Problem 1a:

Bayes' theorem gives for x that

$$p(\theta|x) \propto p(x|\theta)p(\theta) \propto \theta^{\sum_{i=1}^{n} x_i + 2 - 1} \cdot (1-\theta)^{n \cdot N - \sum_{i=1}^{n} x_i + 3 - 1},$$

which can be identified as the form of a beta distribution with parameters $\alpha' = \sum_{i=1}^{n} x_i + 2$ and $\beta' = n \cdot N - \sum_{i=1}^{n} x_i + 3$

Problem 1b:

The Bayes point estimate is the posterior mode given by

$$\frac{\partial \ln p(\theta|x)}{\partial \theta} = \frac{\alpha' - 1}{\theta} - \frac{\beta' - 1}{1-\theta} = 0$$

Solving for θ gives $\hat{\theta} = \frac{\alpha' - 1}{\alpha' + \beta' - 2} = \frac{41 - 1}{41 + 44 - 2} \approx 0.482$

Problem 1c:

The posterior predictive distribution of a new observation x_{n+1} is

$$p(x_{n+1}|x) = \int p(x_{n+1}|\theta) p(\theta|x) \, d\theta$$

$$= \binom{N}{x_{n+1}} \cdot \frac{\Gamma(\alpha' + \beta')}{\Gamma(\alpha')\Gamma(\beta')} \cdot \underbrace{\int \theta^{x_{n+1}} (1-\theta)^{N - x_{n+1}} \cdot \theta^{\alpha' - 1} (1-\theta)^{\beta' - 1} \, d\theta}_{= \frac{\Gamma(\alpha' + x_{n+1}) \Gamma(\beta' + N - x_{n+1})}{\Gamma(\alpha' + \beta' + N)}}$$

Problem 2a:

Bayes' theorem gives for $y = (y_1, \ldots, y_n)$ that

$$p(\theta|y) \propto p(y|\theta)\,p(\theta) \propto \theta^{\sum_{i=1}^{n} y_i + 40 - 1} e^{-(n+2)\theta},$$

which can be identified as the form of a gamma distribution with parameters $\alpha' = \sum_{i=1}^{n} y_i + 40$ and $\beta' = n+2$