Semimetric properties of Sørensen-Dice and Tversky Indices

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Short overview

Jaccard-Tanimoto index

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$$S_{RJ,\alpha}(X,Y) = \frac{\alpha\,|X \cap Y|}{|X \cup Y| + (\alpha - 1)|X \cap Y|}$$

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Metric **(distance)**

1. d(x,y) ≥ 0
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Can guarantee efficiency of several machine learning algorithms if ρ is small

METRIC

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$$S_J(X,Y) = \frac{|X \cap Y|}{|X \cup Y|}$$

NEAR-METRIC

Sørensen-Dice index (ρ = 1.5)

$$S_{SD}(X,Y) = \frac{2\,|X \cap Y|}{|X \cup Y| + |X \cap Y|}$$

Robust Jaccard index (ρ = (α + 1)/2)

$$S_{RJ,\alpha}(X,Y) = \frac{\alpha\,|X \cap Y|}{|X \cup Y| + (\alpha - 1)|X \cap Y|}$$

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1. Similarity indexes
2. Near-metricness
3. Our results
4. Proof outline

Similarity indexes

Jaccard-Tanimoto index (α = 1)

$$S_J(X,Y) = \frac{|X \cap Y|}{|X \cup Y|}$$

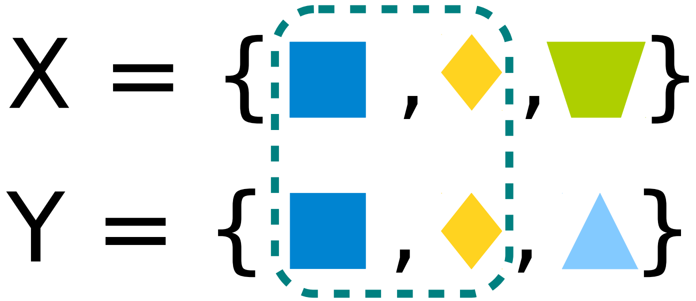
Sørensen-Dice index (α = 2)

$$S_{SD}(X,Y) = \frac{2\ |X \cap Y|}{|X \cup Y| + |X \cap Y|}$$

Robust Jaccard index **[Our proposal]**

$$S_{RJ,\alpha}(X,Y) = \frac{\alpha |X \cap Y|}{|X \cup Y| + (\alpha - 1)|X \cap Y|}$$

Similarity indexes



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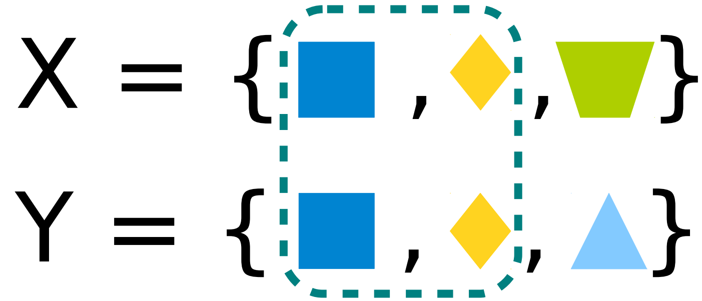

$$|X \cap Y| = 2$$
$$|X \cup Y| = 4$$

$$S_J(X,Y) = \frac{2}{4} = 0.5$$

$$S_{SD}(X,Y) = \frac{2 \cdot 2}{4+2} = 0.66$$

$$S_{RJ,4}(X,Y) = \frac{4 \cdot 2}{4+3 \cdot 2} = 0.8$$

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Similarity indexes

Robust Jaccard index **[Our proposal]**

$$S_{RJ,\alpha}(X,Y) = \frac{\alpha|X \cap Y|}{|X \cup Y| + (\alpha - 1)|X \cap Y|}$$

Tversky index family

$$S_T(X,Y) = \frac{|X \cap Y|}{|X \cap Y| + \beta|X - Y| + \gamma|Y - X|}$$

with β,ɣ ∈ [0,1]

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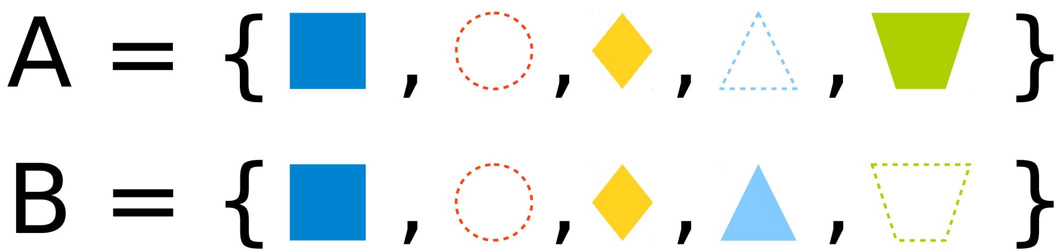
- Robust Jaccard index is equal to Jaccard-Tanimoto index when α = 1.
- Robust Jaccard index is equal to Sørensen-Dice index α = 2.
- Robust Jaccard index is a subset of the Tversky index family when β = ɣ.

Why is 'Robustness' needed?

Larger α values are the more robust it becomes.



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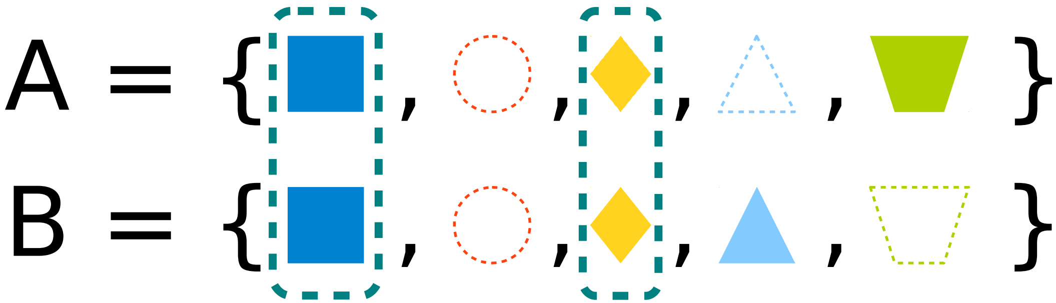
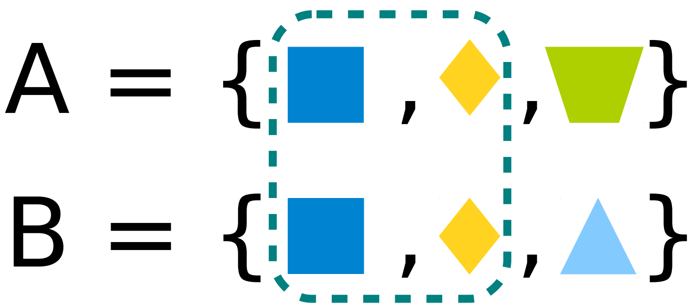
Why is 'Robustness' needed?

When measuring similarity, context (instrumental error, missing information, ...) matters.

It is not the same:



Than:



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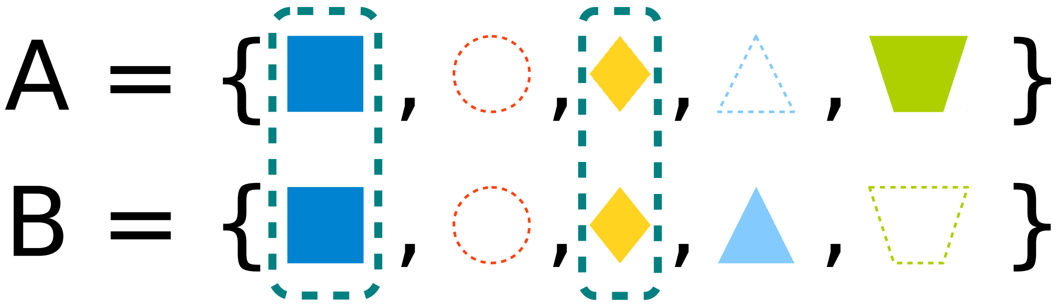
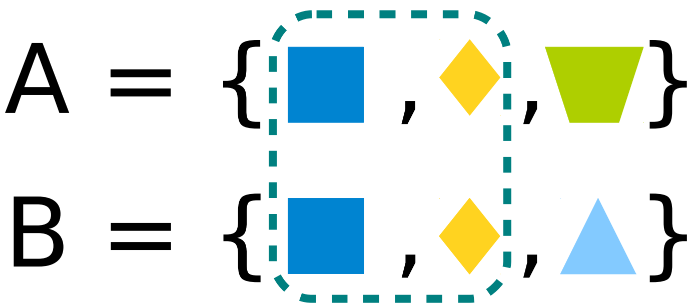


$S_J = 0.5$

$S_{SD} = 0.66$

Than:



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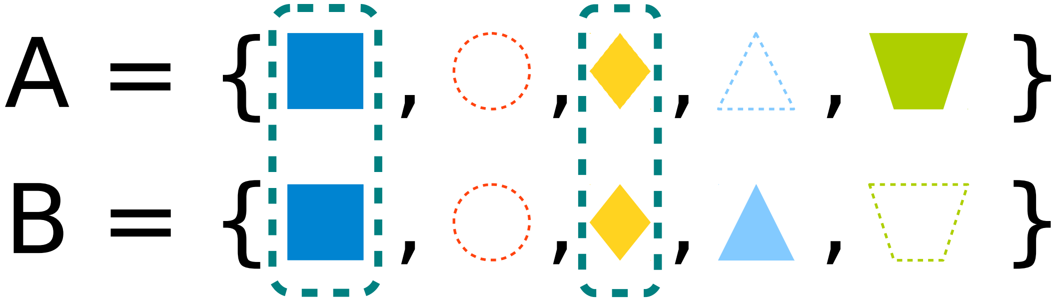
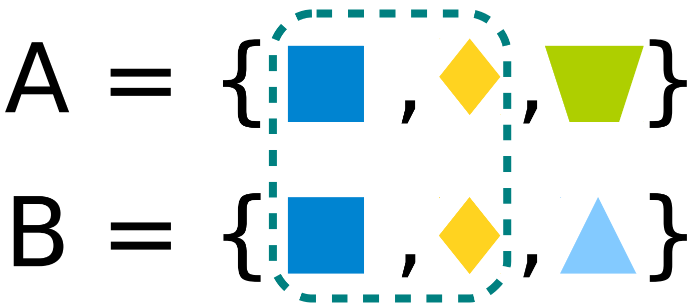
But if we have an insight of how good is the observation at each time, then we can adjust the granularity of our similarity measure.



Or



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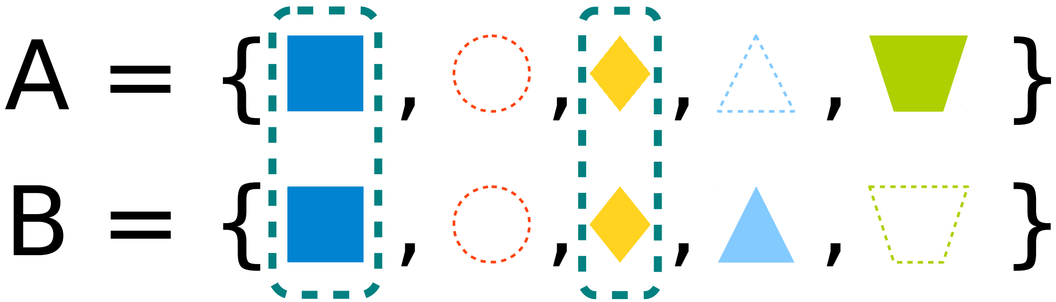
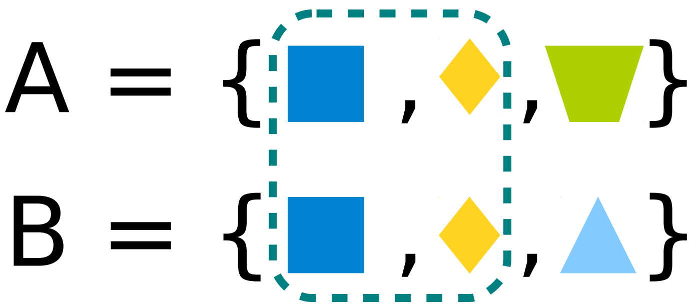


$S_{RJ,4} = 0.8$

Or

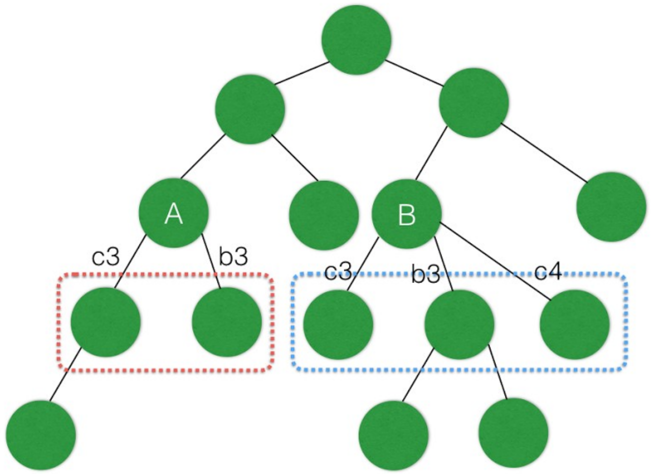


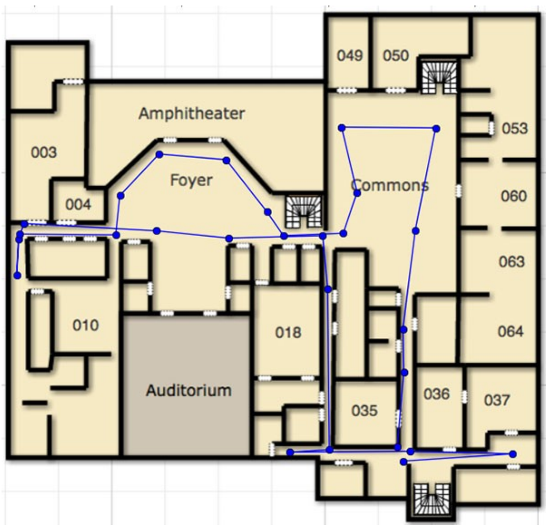
$S_{RJ,1} = 0.5$

Why is 'Robustness' needed?

Computer Go
In Monte-Carlo Tree Search algorithm the quality of the moves for each board position depends on the number of simulations performed so far.

[Gragera, 2015]





Self Localization and Mapping
In wifi-SLAM for robotics the signal presence and intensity depends on the noise of the environment.

[Miyagusuku, 2016]

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2. Near-metricness
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Metric, Semi-metric and Near-metric

Metric properties *(or distance)*

1. $d(x,y) \geq 0$ — N*on-negativity*
2. $d(x,y) = 0 \Leftrightarrow x = y$ — *Identity of indiscernibles*
3. $d(x,y) = d(y,x)$ — *Symmetry*
4. $d(x,z) \leq d(x,y) + d(y, z)$ — *Triangle inequality*

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Semi-metric properties

4. $\cancel{d(x,z) \leq d(x,y) + d(y, z)}$

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Semi-metric properties

4. ~~$d(x,z) \leq d(x,y) + d(y, z)$~~

Near-metric properties *(or ρ-relaxed semi-metric properties)*

4. $d(x,z) \leq \rho(d(x,y) + d(y, z))$ — Approximate triangle inequality

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Proving algorithms efficiency

For many recent algorithms, computation time depends on the value of ρ.

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Approximation ratio depends on the value of ρ of the near-metric used.

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Several properties of well-studied metric problems can be generalized for the near-metric cases.

[Xia, 2008]

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[Xia, 2008] ← Geodesic problem

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Summary of our paper results

Dissimilarity		Result	
Jaccard-Tanimoto	$\frac{\lvert X \cap Y \rvert}{\lvert X \cup Y \rvert}$	ρ* = 1	(metric) *[Lipkus, 1999]*
Sørensen-Dice	$\frac{2\,\lvert X \cap Y \rvert}{\lvert X \cup Y \rvert + \lvert X \cap Y \rvert}$	ρ* = 1.5	(near-metric) *[Our Result]*
Robust-Jaccard	$\frac{\alpha \lvert X \cap Y \rvert}{\lvert X \cup Y \rvert + (\alpha - 1) \lvert X \cap Y \rvert}$	ρ* = (α + 1)/2	(near-metric) *[Our Result]*
Tversky	$\frac{\lvert X \cap Y \rvert}{\lvert X \cap Y \rvert + \beta \lvert X - Y \rvert + \gamma \lvert Y - X \rvert}$	*If β = γ = 1/α* ρ* = (α + 1)/2	(near-metric) *[Our Result]*
		If β ≠ γ	

Near-metricness is proven and tight ρ values are also given.

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Robust-Jaccard	$\frac{\alpha \lvert X \cap Y \rvert}{\lvert X \cup Y \rvert + (\alpha - 1)\lvert X \cap Y \rvert}$	$\rho^* = (\alpha + 1)/2$	(near-metric) *[Our Result]*
Tversky	$\frac{\lvert X \cap Y \rvert}{\lvert X \cap Y \rvert + \beta \lvert X - Y \rvert + \gamma \lvert Y - X \rvert}$	*If $\beta = \gamma = 1/\alpha$* $\rho^* = (\alpha + 1)/2$	(near-metric) *[Our Result]*
		If $\beta \neq \gamma$ Near-quasimetric	

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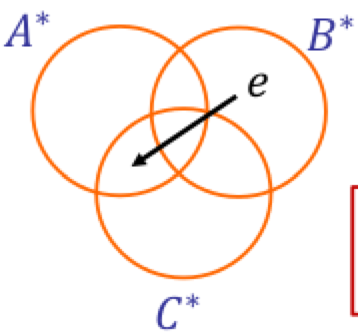
Proof Outline

$$\rho^* = \max_{A,B,C} \frac{d(A,C)}{d(A,B)+d(B,C)}$$

Find $A^*, B^*, C^* = \max_{A,B,C} \frac{d(A,C)}{d(A,B)+d(B,C)}$



$$d(A', B') \leq d(A^*, B^*)$$
$$d(B', C') \leq d(B^*, C^*)$$
$$d(A', C') \geq d(A^*, C^*)$$

A', B', C' can also be the optimization results.

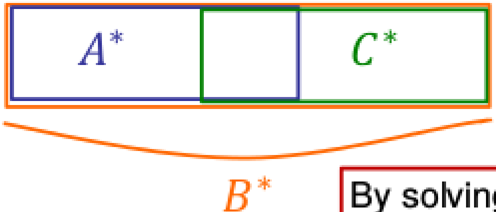
$$d(A', C') = \frac{\alpha(|A^* \cap C^*| + 1)}{|A^* \cup C^*| + 1 + (\alpha - 1)(|A^* \cap C^*| + 1)}$$
$$= \frac{\alpha|A^* \cap C^*| + \alpha}{|A^* \cup C^*| + (\alpha - 1)|A^* \cap C^*| + \alpha}$$
$$\geq \frac{\alpha|A^* \cap C^*|}{|A^* \cup C^*| + (\alpha - 1)|A^* \cap C^*|} = d(A^*, C^*)$$

To find ρ is equivalent to find A^*,B^*,C^*

Adding certain elements doesn't improve to A^*,B^* and C^*

A^* and C^* should of the same size and disjoint

B^* is the union of A^* and C^*



By solving optimization problem, we have $A^* = \{1\}$, $B^* = \{1,2\}$, $C^* = \{2\}$

Concluding remarks

- Robustness is very interesting property of Robust Jaccard index for dynamic problems.
- Sørensen-Dice, Robust Jaccard and Tversky are now included the indexes that can be safely used as near-metrics.
- We now know their value of ρ, that is critical to guarantee the performance of several algorithms.

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Thank you!

Q&A