A linear formulation for the Synchronous Management of Energy Production and Consumption

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Abstract Renewable energies presently appear to be an obvious and efficient solution to fight against climate change. Among these, hydrogen has all the attributes to be the most renewable, clean, efficient and non-toxic form of energy. In this work, we are interested in modeling the synchronization of hydrogen refueling of a vehicle during its pick-up and delivery tour with the production of hydrogen by photolysis and solar heat within a micro-plant. A Mixed Integer Linear Program is provided along with families of strengthening valid inequalities. A structural study of two families of matching in a bipartite graph are described and as a result, path constraints are generated. Experimental results show that the linear relaxation is deeply improved by introducing some valid inequalities whereas the mixed integer linear program seems to resist.

Keywords: Mixed Integer Linear Program, Synchronization, Hydrogen production, Vehicle consumption, Matching, Path polytope.

1 Introduction

Undoubtedly, traditional transport and electricity generation have their share of responsibility in the climate warming emissions. OR community is very active on green transportation and deep reflection is conducted about the optimal uses of renewable energy [2, 3, 4, 8]. Hydrogen does not emit polluting gases during production and is easy to store. However, its generating cost through electrolysis process is expensive compared to that of fossil fuel and its high volatility requires extensive safety measures.

In this line of thought are inserted activities of Labex IMOBS3 program, at Clermont-Ferrand (France), devoted to Innovative Mobility. To avoid costly electrolytic techniques, researchers rely on solar power and photolysis [5, 7], which naturally makes the productivity of the process deeply dependent on the sun intensity. This work mixes the control of a micro-plant producing hydrogen with the search of pick-up and delivery tours for autonomous hydrogen-powered vehicles. Taken as a whole, the problem involves forecasting, safety management and scheduling. Still, since our purpose is to focus on algorithmic features of *synchronization*, we restrict ourselves to the last issue, and set a simplified Energy Production and Consumption model restricted to the case of *one* vehicle required to perform tasks according to a prefixed order as in [6], periodically going back to the micro-plant in order to refuel. The vehicle starts its route with some H_2 fuel load, and its tank has a limited capacity. The micro-plant has a limited production/storage capacity, which depends on solar illumination. Our goal is to synchronize both the refueling transactions

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of the vehicle and the production/storage activity of the micro-plant while minimizing economical production cost and the duration of the vehicle tour. This problem is NP-Hard. We have proposed in [1] a model based on dynamic programming. The algorithm involves a 2D time space which links energy consumption by the vehicle and the production by the micro-plant. The number of states induced by the dynamic programming procedure becomes an issue as soon as the size of the problem increases, although several filtering rules have been defined.

In this paper, a Mixed Integer Linear Programming formulation ($MILP_{SMEPC}$) for the Synchronous Management Energy Production and Consumption problem ($SMEPC$) is presented. Its linear relaxation ($RMILP_{SMEPC}$) is shown to have a positive objective value and some valid inequalities are given to strengthen the linear formulation. Moreover, two families of $\{0,1\}$-variables are highlighted to drive any feasible solution of $SMEPC$. As a consequence, the structural properties of these variables are studied. We find that the synchronization of the energy consumption of the vehicle and the energy production by the micro-plant can be associated to matchings in a bipartite graph, characterized by solutions of a chain polyhedron. Some numerical experiments are carried out with CPLEX solver that emphasize the efficiency of some of valid inequalities while some others fail to improve the results.

The remainder of this paper is organized as follows. Section 2 formally introduces the *SMEPC* model. Section 3 describes a MILP formulation of the problem. Section 4 presents theoretical properties of the linear relaxation as well as some new valid inequalities. Structural properties and separation procedures are given in Section 5 and computational results are proposed in Section 6. Finally, Section 7 concludes and outlines research perspectives.

2 Problem description, definitions and notations

2.1 *The model*

We define the *SMEPC* as follows. A vehicle powered by H^2 hydrogen fuel, has to visit a set of stations that need to be served, following a route which begins and ends at a special station: the *Depot*. Close to it, there is a Micro-Plant (MP) which is a charging station. The vehicle performs a tour starting from the *Depot* (labelled 0), passing through the sequence of stations $j = 1, \ldots, M$ and returning to the *Depot* (labelled $M+1$). It leaves the depot at the date 0 and must return before the date T_{Max}. Travelling from one station j to the next one $j+1$ incurs a driving time t_j that corresponds to the shortest path in time from j to $j+1$ and an energy consumption e_j associated to this path. As the capacity of its tank is limited to C^{Veh}, the vehicle may need to periodically refuel between two stations. The initial and final quantity of hydrogen in the vehicle's tank is denoted by E_0. Going from station j to the Micro-Plant takes a time (resp. an energy consumption) d_j (resp. ϵ_j), whereas the time (resp. the energy consumption) to return from MP to j is denoted by d^*_j (resp. ϵ^*_j), see Figure 1. We consider that the time values and energy ones are non null and satisfy the triangle inequality.

The micro-plant produces H^2 from water through a combination of photolysis and electrolysis. Resulting H^2 is stored inside a tank located directly beside the micro-plant, whose capacity (in energy units) is denoted by C^{MP}. We suppose that:

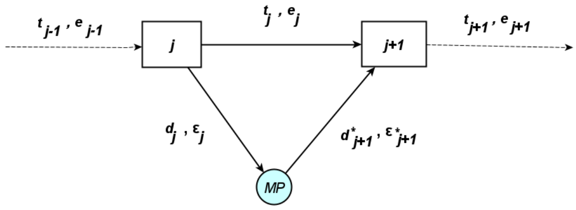

Figure 1: Time and Energy symbols between stations j, $j+1$ and the micro-plant

- The time space $\{0, 1, 2, \ldots, T_{\text{Max}}\}$ is divided into periods $P_i = [p.i, p.(i+1)[$, for $i \in \{0, \ldots, N-1\}$, all with a same length equal to p such that $T_{\text{Max}} = N.p$. For the sake of simplicity, we identify index i and period P_i.
- If the micro-plant is active at some time during period i, then it is active during the whole period i, and produces R_i hydrogen fuel units, where production rate R_i depends on period i.
- At time 0, the current load of the micro-plant tank is equal to $H_0 \leq C^{\text{MP}}$ and the micro-plant is not active. We impose that the same situation holds at time T_{Max}.
- Because of safety concerns, the vehicle cannot refuel while the micro-plant is producing. Any vehicle refueling transaction should start at the beginning of some period $i \in \{0, 1 \ldots, N-1\}$, and finish at the end of period i (which means that each refueling operation lasts p units of time). Since vehicle refueling and energy production are mutually exclusive, the vehicle may wait at the micro-plant before being allowed to refuel.

Producing H^2 fuel has a cost, which may be decomposed into *a fixed activation cost* denoted by $Cost^F$, charged every time the micro-plant is activated and *a time-dependent production cost* $Cost_i^V$ which corresponds to the power consumed during period i provided that the micro-plant is active during this period.

A solution of the SMEPC consists in a plant activity schedule that indicates the periods when the H^2 production happens, a vehicle timetable which specifies the arrival dates at each station and a refueling policy which provides the energy quantities needed by the vehicle and the stations from which it will recharge.

2.2 *The objectif*

The Synchronized Management of Energy Production and Consumption problem ($SMEPC$) consists in scheduling both the vehicle and the micro-plant in such a way that:

- The vehicle starts from $Depot = 0$, visits all stations $j = 1, \dots, M$ and comes back to $Depot = M + 1$ at some time $T_{M+1} \in [0, T_{\text{Max}}]$, refueling as many times as necessary at the micro-plant ;
- The micro-plant produces and stores in time the H^2 fuel needed by the vehicle;
- Both induced H^2 production cost $Cost$ and time T_{M+1} are the smallest possible. The two previous criteria are merged together into a unique one of the form: $Cost + \alpha.T_{M+1}$, where α is a conversion factor from time into economical cost.

3 A Mixed Integer Linear Programming formulation

We propose here a mathematical formulation by a linear program with mixed integer variables called $MILP_{SMEPC}$. For the sake of clarity, it will be defined on the basis of the following three elements:

- Hydrogen production: The Micro-Plant problem,
- Hydrogen consumption: The vehicle tour problem,
- Synchronization of consumption and production: The synchronization problem.

3.1 Input for $MILP_{SMEPC}$

1. **Vehicle related input**
   - M: number of stations ($Depot$ excluded)
   - $\Gamma = (Depot = 0, 1, \dots, M, Depot = M + 1)$: vehicle tour (without refueling)
   - T_{Max}: maximal time for the vehicle to achieve its tour
   - C^{Veh}: vehicle tank capacity
   - E_0: initial vehicle hydrogen load
   - t_j: required time to go from station j to station $j + 1$
   - d_j: required time to go from station j to the micro-plant
   - d_j^*: required time to go from the micro-plant to station j
   - e_j: required energy to go from station j to station $j + 1$
   - ϵ_j: required energy to go from station j to the micro-plant
   - ϵ_j^*: required energy to go from the micro-plant to station j
2. **Micro-plant production related input**
   - N: number of production periods
   - p: duration (in time units) of one production period, $N.p = T_{\text{Max}}$
   - C^{MP}: micro-plant tank capacity

- H_0: initial micro-plant hydrogen load
- $Cost^F$: activation cost
- $P_i = [p.i, p.(i+1)[$: time interval related to production period i
- R_i: production rate related to period i
- $Cost_i^V$: production cost related to period i

The data above are supposed to be integer.

3.2 Variables and constraints

We first propose a Mathematical Programming oriented formulation, which allows to clearly identify main variables and constraints.

3.2.1 Variables

Vehicle: When the vehicle travels from a station $j \in \{0, \dots, M\}$ to the next one, we define

- a decision variable $x_j \in \{0, 1\}$, such that $x_j = 1$ when the vehicle refuels between j and $j+1$;
- the refueling time $T_j^*, T_j^* \in \{0, p, 2p, \dots, (N-1)p\}$, when the vehicle starts to refuel, if $x_j = 1$;
- the H^2 quantity $L_j, L_j \geq 0$ and integer , loaded by the vehicle, if $x_j = 1$.

We also consider, when the vehicle arrives at any station $j \in \{0, \dots, M+1\}$,

- its arrival time $T_j, T_j \in \{0, 1, 2, \dots, T_{\text{Max}}\}$;
- the H^2 load $V_j^{\text{Veh}}, V_j^{\text{Veh}} \geq 0$ and integer , of its tank.

Production: For a period $i \in \{0, \dots, N-1\}$,

- $y_i \in \{0, 1\}$ depicts whether the micro-plant is activated at the beginning of the period;
- $z_i \in \{0, 1\}$ takes the value one if the micro-plant is active along the period;
- $\delta_i \in \{0, 1\}$ indicates if the vehicle is refueling during the period;
- $L_i^*, L_i^* \geq 0$ and integer, is the quantity of H^2 loaded by the vehicle during the period in case $\delta_i = 1$.
- $V_i^{\text{MP}}, V_i^{\text{MP}} \geq 0$ and integer, is the H^2 load of the micro-plant tank at the beginning of the period (a fictitious period N is added).

Synchronization: The meeting point between the vehicle and the plant is the time when the refueling takes place. Thus we consider

- the binary variable $U_{i,j}$ which is equal to one when the vehicle refuels during a period $i, i = 0, \dots, N-1$, while travelling from a station $j, j = 0, \dots, M$, to the next one;
- the quantity of fuel $m_{i,j}$ given to the vehicle while its travel from station j to $j+1$, at the period i, if $U_{i,j} = 1$.

3.2.2 Constraints

The previous variables are used in a mixed Integer Linear Program. We explain the way some of those constraints must be understood, specially if they result from a linearization of logical implications by the Big M technique.

Production: Let i be a period in $\{1, \ldots, N-1\}$.

- The Micro-Plant is *activated* if the production begins whereas the micro-plant was shut down at the preceding period. The following constraints are obtained.

$$\begin{array}{rrrl} y_i & -z_i & & \leq 0 \\ y_i & & +z_{i-1} & \leq 1 \\ -y_i & +z_i & -z_{i-1} & \leq 0. \end{array}$$

- Production and recharge cannot be carried out simultaneously during a period. This implies that

$$z_i \quad +\delta_i \quad \leq 1.$$

  We also add the initial conditions $y_0 - z_0 = 0$ and $\delta_0 = 0$.
- The storage capacity induces

$$V_i^{\text{MP}} \leq C^{\text{MP}},$$

  with the initial and final assumptions $V_0^{\text{MP}} = H_0$ and $V_N^{\text{MP}} \geq H_0$ (recall that N is a fictitious period).
- The production of the micro-plant generates flow type equation

$$V_{i+1}^{\text{MP}} = V_i^{\text{MP}} + R_i \cdot z_i - L_i^*.$$

- The energy L_i^* given by the plant to the vehicle must be non-negative and smaller than the capacity of the plant tank, that is $L_i^* \geq 0$ and $L_i^* \leq C^{\text{MP}} \cdot \delta_i$.

Vehicle: Assume that j is a station in $\{0 \ldots M\}$.

- Taking into account storage capacity of the vehicle, we have

$$V_j^{\text{Veh}} \leq C^{\text{Veh}},$$

  with the particular conditions at the Depot $V_0^{\text{Veh}} = E_0 \geq \epsilon_0$ and $V_{M+1}^{\text{Veh}} \geq E_0$ (the indices 0 and $M+1$ are assigned to the depot).
- The following set of constraints means that at any time, the vehicle must be able to go to the micro-plant and refuel, and relies on the Triangle Inequality for energy coefficients e_j and ϵ_j

$$V_j^{\text{Veh}} \geq \epsilon_j.$$

- The load of the vehicle verifies a flow type equation

$$V_{j+1}^{\text{Veh}} = V_j^{\text{Veh}} - e_j + (e_j - \epsilon_j - \epsilon_{j+1}^*) \cdot x_j + L_j.$$

- The quantity L_j loaded by the vehicle during its travel from j to $j+1$, if $x_j = 1$, is subject to:
  - $L_j \geq 0$,
  - $L_j \leq C^{\text{Veh}} \cdot x_j$,
  - $L_j \leq C^{\text{Veh}} + \epsilon_j - V_j^{\text{Veh}}$. This last inequality expresses the residual capacity of the vehicle when it arrives at the plant.
- At any station, the time needed to reach the next one depends on the choice of the trajectory, a direct route or with a deviation through the plant. Thus, we have first

  $$T_j^* \geq T_j + d_j.$$

  Moreover, the arrival date T_{j+1} at station $j+1$ and the time T_j^* when a refueling is eventually done are linked by the non linear inequality below.

  $$T_{j+1} \geq (1 - x_j) \cdot (T_j + t_j) + x_j \cdot (T_j^* + p + d_{j+1}^*)$$

  Either the vehicle takes the straight road to the next stage or the vehicle deviates through the plant. In that case, the next arrival time takes into account the waiting time at the tank before refueling. This can be linearized according to the Big M technique:

  $$T_{j+1} - T_j - t_j \geq -2T_{\text{Max}} \cdot x_j$$

  and

  $$T_{j+1} - (T_j^* + p + d_{j+1}^*) \geq -2T_{\text{Max}} \cdot (1 - x_j).$$

  The boundary conditions are $T_0 = 0$ and $T_{M+1} \leq T_{\text{Max}}$.

Synchronization: In order to get a complete formulation, we must explain the way the activities of both the vehicle and the micro-plant are synchronized. We use the synchronization variable $U_{i,j}$ which will tell us, in case the vehicle decides to refuel between a station j and a station $j+1$, during which period i it will do it.

- For any station j, if the vehicle decides to refuel before going to the following station i.e. $x_j = 1$, exactly one period is concerned by this activity:

  $$\sum_{i=0}^{N-1} U_{i,j} = x_j$$

- If any period i is corresponds to a refueling, that is $\delta_i = 1$, then the vehicle comes from a unique station j:

  $$\sum_{j=0}^{M} U_{i,j} = \delta_i$$

- The date to reach the micro-plant from a station j is at most the beginning of a period. This can be expressed by the next non linear inequality

$$\sum_{i=0}^{N-1} p.i.U_{i,j} \geq x_j \cdot (T_j + d_j).$$

This logical constraint can be linearized by:

$$\sum_{i=0}^{N-1} pi \cdot U_{i,j} \geq T_j + d_j - 2T_{\text{Max}} \cdot (1 - x_j)$$

- The date for refueling between stations j and $j+1$ is at least the beginning of a period:

$$\sum_{i=0}^{N-1} pi \cdot U_{i,j} \leq T_j^*$$

- The amount of fuel distributed at a period corresponds to the quantity of fuel loaded during a travel, specifically at a period i that is matched with a station j:

$$L_i^* = \sum_{j=0}^{M} U_{i,j} \cdot L_j,$$

And symmetrically,

$$L_j = \sum_{i=0}^{N-1} U_{i,j} \cdot L_i^*.$$

To linearize this constraint we use variables $m_{i,j}$. Hence,

$$\begin{array}{lcl} L_i^* & = & \sum_{j=0}^{M} m_{i,j} \\ L_j & = & \sum_{i=0}^{N-1} m_{i,j} \\ m_{i,j} & \leq & C^{\text{Veh}} \cdot U_{i,j} \\ m_{i,j} & \geq & 0. \end{array}$$

3.3 The Mixed integer linear formulation $MILP_{SMEPC}$

$$\textbf{Minimize} \sum_{i=0}^{N-1} ((Cost^F.y_i) + (Cost_i^V.z_i)) + \alpha.T_{M+1} \quad (1)$$

1. **Production constraints:**

$$
\begin{aligned}
y_0 - z_0 = 0, \delta_0 = 0, & \qquad & (2)\\
y_i - z_i \leq 0, &\quad \text{For } i = 1, \dots N-1, & (3)\\
y_i + z_{i-1} \leq 1, &\quad \text{For } i = 1, \dots N-1, & (4)\\
z_i - z_{i-1} - y_i \leq 0, &\quad \text{For } i = 1, \dots N-1, & (5)\\
z_i + \delta_i \leq 1, &\quad \text{For } i = 1, \dots N-1, & (6)\\
V_0^{\text{MP}} = H_0, & & (7)\\
V_i^{\text{MP}} \leq C^{\text{MP}}, &\quad \text{For } i = 0, \dots N-1, & (8)\\
V_N^{\text{MP}} \geq H_0, & & (9)\\
V_{i+1}^{\text{MP}} = V_i^{\text{MP}} + R_i \cdot z_i - L_i^*, &\quad \text{For } i = 0, \dots N-1, & (10)\\
V_i^{\text{MP}} \geq 0, L_i^* \geq 0, &\quad \text{For } i = 0, \dots N-1, & (11)\\
L_i^* \leq C^{\text{MP}} \cdot \delta_i, &\quad \text{For } i = 0, \dots N-1. & (12)
\end{aligned}
$$

2. **Vehicle constraints:**

$$
\begin{aligned}
V_0^{\text{Veh}} = E_0, & & (13)\\
V_j^{\text{Veh}} \leq C^{\text{Veh}}, &\quad \text{For } j = 0, \dots M, & (14)\\
V_{M+1}^{\text{Veh}} \geq E_0, & & (15)\\
V_j^{\text{Veh}} \geq \epsilon_j, &\quad \text{For } j = 0, \dots M, & (16)\\
V_{j+1}^{\text{Veh}} = V_j^{\text{Veh}} - e_j + (e_j - \epsilon_j - \epsilon_{j+1}^*) \cdot x_j + L_j, &\quad \text{For } j = 0, \dots M, & (17)\\
L_j \geq 0, &\quad \text{For } j = 0, \dots M, & (18)\\
L_j \leq C^{\text{Veh}} \cdot x_j, &\quad \text{For } j = 0, \dots M, & (19)\\
L_j \leq C^{\text{Veh}} + \epsilon_j - V_j^{\text{Veh}}, &\quad \text{For } j = 0, \dots M, & (20)\\
T_0 = 0, & & (21)\\
T_j \geq 0, &\quad \text{For } j = 0, \dots M, & (22)\\
T_{j+1} - T_j - t_j \geq -2T_{\text{Max}} \cdot x_j, &\quad \text{For } j = 0, \dots M, & (23)\\
T_{j+1} - (T_j^* + p + d_{j+1}^*) \geq -2T_{\text{Max}} \cdot (1 - x_j), &\quad \text{For } j = 0, \dots M, & (24)\\
T_j^* \geq T_j + d_j, &\quad \text{For } j = 0, \dots M, & (25)\\
T_{M+1} \leq T_{\text{Max}}. & & (26)
\end{aligned}
$$

3. **Synchronisation constraints:**

$$\sum_{i=0}^{N-1} U_{i,j} = x_j, \qquad \text{For } j = 0, \ldots M, \tag{27}$$

$$\sum_{j=0}^{M} U_{i,j} = \delta_i, \qquad \text{For } i = 0, \ldots N-1, \tag{28}$$

$$\sum_{i=0}^{N-1} pi \cdot U_{i,j} \leq T_j^*, \qquad \text{For } j = 0, \ldots M, \tag{29}$$

$$\sum_{i=0}^{N-1} pi \cdot U_{i,j} \geq T_j + d_j - 2T_{\text{Max}} \cdot (1 - x_j), \qquad \text{For } j = 0, \ldots M, \tag{30}$$

$$\sum_{j=0}^{M} m_{i,j} = L_i^*, \qquad \text{For } i = 0, \ldots N-1, \tag{31}$$

$$\sum_{i=0}^{N-1} m_{i,j} = L_j, \qquad \text{For } j = 0, \ldots M, \tag{32}$$

$$m_{i,j} \leq C^{\text{Veh}} U_{i,j}, \quad \text{For } i = 0, \ldots, N-1, j = 0, \ldots M, \tag{33}$$

$$m_{i,j} \geq 0, \quad \text{For } i = 0, \ldots, N-1, j = 0, \ldots M. \tag{34}$$

$$U_{i,j} \geq 0, \quad \text{For } i = 0, \ldots, N-1, j = 0, \ldots M. \tag{35}$$

4. **Decision variable constraints:**

$$0 \leq y_i, 0 \leq z_i, 0 \leq \delta_i, \quad \text{For } i = 0, \ldots N-1, \tag{36}$$

$$0 \leq x_j \leq 1, \qquad \text{For } j = 0, \ldots M, \tag{37}$$

and

$$y_i \in \{0,1\}, z_i \in \{0,1\}, \delta_i \in \{0,1\}, \qquad \text{For } i = 0, \ldots N-1, \tag{38}$$

$$x_j \in \{0,1\}, \qquad \text{For } j = 0, \ldots M. \tag{39}$$

$$U_{i,j} \in \{0,1\}, \quad \text{For } i = 0, \ldots, N-1, j = 0, \ldots M. \tag{40}$$

Figure 2 gives a feasible solution in case $p = 2, E_0 = 8, H_0 = 4, TMax = 30, Cost^F = 7, C^{MP} = 15, C^{Veh} = 15$, $\alpha = 1$.The global resulting activation cost is $3 * 7 = 21$. The time T is equal to 30. The time-dependent production costs in the activation periods are $Cost_1^V = Cost_2^V = 1, Cost_5^V = Cost_6^V = Cost_7^V = 2, Cost_{13}^V = 1$. Thus, the global cost is equal to $21+9+30 = 60$.

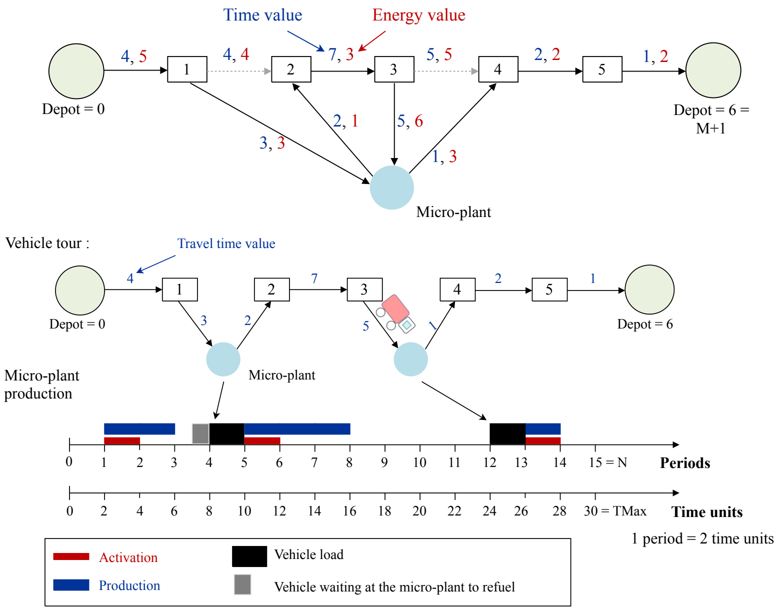

Figure 2: The vehicle pick-up and delivery tour with $M = 5$ and its refueling

Theorem 3.1. $MILP_{SMEPC}$ has a feasible solution if and only if $SMEPC$ has a solution with the same cost.

Proof. Consider an optimal solution $(\bar{z}, \overline{T}, (\bar{L}, \bar{x}))$ of $SMEPC$, which describes the plant activity schedule, the vehicle timetable and the refueling policy respectively. We check that it can be turned into a feasible solution of the MILP with the same cost.

Let $\overline{J}$ be the set of indices $j \in \{0, \dots M\}$ for which x_j takes value one. By applying (13) and (17), the H^2 load $\overline{V}^{\text{Veh}}$ is iteratively calculated in the following way: if $j \in \overline{J}$, then $\overline{V}^{\text{Veh}}_{j+1} = \overline{V}^{\text{Veh}}_j - \epsilon_j + \overline{L}_j - \epsilon^*_{j+1}$, else $\overline{V}^{\text{Veh}}_{j+1} = \overline{V}^{\text{Veh}}_j - e_j$.

As the vehicle directly goes to the station after recharging which takes one period of time, $\overline{T}^*$ can be chosen such that

$$\overline{T}^*_j = \overline{T}_{j+1} - d^*_{j+1} - p \text{ and } \overline{T}^*_j \in \{0, p, \dots (N-1)p\}, \text{ for } j \in \overline{J}.$$

When j does not belong to $\overline{J}$, T^*_j can take any non negative value. The validity of the vehicle's route means that the vectors $\overline{T}$ and $\overline{T}^*$ verify (21)-(26).

Moreover, an index $\hat{\imath}(j)$ can be assigned to each j in $\overline{J}$, by taking $\hat{\imath}(j) \cdot p = \overline{T}^*_j$. Let $\overline{I} = \{\hat{\imath}(j) : j \in \overline{J}\}$. One can assume that $\delta_i = 1, L^*_i = L_j$ for $i \in \overline{I}$, and $\delta_i = L^*_i = 0$ otherwise. Similarly, $U_{i,j} = 1$ if $i = \hat{\imath}(j), j \in \overline{J}$, and $U_{i,j} = 0$ otherwise.

Let $\overline{I}^A = \{i \in [\![0, N-1]\!] : \bar{z}_i = 1\}$. $\overline{I}^A$ can be partitioned into disjoint intervals of the form $[f_k, g_k], k = 1, \dots q$, on which the plant is active. Note than $\bar{z}_{f_k - 1} = 0 = \bar{z}_{g_k+1}$, if $f_k > 0, g_k < N-1, 1 \leq k \leq N-1$. As the micro-plant cannot be active while the vehicle refuels, $\overline{I} \cap \overline{I}^A = \emptyset$; this implies that (6) is satisfied.

Let us define

$$\bar{y}_i = \begin{cases} 1 & \text{if } i = f_k, \text{ for some } k \leq q, \\ 0 & \text{otherwise.} \end{cases}$$

(2)-(5) are therefore verified by $\bar{y}$ and $\bar{z}$. From (7), iteratively applying equation (10) enables us to determine the volume V_i^{MP}, for $i = 0, \dots N-1$. All other inequalities are capacity constraints that are necessary satisfied by a feasible solution of $SMEPC$.

Conversely, let us consider some optimal feasible solution

$$(\bar{y}, \overline{\delta}, \bar{x}, \bar{L}, \bar{z}, \overline{L}^*, \overline{T}, \overline{T}^*, \overline{V}^{\text{Veh}}, \overline{V}^{\text{MP}}, \overline{U})$$

of the above $MILP_{SMEPC}$.

By Constraints ((27)-(28)), the vector $\overline{U}$ defines a matching $j \to \hat{\imath}(j)$ between the sets $\bar{J} = \{j \in \{0, \dots, M\} : \bar{x}_j = 1\}$ and $\bar{I} = \{i \in \{0, \dots, N-1\} : \overline{\delta}_i = 1\}$. This matching is consistent with the standard linear ordering: if $j_1 < j_2$ then $\hat{\imath}(j_1) < \hat{\imath}(j_2)$. Indeed, inequalities (21)-(26) insures that $T_{j+1} > T_j, \forall j$ and $T_{j+1} > T_j^* > T_j, \forall j \in \bar{J}$. Hence $T_{j_2}^* > T_{j_2} \geq T_{j_1+1} > T_{j_1}^*$. From (29)-(30), we derive that $\hat{\imath}(j_2)p > T_{j_2} > T_{j_1}^* \geq \hat{\imath}(j_1)p$. Therefore, $\hat{\imath}(j_1) < \hat{\imath}(j_2)$.

By optimality of the solution, constraints (23), (24) and (29) are tight. Thus $\overline{T}$ defines a vehicle timetable. As $L_{\hat{\imath}(j)}^* = L_j, j \in \bar{J}$ from (31)-(34), the vehicle and the micro-plant tanks have feasible volumes due to flow type and capacity constraints.

We conclude that a solution to the $SMEPC$ can be extracted from the optimal feasible solution of the $MILP$, both of the same cost.

□

4 Linear relaxation

Denote by $RMILP_{SMEPC}$ the linear relaxation of $MILP_{SMEPC}$, i.e. the linear program described by (1)-(37) and by $\bar{Z}_r$ its optimal value. Often, Big M techniques induce very weak linear relaxations. But here we have the following.

Lemma 4.1. If $RMILP_{SMEPC}$ is feasible, then $\bar{Z}_r > 0$.

Proof. Suppose on the contrary that there is an optimal solution

$$(\bar{y}, \overline{\delta}, \bar{x}, \bar{L}, \bar{z}, \overline{L}^*, \overline{T}, \overline{T}^*, \overline{V}^{\text{Veh}}, \overline{V}^{\text{MP}}, \overline{U})$$

of the $RMILP_{SMEPC}$ such that $\bar{Z}_r = 0$. Necessarily

$$\bar{y}_i = \bar{z}_i = 0, \quad \text{for } i = 0, \dots N-1.$$

Then from (10) we obtain that $V_{i+1}^{\text{MP}} = V_i^{\text{MP}} - L_i^*$, for $i = 0, \dots N-1$. By (7) and (9), $L_i^* = 0$, for $i = 0, \dots N-1$. (31) implies that $m_{i,j} = 0$, for $j = 0, \dots M, i = 0, \dots N-1$. By (32), $L_j = 0$, for $j = 0, \dots M$. Henceforth, (17) gives $V_{j+1}^{\text{Veh}} < V_j^{\text{Veh}}$, for $j = 0, \dots M$. But this contradicts the initial and final conditions (13) and (15). □

4.1 Additional constraints

Several constraints can be added for strengthening the linear relaxation. To achieve this objective we introduce the following data.

For any $j = 1, \ldots, M$, we set:

$D_j = \sum_{k=0}^{j-1} t_k + d_j$: the earliest arrival time at the micro-plant for a first refueling after station j.

$D_j^* = d_{j+1}^* + \sum_{k=j+1}^{M} t_k$: the latest time to finish the trip after refueling at station j.

$\tau_m(j) = \lceil \frac{D_j}{p} \rceil$: the earliest period of a possible refueling at station j;

$\tau_M(j) = N - 1 - \lceil \frac{D_j^*}{p} \rceil$: the latest period of a possible refueling at station j;

4.1.1 Simple Time constraints

The inequalities

$$T_{j+1}^* \geq T_j^*, \quad j = 0, \ldots M, \tag{41}$$

$$T_{j+1} \geq T_j + t_j + x_j(d_j + d_{j+1}^* - t_j), \quad j = 0, \ldots M, \tag{42}$$

are straightforwardly seen to be valid for $MILP_{SMEPC}$. (41) and (42) insure that the times form non decreasing sequences. We will see in the experimental results that these constraints are very useful.

4.1.2 Energy constraints

$$L_j \geq x_j(\epsilon_j + \epsilon_{j+1}^* - e_j), \quad j = 0, \ldots M. \tag{43}$$

By considering (43), the vehicle leaving any station will arrive at the next one with more hydrogen after a refueling than if it followed the direct route between the two.

$$U_{i,j} = 0 \quad i < \tau_m(j) \text{ or } i > \tau_M(j) \quad j = 0, \ldots M. \tag{44}$$

Inequalities (44) simply reflect the definitions of $\tau_m(j)$ and $\tau_M(j)$, for any j.

$$\sum_{i=0}^{\tau_M(j)} L_i^* \geq \sum_{k=0}^{j} L_k, \quad j = 0, \ldots M. \tag{45}$$

(45) expresses the fact that the total quantity refuelled by the vehicle at station j does not exceed the total quantity of hydrogen given by the micro-plant till $\tau_M(j)$.

(45) has the following consequences.

Let $F_{j+1} = F_j + e_j + x_j(\epsilon_j + \epsilon_{j+1}^* - e_j)$ with $F_0 = 0$. F_j is the energy used by the vehicle from 0 to j.

From (10) and (17)

$$
\begin{aligned}
V_{j+1}^{\text{Veh}} &= E_0 - F_{j+1} + \sum_{k=0}^{j} L_k, \\
V_{\tau_M(j)+1}^{\text{MP}} &= H_0 + \sum_{i=0}^{\tau_M(j)} R_i \cdot z_i - \sum_{i=0}^{\tau_M(j)} L_i^*,
\end{aligned}
$$

for any $j = 0, \ldots M$.

Hence

$$
\begin{aligned}
E_0 + \sum_{k=0}^{j} L_k &\geq F_{j+1}, \\
H_0 + \sum_{i=0}^{\tau_M(j)} R_i \cdot z_i &\geq \sum_{i=0}^{\tau_M(j)} L_i^*,
\end{aligned}
$$

for any $j = 0, \ldots M$.

Thus we obtain:

EC1:

$$
\sum_{i=1}^{\tau_M(j)} R_i z_i \geq F_{j+1} - E_0 - H_0 \quad j = 0, \ldots M, \tag{46}
$$

$$
\sum_{i=1}^{N-1} R_i z_i \geq F_{M+1}. \tag{47}
$$

The binary variable y_i is equal to 1 when the micro-plant is activated at period i. Thus the value $\sum_{i=0}^{\tau_M(j)} y_i$ indicates the number of production intervals between the periods 0 and $\tau_M(j)$. During each of these intervals, the H^2 production cannot exceed the capacity of the micro-plant. Hence

$$
C^{\text{MP}} \textstyle\sum_{i=0}^{\tau_M(j)} y_i \geq \sum_{i=0}^{\tau_M(j)} R_i z_i \quad j = 0, \ldots M.
$$

From (46), this implies that

EC2:

$$
C^{\text{MP}} \cdot \sum_{i=0}^{\tau_M(j)} y_i \geq F_j - E_0 - H_0 \quad j = 0, \ldots M, \tag{48}
$$

$$
C^{\text{MP}} \cdot \sum_{i=0}^{N-1} y_i \geq F_{M+1}. \tag{49}
$$

In a same way, from (12),

$$
\sum_{i=0}^{\tau_M(j)} C^{\text{MP}} \cdot \delta_i \geq \sum_{i=0}^{\tau_M(j)} L_i^*, \quad \text{For } j = 0, \ldots M.
$$

Therefore we get
EC2':

$$C^{\text{MP}} \cdot \sum_{i=0}^{\tau_M(j)} \delta_i \geq \quad F_j - E_0 \quad j = 0, \dots M, \tag{50}$$

$$C^{\text{MP}} \cdot \sum_{i=0}^{N-1} \delta_i \geq \quad F_{M+1}. \tag{51}$$

Similarly to (45), it can be seen that

$$\sum_{i=\tau_m(j)}^{N-1} L_i^* \geq \quad \sum_{k=j}^{M} L_k, \quad j = 0, \dots M. \tag{52}$$

(52) expresses the fact that the total quantity refuelled by the vehicle from station j does not exceed the total quantity of hydrogen given by the micro-plant from $\tau_m(j)$. It could be also used for generating inequalities similar to those of type (46) and (48).

Finally some cover inequalities are proposed. First, let $\mu_j^0 = \sum_{k=0}^{j-1} e_k + \epsilon_j$, for any $j = 1, \dots M$. μ_j^0 represents the energy consumption of the vehicle starting at the $Depot$ and finishing at the micro-plant before refueling at station j. Next, $\mu_j^* = \epsilon_{j+1}^* + \sum_{k=j+1}^{M} e_k$, $j = 0, \dots, M$. μ_j^* is the energy consumption of the vehicle starting at the micro-plant after a refueling at j and finishing at the $Depot$. Depending of the fact that the minimal consumption of the vehicle exceeds the initial amount of hydrogen or the capacity of its tank, the following constraints can be obtained.

EC3:

$$\sum_{k=0}^{j} x_k \geq \quad 1, \quad j = 0, \dots M, \text{ such that } \mu_j^0 > E_0, \tag{53}$$

$$\sum_{k=j+1}^{M} x_k \geq \quad 1, \quad j = 0, \dots M, \text{ such that } \mu_j^* > C^{\text{Veh}} - E_0. \tag{54}$$

5 Structural study

5.1 Principal variables

Among all unknowns, the decision variables $(z_i)_{0 \leq i \leq N-1}$ and $(U_{i,j})_{0 \leq j \leq M, 0 \leq i \leq N-1}$ play a central role.

Indeed, assume that feasible (z_i) and $(U_{i,j})$ are given. Then, by applying (27) and (28), (x_j) and (δ_i) are determined, and in consequence, the compatibility of (z_i) and $(U_{i,j})$ is checked with (6). Next, the (y_i) which indicate the activation of the micro-plant are simply calculated from (2)-(5).

Thus, we obtain two flow-models, (7)-(12) for managing the plant and (13)-(20) for the evolution of the vehicle tank. These two systems are synchronized through the variables $(m_{i,j})$ within (31)-(34) and the schedule is sequentially calculated by (21)-(26), (29)-(30).

These remarks endorse the fact that (z_i) and $(U_{i,j})$ are the principal variables of our formulation. So we believe that any inequality involving these variables could improve the linear relaxation.

In the following, the complete bipartite graph $G = (I + J, E)$ is used where the vertices can be partitionned into two independent sets I and J with $I = \{0, 1, \ldots, N-1\}$ and $J = \{0, 1, \ldots, M\}$ and the set of edges $E = \{(i, j) : 0 \leq i \leq N-1, 0 \leq j \leq M\}$.

5.2 Non-crossing matchings

We say that two edges (i, j) and (i', j') of a matching in G are crossing if $j < j'$ and $i > i'$. A matching C is called *non-crossing* if C has no crossing pair of edges.

Let (z_i) and $(U_{i,j})$ be a feasible solution of $MILP_{SMEPC}$. By (27) and (28), the edge set $C(U) = \{(i, j) : U_{i,j} = 1, 0 \leq j \leq M, 0 \leq i \leq N-1\}$ is a matching in G. Moreover, due to the time constraints (21)-(26) and (29)-(30), the matching $C(U)$ has no crossing pair of edges.

Thus, we aim to search valid constraints for non-crossing matchings.

We associate to G an acyclic digraph $H = (V(H), A(H))$ as follows.

$V(H) = \{\{i, j) : 0 \leq i \leq N-1, 0 \leq j \leq M\}$,

The edge set $A(H)$ contains the arcs of the form $((i, j), (i, j-1)),$ for $1 \leq j \leq M, 0 \leq i \leq N-1$ and $((i, j), (i+1, j)),$ for $0 \leq i \leq N-2, 0 \leq j \leq M$.

Note that $V(H) = E(G)$, and the vertices $(0, M)$ and $(N-1, 0)$ are the sink and the tank of H, respectively. All maximal paths in H are of length $N + M - 1$. Denote by $\mathcal{P}$ the set of all maximal paths in H and by $E(P)$ the edges of E that correspond to vertices of P, for $P \in \mathcal{P}$.

The graph H induces an order $\preceq$ on the vertices of $V(H)$ so that $(i_1, j_1) \preceq (i_2, j_2)$ if (i_2, j_2) is reachable in H from (i_1, j_1)

Given a maximal path $P \in \mathcal{P}$, the constraint

$$\sum_{(i,j) \in E(P)} U_{i,j} \leq 1 \tag{55}$$

is called the *antagonistic* constraint associated to P.

Lemma 5.1. For any maximal path P in H, the antagonistic constraint induced by P is valid for $MILP_{SMEPC}$.

Proof. Consider a maximal path $P = ((i_1, j_1), (i_2, j_2), \ldots (i_{N+M-1}, j_{N+M-1}), (i_{N+M}, j_{N+M}))$ of $\mathcal{P}$ with $(i_1, j_1) = (0, M)$ and $(i_{N+M}, j_{N+M}) = (N-1, 0)$. Suppose that a non crossing matching $C(U)$ has a non empty intersection with $E(P)$, for some feasible U of $MILP_{SMEPC}$. Let $(i_{k_0}, j_{k_0}), 1 \leq k_0 \leq N + M$, be the first edge of $E(P)$ belonging to $C(U)$. If there exists an other edge $(i_{k_1}, j_{k_1}), 1 \leq k_0 < k_1 \leq N + M$, then this implies that $i_{k_0} \neq i_{k_1}$ and $j_{k_0} \neq j_{k_1}$. Furthermore, as the two nodes are in P, by construction of $A(H)$, we have $i_{k_0} < i_{k_1}$ and $j_{k_0} > j_{k_1}$. Hence, the edges (i_{k_0}, j_{k_0}) and (i_{k_1}, j_{k_1}) are crossing in $C(U)$, a contradiction. $\square$

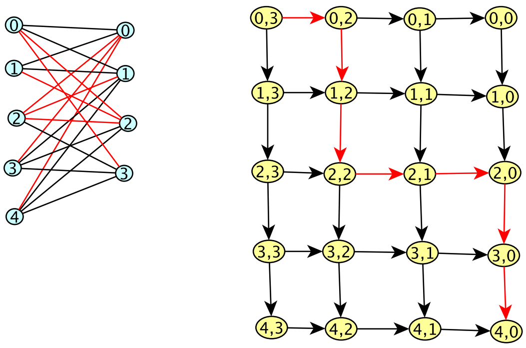

Figure 3: A path in H associated to a set of crossing edges in G.

5.3 Non-crossing matching polytope

Let Q be the polytope defined by

$$0 \leq U_{i,j} \leq 1, \quad \text{for all } 0 \leq i \leq N-1, 0 \leq j \leq M \tag{56}$$

$$\sum_{(i,j)\in E(P)} U_{i,j} \leq 1, \quad \text{for all } P \in \mathcal{P}. \tag{57}$$

An antichain A of $(V(H), \preceq)$ is a subset of pairwise incomparable elements of $V(H)$, i.e. A is the set of vertices of $V(H)$ such that either $(i_1, j_1) \not\preceq (i_2, j_2)$ or $(i_2, j_2) \not\preceq (i_1, j_1)$ for all $(i_1, j_1) \in A, (i_2, j_2) \in A, (i_1, j_1) \neq (i_2, j_2)$.

The polytope Q is precisely the chain polytope whose vertices are the characteristic vectors of antichains of $(V(H), \preceq)$ ([9]).

Lemma 5.2. An antichain A of $(V(H), \preceq)$ corresponds to a non-crossing matching set of G.

Proof. Consider an antichain A of $(V(H), \preceq)$.

Note first, that $(i, j) \preceq (i+k, j)$, for any $1 \leq k \leq N-1-i, 0 \leq i \leq N-2$, thanks to the path $(i, j), (i+1, j), (i+k, j)$ in H. In the same way, $(i, j) \preceq (i, j-k)$, for any $1 \leq k \leq j, 1 \leq j \leq M$. Thus A induces a matching in G.

Let (i_1, j_1) and (i_2, j_2) be two elements of A that correspond to a crossing pair of edges of G. Without lost of generality, suppose that $j_1 < j_2$ and $i_1 > i_2$. But then $(i_2, j_2) \preceq (i_1, j_1)$ due to the path in $H : (0, j_2), \ldots, (i_2, j_2), (i_2 + 1, j_2), \ldots, (i_1, j_2), \ldots, (i_1, j_2 - 1), \ldots, (i_1, j_1)$. This contradicts the fact that A is an antichain.

The elements of A constitute a non crossing matching of G. □

From Lemma (5.1) and (5.2), vertices of the polytope Q are characteristic vectors of non-crossing matchings of G.

Moreover, the antagonistic constraints can be used to strengthen the linear relaxation $RMILP_{SMEPC}$ of Section 4. Testing whether a feasible solution U of the linear relaxation satisfies inequalities of type (55) can be done in polynomial time by searching a longest path on the acyclic digraph H with Bellman's algorithm.

5.4 Time-consistent matching

This section focuses on a subclass of non-crossing matchings. Given the complete bipartite graph $G = (I + J, E(G))$ and a weight function $w : J \times J \to \mathbb{Z}_+$, we say that two edges (i, j) and (i', j'), with $j < j'$, of a matching in G are *time-consistent* (resp. *time-inconsistent*) if $i + w(j, j') \leq i'$ (resp. $i + w(j, j') > i'$). A matching C is called *time-consistent* if C is a subset of pairwise time-consistent edges.

Note that, as $w(j, j') \geq 0$, for all $0 \leq j < j' \leq M$, a time-consistent matching C is also a non-crossing matching of G. Indeed, if two edges (i, j) and (i', j') with $j < j'$ belong to C, then we have $i' \geq i$, and the two edges are necessarily non-crossing.

In a tour, the vehicle moving from a station j to another station j' cannot reach j' more quickly than the time required for the straight trip between j and j'. The integer values $w(j, j'), 0 \leq j < j' \leq M$, reflect this incompressible time.

Lemma 5.3. Given a complete bipartite graph $G = (I + J, E(G))$, a weight function $w : J \times J \to \mathbb{Z}_+$ and an integer k, the problem of finding a time-consistent matching of size k in G is NP-complete.

Proof. Consider a connected undirected graph $\hat{G} = (V(\hat{G}), E(\hat{G}))$ and an integer k. It is well known that determining if there exists a stable set in $\hat{G}$ of size at least k is an NP-complete problem.

Let $N - 1 = M = |V(\hat{G})| - 1$ and $V(\hat{G}) = \{v_0, v_1 \ldots v_M\}$. We construct a complete bipartite graph $G = (I + J, E(G))$, where I and J are two ordered subsets such that $I = \{0, 1, \ldots, N - 1\}$ and $J = \{0, 1, \ldots, M\}$. The weights are given by $w(j, j') = N$, if the edge $\{v_j, v_{j'}\}$ is in $E(\hat{G})$ and $w(j, j') = 1$ otherwise, for any $j < j'$.

We claim that $\hat{G}$ has a stable set of size at least k if and only if G has a time-consistent matching of cardinality k.

Let C be a time-consistent matching in G. Denote by $S(C)$ the set $\{j \in J : \exists i \in I, (i, j) \in C\}$. $S(C)$ is a stable set in $\hat{G}$. On the contrary, suppose that there is an edge between j and j' in $S(C)$. Let i and i' be the vertices associated respectively with j and j' such that the two edges (i, j) and (i', j') are in C. Hence, we should have $i + w(j, j') = i + N \leq i' \leq N - 1$, a contradiction.

Now suppose that $S = \{v_{j_1}, \ldots, v_{j_k}\}$ is a stable set of cardinality k in $\hat{G}$. Let $C(S) = \{(j, j) : v_j \in S\}$. It is straightforwardly seen that $j_q + w(j_q, j_{q'}) = j_q + 1 \leq j_{q'}$ for all $1 \leq q < q' \leq k$. Therefore $C(S)$ is a time-consistent matching of G. ☐

The preceding lemma suggests that a complete description of the polytope of the time-consistent matchings is difficult to obtain for any weight function w.

In the last part of this subsection, assume that the weight function satisfies the *(SA)* property:

$$\forall j_1, j_2, j_3 \in [\![0, M]\!], j_1 < j_2 < j_3 \Rightarrow w(j_1, j_2) + w(j_2, j_3) \leq w(j_1, j_3) \quad \text{(SA)}$$

Then, if the pairs $\{(i_1, j_1), (i_2, j_2)\}$ and $\{(i_2, j_2), (i_3, j_3)\}$ are time-inconsistent for some $0 \leq j_1 < j_2 < j_3 \leq M$ and $0 \leq i_1 < i_2 < i_3 \leq N - 1$, then the pair $\{(i_1, j_1), (i_3, j_3)\}$ is also time-inconsistent.

With respect to the (SA)-property, valid constraints can be obtained as in Section 5.2. First, let us complete the directed acyclic graph H of Section 5.2 as follows. For any $j < j'$ such that

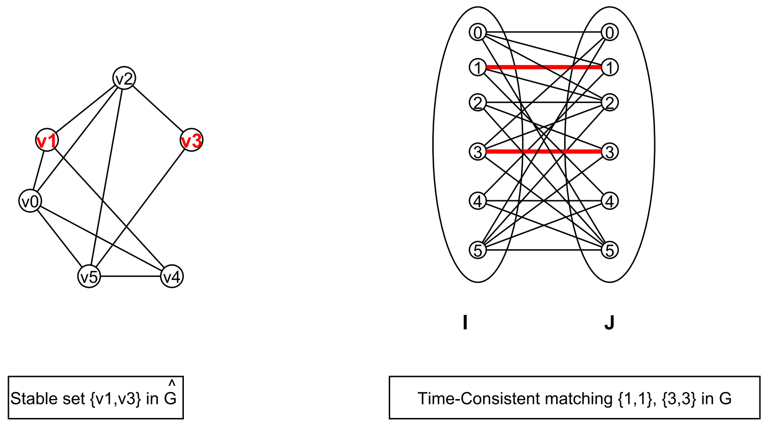

Figure 4: Transformation from a stable set in H to a Time-consistent matching in G

$w(j, j') \geq 2$, we add the arcs $((i', j'), (i, j))$ for $\max(0, i' - w(j, j') + 1) \leq i \leq i' - 1, 1 \leq i' \leq N - 1$. Denote by H_{tc} this modified graph, by $A^+(H_{\text{tc}})$ the set of the new arcs, and by $\mathcal{P}_{\text{tc}}$ the set of paths P of origin $(0, M)$ and end $(N - 1, 0)$ in H_{tc}.

So, for a maximal path $P \in \mathcal{P}_{\text{tc}}$, the constraint

$$\sum_{(i,j) \in E(P)} U_{i,j} \leq 1 \tag{58}$$

is called the *time-consistent* constraint associated to P.

Lemma 5.4. Let (i'_0, j'_0) and (i_0, j_0) be two vertices of H_{tc} such that $i'_0 > i_0$ and $j'_0 > j_0$. If there is a path in H_{tc} from (i'_0, j'_0) to (i_0, j_0), then the arc $((i'_0, j'_0), (i_0, j_0))$ exists in $A^+(H_{\text{tc}})$.

Proof. Consider two vertices (i'_0, j'_0) and (i_0, j_0) with $i'_0 > i_0$ and $j'_0 > j_0$ that are connected by a path P in H_{tc}. By construction of H_{tc}, it can be seen that $j'_0 \geq j \geq j_0$ for all node (i, j) in P.

Now, moving backwards along P from (i_0, j_0), we collect any arc $a = ((i', j'), (i, j))$ of $A^+(H_{\text{tc}})$ which satisfies $i' < i'_0$. The collecting process is stopped when an arc $a_1 = ((i'_1, j'_1), (i_1, j_1))$ is found with $i'_1 \geq i'_0$ for the first time.

Consider the sequence $((i'_1, j'_1), (i_1, j_1)), ((i'_2, j'_2), (i_2, j_2)), \ldots ((i'_q, j'_q), (i_q, j_q))$ of the arcs selected by the above procedure for some integer $q \geq 1$. We have that

- $i'_1 \geq i'_0$, $i'_k < i'_0$, $k = 2, \ldots q$, $i_q \leq i_0$,
- $i_k < i'_k$, $k = 1, \ldots q$,
- the intervals $[i_k, i'_k]$, $k = 1, \ldots q$ cover the interval $[i_0, i'_0]$.

By construction of $A^+(H_{\text{tc}})$, it is known that

$$\begin{array}{rcl} i_1 + w(j_1, j'_1) & > & i'_1, \\ i_2 + w(j_2, j'_2) & > & i'_2, \\ \ldots & & \\ i_q + w(j_q, j'_q) & > & i'_q. \end{array}$$

Thus

$$\sum_{k=1}^{q} w(j_k, j'_k) > i'_1 - i_1 + i'_2 - i_2 + \dots i'_q - i_q \geq i'_0 - i_0.$$

As $j'_0 \geq j'_1 > j_1 > j'_2 > j_2 \cdots > j'_q > j_q \geq j_0$, by the (SA) property, we deduce that $w(j_0, j'_0) > i'_0 - i_0$. This implies that the arc $((i'_0, j'_0), (i_0, j_0))$ is in $A^+(H_{\text{tc}})$. □

Lemma 5.5. For any maximal path P in H_{tc}, the time-consistent constraint induced by P is valid for $MILP_{SMEPC}$.

Proof. Consider a maximal path $P = ((i_1, j_1), (i_2, j_2), \dots (i_{Q-1}, j_{Q-1}), (i_Q, j_Q))$ of $\mathcal{P}_{\text{tc}}$ with $(i_1, j_1) = (0, M)$ and $(i_Q, j_Q) = (N-1, 0)$. Suppose that a time-consistent matching $C(U)$ has a non empty intersection with $E(P)$, for some feasible U of $MILP_{SMEPC}$. Let $(i_{k_0}, j_{k_0}), 1 \leq k_0 \leq Q$, be the first edge of $E(P)$ belonging to $C(U)$. Suppose that there is an other edge $(i_{k_1}, j_{k_1}), 1 \leq k_0 < k_1 \leq Q$ in $C(U)$. This implies that $i_{k_0} \neq i_{k_1}$ and $j_{k_0} > j_{k_1}$ since $C(U)$ is a matching. Moreover, as $C(U)$ is also non-crossing, by construction of $A^+(H_{\text{tc}})$, we have $i_{k_0} > i_{k_1}$. Thus, the two nodes are in P and, by Lemma 5.4, the arc $a = ((i_{k_0}, j_{k_0}), (i_{k_1}, j_{k_1}))$ belongs to $A^+(H_{\text{tc}})$. Therefore, the edges (i_{k_0}, j_{k_0}) and (i_{k_1}, j_{k_1}) are time-inconsistent, a contradiction. □

In the same way as Section 5.3, it can be seen that the polytope Q_{t_c}, defined by inequalities (56) and (58), for any path $P \in \mathcal{P}_{\text{tc}}$, describes the convex hull of characteristic vectors of time-consistent matchings, when (SA) property is satisfied.

For $0 \leq j < j' \leq M$, let $\tilde{w}(j, j') = \sum_{k=j+1}^{j'-1} t_k, j' > j+1$ and 0 otherwise. It is obvious that the values $\tilde{w}(j, j')$ satisfy the (SA) property and allow a time-consistent constraint generation, which is tested in the experimental part.

6 Computational results

6.1 Computation environment

The experiments are performed on a computer with AMD EPYC 7H12 64-Core processor, and are running under Gnu/linux Ubuntu 20.04.2. CPLEX 12.10 is used in a single-thread mode with all its parameters set to their default values. The maximum CPU time is fixed to 1 hour.

6.2 Description of instances

All series of experiments concern the instances presented in Table 1 with the following notations:

- num : the number of the instance,
- M: number of stations,
- N: number of production periods,
- p: duration of one production period,
- H_0: initial micro-plant hydrogen load,

- C^{MP}: micro-plant tank capacity,
- E_0: initial vehicle hydrogen load,
- C^{Veh}: vehicle tank capacity,
- LTour : duration of the tour without refueling,
- ETour : consumed volume without refueling.

The instances are generated as follows:

- N, M and p are fixed.
- The stations, $Depot$ and the Micro-Plant are randomly generated as points of a square in $\mathbb{R}^2$.
- d_j, d_j^* and t_j, e_j, ϵ_j, ϵ_j^*, respectively correspond to Euclidean distance and Manhattan distance rounded, in such a way they take integral values.
- C^{MP}, C^{Veh} and $TMax$ are fixed in such a way the existence of a feasible solution is ensured.
- Finally, the activation cost $Cost^F$ is set to 20. Values of production costs $Cost_i^V$ $i = 0, \ldots, N-1$, are independently uniformly generated in $[\![1, 10]\!]$.

Table 1: Instances

num	M	N	p	H_0	C^{MP}	E_0	C^{Veh}	LTour	Etour
1	8	20	4	6	25	8	12	20	20
2	8	25	4	5	20	8	10	25	26
3	8	40	5	10	70	20	30	44	50
4	10	36	2	8	25	9	12	36	38
5	10	50	4	6	40	10	20	50	54
6	10	94	1	3	30	4	15	47	52
7	12	32	4	0	50	4	25	63	68
8	12	50	4	3	36	3	18	50	58
9	15	160	4	8	240	20	120	426	556
10	20	108	10	20	390	10	190	716	894
11	10	80	2	10	40	10	20	38	38
12	15	327	4	20	400	11	140	653	790
13	20	180	6	3	500	20	170	718	834
14	20	440	5	100	350	50	170	910	1164
15	30	177	8	6	420	10	200	944	1188
16	30	260	6	50	400	50	200	1034	1320
17	30	544	4	10	560	15	250	1088	1382
18	50	265	10	6	400	9	200	1764	2216
19	50	500	8	100	420	50	180	1565	2000
20	70	328	10	15	550	25	270	2183	2766
21	30	1100	2	100	400	80	200	1097	1342
22	30	1200	2	100	400	70	200	957	1200
23	50	634	6	50	300	50	150	1542	1938
24	50	850	4	20	400	50	180	1941	2500
25	50	1125	4	20	700	50	280	1718	2128
26	70	383	8	20	760	15	240	2039	2590
27	70	683	8	8	850	14	370	2185	2826
28	70	984	6	50	750	40	250	2340	2926
29	100	651	8	100	600	60	200	3471	4360
30	100	800	10	100	950	100	430	3095	3796

6.3 Results for the linear relaxation $RMILP_{SMEPC}$

Our first experiments concern the linear relaxation $RMILP_{SMEPC}$. We study the contribution of valid inequalities given in Section 4. In each case, the optimal objective value, the CPU time and the gap to the optimal solution of the $MILP_{SMEPC}$, if it exists or to the best bound, otherwise, are given. Table 2 presents the results for:

- (LR): $RMILP_{SMEPC}$, the linear relaxation alone.
- (LR)+(STC): $RMILP_{SMEPC}$, when only the simple time constraints are added.
- (EC1+EC2): incorporate the energy constraints EC1 and EC2 to (LR)+(STC).
- (EC3): (LR)+(STC) with energy constraints EC3.

- All energy: (LR)+(STC) with energy constraints EC1, EC2 and EC3.

For each case, we give the following results are specified

- TT : total CPU time in seconds.
- obj : optimal value of the linear program.
- $GapF$: percentage of the relative error between the best feasible solution (optimal solution if the problem has been solved to optimality) and the optimal objective value of the linear relaxation. The instances indicated with ”**” are those whose CPU time has exceeded 1h.

Table 2: The linear relaxation $RMILP_{SMEPC}$ and the effects of simple time constraints

	LR			LR + STC		
num	**obj**	**TT**	**GapF**	**obj**	**TT**	**GapF**
1	44.15	0.0	66.29	68.18	0.0	47.95
2	36.94	0.0	75.53	64.89	0.1	57.02
3	30.93	0.1	78.52	79.51	0.1	44.79
4	45.34	0.0	67.62	82.52	0.0	41.06
5	46.42	0.1	71.17	97.27	0.0	39.58
6	41.26	0.1	76.82	97.79	0.1	45.06
7	52.00	0.1	76.57	124.84	0.0	43.77
8	39.97	0.0	79.18	96.43	0.0	49.78
9	80.16	0.1	87.55	523.61	0.1	18.69
10	59.41	0.1	94.78	883.52	0.1	22.43
11	27.60	0.1	79.40	78.80	0.0	41.19
12	37.90	0.2	95.84	717.05	0.4	21.38
13	33.34	0.1	96.51	794.71	0.2	16.87
14	69.60	0.4	94.93	1049.27	1.3	23.52
15	68.84	0.2	94.89	1114.54	0.3	17.20
16	62.16	0.3	95.19	1120.57	0.4	13.20
17	49.10	0.5	*96.18*	1155.17	2.4	10.10
18	125.35	0.6	*94.72*	1942.44	0.8	18.14
19	137.60	1.4	*93.84*	1826.41	5.0	18.17
20	106.04	1.0	*96.02*	2311.54	1.5	13.30
21	70.42	3.3	*95.01*	1240.39	10.6	12.15
22	60.27	2.7	*95.51*	1106.55	7.0	17.54
23	134.63	2.2	*94.80*	1910.07	7.6	26.28
24	135.81	3.2	*95.18*	2268.60	10.8	19.55
25	75.16	7.3	*96.35*	1864.20	17.3	9.37
26	15.78	2.8	*99.33*	2088.61	4.0	1 1.95
27	76.73	2.9	*96.94*	2268.94	7.7	9.50
28	90.08	7.3	*98.02*	2661.38	21.2	41.43
29	175.85	6.2	**	3759.44	13.9	**
30	72.49	3.4	98.62	3191.14	8.4	39.07

Table 3: The linear relaxation $RMILP_{SMEPC}$ and the effects of energy constraints

	LR + STC			EC1+EC2			EC3			All Energy		
num	**obj**	**TT**	**GapF**	**obj**	**TT**	**GapF**	**obj**	**TT**	**GapF**	**obj**	**TT**	**GapF**
1	68.18	0.0	47.95	68.18	0.1	47.95	79.5258	0.1	39.29	81.77	0.0	37.58
2	64.89	0.1	57.02	77.25	0.0	48.84	73.6146	0.1	51.25	87.58	0.1	42.00
3	79.51	0.1	44.79	85.18	0.1	40.85	80.7294	0.1	43.94	86.48	0.1	39.94
4	82.52	0.0	41.06	95.13	0.0	32.05	89.1597	0.1	36.31	103.15	0.1	26.32
5	97.27	0.0	39.58	102.97	0.1	36.04	102.194	0.1	36.53	108.52	0.1	32.60
6	97.79	0.1	45.06	117.31	0.1	34.09	115.567	0.1	35.07	134.76	0.1	24.29
7	124.84	0.0	43.77	136.46	0.0	38.53	142.594	0.1	35.77	155.67	0.1	29.88
8	96.43	0.0	49.78	109.33	0.1	43.06	110.698	0.1	42.34	125.99	0.1	34.38
9	523.61	0.1	18.69	536.21	0.1	16.74	570.346	0.2	11.44	584.29	0.3	9.27
10	883.52	0.1	22.43	909.99	0.1	20.11	963.463	0.2	15.41	984.70	0.3	13.55
11	78.80	0.0	41.19	88.46	0.0	33.99	91.3411	0.1	31.84	102.31	0.1	23.65
12	717.05	0.4	21.38	732.63	0.2	19.67	830.481	0.9	8.94	846.13	0.7	7.22
13	794.71	0.2	16.87	807.91	0.2	15.49	854.829	0.4	10.58	863.65	0.2	9.66
14	1049.27	1.3	23.52	1073.81	0.5	21.73	1218.37	2.8	11.20	1246.31	1.8	9.16
15	1114.54	0.3	17.20	1135.81	0.2	15.62	1183.52	0.4	12.07	1200.46	0.5	10.81
16	1120.57	0.4	13.20	1149.03	0.4	11.00	1160.43	1.7	10.11	1189.91	1.7	7.83
17	1155.17	2.4	10.10	1172.22	1.2	8.78	1196.6	3.9	6.88	1209.30	7.0	5.89
18	1942.44	0.8	18.14	1995.63	0.8	15.90	2059.16	1.1	13.23	2110.38	1.7	11.07
19	1826.41	5.0	18.17	1856.81	3.7	16.81	1902.51	11.0	14.76	1932.25	21.6	13.43
20	2311.54	1.5	13.30	2351.12	0.9	11.81	2342.78	7.1	12.12	2382.36	16.3	10.64
21	1240.39	10.6	12.15	1263.48	2.3	10.52	1316.83	18.7	6.74	1340.30	7.6	5.08
22	1106.55	7.0	17.54	1129.40	4.4	15.84	1228.15	20.0	8.48	1252.20	34.8	6.69
23	1910.07	7.6	26.28	1971.99	4.5	23.89	2316.46	8.7	10.60	2387.33	12.3	7.86
24	2268.60	10.8	19.55	2325.79	2.4	17.53	2561.57	16.7	9.16	2621.32	14.1	7.05
25	1864.20	17.3	9.37	1876.85	3.9	8.76	1944.81	42.0	5.45	1956.51	83.2	4.89
26	2088.61	4.0	11.95	2151.98	1.0	9.28	2115.79	8.3	10.80	2176.70	8.3	8.23
27	2268.94	7.7	9.50	2289.61	6.2	8.67	2292.71	17.3	8.55	2309.62	27.9	7.87
28	2661.38	21.2	41.43	2685.29	15.3	40.90	2815.69	52.5	38.03	2839.81	119.2	37.50
29	3759.44	13.9	**	3813.40	12.1	**	3914.15	30.8	**	3970.15	101.5	**
30	3191.14	8.4	39.07	3224.40	7.6	38.43	3233.22	8.0	38.26	3265.79	64.1	37.64

6.4 Results for Mixed integer linear programming $MILP_{SMEPC}$

- TT: total CPU time in seconds.
- LB and UB: Lower and Upper bounds of the mixed integer linear program respectively. The upper bound is the best integer feasible solution.
- $GapMI$: percentage of the relative error between the lower bound and the upper bound. The instances indicated with "**" are those whose CPU time has exceeded 1h and no upper bound was obtained after this time. The instances that could not be solved to optimality but for which an upper bound is reached within 1h have their gap indicated in italic.

Table 4: Cplex +STC, time limit = 1h

	STC				
num	**LB**	**UB**	**nodes**	**TT**	**GapMI**
1	131	131	1123	1.0	0.00
2	151	151	2701	2.3	0.00
3	144	144	1699	2.3	0.00
4	140	140	3896	3.9	0.00
5	161	161	1252	1.7	0.00
6	178	178	20527	33.8	0.00
7	222	222	4683	3.5	0.00
8	192	192	12263	25.5	0.00
9	644	644	7693	17.7	0.00
10	1139	1139	46518	37.7	0.00
11	134	134	5870	7.6	0.00
12	912	912	97686	458.2	0.00
13	956	956	24101	48.6	0.00
14	1372	1372	487725	2629.5	0.00
15	1346	1346	678590	1188.2	0.00
16	1291	1291	620432	1620.9	0.00
17	1246.49	1285	242500	3600.1	*3.00*
18	2208.43	2456	198780	3600.1	*10.08*
19	2038.48	2275	38496	3600.1	*10.40*
20	2416.31	2666	165044	3600.1	*9.37*
21	1353.44	1424	81490	3600.2	*4.96*
22	1253.15	1363	28400	3600.3	*8.06*
23	2237.48	**	30474	3600.1	**
24	2572.50	**	33718	3600.2	**
25	1946.27	**	6269	3600.2	**
26	2175.70	2458	43231	3600.1	*11.48*
27	2335.89	2507	23987	3600.2	*6.83*
28	2768.95	4544	5681	3600.5	*39.06*
29	3941.23	**	18958	3600.3	**
30	3269.36	**	2726	3600.3	**

Table 5: Energy constraints

	EC1, EC2				EC3				All energy			
num	**LB**	**UB**	**TT**	**gapMI**	**LB**	**UB**	**TT**	**gapMI**	**LB**	**UB**	**TT**	**gapMI**
1	131	131	0.5	0	131	131	0.5	0	131	131	0.6	0
2	151	151	1.3	0	151	151	1.9	0	151	151	1.3	0
3	144	144	1.4	0	144	144	1.6	0	144	144	1.3	0
4	140	140	5.8	0	140	140	2.2	0	140	140	2.3	0
5	161	161	2.4	0	161	161	2.3	0	161	161	1.4	0
6	178	178	29.8	0	178	178	64.2	0	178	178	39.9	0
7	222	222	1.7	0	222	222	0.8	0	222	222	0.9	0
8	192	192	33.5	0	192	192	29.4	0	192	192	21.6	0
9	644	644	26.2	0	644	644	63.1	0	644	644	37.4	0
10	1139	1139	26.9	0	1139	1139	35.7	0	1139	1139	8.7	0
11	134	134	7.1	0	134	134	6.7	0	134	134	4.9	0
12	912	912	326.7	0	912	912	259.2	0	912	912	231.4	0
13	956	956	58.1	0	956	956	64.1	0	956	956	32.2	0
14	1372	1372	192.1	0	1372	1372	1686.1	0	1372	1372	885.8	0
15	1346	1346	1321.3	0	1346	1346	1675.0	0	1346	1346	677.5	0
16	1291	1291	2543.8	0	1291	1291	3451.9	0	1291	1291	869.8	0
17	1250.30	1288	3600.1	*2.93*	1267.42	1285	3600.1	*1.37*	1272.09	1288	3600.1	*1.24*
18	2252.20	2408	3600.1	*6.47*	2317.26	2382	3600.1	*2.72*	2322.5	2373	3600.1	*2.13*
19	2044.24	2374	3600.2	*13.89*	2083.45	2232	3600.1	*6.66*	2105.92	2247	3600.3	*6.28*
20	2446.70	2993	3600.2	*18.25*	2433.72	2713	3600.1	*10.29*	2462.32	2780	3600.2	*11.43*
21	1364.40	1412	3600.3	*3.37*	1363.56	1412	3600.2	*3.43*	1377.03	**	3600.2	**
22	1288.58	1352	3600.2	*4.69*	1297.89	1342	3600.2	*3.29*	1310.74	**	3600.3	**
23	2417.76	2648	3600.2	*8.69*	2486.83	2591	3600.2	*4.02*	2502.81	**	3600.2	**
24	2635.35	3132	3600.3	*15.86*	2682.25	2820	3600.2	*4.88*	2723.43	2861	3600.3	*4.81*
25	1975.89	2131	3600.4	*7.28*	2010.29	2061	3600.3	*2.46*	2020.03	2057	3600.4	*1.80*
26	2245.61	2396	3600.2	*6.28*	2250.90	2372	3600.1	*5.11*	2299.79	2380	3600.2	*3.37*
27	2349.76	**	3600.4	**	2349.96	2531	3600.3	*7.15*	2349.6	**	3600.4	**
28	2812.58	**	3600.5	**	2846.28	**	3600.3	**	2870.28	**	3600.6	**
29	3953.29	**	3600.5	**	4044.89	**	3600.3	**	4113.51	**	3600.5	**
30	3291.77	**	3600.8	**	3287.42	5237	3600.4	*37.23*	3311.99	**	3600.8	**

Preliminary trials were done with the basic formulation of Section 4 only. Its weakness rapidly appeared since CPLEX could not find a feasible solution to the last fifteen instances of Table 1 after one hour of CPU time. We then observe that with (STC) constraints, CPLEX can reach a feasible solution on half of these data sets. This explains why the reference formulation will incorporate (STC) constraints.

Let us examine before the linear relaxation. Tests are made with different sets of constraints to explore their efficiency in solving these instances. In Table 2, the first column confirms the deficiency of the initial formulation due to the large gap with respect to the optimum integer solution. The second column shows us that the objective value of $RMILP_{SMEPC}$ is less than 50% of the optimum for most of 50% of the instances with regard to the reference formulation. The gap to the best upper bound for the difficult instances 17 to 30 even drops to 20%. Table 3 shows that these results are improved by adding the energy constraints. Specially, the three sets

(EC1), (EC2) and (EC3) of energy constraints put all together, give the best bounds in return for a moderate running time increase.

In the integer resolution framework of the reference model, Tables 4 and 5 show that sixteen instances out of the thirty tested are solved to optimality. These data sets can be considered as *easy* instances. The different sets of inequalities are successively put at stake for testing their contributions. We can note from Table 5, that (EC1) and (EC2) energy constraints fail on only four instances whereas the (EC3) constraints alone fail on two instances. Surprisingly, other constraints that seemed promising such as (44) and the antagonistic constraints, have no effect in improving the gap. A Branch-and-Cut procedure has been tested where the polynomial time separation algorithms proposed in Section 5 have been implemented. But the results indicate that neither antagonistic constraints nor time-consistent inequalities even violated at 0.1 have no impact on the objective value. The combinatorial nature of these inequalities does not help to solve the mixed integer linear program.

7 Conclusion

A mixed integer linear programming formulation for the Synchronized Management Energy Production and Consumption problem ($SMEPC$) is given and several families of valid inequalities are proposed. The experimental tests show the necessity of adding some of these constraints to enhance the basic model.

However, we report that structural constraints do not allow us to obtain an efficient Branch-and-Cut algorithm without knowing the reason of this failure.

The future work consists in introducing the optimization of vehicle tour and integrating several vehicles.

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