Computation of the fourth cumulant, 2D flat channel without flow

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REMINDERS

We consider the following stochastic process:

$$\mathrm{d}x_t = \sqrt{2D_\parallel(z)}\mathrm{d}B_x \tag{1}$$

$$\mathrm{d}z_t = \sqrt{2D_\perp(z)}\mathrm{d}B_z + D'_\perp(z_t)\mathrm{d}t - \beta D_\perp(z_t)V'(z_t)\mathrm{d}t \tag{2}$$

FOURTH CUMULANT

The fourth cumulant reads

$$\langle x_t^4\rangle_C = \langle x_t^4\rangle - 4\langle x_t\rangle\langle x_t^3\rangle + 12\langle x_t^2\rangle\langle x_t\rangle^2 - 3\langle x_t^2\rangle^2 - 6\langle x_t\rangle^4 \tag{3}$$

with

$$\langle x_t^4\rangle = 12\left\langle\left(\int_0^t \mathrm{d}s D_\parallel(z_s)\right)^2\right\rangle \tag{4}$$

$$\frac{\langle x_t^4\rangle}{4!} = \int \mathrm{d}z\mathrm{d}z' \int_0^t \mathrm{d}s \int_0^s \mathrm{d}s' D_\parallel(z)D_\parallel(z')p_0(z')\left(p(z,s-s'|z') - p_0(z)\right) \tag{5}$$

$$\frac{\langle x_t^4\rangle}{4!} = \int \mathrm{d}z\mathrm{d}z' D_\parallel(z)D_\parallel(z')p_0(z')\sum_{\lambda>0}\psi_{R\lambda}(z)\psi_{L\lambda}(z')\int_0^t \mathrm{d}s\int_0^s \mathrm{d}s' e^{-\lambda(s-s')} \tag{6}$$

$$\frac{\langle x_t^4\rangle}{4!} = \int \mathrm{d}z\mathrm{d}z' D_\parallel(z)D_\parallel(z')p_0(z')\sum_{\lambda>0}\frac{\psi_{R\lambda}(z)\psi_{L\lambda}(z')}{\lambda}\left[t + \frac{1}{\lambda}\left(e^{-\lambda t} - 1\right)\right] \tag{7}$$

In the limit where $t \to 0$, we find

$$\frac{\langle x_t^4\rangle}{4!} = \frac{t^2}{2}\int \mathrm{d}z\mathrm{d}z' D_\parallel(z)D_\parallel(z')p_0(z')\sum_{\lambda>0}\psi_{R\lambda}(z)\psi_{L\lambda}(z') \tag{8}$$

$$= \frac{t^2}{2}\left(\left\langle D_\parallel^2\right\rangle - \langle D_\parallel\rangle^2\right) \tag{9}$$

In the limit where $t \to +\infty$, we have

$$\frac{\langle x_t^4\rangle}{4!} = \int \mathrm{d}z\mathrm{d}z' D_\parallel(z)D_\parallel(z')p_0(z')\sum_{\lambda>0}\psi_{R\lambda}(z)\psi_{L\lambda}(z')\left(\frac{t}{\lambda} - \frac{1}{\lambda^2}\right) \tag{10}$$

So

$$\frac{\langle x_t^4\rangle_C}{4!} \underset{t\to+\infty}{\simeq} A_4 t - B_4 \tag{11}$$

with

$$A_4 = \int \mathrm{d}z\mathrm{d}z' D_\parallel(z) D_\parallel(z') p_0(z') G(z,z') \tag{12}$$

and

$$B_4 = \int \mathrm{d}z\mathrm{d}z' D_\parallel(z) D_\parallel(z') p_0(z') G^2(z,z') \tag{13}$$

$$A_4 = \int \mathrm{d}z D_\parallel(z) g(z) \tag{14}$$

$$B_4 = \int \mathrm{d}z \frac{g^2(z)}{p_0(z)} \tag{15}$$

with

$$g(z) = \int_{-H}^{z} p_0(z') D_\parallel(z') G(z,z') \tag{16}$$

$$G(z,z') = \sum_{\lambda>0} \frac{\psi_{R\lambda}(z)\psi_{L\lambda}(z')}{\lambda} \tag{17}$$

We can compute this integral and we find

$$g(z) = \frac{e^{-\beta V(z)}}{Z_0} \left(\langle Q \rangle_0 - Q(z)\right) \tag{18}$$

$$Q(z) = \int_{-H}^{z} \mathrm{d}z' \frac{K(z') e^{\beta V(z')}}{D_\perp(z')} \tag{19}$$

$$K(z) = \int_{-H}^{z} \mathrm{d}z' e^{-\beta V(z')} \left(D_\parallel(z') - \left\langle D_\parallel \right\rangle_0\right) \tag{20}$$

Using an integration by part for A_4 exactly as in [1], we find

$$A_4 = \frac{1}{Z_0} \int_{-H}^{H} \mathrm{d}z \frac{K^2(z) e^{\beta V(z)}}{D_\perp(z)} \tag{21}$$

and

$$B_4 = \left\langle Q^2 \right\rangle_0 - \langle Q \rangle_0^2 \tag{22}$$

[1] Arthur Alexandre, Thomas Guérin, and David S Dean. Generalized taylor dispersion for translationally invariant microfluidic systems. *arXiv preprint arXiv:2105.06212*, 2021.