Efficient Ray-Torus Intersection for Ray Tracing

Start by representing a torus as 2 vectors and 2 scalars as follows:

C – vector position of the center of the torus.
A – the axis of revolution of the torus – i.e. a unit vector pointing through the middle.
R – the major radius of the torus
r – the minor radius of the torus

And the parametric equation for a ray:

1) $P = P_0 + P_1 t$ where $P_0 = (x_0, y_0, z_0)$ $P_1 = (x_1, y_1, z_1)$

Any point along the ray can be represented in cylindrical coordinates relative to the torus. We are not interested in the angle, but only the position along the axis and the radius. The distance along the axis is given by:

2) $y = A \bullet (P_0 - C + P_1 t) = A \bullet (P_0 - C) + A \bullet P_1 t$ where $\bullet$ is the standard dot product

The radial position of the ray relative to the torus can be represented as a vector D by subtracting the component of P–C that lies along A.

3) $D = P_0 + P_1 t - (C + Ay) = P_0 - C + P_1 t - A \bullet (P_0 - C + P_1 t) A$

Let $Q = P_0 - C$
$u = A \bullet Q$
$v = A \bullet P_1$

This results in:

4) $y = u + vt$

4) $D = Q + P_1 t - A(u + vt)$
$= Q + P_1 t - uA - vtA$

The magnitude of D is the radial distance from the torus axis A, to the ray, so the radius squared is:

$$z^2 = D \bullet D = (Q + P_1 t - A(u + vt)) \bullet (Q + P_1 t - A(u + vt))$$

$$z^2 = Q \bullet Q + 2Q \bullet P_1 t - 2u(u + vt) + P_1 \bullet P_1 t^2 - 2v(u + vt)t + A \bullet A(u + vt)^2$$

$$z^2 = Q \bullet Q + 2Q \bullet P_1 t - 2u^2 - 2uvt + P_1 \bullet P_1 t^2 - 2uvt - 2v^2 t^2 + u^2 + 2uvt + v^2 t^2$$

$$z^2 = Q \bullet Q + 2Q \bullet P_1 t - u^2 + P_1 \bullet P_1 t^2 - 2uvt - 2v^2 t^2 + v^2 t^2$$

6) $z^2 = Q \bullet Q - u^2 + 2(Q \bullet P_1 - uv)t + (P_1 \bullet P_1 - v^2)t^2$

therefore $z^2 = at^2 + bt + c$ where:

7) $a = (P_1 \bullet P_1 - v^2)$ If P_1 has been normalized then $a = 1 - v^2$
8) $b = 2(Q \bullet P_1 - uv)$
9) $c = Q \bullet Q - u^2$

Many interesting surfaces of revolution can be defined in terms of y, y^2 and z^2. These include cylinders, paraboloids, and cones. In fact, we can use a bounding cylinder around the torus for easy rejection of most rays without resorting to solving a quartic equation. The equation of the bounding cylinder is:

$$at^2 + bt + c - (R+r)^2 = 0$$

There is no need to completely solve this equation, only to check if a solution exists. If there is no solution, the ray does not intersect the bounding cylinder and therefore doesn't intersect the torus either.

Another test is to solve $at^2 + bt + c - (R+r)^2 = 0$ for t and find the corresponding values of y. These are the y axis positions of where the ray enters and exits the bounding cylinder for the torus. Since the torus only exists for y values in the range $[-r, +r]$ we can test. If both of these roots are greater than r or both are less than $-r$ then the ray does not intersect the torus because the range of y values where the ray is inside the cylinder does not intersect the range where the torus exists.

When using torii that have a major radius much larger than the minor radius, further testing can be done using the cylinder of radius $(R-r)$. This would provide an even tighter bounding volume involving only second order surfaces.

When these simpler tests are unable to determine that there is no intersection, we must resort to solving for the intersection points. Use the following to define a locus of points on the torus:

10) $y^2 + z^2 - 2zR + R^2 = r^2$

Since we do not know z in terms of t we need to isolate the z term and square both sides so we are left with only a function of z^2.

11) $y^2 + z^2 + R^2 - r^2 = 2zR$

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9 $[y^2 + z^2 + (R^2 - r^2)]^2 = 4R^2z^2$

$$[(u + vt)^2 + at^2 + bt + c + (R^2 - r^2)]^2 = 4R^2(at^2 + bt + c)$$

$$[u^2 + 2uvt + v^2t^2 + at^2 + bt + c + (R^2 - r^2)]^2 = 4R^2(at^2 + bt + c)$$

$$[(a + v^2)t^2 + (b + 2uv)t + (c + u^2 + R^2 - r^2)]^2 = 4R^2(at^2 + bt + c)$$

This can be written as a 4th order polynomial in t with the following coefficients:

12)
$$A = (a + v^2)^2$$
$$B = 2(a + v^2)(b + 2uv)$$
$$C = 2(a + v^2)(c + u^2 + R^2 - r^2) + (b + 2uv)^2 - 4R^2a$$
$$D = 2(b + 2uv)(c + u^2 + R^2 - r^2) - 4R^2b$$
$$E = (c + u^2 + R^2 - r^2)^2 - 4R^2c$$

By substituting (7),(8),(9) this becomes simpler.

13)
$$A = (P_1 \bullet P_1)^2$$
$$B = 2(P_1 \bullet P_1)(2Q \bullet P_1)$$
$$C = 2(P_1 \bullet P_1)(Q \bullet Q + R^2 - r^2) + (2Q \bullet P_1)^2 - 4R^2a$$
$$D = 2(2Q \bullet P_1)(Q \bullet Q + R^2 - r^2) - 4R^2b$$
$$E = (Q \bullet Q + R^2 - r^2)^2 - 4R^2c$$

Assuming the ray direction vector P_1 has been normalized such that $P_1 \bullet P_1 = 1$ we get:

14) $A = 1$
$B = 4Q \bullet P_1$
$C = 2(Q \bullet Q + R^2 - r^2) + 4(Q \bullet P_1)^2 - 4R^2a$
$D = 4(Q \bullet Q + R^2 - r^2)(Q \bullet P_1) - 4R^2b$
$E = (Q \bullet Q + R^2 - r^2)^2 - 4R^2c$

Making the substitution: $d = (Q \bullet Q + R^2 - r^2)$ yeilds:

15) $A = 1$
$B = 4Q \bullet P_1$
$C = 2d + \frac{1}{4}B^2 - 4R^2a$
$D = Bd - 4R^2b$
$E = d^2 - 4R^2c$

Finding the normal vector at a point P on the torus

First find the cylindrical coordinate y,D of P relative to the torus.

$y = (P - C) \bullet A$
$D = (P - C) - yA$

$X = D(1/\mathrm{sqrt}(D \bullet D))R$

$N = P - C - X$

Using a more complex torus

A slightly more useful shape can be obtained by allowing the cross section of the torus to be an ellipse instead of just a circle. This involves a change to equation 10 to include another parameter.

16) $y^2/r_2^2 + (z^2 - 2zR + R^2)/r_1^2 = 1$

$y^2 + (z^2 - 2zR + R^2)r_2^2/r_1^2 = r_2^2$

Now let $e = r_2/r_1$

17) $y^2 + z^2e^2 + R^2e^2 - r_2^2 = 2zRe^2$

10 $[y^2 + z^2e^2 + (R^2e^2 - r_2^2)]^2 = 4R^2z^2e^4$

Let $e = r_1/r_2$

$[(u + vt)^2 + (at^2 + bt + c)e^2 + (R^2e^2 - r_2^2)]^2 = 4R^2(at^2 + bt + c)e^4$

Now redefine a,b,c as ae^2, be^2, ce^2

$[(u + vt)^2 + at^2 + bt + c + (R^2e^2 - r_2^2)]^2 = 4R^2(at^2 + bt + c)e^2$

$[u^2 + 2uvt + v^2t^2 + at^2 + bt + c + (R^2e^2 - r_2^2)]^2 = 4R^2(at^2 + bt + c)e^2$

$[(a + v^2)t^2 + (b + 2uv)t + (c + u^2 + R^2e^2 - r_2^2)]^2 = 4R^2(at^2 + bt + c)e^2$

Let $p = a + v^2$

$q = b + 2uv$
$r = c + u^2 + R^2e^2 - r_2^2$

to form:

$$[pt^2 + qt + r]^2 = 4R^2(at^2 + bt + c)e^2$$

This can be written as a 4^{th} order polynomial in t with the following coefficients:

12) $A = p^2$
$B = 2pq$
$C = 2pr + q^2 - 4R^2ae^2$
$D = 2qr - 4R^2be^2$
$E = r^2 - 4R^2ce^2$

Normal vector for the complex torus

First find the cylindrical coordinate y,D of P relative to the torus.

$y = (P - C)\bullet A$
$D = (P - C) - yA$

$x = \mathrm{sqrt}(D\bullet D)$

The torus equation in X and Y is:

$y^2/r_2^2 + (x - R)^2/r_1^2 = 1$

$y^2/r_2^2 + (x^2 - 2xR + R^2)/r_1^2 = 1$

$y^2(r_1^2/r_2^2) + x^2 - 2xR + R^2 = r_1^2$

The partials of this function with respect to x and y are:

$dx = 2(x - R)$

$dy = 2y(r_1^2/r_2^2)$

$N = 2(1/x)(x - R)D + 2y(r_1^2/r_2^2)A$

Or without the 2's

$N = (1/x)(x - R)D + y(r_1^2/r_2^2)A$