LINKÖPING UNIVERSITY
Dept. of Computer and Information Science
Division of Statistics and Machine Learning
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2021-08-18
Bayesian Learning, 6 hp
732A73/732A91/TDDE07

Distance Exam
Bayesian Learning (732A73/732A91/TDDE07), 6 hp

Time:	14-18
Allowable material:	All aids are permitted during the exam with the following exceptions: • You may not communicate with anyone else except the responsible teacher • You may not look at the solutions of any other students.
Teacher:	Bertil Wegmann. Contact via email at bertil.wegmann@liu.se
Exam scores:	Maximum number of credits on the exam: 40.
Grades (732A73/732A91):	A: 36 points B: 32 points C: 24 points D: 20 points E: 16 points F: $<$16 points
Grades (TDDE07):	5: 34 points 4: 26 points 3: 18 points U: $<$18 points

INSTRUCTIONS:
For full information, see the document *Info distance exam.pdf.*
You should submit your solutions via the Submission in LISAM.
Your submitted solutions must contain the following two things:

- A PDF named *ComputerSol.pdf* containing your computer-based solutions
- A PDF named *PaperSol* containing your hand-written solutions (when asked to answer on **Paper**)

Full score requires clear and well motivated answers.

1. Maximal weight for an escalator (*Total credits: 13p*)

   Problem (a) should only be solved on **Paper**. In a large indoor amusement park the owners are planning to replace a staircase with an escalator. During five weeks, taken at random, the maximal weight y (in thousands of kilos) on the staircase has been collected: $y_1 = 2.32$, $y_2 = 1.82$, $y_3 = 2.40$ $y_4 = 2.08$ and $y_5 = 2.13$. Assume that these measurements are an independent sample $y_1, \ldots, y_5|\theta$ from the gamma distribution with the following density:

   $$f(y_i|\theta) = \theta^2 y_i \exp[-\theta y_i], \; y_i > 0, \theta > 0 \; i = 1, \ldots, 5.$$

   Assume that the prior for θ is $p(\theta) = 0.5 \exp[-0.5\theta]$.

   (a) *Credits: 3p.* Derive the posterior distribution for θ.

   (b) *Credits: 3p.* Simulate 10000 draws from the predictive distribution of the maximal weight in a given future week, and plot the draws. Calculate $\Pr(Y_6 < 1.9|y_1, ..., y_5)$.

   (c) *Credits: 2p.* Use simulation to approximate the expected number of weeks out of the future 30 weeks in which the maximal weight will exceed 2.4 thousands of kilos, based on the predictive distribution.

   (d) *Credits: 5p.* The weight in thousands of kilos that the escalator can hold at any given time is given by

   $$0.9 \ln a,$$

   where a is the cost of building the escalator. If the weight is exceeded the escalator breaks and has to be repaired. The loss function for the indoor amusement park is

   $$L(a, \theta) = a + w(a, \theta),$$

   where $w(a, \theta)$ is the number of weeks out of the future 30 in which the escalator breaks. Compute the optimal build cost (a) using a Bayesian approach.

2. Regression problems (*Total credits: 13p*)

   The file `SoldApartments.RData`, which is loaded by the code in `ExamData.R` contains data on 52 apartments sold in the Stockholm area. For each apartment i we have observed y_i = *selling price for apartment i*, x_{1i} = *area in square metres for apartment i*, x_{2i} = *number of rooms for apartment i*, x_{3i} = *monthly fee for apartment i*, $x_{4i} = 1$ *if apartment i was sold in the inner city* (0 otherwise), $x_{5i} = 1$ *if apartment i was sold on the south side of the city* (0 otherwise). All variables (except the dummy variables) have been standardized to mean 0 and variance 1. The data matrix also contains a column *const* with ones to include an intercept in the model. Now, use `BayesLinReg.R` in `ExamData.R` to sample from the joint posterior distribution of the model parameters in the following Gaussian linear regression

   $$y = \mu + \varepsilon, \quad \varepsilon \sim N(0, \sigma^2)$$
   $$\mu = \beta_0 + \beta_1 \cdot x_1 + \beta_2 \cdot x_2 + \beta_3 \cdot x_3 + \beta_4 \cdot x_4 + \beta_5 \cdot x_5 + \beta_6 \cdot x_1 \cdot x_4 + \beta_7 \cdot x_1 \cdot x_5$$

   Simulate 10000 draws from the joint posterior. Use the following prior distributions for $\beta = (\beta_0, \beta_1, \ldots, \beta_7)$ and σ^2:

   $$\beta|\sigma^2 \sim N(0, 3^2\sigma^2 I_8)$$
   $$\sigma^2 \sim Inv - \chi^2(1, 3^2)$$

   (a) *Credits: 1p.* Compute the 99% equal tail credible interval for β_1 and interpret it.

   (b) *Credits: 2p.* Compute the median of the coefficient of variation (CV), given by $CV = \frac{\sigma}{\mu}$, for apartments on the south side of the city and with $x_1 = x_2 = 1$ and $x_3 = 0.5$.

   (c) *Credits: 4p.* Consider apartments with $x_2 = x_3 = 0$. Investigate first if the expected selling price μ is different for apartments in the inner city compared to apartments on the south side of the city when $x_1 = 1$. Then, investigate if the effect on the selling price y from x_1 is different for apartments on the south side of the city compared to apartments which are neither in the inner city nor on the south side of the city.

(d) *Credits: 3p.* Plot the posterior distribution of μ for apartments on the south side of the city with $x_1 = x_2 = -0.5$ and $x_3 = 0$ and compute the posterior probability that $\mu > 0$ for such apartments.

(e) *Credits: 3p.* Compute 95 % equal tail posterior predictive intervals for y on a grid of values of x_1 for apartments in the inner city and with $x_2 = 1$ and $x_3 = 0.5$. The grid of values of x_1 shall span between the lowest and highest values of x_1 with the distance 0.01 between any pair of values on the grid. Plot the lower and upper limits of the posterior predictive intervals of y as a function of x_1.

3. DERIVATIONS AND COMPARING POSTERIOR DISTRIBUTIONS (*Total credits: 14p*)

Problems (a), (b) and (c) should only be solved on **Paper**. Let $x_1, \ldots, x_n|\theta$ be an independent sample from a distribution with the following density:

$$f(x_i|\theta) = \frac{1}{x_i\sqrt{\pi}} \exp\left[-(\ln x_i - \theta)^2\right],\ x_i > 0, -\infty < \theta < \infty,\ i = 1, \ldots, n.$$

Assume a uniform prior for θ, i.e. $p(\theta) \propto c$, where c is a constant.

(a) *Credits: 3p.* Derive the posterior distribution for θ.

(b) *Credits: 2p.* Compute the Bayes point estimate of θ for $n = 5, \sum_{i=1}^{n} \ln x_i = 2$, by assuming the linear loss function.

(c) *Credits: 5p.* Derive the marginal likelihood for an independent sample $x_1, \ldots, x_n|\theta$ from the model above. An expression is enough, you don't need to recognize any distributional family from the end result.

(d) *Credits: 4p.* Write a function in R that computes the log posterior distribution of θ for an independent sample $x_1, \ldots, x_n|\theta$. Then, use this function to plot the posterior distribution of θ based on the data in 3(b).

GOOD LUCK!
BEST, BERTIL