Sudoku: Counting boards

Gröbner Basis Representations of Sudoku by E.Arnold, S.Lucas, and L.Taalman

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Shidoku

A friendly simplified and illustrative version of Sudoku

- A **Region** of a Shidoku grid is either a complete row, a complete column, or a 2×2 corner sub-matrix.
- A **Board of Shidoku** is a 4×4 matrix with all the integers $1, 2, 3, 4$ in region.
- A **Puzzle** is an array incompletely filled that can be completely filled uniquely. In the following solutions, a puzzle involves extra equations of the form v̈ariable = actual value ;̈ which leads to a unique solution of the rest of variables.

Different Shidoku Boards

Counting Methods

The Sum-Product Shidoku-System: $40 - 16$

- 16 variables: one per cell with values in $\{1, 2, 3, 4\}$.
- 16 equations: one per variable $(a-1)(a-2)(a-3)(a-4) = 0$.
- 24 equations: two per region $a + b + c + d = 10$ and $abcd = 24$.

The roots of unity Shidoku-System $88 - 16$

- 16 variables: one per cell with values in $\{1, \zeta_4, \zeta_4^2, \zeta_4^3\}$.
- 16 equations: one per variable $w^4 - 1 = 0$.
- 72 equations: one per pair of variables in the same region: $\frac{w^4 - x^4}{w - x} = 0$.

The Boolean Shidoku-System $136 - 64$

- 64 equations: one per variable $w_k(w_k - 1) = 0$: $w_i = 1 \Leftrightarrow w = i$.
- 16 equations: one per cell $w_1 + w_2 + w_3 + w_4 = 1$.
- 56 equations: one per pair of cells in the same region:

Different Sudoku Boards

Can these methods generalize?

The Sum-Product Sudoku-System: $135 - 81$

Does $a+b+c+d+e+f+g+h+i = 45$ and $abcdefghi = 9!$ have a unique solution up to permutations? No. Replace them by:

$$(w+2)(w+1)\prod_{i=1}^{7}(w-i) = 0$$

$$\sum_{k=1}^{9} x_k = 25$$

$$\prod_{k=1}^{9} x_k = 2 * 7!$$

Different Sudoku Boards

Can these methods generalize?

The roots of unity Sudoku-system $972 - 81$

$$w^9 = 1$$

$$\frac{w^9 - x^9}{w - x} = 0$$

This method can be understood as a graph: Every cell is a vertex, and two vertices are joint by an edge if and only if the cells are in the same region. In how many ways can we color the vertices with 9 colors so that adjacent vertices have different colors?

Different Sudoku Boards

Can these methods generalize?

The Boolean Sudoku-System $1782 - 729$

$$w_k(w_k - 1) = 0$$

Variables are idempotents: $w_k^2 = w_k$.

$$\sum_{i=1}^{9} w_i = 1$$

$$\sum_{i=1}^{9} x_i w_i = 0$$

Bernasconi [2] and Sato [3] suggest that the computational cost of finding Gröbner bases in the Boolean case is greatly reduced.

Related Problems

Theorem (The Four Color Problem -1976 by K. Appel and W. Haken)

Given any separation of a plane into contiguous regions, producing a figure called a **map***, no more than four colors are required to color the regions of the map so that no two adjacent regions have the same color.*

In 2005, it was proven by G. Gonthier [4] using theorem proving/interactive theorem software.

Also Related

The **Exact Cover** problem: Cover the universe with disjoint subsets in a given subsets of the power set of the universe.
The matrix version: Given a matrix A with entries 0's and 1's, does it have a set of rows containing exactly one 1 in each column?

Theorem (Algorithm X - by D. Knuth, Dancing Links [5])

If A is empty, the problem is solved; terminate successfully.
Otherwise choose a column c (deterministically);
Choose a row r with $A[r, c] = 1$ (nondeterministically);
Include r in the partial solution;
For each column j with $A[r, j] = 1$:
    For each row i with $A[i, j] = 1$:
        delete row i from matrix A;
    Delete column j from A;
Repeat this algorithm recursively on the reduced matrix A.

Results

1. The authors used Maple 12 for the computations on Shidoku boards, which already requires great computer power!
2. There are 288 different boards of Shidoku.
3. B.Felgenhauer and F.Jarvis [6] showed that there are 6.67×10^{21} different Sudoku boards.
4. Russell and Jarvis [7] showed that there are 5.47×10^{9} essentially different non-equivalent boards.
   - Relabeling: Permuting $\{1, \ldots, 9\}$ in a given solution. This divides by 9! the previous result.
   - Lexicographical reduction: The first row of the Top-middle and Top-Right 3×3 regions are ordered increasingly. The first column of the Middle-Left and Lower-Left 3×3 regions are ordered increasingly. This divides by 72^2 the previous result.

Further Problems

The **Minimum Givens** Problem, which aks for the smallest number of given values that can completely determine a Sudoku board. It is conjectured that it is 17. [8]

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