ODEs
- derivation of RK2 family
- RK4
- Bulirsch-Stoer (ODE equivalent of Romberg)
- Stiff equations / implicit methods

$$\frac{dy}{dx} = f(x, \vec{y}) \quad , \quad y(x_0 + h) = y(x_0) + h f(x_0, \vec{y})$$

integration to 2nd order

we want a solution accurate to 2nd order

$k_0 = hf(t, y)$

$k_1 = hf(t + \alpha h, y + \beta k_0)$

$y(t+h) = y(t) + a k_0 + b k_1$

$$y(t+h) = y(t) + h\frac{dy}{dt} + \frac{h^3}{2}\left(\frac{d^2y}{dt^2}\right) + O(h^3) \ldots$$

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dt}\right) \qquad \frac{dy}{dt} = f$$

$(a+b)hf = hf$, or $a+b=1$

$b\alpha h^2 = \frac{h^2}{2}$ or $\alpha b = \frac{1}{2}$

$$= \frac{d}{dt}(f) = \frac{\partial}{\partial t}(f) + \frac{\partial}{\partial y}(f)\frac{dy}{dt}$$

$\beta h^2 b = \frac{h^2}{2}$ or $\beta b = \frac{1}{2}$

$$= \frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y}$$

$$\boxed{\begin{aligned} a+b&=1\\ \alpha b&=\tfrac{1}{2}\\ \beta b&=\tfrac{1}{2}\end{aligned}}$$

$$y(t+h) = y(t) + hf + \frac{h^2}{2}\left(\frac{\partial f}{\partial t} + f\frac{\partial f}{\partial y}\right)$$

$$k_1 = hf(t+\alpha h, y+\beta hf)$$

$$\simeq h\left(f(t,y) + \alpha h^2\frac{\partial f}{\partial t} + \beta h^2 f\frac{\partial f}{\partial y}\right)$$

$$y(t+h) = y_0 + a\,hf + b\left(hf + \alpha h^2\frac{\partial f}{\partial t} + \beta h^2 f\frac{\partial f}{\partial y}\right)$$

$a+b=1$, $a=\frac{1}{2}$, $b=\frac{1}{2}$

then $\alpha b=\frac{1}{2} \Rightarrow \alpha=1$

$\beta b=\frac{1}{2} \Rightarrow \beta=1$

$k_0 = hf$

$k_1 = hf(t+h, y+hf)$

$y(t+h) = \frac{1}{2}(hf + hf(t+h, y+hf)) \Rightarrow$ 1 choice

other sensible options

$b=1, a=0$ $\alpha b=\frac{1}{2} \Rightarrow \alpha=\frac{1}{2}$

$\beta b=\frac{1}{2} \Rightarrow \beta=\frac{1}{2}$

$y(t+h) = \frac{1}{2}(hf(t+\frac{h}{2}, y+\frac{hf}{2})$

$y' = -cy$ write down 1st order solution

$y(t+h) \Rightarrow y_{n+1} = y_n - chy_n$

$= y_n(1-ch)$

$y_{n+2} = (1-ch)\,y_{n+1} = (1-ch)^2 y_n$

$y_{n+3} = (1-ch)^3 y_n$

$\Rightarrow y_n = (1-ch)^n y_0$

$1-ch = \frac{1}{2}$, $y_n = \frac{1}{2^n} y_0$

$ch = 3$ (pick h)

$y_n = (1-3)^n y_0$

$= (-2)^n y_0 \Rightarrow$ we want $e^{-n} y_0$

how could we fix? $\Rightarrow$ we may not be super accurate
but we don't want solution to explode

$$y_{n+1} = y_n - ch\,y_n$$

$$y_{n+1} = y_n - ch\,y_{n+1}$$

$$y_{n+1}(1+ch) = y_n$$

$$y_{n+1} = \frac{y_n}{1+ch}$$

$$y_n = \frac{y_0}{(1+ch)^n}$$

if $ch = 3$

then $y_n = \dfrac{y_0}{(4)^n}$ $\Rightarrow$ stable

$\Rightarrow$ implicit methods

ϑ_a decays into ϑ_b $\Rightarrow$ with rate τ_a

ϑ_b decays into ϑ_c $\Rightarrow$ with rate τ_b

$$\frac{d\vartheta_a}{dt} = -\tau_a \vartheta_a \qquad \Rightarrow \frac{d\vartheta}{dt} = -\tilde{C}\vartheta$$

$$\frac{d\vartheta_b}{dt} = \tau_a \vartheta_a - \tau_b \vartheta_b \qquad \begin{pmatrix} -\tau_a & 0 & 0 \\ \tau_a & -\tau_b & 0 \\ 0 & +\tau_b & 0 \end{pmatrix} \begin{pmatrix} \vartheta_a \\ \vartheta_b \\ \vartheta_c \end{pmatrix}$$

$$\frac{d\vartheta_c}{dt} = \tau_b \vartheta_b$$

$$\begin{pmatrix} -\tau_a & 0 & 0 \\ \tau_a & -\tau_b & 0 \\ 0 & \tau_b & 0 \end{pmatrix} \Rightarrow \vartheta(t) = V e^{-tS} V^{-1} \vartheta_0$$