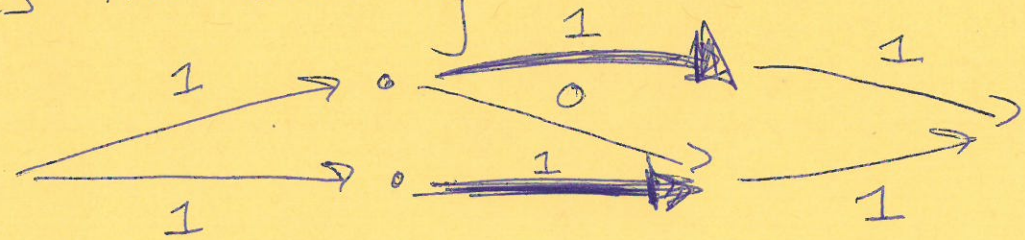
How many "total" kids get a toy in B.

happiness = 1

Residual graph

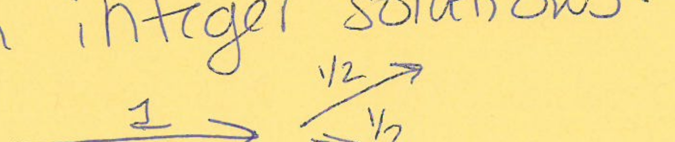


Gives the following flow:



Bolded edges give matching

Lemma: If capacities are integers, ~~we can~~ ~~can we choose a solution~~ then there exists a max flow with integer solutions.



Theorem (that proves max flow = min cut & FF works)

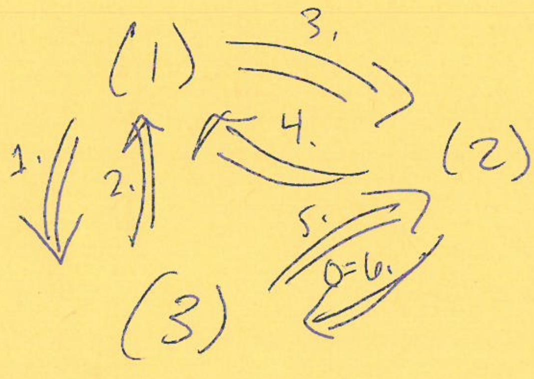
Let f be a flow in $G = (V, E, c)$

Then, the following are equivalent:

(1) f is a max flow

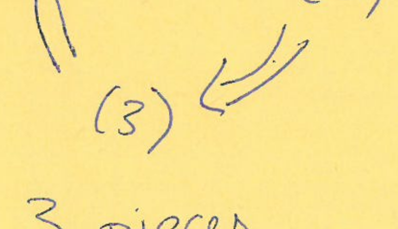
(2) The residual graph has no augmenting path

(3) $\exists$ cut (S, T) s.t. $|f| = c(S, T)$



"6 pieces"

suff to prove



3 pieces

OR

$(1) \Longleftrightarrow (2) \Longleftrightarrow (3)$

4 pieces

$(3) \Rightarrow (1)$

$\exists$ cut (S,T) s.t.

$|f| = c(S,T)$

By cor,

$|f| \leq \min c(S,T)$

$\Rightarrow f$ is a max flow. $\boxplus$

Recall

Lemma 1: f flow on G, (S,T) cut

$\Rightarrow f(S,T) = |f|$

Cor: $\underbrace{|f|}_{\text{any flow}} \leq c\underbrace{(S,T)}_{\text{any cut}}$

$$|f| = f(S,T)$$
$$= \sum\sum f(u,v) - \sum\sum f(v,u)$$
$$\leq \sum\sum f(u,v)$$
$$\leq \sum\sum c(u,v)$$
$$= c(S,T)$$

$\boxplus$