Report on Free-Fall Through Trans-Planetary

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I. Introduction

As many of us have noticed, what goes up comes down. This phenomenon is due to the natural force of gravity. The strength of gravity is directly proportional to mass. So for an object with a mass like Earth, the gravitational force is far greater than a human. So when an object enters a free-fall state, the driving acceleration is the gravitational force with Earth. If you are planning to mine at a significant depth, we must consider the free-fall of objects falling into such a hole. For a simple case, the free-fall time over some distance can be found with the equation below.

$$t = \sqrt{\frac{2x}{g}}$$

Where x is the distance of the fall and g is the surface gravity.

The equation above gives the free-fall time over some distance without drag. However, the equation can further include other factors like air resistance and drag can be included. This is best described in the second order differential equation below.

$$\frac{d^2y}{dt^2} = -g + \alpha\left|\frac{dy}{dt}\right|^{\gamma}$$

This second order differential equation can be broken down to a system of coupled first order equations:

$$\frac{dy}{dt} = v$$

$$\frac{dv}{dt} = -g + \alpha|v|^{\gamma}$$

Where v is the velocity, α is the air resistance coefficient, and γ is the drag exponent.

Most differential equations can be difficult to calculate by hand, but by using Python's scipy library, we can make use of the 'solve_ivp' command to solve the differential equation and detect when it reaches a certain distance. By using this feature, conditions can be set up to detect and find the time and distance for an object in free-fall. Furthermore, we can use Python's matplotlib library twin axis feature to plot such a scenario to see how an object's position and velocity change as it falls.

II. The Ideal Case vs. Drag vs. Variable g

To compare each scenario, I perform the calculation in the case of a test mass falling down a 4 kilometers mine shaft. For an ideal free-fall case, the only driving factor is acceleration due to gravity, which is about $9.81\ m/s^2$, while α and γ will be set to 0. After performing the computation, it would take approximately 28.6 seconds. In this ideal case, the differential equation gave the same time for the simpler expression for free-fall time, which was 28.6 seconds. As mentioned before, an object in free fall is most likely to experience drag, so for this case, α and γ will have some values. For most cases with drag, $\gamma = 2$, but the air resistance coefficient may vary. To calibrate α, we can use the terminal velocity of the object. Setting all variables into the equation, the free-fall time with drag is 83.3 seconds, almost 3 times longer than the ideal case. However, a big factor that is left out is that the gravitational acceleration g is not constant throughout the free-fall. This relationship between g and distance traveled is:

$$g(r) = g_o\left(\frac{r}{r_{Earth}}\right)$$

Where g_0 is the surface gravity of Earth, r is the distance from the center of the Earth.

By introducing the varying g function, the free-fall time of the test mass is 28.6 seconds, but it is slightly longer than the ideal case.. When comparing to the ideal case, the free-fall time differs by 0.0015 seconds. This makes sense because as the test mass approaches the center of the Earth, g becomes less.

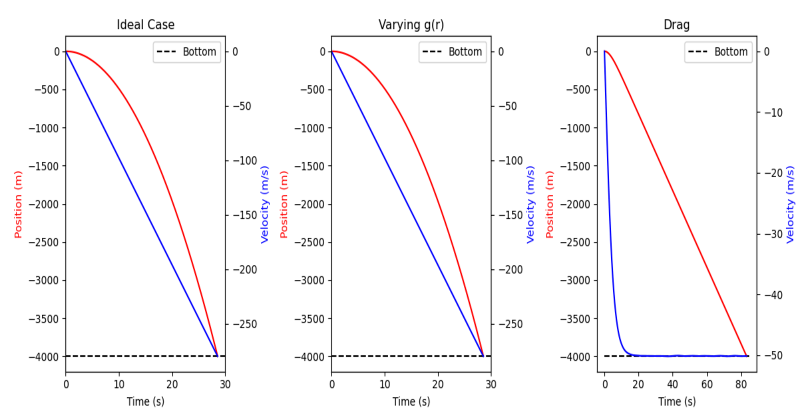

Figure 1: Various plots of the position and velocity as a function of time for each case.For the drag case, $\gamma = 2$ and $\alpha = 0.0039$.

III. The Coriolis Force

In addition to varying g and drag, we must also consider the Coriolis force due to Earth's rotation. The equation for the Coriolis force is as follows:

$$\vec{F}_c - 2m(\vec{\Omega} \times \vec{v})$$

Where m is the mass the object, Ω is the Earth rotation speed, and v is the velocity.

To simplify this, I will consider dropping the test mass in the same 4 kilometers mine shaft, but at the equator. In this case, the Coriolis force can only be considered for the x and y components of the direction. Thus giving the following components for the Coriolis force:

$$F_{c,x} = 2m\Omega v_y, \quad F_{c,y} =- 2m\Omega v_x, \quad F_{c,z} = 0$$

Knowing that $F = ma$, we can add the Coriolis force to the system of coupled first order differential equations. Updating the previous equation, the first order differential equations now becomes:

$$\frac{dy_x}{dt} = v_x, \quad \frac{dv_x}{dt} = \frac{F_{c,x}}{m}$$

$$\frac{dy_y}{dt} = v_y, \quad \frac{dv_x}{dt} =- g(r) + \alpha\left|v_y\right|^{\gamma} + \frac{F_{c,y}}{m}$$

Where x components represent the transverse direction (side-to-side) and the y components represent vertical direction.

By introducing the Coriolis force, I am able to track how the test mass moves side-to-side and determine if it can reach the bottom before hitting the walls. So for this scenario, the mine shaft will be 4 kilometers deep and 5 meters wide. Without drag, the test mass hits the wall first before reaching the bottom. The test mass hits the wall in 27.6 seconds, an entire second before reaching the bottom. When the test mass hits the wall, it has only traveled approximately 3736.7 meters deep. To further examine this effect, drag was reintroduced. Again, the test mass hits the wall much sooner before reaching the bottom. In the end, the Coriolis force heavily affects the free-fall. This shows the important consideration of the Coriolis force when designing a mine shaft with the intent to drop objects.

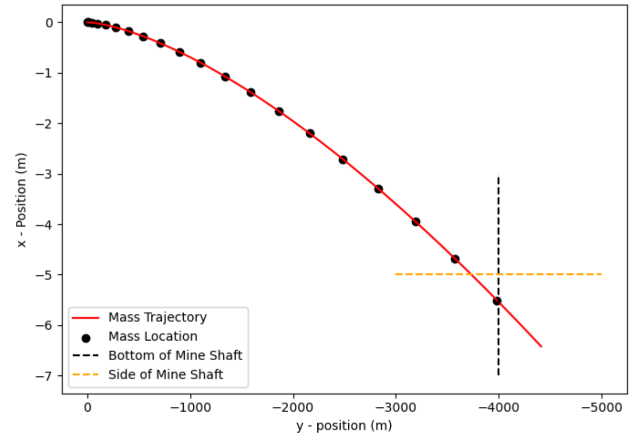

Figure 2: Plot the vertical and horizontal position of the test mass. The black circles represent the test mass location at different times.

IV. Crossing Time of Homogeneous and Non-Homogeneous Earth

As previously mentioned, the driving force of acceleration is gravity. Although I have calculated the free-fall time for some varying g, the non-homogeneous nature of Earth plays a larger role. The mass of the Earth is not constant, so the best way to find mass at some distance would be to look at the density. The Earth's density increases as the deeper we go, with the values ranging from 3000 to 12000 kg/m^3. However, there is a model that can describe the changes in density. The model equation is as follows:

$$\rho(r) = \rho_n\left(1 - \frac{r^2}{r_{Earth}^2}\right)^n$$

Where ρ_n is some normalization constant, r is the distance from the center of Earth, and n is some exponent representing the density distribution of Earth. The ideal density distribution of Earth is $n = 2$. Knowing the density, we can integrate it to find the mass. The integral is as followed:

$$M = 4\pi\int_0^r \rho(r)r^2dr$$

Before we can begin plotting, we need to find the normalization constant for each case where n =0, 1, 2, and 9. To find the normalization constant, I use the fact that gravitational force should equal the surface force. I can easily solve for the normalization constants by using the capabilities of Python, through the usage of loops. After finding the normalization constant, I can plot gravitational force as a function of distance for each n value. From there, I use the force to get the acceleration and substitute it in the free-fall equation. For simplicity, I will be assuming that the test mass is dropped from the pole of the Earth to avoid the Coriolis force.

For $n = 0$, or constant density, it takes about 1266.1 seconds for the test mass to reach the center of the Earth. For comparison, using the varying g, it takes the mass 1266.6 seconds to reach the center. Coming to a surprise, the two times are similar, suggesting that the varying g approach considers a constant density Earth. On the other hand, when having n =1, 2, and 9, it takes the mass 1095.6 seconds, 1034.3 seconds, and 943.2 seconds respectively, to reach the center. By changing the density distribution of Earth through n, the free-fall time shortens. This makes sense because the gravitational force is proportional to mass, and if the density is greater towards the center, then acceleration becomes greater.

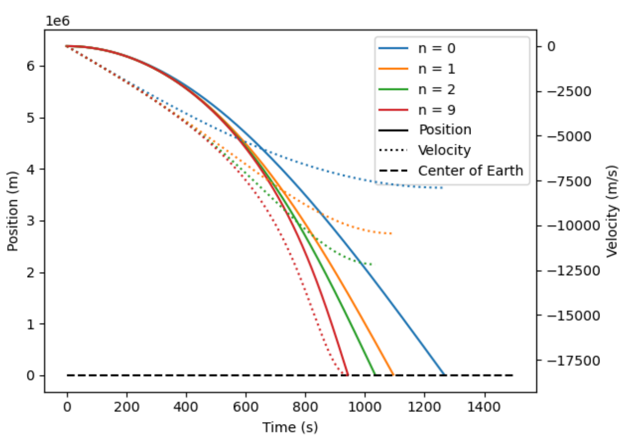

Figure 3: Plot showing the position and velocity as function of time for the various n values. Solid lines represent position and dotted lines represent velocity.

Moving further, I calculated the free-fall time for the test mass on the Moon. Since the Moon has a thin atmosphere and the density is homogeneous, the drag terms were set to zero, and the only driving factor is gravity. Furthermore, I will again be assuming that the test mass is dropped from the pole of the Moon to avoid Coriolis force. I first calculated the surface gravity of the moon, which is approximately $\frac{1}{6}$ of Earth's surface gravity. Afterward, I used the same differential equation. Doing so, the free-fall time for the test mass to reach the center of the Moon is 1625.1 seconds. Assuming constant density, the Moon's density is 3341.8 kg/m^3 and the Earth's density is 5494.9 kg/m^3. Knowing that gravitational acceleration is proportional to density through mass, and the Earth's density is about 1.5 times greater than the Moon's density, the free-fall time is expected to be shorter for Earth than for the Moon. Although the Earth is about 100 times larger than the Moon, the gravitational acceleration of Earth is significantly greater than the Moon.

V. Discussion and Future Work

In conclusion, I have gone over various scenarios and cases to determine the effects of some of the major factors for free-fall. I covered simple free-fall, with drag, with Coriolis force, with different density distribution, and free-fall on the Moon. As expected, by introducing drag terms to the equation, the test mass travels significantly slower. One thing to note is that drag exponent and air resistance coefficients vary. Then, for the Coriolis force, I assumed that the mass is dropped at the equator and the poles for simplicity.

Moving forward, these calculations can be further improved by introducing additional factors to the problem. For example, Earth was assumed to be spherical for these calculations, but that is not the case. The Earth is not a perfect sphere and is bulged at the equator due to conservation of angular momentum. The equatorial bulge can affect the gravitational field by slight increasing of distance from the center and the rotational speed. Additionally, there is fluid movement. Water and other fluids on Earth can alter the density on the surface, and the friction generated by crashing tides slows the Earth's rotational speed. Then there is the difference in rotational speed at varying locations, which was studied when looking at the Coriolis force.