Deep Generative Models: Discrete Latent Variables

Philip Schulz and Wilker Aziz

https://github.com/philschulz/VITutorial

What we know so far

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- Because the ELBO cannot be computed exactly, we need to sample latent values
- Main problem: the MC estimator is not differentiable
- Solution: reparametrisation gradient

Reparametrisation Gradient

Model Gradient

$$\frac{\partial}{\partial\theta}\mathbb{E}_{q(z|\lambda)}\left[\log p(x|z,\theta)\right] - \frac{\partial}{\partial\theta}\mathrm{KL}\left(q(z|\lambda) \mid\mid p(z|\theta)\right)$$

Reparametrisation Gradient

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Inference Network Gradient

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$$\mathbb{E}_{\phi(\epsilon)} \left[\frac{\partial}{\partial z} \log p(x| \overbrace{h^{-1}(\epsilon, \lambda)}^{z}, \theta) \times \frac{\partial}{\partial \lambda} \overbrace{h^{-1}(\epsilon, \lambda)}^{z}\right]$$

Reparametrisation for Discrete Variables?

Revisiting the Inference Gradient

Control Variates and Baselines

Semisupervised Learning

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Reparametrisation

In order to tranform variables, we need to compute the Jacobian (matrix of partial derivatives).

$$p(z) = \phi(h(z)) \left| \det J_{h(z)} \right|$$

The Jacobian

$$J_{ij} = \frac{\partial h_i(z)}{\partial z_j} \tag{1}$$

is generally not available for discrete variables.

Continuity

The outcome space of discrete variables is non-continuous. Thus, we cannot take derivatives with respect to real variables.

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$$
\begin{aligned}
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\end{aligned}
$$

Back to Basic Calculus

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \log f(\lambda)$$

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$$\frac{\mathrm{d}}{\mathrm{d}\lambda} \log f(\lambda) = \frac{\frac{\mathrm{d}}{\mathrm{d}\lambda} f(\lambda)}{f(\lambda)}$$

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Consequence

$$\frac{\mathrm{d}}{\mathrm{d}\lambda} f(\lambda) = \frac{\mathrm{d}}{\mathrm{d}\lambda} \log f(\lambda) \times f(\lambda)$$

Score Function Estimator

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$$\mathbb{E}_{q(z|\lambda)} \left[\frac{\partial}{\partial \lambda} \log q(z|\lambda) \times \log p(x|z, \theta) \right]$$

Comparison Between Estimators

- Score function gradient

$$\mathbb{E}_{q(z|\lambda)}\left[\frac{\partial}{\partial\lambda}\log q(z|\lambda) \times \log p(x|z,\theta)\right]$$

- Reparametrisation gradient

$$\mathbb{E}_{\phi(\epsilon)}\left[\frac{\partial}{\partial\lambda}\log p(x|h^{-1}(\epsilon,\lambda),\theta)\right]$$

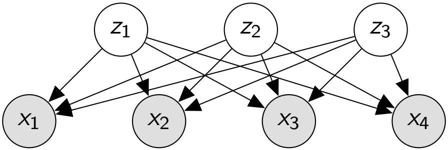
Example Model

Let us consider a latent factor model for topic modelling. Each document x consists of n i.i.d. categorical draws from that model. The categorical distribution in turn depends on the binary latent factors $z = (z_1, \ldots, z_k)$ which are also i.i.d.

$$\begin{aligned} Z_j &\sim \text{Bernoulli}(\phi) && (1 \leq j \leq k) \\ X_i | z &\sim \text{Categorical}(g(z)) && (1 \leq i \leq n) \end{aligned}$$

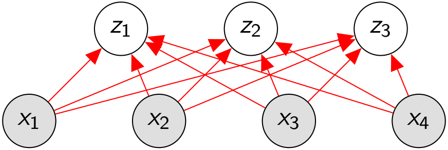
Here $g(\cdot)$ is a function computed by neural network with softmax output.

Example Model



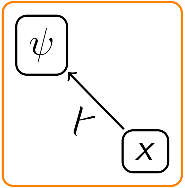
At inference time the latent variables are marginally dependent. For our variational distribution we are going to assume that they are not (recall: mean field assumption).

Inference Network



The inference network needs to predict k Bernoulli parameters ψ. Any neural network with sigmoid output will do that job.

Computation Graph



inference model

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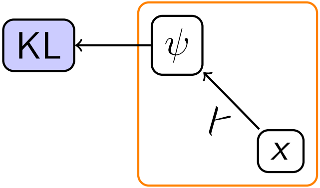

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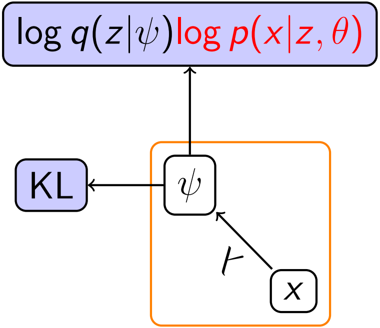

inference model

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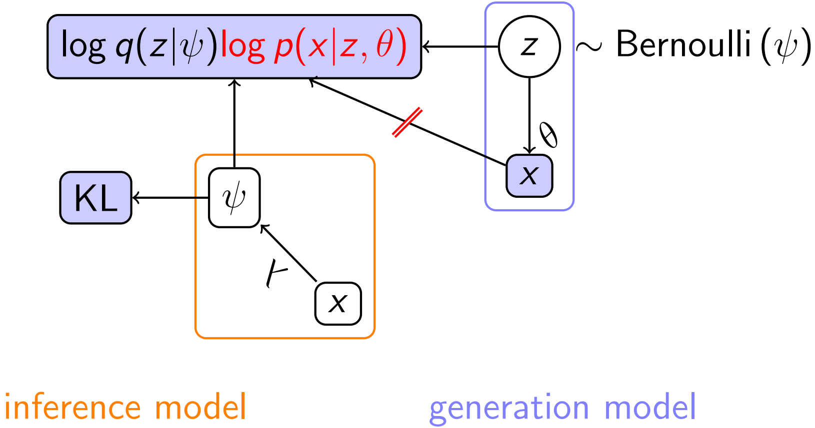
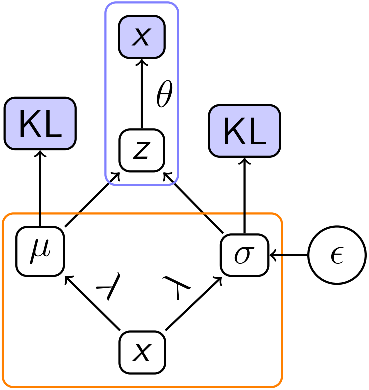


Reparametrisation Gradient

generation model

inference model



$\sim \mathcal{N}(0,1)$

Pros and Cons

- Pros
  - Applicable to all distributions
  - Many libraries come with samplers for common distributions

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  - Many libraries come with samplers for common distributions
- Cons
  - High Variance!

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Baselines

Fact

The Expectation of the score function is 0.

Baselines

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$$\mathbb{E}_{q(z|x,\lambda)}\left[\frac{\partial}{\partial\lambda}\log q(z|x,\lambda)\right] = 0$$

Baselines

We attempt to centre the gradient estimate. To do this we learn a quantity C that we subtract from the reconstruction loss.

$$\mathbb{E}_{q(z|\lambda)}\left[\log q(z|\lambda)\left(\log p(x|z,\theta) - C\right)\right]$$

We call C a baseline. It does not change the expected gradient (Williams, 1992).

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Baselines

We can make baselines input-dependent to make them more flexible.

$$\log q(z|\lambda) \left(\log p(x|z,\theta) - C(x;\omega)\right)$$

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$$\log q(z|\lambda)\,(\log p(x|z,\theta) - C(x;\omega))$$

However, baselines may not depend on the random value z! Quantities that may depend on the random value ($C(z)$) are called **control variates**.

See Blei et al. (2012); Ranganath et al. (2014); Gregor et al. (2014).

Baselines

Baselines are predicted by a regression model (e.g. a neural net).

The model is trained using an L_2-loss.

$$\min_{\omega} \left(C(x; \omega) - \log p(x|z, \theta) \right)^2$$

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- Reparametrisation not available for discrete variables.
- Use score function estimator.
- High variance.
- Always use baselines for variance reduction!

Putting it all together

We now know how to handle continuous and discrete latent variables. Let us combine these two treat partially observed data.

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Transform an inflected form of a verb into another.

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Morphological Reinflection

Transform an inflected form of a verb into another.

- plays $\rightarrow$ played
- walking $\rightarrow$ walks

A Simple Model (Zhou and Neubig, 2017)

What do we need to correctly inflect a word?

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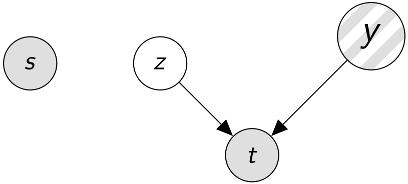
What do we need to correctly inflect a word?

- lemma (real vector)
- morphological information (discrete vector)

A Simple Model (Zhou and Neubig, 2017)



- z = lemma (continuous)
- y = morphological features (discrete)
- s = source form (inflected)
- t = target form (inflected)

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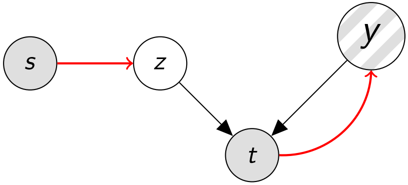

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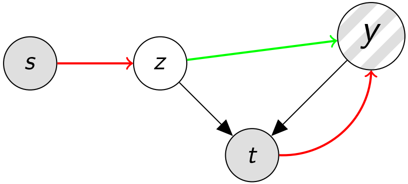

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David M. Blei, Michael I. Jordan, and John W. Paisley. Variational bayesian inference with stochastic search. In *ICML*, 2012. URL http://icml.cc/2012/papers/687.pdf.

Karol Gregor, Ivo Danihelka, Andriy Mnih, Charles Blundell, and Daan Wierstra. Deep autoregressive networks. In Eric P. Xing and Tony Jebara, editors, *ICML*, pages 1242–1250, 2014. URL http://proceedings.mlr.press/v32/gregor14.html.

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Chunting Zhou and Graham Neubig. Multi-space variational encoder-decoders for semi-supervised labeled sequence transduction. In *ACL*, pages 310–320, 2017. doi: 10.18653/v1/P17-1029. URL http://www.aclweb.org/anthology/P17-1029.

Goodbye and thank you!

What we covered today

- Derivation of VI
- Continuous VAE
- Discrete VAE
- Semisupervised Models

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Next steps

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- Stay in touch
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  - phschulz@amazon.com