Problem 2a:

Bayes' theorem gives

$$p(\theta|x) \propto p(x|\theta)\,p(\theta) \propto \theta^n e^{-\sum_{i=1}^{n} x_i^2 \cdot \theta} \cdot e^{-\frac{1}{2}\theta} = \theta^n e^{-\left(\frac{1}{2}+\sum_{i=1}^{n} x_i^2\right)\theta},$$

which can be identified as the form of a gamma distribution with parameters $\alpha = n+1$ and $\beta = \frac{1}{2} + \sum_{i=1}^{n} x_i^2$

Problem 2b:

The Bayes point estimate is the posterior mode given by $\frac{\alpha-1}{\beta}$, i.e.

$$\hat{\theta} = \frac{13+1-1}{\frac{1}{2}+2{,}8} \approx 3{,}94$$

Problem 2c:

The posterior predictive density of x_{n+1} is given by

$$p(x_{n+1}|x) = \int_\theta p(x_{n+1}|\theta)\,p(\theta|x)\,d\theta$$

$$= 2x_{n+1} \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \int_\theta \underbrace{\theta e^{-\theta x_{n+1}^2} \cdot \theta^{\overset{\alpha-1}{n}} e^{-\beta\theta}}_{\theta^\alpha e^{-\underbrace{(x_{n+1}^2+\beta)}_{\beta'}\theta}}\,d\theta$$

$$= 2x_{n+1} \cdot \frac{\beta^\alpha}{\Gamma(\alpha)} \cdot \frac{\Gamma(\alpha+1)}{(\beta')^{\alpha+1}} = 2x_{n+1} \cdot \frac{\alpha}{\beta'} \cdot \left(\frac{\beta}{\beta'}\right)^\alpha$$