Problem 1a:

Bayes' theorem gives for x that

$p(\theta|x) \propto p(x|\theta)\,p(\theta) \propto \theta^{2+\sum_{i=1}^{n} x_i} \cdot \exp\left[-(0.5+n)\theta\right]$, which can be identified as the form of a gamma distribution with parameters

$\alpha = 3 + \sum_{i=1}^{n} x_i$ and $\beta = 0.5 + n$

Problem 1b:

The Bayes point estimate is the posterior mean given by

$$\hat{\theta} = \frac{\alpha}{\beta} = \frac{78}{15.5} \approx 5.03$$

Problem 1c:

The marginal likelihood for x is given by

$$p(x) = \int_0^\infty p(x|\theta)\,p(\theta)\,d\theta = \frac{1}{16 \cdot \prod_{i=1}^{n} x_i!} \cdot \underbrace{\int_0^\infty \theta^{\alpha-1} \exp\left[-\beta\theta\right] d\theta}_{= \frac{\Gamma(\alpha)}{\beta^{\alpha}}}$$

Problem 3a:

From lecture 2, slide 4 we have that

$$\frac{1}{\tau_n^2} = \frac{n}{\sigma^2} + \frac{1}{\tau_0^2} \iff \frac{1}{2^2} = \frac{10}{50} + \frac{1}{\tau_0^2},$$

which gives $\tau_0^2 = 20$ and $w = \dfrac{\frac{10}{50}}{\frac{10}{50} + \frac{1}{20}} = 0.8$, so that

$\mu_n = 92 = 0.8 \cdot 90 + (1 - 0.8) \cdot \mu_0$ gives $\mu_0 = 100$

Hence, $\mu \sim N(\mu_0 = 100, \tau_0^2 = 20)$