Problem 1a:

Bayes' theorem gives for x that

$$p(\theta|x) \propto p(x|\theta)p(\theta) \propto \theta^{4+3n-1} \cdot \exp\left[-\left(2+\sum_{i=1}^{n}\frac{1}{x_i}\right)\theta\right],$$

which can be identified as the form of a gamma distribution with parameters $\alpha' = 4+3n$ and $\beta' = 2+\sum_{i=1}^{n}\frac{1}{x_i}$

Problem 1b:

The marginal likelihood for x is given by

$$p(x) = \int_0^\infty p(x|\theta)p(\theta)\, d\theta = \frac{\beta^\alpha}{\Gamma(\alpha)\cdot 2^n \prod_{i=1}^{n} x_i^4} \cdot \underbrace{\int_0^\infty \theta^{\alpha'-1} \exp[-\beta'\theta]\, d\theta}_{=\frac{\Gamma(\alpha')}{\beta'^{\alpha'}}},$$

which gives the result.

This gives for $x_1 = 0.7, \dots, x_4 = 1.5$ that

$$p(x) = \frac{\overbrace{\Gamma(4+3\cdot 4)}^{=15!}\, 2^4}{\underbrace{\Gamma(4)}_{=3!=6}\cdot(2+4.115)^{4+3\cdot 4}\cdot 2^4 \cdot 1.168} \approx 0.0488$$

Problem 1c:

The posterior mode is given by

$$\frac{\partial \ln p(\theta|x)}{\partial\theta} = \frac{3+3n}{\theta} - \left(2+\sum_{i=1}^{n}\frac{1}{x_i}\right) = 0$$

Solving for θ gives $\hat{\theta} = \frac{3+3n}{2+\underbrace{\sum_{i=1}^{n}\frac{1}{x_i}}_{=4.115}} \approx 2.453$

$$\ln p(x|\hat{\theta}) = 4\cdot 3\cdot \ln\hat{\theta} - \hat{\theta}\cdot\sum_{i=1}^{n}\frac{1}{x_i} - 4\cdot\ln 2 - \underbrace{\sum_{i=1}^{n}\ln x_i^4}_{=0.155} = -2.255$$

$$\ln p(\hat{\theta}) = 4\cdot\ln 2 + 3\cdot\ln(\hat{\theta}) - 2\hat{\theta} - \ln 6 = -1.233$$

$$J_{\hat{\theta},x} = \frac{3+3n}{\hat{\theta}^2} \Rightarrow \frac{1}{2}\ln\left|J^{-1}_{\hat{\theta},x}\right| = -0.457, \quad \frac{1}{2}\ln(2\pi) = 0.9189$$

Summing all together gives $\ln \hat{p}(x) = -3.0261$, i.e. $\hat{p}(x) \approx 0.0485$, so quite accurate.

Problem 3a:

$$E[\text{buy the option}] = 60\cdot\theta - 20\cdot(1-\theta) = 80\cdot 0.6 - 20 = 28$$

$$\vee$$

$$E[\text{Don't buy the option}] = 180\cdot\theta - 240\cdot(1-\theta) = 420\cdot 0.6 - 240 = 12$$

The bank should buy the option

Problem 3b:

$$p(x_{n+1}|x_{1:n}) = \int p(x_{n+1}|\theta)\,p(\theta|x_{1:n})\,d\theta$$

$$= \underbrace{\frac{\Gamma(\alpha+\beta+n)}{\Gamma(\alpha+s)\,\Gamma(\beta+f)}}_{\text{constant with respect to }\theta} \cdot \int \underbrace{\theta^{x_{n+1}}(1-\theta)^{1-x_{n+1}}\,\theta^{\overbrace{\alpha+s}^{6+32}-1}\,(1-\theta)^{\overbrace{\beta+f}^{4+18}-1}}_{=\theta^{x_{n+1}+38-1}(1-\theta)^{1-x_{n+1}+22-1}}\,d\theta$$

$$= \frac{\Gamma(60)}{\Gamma(38)\,\Gamma(22)} \cdot \frac{\Gamma(x_{n+1}+38)\,\Gamma(1-x_{n+1}+22)}{\underbrace{\Gamma(61)}_{=60\cdot\Gamma(60)}}$$

using $\Gamma(y+1) = y\,\Gamma(y)$

$$= \frac{\Gamma(x_{n+1}+38)\,\Gamma(1-x_{n+1}+22)}{60\cdot\Gamma(38)\cdot\Gamma(22)}$$

So, $p(x_{51}=1|x_{1:50}) = \dfrac{\Gamma(39)\,\Gamma(22)}{60\,\Gamma(38)\,\Gamma(22)} = \dfrac{38}{60} = \dfrac{19}{30}$

and conversely $p(x_{51}=0|x_{1:50}) = \dfrac{11}{30}$, which gives that

$$x_{51}|x_{1:50} \sim \text{Bern}(19/30)$$