Deep Generative Models

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Generative Models

First Attempt: Log-linear Models

Second Attempt: Wake-Sleep

This is how we do: Variational Autoencoders

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Recap: Generative Models

Joint distribution over observed data x and latent variables Z.

$$p(x, z|\alpha) = \overbrace{p(x|z, \alpha)}^{\text{likelihood}} \underbrace{p(z|\alpha)}_{\text{prior}}$$

The likelihood and prior are often standard distributions (Gaussian, Bernoulli) with little dependence on side information.

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Feature-rich Generative Models

Let us assume that z has internal structure (features). How can we exploit that?

First Idea

Make $p(x|z, \alpha)$ a log-linear model.

- Only discrete data
- Trainable with EM if we can efficiently enumerate $\mathcal{X}$ and $\mathcal{Z}$.

Log-linear Model

Let us treat z as observed.

$$p(x|z, \alpha = w) = \frac{\exp\left(w^\top f(x, z)\right)}{\sum_{x \in \mathcal{X}} \exp\left(w^\top f(x, z)\right)}$$

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Weight Gradient

$$\frac{d}{dw} \log p(x|z, w) = f(x, z) - \mathbb{E}\left[f(X, z)|z, w\right]$$

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Updates need to be performed iteratively.

Log-linear model with latent variables

Now let us treat z as latent.

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Model

$$p(x, z|w) = \underbrace{\frac{\exp\left(w^\top f(x,z)\right)}{\sum_{x \in \mathcal{X}} \exp\left(w^\top f(x,z)\right)}}_{p(x|z,w)} \times \underbrace{p(z)}_{arbitrary}$$

Log-linear model with latent variables

Posterior

$$p(z|x,w) = \frac{p(x,z|w)}{p(x|w)} = \frac{p(x,z|w)}{\sum_z p(x,z|w)} = \frac{\frac{\exp\left(w^\top f(x,z)\right)}{\sum_{x\in\mathcal{X}} \exp(w^\top f(x,z))} \times p(z)}{\sum_z \frac{\exp(w^\top f(x,z))}{\sum_{x\in\mathcal{X}} \exp(w^\top f(x,z))} \times p(z)}$$

Log-linear model with latent variables

Weight Gradient

$$\frac{d}{dw}\mathbb{E}_{p(z|x,w)}\left[\log p(x,z|w)\right] =$$

$$\frac{d}{dw}\sum_z p(z|x,w)\log p(x,z|w) =$$

$$\sum_z p(z|x,w)\frac{d}{dw}\log p(x,z|w)$$

Log-linear model with latent variables

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$$\frac{d}{dw}\mathbb{E}_{p(z|x,w)}\left[\log p(x,z|w)\right] =$$

$$\frac{d}{dw}\sum_z p(z|x,w)\log p(x,z|w) =$$

$$\sum_z p(z|x,w)\ \underbrace{\frac{d}{dw}\log p(x,z|w)}_{\text{We've already solved this!}}$$

Log-linear model with latent variables

Weight Gradient

$$\frac{d}{dw}\mathbb{E}_{p(z|x,w)}\left[\log p(x, z|w)\right] =$$
$$\mathbb{E}_{p(z|x,w)}\left[f(x, Z)|x, w\right] - \mathbb{E}_{p(z|x,w)}\left[\mathbb{E}\left[(f(X, Z)|Z, w\right]\right]$$

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Procedurally

$$\text{E\_count}(x,z) \text{ - } \text{E\_count}(x,z) \times \mathbb{E}\left[X|z,w\right]$$

EM

E-step $p(z|x,w) = \frac{p(x,z|w)}{\sum_z p(x,z|w)}$ in $\mathcal{O}(|\mathcal{X}| \times |\mathcal{Z}|)$

M-step Iteratively optimise w to match E_count(x,z)
with E_count(x,z) $\times \mathbb{E}[X|z,w]$

Restrictions

- Only log-linear models
- Scales badly

Wake-sleep Algorithm

- Generalise latent variables to Neural Networks
- Train generative neural model
- Use variational inference! (kind of)

Wake-sleep Architecture

2 Neural Networks:

- A generation network to model the data (the one we want to optimise) – parameters: θ
- An inference (recognition) network (to model the latent variable) – parameters: λ
- Original setting: binary hidden units

Wake-sleep Architecture

2 Neural Networks:

- A generation network to model the data (the one we want to optimise) – parameters: θ
- An inference (recognition) network (to model the latent variable) – parameters: λ
- Original setting: binary hidden units
- Training is performed in a “hard EM” fashion

Wake-sleep Training

Wake Phase

- Use inference network to sample hidden unit setting z from $q(z|x, \lambda)$
- Update generation parameters θ to maximize liklelihood of data given latent state $p(x|z, \theta)$

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Sleep Phase

- Produce dream sample $\tilde{x}$ from random hidden unit z
- Update inference parameters λ to maximize probability of latent state $q(z|\tilde{x}, \lambda)$

Wake Phase Objective

Assumes latent state z to be fixed random draws from $q(z|x, \lambda)$.

$$\max_{\theta} \log p(x|z, \theta)$$

This is simply supervised learning with imputed latent data!

Sleep Phase Objective

Assumes fake data $\tilde{x}$ and latent variables z to be fixed random draw from $p(x, z|\theta)$.

$$\min_{\lambda} \; \mathbb{E}_{q(z|\tilde{x},\lambda)} [\log p(\tilde{x}, z|\theta)] + \mathbb{H}\left(q(z|\tilde{x}, \lambda)\right)$$

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Advantages

- Backprop can be used without modification
- Amortised inference: all latent variables are inferred from the same weights λ

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Drawbacks

- Inference and generative networks are trained on different objectives
- Inference weights λ are updated on fake data $\tilde{x}$
- Generative weights are bad initially, giving wrong signal to the updates of λ

Generative Model with NN Likelihood

Goal

Define model $p(x, z|\theta) = p(x|z, \theta)p(z)$ where the likelihood $p(x|z, \theta)$ is given by a neural network. (We fix $p(z)$ for simplicity.)

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Define model $p(x, z|\theta) = p(x|z, \theta)p(z)$ where the likelihood $p(x|z, \theta)$ is given by a neural network. (We fix $p(z)$ for simplicity.)

Problem

$p(x) = \int p(x|z, \theta)p(z)\mathrm{dz}$ is hard to compute.

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Define model $p(x, z|\theta) = p(x|z, \theta)p(z)$ where the likelihood $p(x|z, \theta)$ is given by a neural network. (We fix $p(z)$ for simplicity.)

Problem

$p(x) = \int \underbrace{p(x|z, \theta)}_{\text{highly non-linear!}} p(z)\mathrm{dz}$ is hard to compute.

Generative Model with NN Likelihood

Solution: VI

$$
\begin{aligned}
\log p(x) &\geq \overbrace{\mathbb{E}_{q(z|x,\lambda)}\left[\log p(x, z|\theta)\right] + \mathbb{H}\left(q(z|x,\lambda)\right)}^{\text{ELBO}} \\
&= \mathbb{E}_{q(z|x,\lambda)}\left[\log p(x|z,\theta)\right] + \mathrm{KL}\left(p(z) \mid\mid q(z|x,\lambda)\right)
\end{aligned}
$$

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$$= \mathbb{E}_{q(z|x,\lambda)}\left[\log p(x|z,\theta)\right] + \underbrace{\mathrm{KL}\left(p(z) \mid\mid q(z|x,\lambda)\right)}_{\substack{\text{assume analytical} \\ \text{(true for exponential families)}}}$$

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Solution: VI

$$\log p(x) \geq \overbrace{\mathbb{E}_{q(z|x,\lambda)}\left[\log p(x, z|\theta)\right] + \mathbb{H}\left(q(z|x,\lambda)\right)}^{\text{ELBO}}$$

$$= \underbrace{\mathbb{E}_{q(z|x,\lambda)}\left[\log p(x|z,\theta)\right]}_{\text{approximate by sampling}} + \underbrace{\mathrm{KL}\left(p(z) \,||\, q(z|x,\lambda)\right)}_{\substack{\text{assume analytical} \\ \text{(true for exponential families)}}}$$

Generation Network Gradient

$$
\begin{aligned}
&\frac{d}{d\theta}\mathbb{E}_{q(z|x,\lambda)}\left[\log p(x|z,\theta)\right] \\
&= \mathbb{E}_{q(z|x,\lambda)}\left[\frac{d}{d\theta}\log p(x|z,\theta)\right] \\
&\overset{\text{MC}}{\approx} \frac{1}{S}\sum_{i=1}^{S}\frac{d}{d\theta}\log p(x|z_i,\theta)
\end{aligned}
$$

Note: $q(z|x,\lambda)$ does not depend on θ.

Inference Network Gradient

$$
\begin{aligned}
&\frac{d}{d\lambda}\left[\mathbb{E}_{q(z|x,\lambda)}\left[\log p(x|z,\theta)\right] + \mathrm{KL}\left(p(z) \mid\mid q(z|x,\lambda)\right)\right] \\
=&\frac{d}{d\lambda}\mathbb{E}_{q(z|x,\lambda)}\left[\log p(x|z,\theta)\right] + \underbrace{\frac{d}{d\lambda}\mathrm{KL}\left(p(z) \mid\mid q(z|x,\lambda)\right)}_{\text{analytical computation}}
\end{aligned}
$$

The first term again requires approximation by sampling

Inference Network Gradient

$$\frac{d}{d\lambda}\mathbb{E}_{q(z|x,\lambda)}\left[\log p(x|z,\theta)\right]$$
$$= \frac{d}{d\lambda}\int q(z|x,\lambda)\log p(x|z,\theta)\mathrm{dz}$$

Problems for MC

- Sampling z neglects $\frac{d}{d\lambda}q(z|x,\lambda)$
- Differentiating $q(z|x,\lambda)$ breaks the expectation

Inference Network Gradient

$$= \frac{d}{d\lambda} \int q(z|x,\lambda) \log p(x|z,\theta) \mathrm{dz}$$

$$= \frac{d}{d\lambda} \int q(\epsilon) \log \left(p(x|\overbrace{h(\epsilon,\lambda)}^{=z},\theta) \times \underbrace{\left| \frac{d}{d\epsilon} h(\epsilon,\lambda) \right|}_{\text{constant if h linear}} \right) \mathrm{d}\epsilon$$

$$= \int q(\epsilon) \frac{d}{d\lambda} \log p(x|h(\epsilon,\lambda),\theta) \mathrm{d}\epsilon$$

$$= \mathbb{E}_{p(\epsilon)} \left[\frac{d}{d\lambda} \log p(x|h(\epsilon,\lambda),\theta) \right] \overset{\text{MC}}{\approx} \frac{1}{S} \sum_{i=1}^{S} \frac{d}{d\lambda} \log p(x|h(\epsilon_i,\lambda),\theta)$$

Gaussian Transformation

Affine property

$$Ax + b \sim \mathcal{N}\left(\mu + b, A\Sigma A^T\right) \text{ for } x \sim \mathcal{N}\left(\mu, \Sigma\right)$$

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Special case

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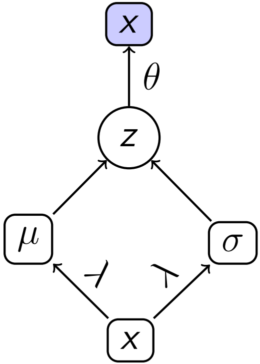
Gaussian transformation

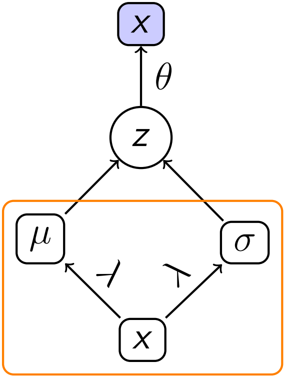
$$h(\epsilon, \lambda) = \mu(x, \lambda) + \mathrm{diag}(\sigma(x, \lambda)) \odot \epsilon \quad \epsilon \sim \mathcal{N}\left(0, \mathrm{I}\right)$$

Computation Graph



Computation Graph

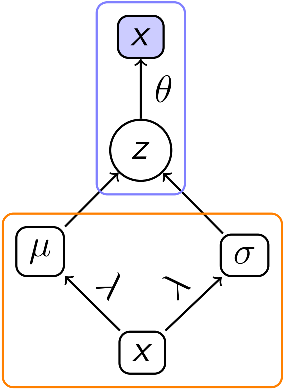
inference model



Computation Graph

generation model

inference model



Computation Graph

generation model

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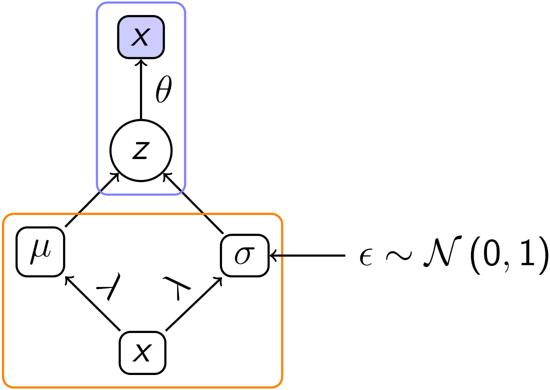

Example

- Data: binary mnist
- Likelihood: product of Bernoullis
  - Let $\phi = \sigma(\mathrm{NN}(z))$
  - $\prod_{i=1}^{N} p(x_i|\phi) = \prod_{i=1}^{N} \phi^{x_i} \times (1-\phi)^{1-x_i}$
- Prior over z: $\mathcal{N}(0, 1)$
- $q(z|x, \lambda) = \mathcal{N}\left(\mu(x, \lambda), \sigma(x, \lambda)^2\right)$
- $\mu(x, \lambda) = \mathrm{NN}_{\mu}(x; \lambda)$
- $\sigma(x, \lambda) = \mathrm{NN}_{\sigma}(x; \lambda)$

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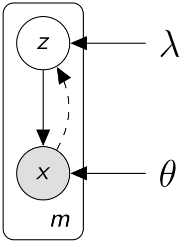
- $q(z|x,\lambda) = \mathcal{N}\left(\mu(x,\lambda), \sigma(x,\lambda)^2\right)$
- $\mu(x,\lambda) = \mathrm{NN}_\mu(x;\lambda)$
- $\sigma(x,\lambda) = \mathrm{NN}_\sigma(x;\lambda)$

Mean Field assumption

Variational approximation factorises over latent dimensions.

Graphical Model



- approximate posterior
  $q(z|x, \lambda) = \mathcal{N}\left(\mu(x, \lambda), \sigma(x, \lambda)^2\right)$

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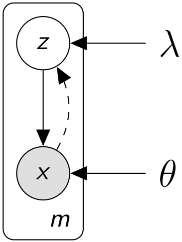

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    e.g. $\mu(x, \lambda) = W^{(u)}x + b^{(u)}$

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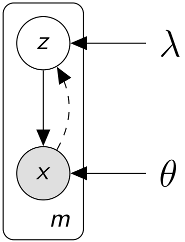

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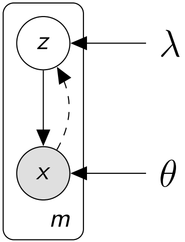
  - $\sigma(x,\lambda) = \exp(\mathrm{NN}_\sigma(x;\lambda))$
    e.g. $\sigma(x,\lambda) =$
    $\exp\left(\tanh(W^{(v)}x + b^{(v)})\right)$

Graphical Model



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  - $\sigma(x,\lambda) = \exp(\mathrm{NN}_\sigma(x;\lambda))$
    e.g. $\sigma(x,\lambda) =$
    $\exp\left(\tanh(W^{(v)}x + b^{(v)})\right)$
  - $\lambda = \left(W^{(u)}, W^{(v)}, b^{(u)}, b^{(v)}\right)$

Variational Autoencoder

Advantages

- Backprop training
- Easy to implement
- Posterior inference possible
- One objective for both NNs

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Drawbacks

- Discrete latent variables are difficult
- Optimisation may be difficult with several latent variables

Summary

- When $|\mathcal{X}|$ and $|\mathcal{Z}|$ are not too large, we can do EM with features
- Otherwise use VI with simple approximation
- Wake-Sleep: train inference and generation networks with separate objectives
- VAE: train both networks with same objective
- Reparametrisation
  - Transform parameter-free variable ϵ into latent value z
  - Update parameters with stochastic gradient estimates

Literature I

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