Problem 1a,

Bayes' theorem can be rewritten for $Q = (Q_1, \dots, Q_5)$ as

$$p(\theta) \propto \frac{p(\theta|Q)}{p(Q|\theta)} \propto \frac{\theta^{\alpha_n - 1} e^{-\beta_n \theta}}{\prod_{i=1}^{5} \theta^{Q_i} e^{-\theta}} = \theta^{\alpha_n - \sum_{i=1}^{5} Q_i - 1} e^{-(\beta_n - 5)\theta},$$

which can be identified as the form of a gamma distribution with parameters

$$\alpha = \alpha_n - \sum_{i=1}^{5} Q_i = 720 \quad \text{and} \quad \beta = \beta_n - 5 = 2$$

Problem 3a,

Bayes' theorem gives for x that

$$p(\theta|x) \propto p(x|\theta)\, p(\theta) \propto \theta^{2+n} \cdot \exp\left[-\left(4 + \sum_{i=1}^{n} x_i^3\right)\theta\right],$$

which can be identified as the form of a gamma distribution with parameters

$$\alpha = 3 + n \quad \text{and} \quad \beta = 4 + \sum_{i=1}^{n} x_i^3$$

Problem 3b,

The Bayes point estimate is the posterior mode given by

$$\frac{\partial \ln p(\theta|x)}{\partial \theta} = \frac{2+n}{\theta} - \left(4 + \sum_{i=1}^{n} x_i^3\right) = 0$$

Solving for θ gives

$$\hat{\theta} = \frac{2+n}{4 + \underbrace{\sum_{i=1}^{n} x_i^3}_{4.084}} \approx 0.866$$

Problem 3c,

$$\ln p(x|\hat{\theta}) = 5 \cdot \ln 3 + 5 \cdot \ln \hat{\theta} + \underbrace{\sum_{i=1}^{5} \ln x_i^2}_{-0.913} - \hat{\theta} \cdot \sum_{i=1}^{5} x_i^3$$

$$= 0.3238$$

$$\ln p(\hat{\theta}) = \ln 32 + 2 \cdot \ln \hat{\theta} - 4\hat{\theta} = -0.2858$$

$$J_{\hat{\theta},x} = \frac{2+n}{\hat{\theta}^2} \implies \frac{1}{2} \ln \left| J^{-1}_{\hat{\theta},x} \right| = -1.1169$$

$$\frac{1}{2} \ln (2\pi) = 0.9189$$

Summing all together gives $\ln \hat{p}(x) = -0.16$, i.e.

$$\hat{p}(x) = 0.852$$