EPC Linear Model.

Let us recall the input data for the **EPC** problem:

Vehicle related input
M: number of stations (*Depot* excluded)
$\Gamma = (Depot = 0, 1, ..., M, Depot = M + 1)$: vehicle tour (without refueling)
TMax: maximal time for the vehicle to achieve its tour
C^{Veh}: vehicle tank capacity
E_0: initial vehicle hydrogen load
For $j = 0,\ldots, M$, t_j: required time to go from station j to station $j + 1$
For $j = 0,\ldots, M$, d_j: required time to go from station j to the micro-plant
For $j = 0,\ldots, M$, d^*_j: required time to go from the micro-plant to station j
For $j = 0,\ldots, M$, e_j: required energy to go from station j to station $j + 1$
For $j = 0,\ldots, M$, ε_j: required energy to go from station j to the micro-plant
For $j = 0,\ldots, M$, ε^*_j: required energy to go from the micro-plant to station j

Micro-plant production related input
C^{MP}: micro-plant tank capacity
N: number of production periods
p: duration (in time units) of one production period
H_0: initial micro-plant hydrogen load
$Cost^F$: activation cost
For $i = 0,\ldots, N - 1$, $P_i = [p.i, p.(i + 1)[$: time interval related to production period i
For $i = 0,\ldots, N - 1$, R_i: production rate related to period i
For $i = 0,\ldots, N - 1$, $Cost^V_i$: production cost related to period i

Table 1. *Input data for the EPC problem*

I. An Integrated Mathematical Programming (MP) Model

MP is not well-fitted to **EPC**. Still, we may use it in order to formulate our problem in an unambiguous way, based upon 3 main variables:

- **Production variables**:
  - $z = (z_i, i = -1,\ldots, N - 1)$, with $\{0, 1\}$ values: $z_i = 1$ ~ the micro-plant is active during period i ($i = -1$ corresponds to a fictitious period);
  - $y = (y_i, i = 0,\ldots, N - 1)$, with $\{0, 1\}$ values: $y_i = 1$ ~ the micro-plant is activated at the beginning of i;
  - $V^{Tank} = (V^{Tank}_i, i = 0,\ldots, N)$, with non negative integer values: V^{Tank}_i is the hydrogen load of the *micro-plant* tank at the beginning of period i; We involve here a fictitious period N in order to express the fact that the load of micro-plant tank at the end of the process should be at least equal to H_0;
  - $\delta = (\delta_i, i = 0,\ldots, N - 1)$, with $\{0, 1\}$ values: $\delta_i = 1$ ~ the vehicle is refueling during period i;
  - $L^* = (L^*_i, i = 0,\ldots, N - 1)$, with non negative integer values: in case $\delta_i = 1$, L^*_i is the quantity of hydrogen loaded by the vehicle during period i; else, L^*_i may take any non negative value.

- **Vehicle variables**:
  - $x = (x_j, j = 0,\dots, M)$, with $\{0, 1\}$ values: $x_j = 1$ ~ the vehicle refuels while traveling from station j to station $j + 1$;
  - $L = (L_j, j = 0,\dots, M)$, with non negative integer values: if $x_j = 1$, L_j = hydrogen quantity loaded by the vehicle while traveling from j to $j + 1$; else it may take any non negative value;
  - $T = (T_j, j = 0,\dots, M + 1)$, with non negative integer values: T_j = time when the vehicle arrives at j;
  - $T^* = (T^*_j, j = 0,\dots, M + 1)$, with non negative integer values: if $x_j = 1$, T^*_j = time when the vehicle starts refueling while traveling from j to $j + 1$; else it may take any non negative value;
  - $V^{Veh} = (V^{Veh}_j, j = 0,\dots, M + 1)$, with non negative integer values: V^{Veh}_j = hydrogen load of the vehicle tank when the vehicle arrives in j.

- **Synchronization variables**: $U = (U_{i,j}, i = 0,\dots, N - 1, j = 0,..., M)$ with $\{0, 1\}$ values: $U_{i,j} = 1$ ~ the vehicle is going to refuel during period i while traveling from j to $j + 1$.

Constraints come as follows (for a better understanding, we use here a logical formulation, easy to linearize through *Big M* technique):

- **Objective function**: Minimize
  $\Sigma_{i = 0,\dots,N-1} (Cost^F.y_i + Cost^V_i.z_i) + \alpha.T_{M+1}$.

- **Production constraints**:
  - For any $i = 0,\dots, N - 1$: $y_i = 1 \rightarrow (z_i = 1 \wedge z_{i-1} = 0)$;
  - For any $i = 0,\dots, N - 1$: $z_i + \delta_i \leq 1$;
  - $z_{-1} = 0$;
  - $V^{Tank}_0 = H_0$; $V^{Tank}_N \geq H_0$;
  - For any $i = 0,\dots, N - 1$: $V^{Tank}_i \leq C^{MP}$;
  - For any $i = 0,\dots, N - 1$: $V^{Tank}_{i+1} = V^{Tank}_i + z_i.R_i - \delta_i.L^*_i$.

- **Vehicle Constraints**:
  - $T_0 = 0$; $V^{Veh}_0 = E_0$; $V^{Veh}_{M+1} \geq E_0$;
  - For any $j = 1,\dots, M + 1$: $V^{Veh}_j \leq C^{Veh}$;
  - For any $j = 0,\dots, M$: $V^{Veh}_j \geq \varepsilon_j$; (E1)
    (E1) means that at any time, the vehicle must be able to go to the *micro-plant* and refuel, and relies on the *Triangle Inequality* for energy coefficients e_j and ε_j;
  - For any $i = 0,\dots, M$: $L_j \leq C^{Veh} + \varepsilon_j - V^{Veh}_j$; (E2)
    (E2) expresses the fact that the vehicle cannot refuel more than the space which remains inside its tank;
  - For any $j = 0,\dots, M$: $T_{j+1} \geq (1 - x_j). (T_j + t_j) + x_j.(T^*_j + p + d^*_{j+1})$; (E3)
  - For any $j = 0,\dots, M$: $T^*_j \geq T_j + d_j$; (E4)
  - For any $j = 0,\dots, M$: $x_j = 0 \rightarrow V^{Veh}_{j+1} = V^{Veh}_j - e_j$;
  - For any $j = 0,\dots, M$: $x_j = 1 \rightarrow V^{Veh}_{j+1} = V^{Veh}_j - \varepsilon_j - \varepsilon^*_{j+1} + L_j$;
  - $T_{M+1} \leq TMax$.

- **Synchronization constraints**:
  - For any $j = 0,\dots, M$: $\Sigma_{i = 0,\dots, N-1} U_{i,j} = x_j$; (E5)
  - For any $i = 0,\dots, N - 1$, $\delta_i = \Sigma_{j = 0, \dots, M} U_{i,j}$; (E6)

- For any $j = 0,\ldots, M$, $x_j = 1 \rightarrow T^*_j = \Sigma_{i=0,\ldots,N-1}\, p.i.U_{i,j}$; (E7)
- For any $i = 0,\ldots, N-1$: $L^*_i \leq V^{Tank}_i$; (E8)

  (E8) expresses that load L^*_i cannot exceed the current load of the micro-plant tank;
- For any $j = 0, \ldots, M$: $L_j = \Sigma_{i=0,\ldots,N-1}\, U_{i,j}.L^*_i$. (E9)

We may state:

Theorem 1: *Solving above* **MP_EPC** *model also solves our* **EPC** *problem*.

Proof: Checking that a feasible solution (y, x, T, L) of **EPC** can be turned into a feasible solution of above linear model with the same cost comes in a straightforward way. We only need to follow the trajectory induced by (y, δ, x, L) and compute z, L^*, T, T^*, V^{Tank}, V^{Tank}, U, accordingly.

Conversely, let us consider some feasible solution (y, δ, x, T, L, z, L^*, T^*, V^{Tank}, V^{Tank}, U) of above linear model. The key point is that vector U defines a matching $i \to j(i)$ between $I° = \{i \in 0,\ldots, N\text{-}1$, such that $\delta_i = 1\}$ and $J° = \{j \in 0,\ldots, M$, such that $x_j = 1\}$ and that this matching is consistent with standard linear ordering: if $i_1 < i_2$ then $j(i_1) < j(i_2)$. The first point is contained into equations (E5, E6).The second point derives from equations (E7), which fixes values $T^*_{j(i)}$ and inegalities (E3, E4): if i_1, i_2 are consecutive in $I°$ and such that $j(i_1) > j(i_2)$, then we get, by propagating (E3, E4), $T^*_{j1} \geq T^*_{j2}$ and a contradiction with (E7).

It comes that, if (y, δ, x, T, L, z, L^*, T^*, V^{Tank}, V^{Tank}, U) is optimal, we see that (E3, E4) are going to give rise to equalities, which means that T and T^* are going to follow the EPC trajectory induced by (y, δ, x, L). But we also see that related load $L_{j(i)} = L^*_i$ (because of (E9)) are going to be feasible in the sense that they should exceed neither the load of the micro-plant tank at the beginning of period i, nor the difference between the capacity of vehicle tank and its current load when its arrive to the micro-plant, while moving from j to j+1, because (E2) and (E8)). We conclude that our solution (y, δ, x, T, L, z, L^*, T^*, V^{Tank}, V^{Tank}, U) may be interpreted as an **EPC** trajectory, with the same value. □

Let us pay attention now to the linearization **Linear-EPC**, through *Big M* techniques, of above **MP_EPC** model, and its rational relaxation. Let us suppose that we reformulate any implications:

- $X = 0 \to Y \geq 0$
- $X = 1 \to Y \leq 0$

as:

- $X + Y/Big\_M \geq 0$
- $X + Y/Big\_M \leq 1$

where *Big_M* is a very large number.

Then we see that:

Proposition 1: *According to this hypothesis, the optimal value of the rational relaxation of Linear-EPC is null*.

Proof: It is enough to check that, if *Big_M* is choosen large enough, then we get a feasible solution of the rational relaxation of **Linear_EPC** by setting:

- $x_j = ½$ for every j; $\delta_i = (M+1)/2$N for any i;
- $U_{i,j} = 1/2N$ for any i, j;

- $z_i = y_i = 0$ for any i;
- $L_j = L^*_i = 0$ for any i;
- $V^{Tank}_i = H_0$ for any i; $V^{Veh}_j = E_0$ for any j;
- $L_j = 0$ and $T^*_j = d_j$ for any j.

This solution clearly yields a null value. □

Still, we may enhance the quality of such a relaxation by noticing that several additional constraints may be inserted to **MP_EPC**:

- For any j, $T_{j+1} \geq T_j + t_j$;
- $\Sigma_j L_j \geq \Sigma_j e_j$.

II. Additional Constraints.

For any j = 1,…, M, we set:

- $D_j = \Sigma_{k = 0,\dots, j-1} t_k + d_j$; $D_0 = d_0$;
- $D^*_j = \Sigma_{k = j+1,\dots, M} t_k + d^*_{j+1}$; $D^*_0 = \Sigma_{k = 1,\dots, M} t_k + d^*_1$; $D^*_M = d^*_{M+1}$
- $Min_j = \lceil D_j/p \rceil$; $Min_0 = \lceil D_0/p \rceil$;
- $Max_j = N - 1 - \lceil D^*_j/p \rceil$; $Max_0 = N - 1 - \lceil D^*_0/p \rceil$;
- For any pair j1, j2 j1 < j2, $\mu_{j1,j2} = \Sigma_{j1+1 \leq j < j2} e_j + \varepsilon^*_{j1+1} + \varepsilon_{j2}$.
- For any pair j1, j2, j1 < j2, $INT_{j1,j2} = \lceil (\Sigma_{j1+1 \leq j < j2} t_j + d^*_{j1+1} + d_{j2})/p \rceil$;

We say that 2 pairs (i1, j1) an (i2, j2) are *antagonistic* iff i1 < i2 and j1 > j2. A collection Λ of pairwise *antagonistic* pairs (i, j) is called an *antagonistic clique*.
We say that 2 pairs (i1, j1) an (i2, j2) are *time-inconsistent* iff (i2 – i1) ≤ $INT_{j1,j2}$. A collection Λ of pairwise *time-inconsistent* pairs (i, j) is called an *time-inconsistent clique*.

II.1. Simple Time Constraints.

- For any j = 0, …, M: $T_{j+1} \geq T_j + t_j + x_j.(d_j + d^*_j - t_j)$;

II.2. Energy Constraints.

We introduce additional variable $F_j \geq 0$, j = 0,…, M+1= 0, with the meaning that F_j means the energy used by the vehicle from 0 to j. With $F_0 = 0$

- For any j = 0, …, M: $F_{j+1} \geq F_j + e_j + x_j.(\varepsilon_j + \varepsilon^*_{j+1} - e_j)$;
- For any j = 0,…, M: $F_j - E_0 \leq \Sigma_{k < j} L_k$;
- For any j = 1,…, M: $\Sigma_{0 \leq i \leq Max(j-1)-1} R_i.z_i \geq F_j - E_0 - H_0$;
- $\Sigma_{0 \leq i \leq N-1} R_i.z_i \geq F_{M+1}$;
- For any j = 1,…, M: $\Sigma_{0 \leq i \leq Max(j-1)-1} y_i \geq (F_j - E_0 - H_0)/C^{MP}$;
- $\Sigma_{0 \leq i \leq N-1} y_i \geq (F_{M+1})/C^{MP}$;
- For any j1, j2, K such that $\mu_{j1,j2} > C^{Veh} + K.C^{MP}$: $\Sigma_{Minj1 \leq i \leq Maxj2} y_i \geq K$;
- For any j1, j2 such that $\mu_{j1,j2} > C^{Veh}$: $\Sigma_{j1 \leq j \leq j2-1} x_j \geq 1$.

II.3. Structural Constraints.

- For any j, any i such that $i < Min_j$ or $i > Max_j$: $U_{i,j} = 0$;
- For any i, j1, j2: $U_{i,j1} + U_{i+1,j2} \leq 1$;
- For any *antagonistic clique* Λ: $\Sigma_{(i,j) \in \Lambda} U_{i,j} \leq 1$.
- For any *time-inconsistent clique* Λ: $\Sigma_{(i,j) \in \Lambda} U_{i,j} \leq 1$.

III. Separating the Structural Constraints.

III.1. Separating the Antagonistic Cliques.

***SEPARE_ANTAGO*:**
Input: Current vector U, rational.
Output: An antagonistic clique Λ, which violates the antagonistic clique constraint. In case Λ in undefined, then no such a clique exists.

```
j <- 0; i <- N-1;
While j ≤ M+1 do
    Π(i, j) < - 0;
    For j1 ≤ j do
        For i1 ≥ i do
            If (i1 ≠ i) OR (j1 ≠ j) then
                If U_{i,j} + Π(i1, j1) > Π(i, j) then
                    Π(i, j) <- U_{i,j} + Π(i1, j1);
                    Arg(i, j) <- (i1, j1).
    If i  ≥ 1 then i <- i – 1
    Else
        i <- N-1;
        j <- j+1;
If Π(0, M+1) > 1 then Λ <- Reconstruction through Arg(0, M+1) else Λ <- Undefined;
```

III.2. Separating the Time-Inconsistent Cliques.

Preliminary: *Weakening the time-inconsistency*.
Such as it has been defined, the time-inconsistency constraints seems difficult to separate. So we simplify them as follows:

- For any pair j1, j2, j1 < j2, we set $INT^*_{j1,j2} = \lfloor(\Sigma_{j1 \leq j < j2} t_j)/p\rfloor$.; If j1 = j2 then $INT^*_{j1,j2} = 0$.

So we separate the clique in the sense of INT* (instead of INT), while making the assumption that:

- $Inf_j\, t_j + Inf_j\, d_j + Inf_j\, d^*_j \leq p$.

In case this assumption is not satisfied, then our process works as an approximation.

Please also notice that constraint: *For any i, j1, j2:* $U_{i,j1} + U_{i+1,j2} \leq 1$, is a specific case of the time-inconsistency constraint.

***SEPARE_INCONSISTENT*:**

Input: Current vector U, rational.

Output: A time inconsistent clique Λ, which violates the time-inconsistency clique constraint. In case Λ in undefined, then no such a clique exists.

```
j <- 0; i <- 0;
While j ≤ M+1 do
    Π(i, j) < - 0;
    For j1 ≤ j do
        For i1 = i - (1+ INT*_{j1,j2}),…, i do
            If (i1 ≠ i) OR (j1 ≠ j) then
                If U_{i,j} + Π(i1, j1) > Π(i, j) then
                    Π(i, j) <- U_{i,j} + Π(i1, j1);
                    Arg(i, j) <- (i1, j1).
    If i ≤ N-2 then i <- i + 1
    Else
        i <- 0;
        j <- j+1;
If Π(N-1, M+1) > 1 then Λ <- Reconstruction through Arg(N-1, M+1) else Λ <- Undefined;
```