Problem 1a:

Bayes' theorem gives for $y = (y_1, \dots, y_n)$ that

$p(\theta|y) \propto p(y|\theta)\, p(\theta) \propto \theta^{2n} \cdot \exp\left[-\left(\frac{1}{2} + \sum_{i=1}^{n} y_i\right)\theta\right]$, which can be identified as the form of a gamma distribution with parameters

$\alpha = 2n + 1$ and $\beta = \frac{1}{2} + \sum_{i=1}^{n} y_i$

Problem 3a:

Bayes' theorem gives for $x = (x_1, \dots, x_n)$ that

$$p(\theta|x) \propto p(x|\theta)\,p(\theta) \propto \exp\left[-n\theta^2 + 2\theta \sum_{i=1}^{n} \ln x_i\right]$$

If $\theta|x \sim N(\mu_n, \sigma_n^2)$, then

$$p(\theta|x) \propto \exp\left[-\frac{1}{2\sigma_n^2}\cdot\theta^2 + \frac{\mu_n}{\sigma_n^2}\cdot\theta\right].$$

Hence, the posterior distribution for θ is $N(\mu_n, \sigma_n^2)$, where it is identified that

$$\mu_n = \frac{\sum_{i=1}^{n} \ln x_i}{n} \quad \text{and} \quad \sigma_n^2 = \frac{1}{2n}$$

Problem 3b:

The Bayes point estimate is the posterior median given by μ_n, i.e.

$$\hat{\theta} = \mu_n = \frac{2}{5} = 0.4$$

Problem 3c:

The marginal likelihood for $x = (x_1, \dots, x_n)$ and $p(\theta) = \frac{1}{c}$ on an interval with length c is given by

$$p(x) = \int_{-\infty}^{\infty} p(x|\theta)\,p(\theta)\,d\theta = \frac{1}{c}\cdot\pi^{-n/2}\cdot\prod_{i=1}^{n}\left(\frac{1}{x_i}\right)\cdot\exp\left[-\sum_{i=1}^{n}(\ln x_i)^2\right]$$

$$\cdot\underbrace{\int_{-\infty}^{\infty} \exp\left[-n\theta^2 + 2\theta\sum_{i=1}^{n}\ln x_i\right] d\theta}_{= \exp\left[\frac{\mu_n^2}{2\sigma_n^2}\right]\cdot\underbrace{\int_{-\infty}^{\infty}\exp\left[-\frac{1}{2\sigma_n^2}\cdot(\theta-\mu_n)^2\right]d\theta}_{=\sigma_n\cdot\sqrt{2\pi}}}$$