理论力学 Chapter 1

若 $\oint \vec{F}\cdot d\vec{s}=0$ 则 $\vec{F}$ 为保守力

总角动量 = 质心的角动量 + 各点相对质心的角动量

克尼希定理

总动能 = 质心的动能 + 各点相对质心的动能

若力是保守的 则 $T+V$ 守恒

完整约束 $f(q,t)=0$

稳定约束 $\frac{\partial f}{\partial t}=0$ (约束不随时间改变)

达朗贝尔公式

$$m_i\ddot{\vec{r}}_i=\vec{F}_i^{(e)}+\sum_j\vec{F}_{ji}$$

$$\sum_i(\vec{F}_i^{(e)}+\sum_j\vec{F}_{ji}-m_i\ddot{\vec{r}}_i)\delta\vec{r}_i=0$$

$$\sum_i(\vec{F}_i^{(e)}-m_i\ddot{\vec{r}}_i)\delta\vec{r}_i=0 \Rightarrow \sum_i(\vec{F}_i^{(e)}-\dot{\vec{p}}_i)\delta\vec{r}_i=0$$

平衡状态下 = 虚功原理 $\sum_i\vec{F}_i^{(e)}\delta\vec{r}_i=0$

$\delta t=0$ 求变分与求导操作相同

广义力 Q_j

$$\sum_i\vec{F}_i\delta\vec{r}_i=\sum_j\vec{F}_i\frac{\partial\vec{r}_i}{\partial q_j}\delta q_j$$

证明拉格朗日方程（由达朗贝尔原理导出）

$$\sum_i (\vec{F}_i^{(e)} - m_i\ddot{\vec{r}}_i)\delta\vec{r}_i = 0$$

$$\vec{r}_i = \vec{r}_i(q_1 \cdots q_s, t)$$

$$\therefore \delta\vec{r}_i = \sum_{\alpha=1}^{s} \frac{\partial \vec{r}_i}{\partial q_\alpha}\delta q_\alpha \qquad \dot{\vec{r}}_i = \sum_{\alpha=1}^{s}\frac{\partial \vec{r}_i}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial \vec{r}_i}{\partial t}$$

$$\therefore \vec{F}_i^{(e)}\delta\vec{r}_i = \sum_{\alpha=1}^{s} Q_\alpha \delta q_\alpha$$

而对于 $m_i\ddot{\vec{r}}_i\delta\vec{r}_i = m_i\ddot{\vec{r}}_i\sum_{\alpha=1}^{s}\frac{\partial \vec{r}_i}{\partial q_\alpha}\delta q_\alpha$

将求和与 δq_α 提出得

$$\sum_{\alpha=1}^{s}\left(Q_\alpha - m_i\ddot{\vec{r}}_i\frac{\partial \vec{r}_i}{\partial q_\alpha}\right)\delta q_\alpha = 0$$

而下面 $\frac{d}{dt}\left(m_i\dot{\vec{r}}_i\frac{\partial \vec{r}_i}{\partial q_\alpha}\right) = m_i\ddot{\vec{r}}_i\frac{\partial \vec{r}_i}{\partial q_\alpha} + m_i\dot{\vec{r}}_i\frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_\alpha}\right)$

由于 $\frac{\partial \vec{r}_i}{\partial q_\alpha}(q_1 \cdots q_s, t)$ $\quad\therefore \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_\alpha}\right) = \sum_{j=1}^{s}\frac{\partial \vec{r}_i}{\partial q_\alpha\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial q_\alpha\partial t}$

而 $\frac{\partial \dot{\vec{r}}_i}{\partial q_\alpha} = \sum_{j=1}^{s}\frac{\partial^2 \vec{r}_i}{\partial q_j\partial q_\alpha}\dot{q}_j + \frac{\partial^2 \vec{r}_i}{\partial t\partial q_\alpha} = \frac{d}{dt}\left(\frac{\partial \vec{r}_i}{\partial q_\alpha}\right)$

故括号中间 $= Q_\alpha - \frac{d}{dt}\left(m_i\dot{\vec{r}}_i\frac{\partial \vec{r}_i}{\partial q_\alpha}\right) + m_i\dot{\vec{r}}_i\frac{\partial \dot{\vec{r}}_i}{\partial q_\alpha} = 0$

由于 $\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_\alpha} = \sum_{j=1}^{s}\frac{\partial \vec{r}_i}{\partial q_j}\frac{\partial}{\partial \dot{q}_\alpha}\dot{q}_j = \frac{\partial \vec{r}_i}{\partial q_j}\delta_{\alpha j} = \frac{\partial \vec{r}_i}{\partial q_\alpha}$

$\therefore$ 上式 $= Q_\alpha - \frac{d}{dt}\left(m_i\dot{\vec{r}}_i\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_\alpha}\right) + m_i\dot{\vec{r}}_i\frac{\partial \dot{\vec{r}}_i}{\partial q_\alpha}$

而 $T = \frac{1}{2}m_i\dot{\vec{r}}_i\cdot\dot{\vec{r}}_i$

$$\therefore \frac{\partial T}{\partial \dot{q}_\alpha} = m_i\dot{r}_i\frac{\partial \dot{r}_i}{\partial \dot{q}_\alpha}, \quad \frac{\partial T}{\partial q_\alpha} = m_i\dot{\vec{r}}_i\frac{\partial \dot{\vec{r}}_i}{\partial q_\alpha}$$

故原式 $\Rightarrow Q_\alpha - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_\alpha}\right) + \frac{\partial T}{\partial q_\alpha} = 0$

保守力 $Q_\alpha = -\frac{\partial V}{\partial q_\alpha}$

∴原式 ⇒ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_\alpha}\right) - \frac{\partial L}{\partial q_\alpha} = 0$ 其中 $L = T - V$ (不显含速度)

$L' = L + \frac{dF(q,t)}{dt}$ 也满足拉格朗日方程

证明

$$\frac{d}{dt}\left(\frac{\partial L'}{\partial \dot q_\alpha}\right) - \frac{\partial L'}{\partial q_\alpha} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot q_\alpha} + \frac{\partial}{\partial \dot q_\alpha}\left(\frac{dF}{dt}\right)\right) - \frac{\partial L}{\partial q_\alpha} - \frac{\partial}{\partial q_\alpha}\left(\frac{dF}{dt}\right) = 0$$

即 $\frac{dF}{dt} = \sum_i^s \frac{\partial F}{\partial q_i}\dot q_i + \frac{\partial F}{\partial t}$

∴ $\left(\frac{dF}{dt}\right)\frac{\partial}{\partial \dot q_\alpha} = \frac{\partial F}{\partial q_\alpha}$

∴ $\frac{d}{dt}\left(\frac{\partial F}{\partial q_\alpha}\right) = \sum_i^s \frac{\partial^2 F}{\partial q_\alpha \partial q_i}\dot q_i + \frac{\partial^2 F}{\partial t \partial q_\alpha}$ } 相等

而 $\frac{\partial}{\partial q_\alpha}\left(\frac{dF}{dt}\right) = \sum_i^s \frac{\partial^2 F}{\partial q_i \partial q_\alpha}\dot q_i + \frac{\partial^2 F}{\partial t \partial q_\alpha}$

故也满足拉格朗日方程

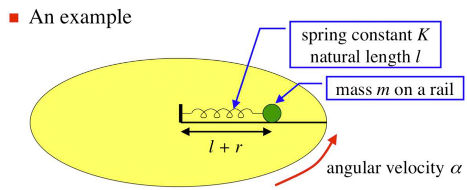
电磁场 $\vec F = q[\vec E + (\vec v \times \vec B)]$

$U = q\phi - q\vec A\cdot\vec v$

∴ $L = \frac{1}{2}mv^2 - q\phi + q\vec A\cdot\vec v$

$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot q_\alpha}\right) - \frac{\partial T}{\partial q_\alpha} = Q_\alpha$ (广义力)

PPT 例题



$x = (l+r)\cos\alpha t \quad y = (l+r)\sin\alpha t$

$\dot x = \dot r\cos\alpha t + (l+r)\,(-\alpha\sin\alpha t)$

$\dot y = \dot r\sin\alpha t + (l+r)\alpha\cos\alpha t$

∴ $T = \frac{1}{2}m(\dot x^2 + \dot y^2) = \frac{1}{2}m(\dot r^2 + (l+r)^2\alpha^2)$

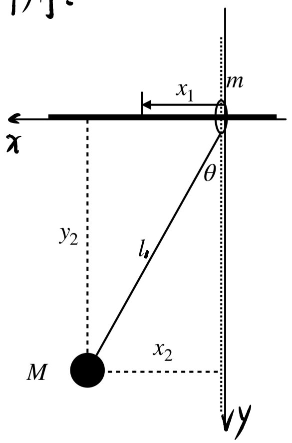
$V = \frac{1}{2}Kr^2$

∴ $L = T - V = \frac{1}{2}m((\dot r)^2 + (l+r)^2\alpha^2) - \frac{1}{2}Kr^2$

∴ $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot r}\right) - \frac{\partial L}{\partial r} = 0$

∴ $m\ddot r - m(l+r)\alpha^2 + Kr = 0$

例：



对 m $T=\frac{1}{2}m\dot{x}_1^2$ $V=0$

对 M $(x,y)=(x_1+l_1\sin\theta,\ l_1\cos\theta)$

$\therefore T=\frac{1}{2}M(\dot{x}^2+\dot{y}^2)=\frac{1}{2}M(\dot{x}_1^2+2\dot{x}_1 l_1\cos\theta\dot{\theta}+l_1^2\cos^2\theta\dot{\theta}^2$
$+l_1^2\sin^2\theta\,\dot{\theta}^2)$

$=\frac{1}{2}M(\dot{x}_1^2+2\dot{x}_1 l_1\cos\theta\,\dot{\theta}+l_1^2\dot{\theta}^2)$

$V=Mgl\cos\theta$

$\therefore L=\frac{1}{2}m\dot{x}_1^2+\frac{1}{2}M(\dot{x}_1^2+2\dot{x}_1 l_1\cos\theta\dot{\theta}+l_1^2\dot{\theta}^2)-Mgl\cos\theta$

$\therefore \frac{d}{dt}(\frac{\partial L}{\partial \dot{x}_1})-\frac{\partial L}{\partial x_1}=[(M+m)\dot{x}_1+Ml_1\cos\theta\,\dot{\theta}]\frac{d}{dt}=0$

$\frac{d}{dt}(\frac{\partial L}{\partial \dot{\theta}})-\frac{\partial L}{\partial \theta}=(l_1^2M\dot{\theta}+M\dot{x}_1 l_1\cos\theta)\frac{d}{dt}+M\dot{x}_1 l_1\sin\theta\dot{\theta}+Mgl\sin\theta=0$

$\therefore (M+m)\dot{x}_1+Ml_1\cos\theta\dot{\theta}=C$

$Ml_1^2\ddot{\theta}+M\ddot{x}_1 l_1\cos\theta-M\dot{x}_1 l_1\cancel{\sin\theta\dot{\theta}}+M\dot{x}_1 l_1\cancel{\sin\theta\dot{\theta}}+Mgl\sin\theta=0$

$=l_1\ddot{\theta}+\ddot{x}_1\cos\theta+g\sin\theta=0$

Chapter 2

利用哈密顿原理得到拉格朗日方程

位形空间 $I=\int_{t_1}^{t_2}L\,dt$

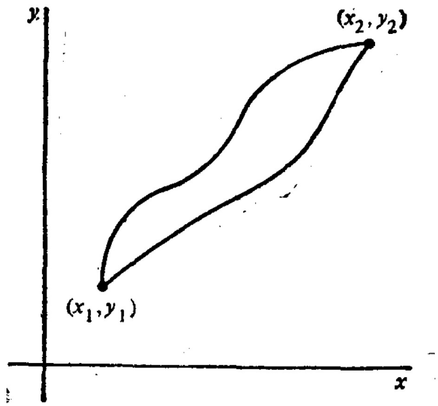
$\therefore \delta I=\int_{t_1}^{t_2}L(q+\delta q,\dot{q}+\delta\dot{q},t)\,dt-\int_{t_1}^{t_2}L(q,\dot{q},t)\,dt$

δq 与 $\delta\dot{q}$ 独立

引进 $\delta q = \alpha\eta(t)$

满足 $\eta(t_1)=\eta(t_2)=0$

$$J = \int_{x_1}^{x_2} f(y,\dot{y},x)\,dx$$



$y_1(x_1)=y_1 \quad y_2(x_2)=y_2$

设 $y(x,\alpha) = y(x,0)+\alpha\eta(x)$

其中 $\eta(x_1)=\eta(x_2)=0$

$$\therefore J(\alpha) = \int_{x_1}^{x_2} f(y(x,\alpha), \dot{y}(x,\alpha), x)\,dx$$

极值点在 $\frac{dJ(\alpha)}{d\alpha} = \int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y}\frac{\partial y}{\partial \alpha} + \frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}\right)dx = 0$ 处

而 $\int_{x_1}^{x_2}\frac{\partial f}{\partial \dot{y}}\frac{\partial \dot{y}}{\partial \alpha}dx = \int_{x_1}^{x_2}\frac{\partial f}{\partial \dot{y}}\underset{\downarrow\ \frac{d}{dx}\left(\frac{\partial y}{\partial \alpha}\right)}{\frac{\partial^2 y}{\partial x\partial \alpha}}dx = \frac{\partial f}{\partial \dot{y}}\frac{\partial y}{\partial \alpha}\Big|_{x_1}^{x_2} - \int_{x_1}^{x_2}\frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\frac{\partial y}{\partial \alpha}dx$

故原式 =

$$\int_{x_1}^{x_2}\left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial \dot{y}}\right)\frac{\partial y}{\partial \alpha}dx$$

$$\therefore \int_{x_1}^{x_2}\left[\frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right)\right]\left(\frac{\partial y}{\partial \alpha}\right)\Big|_{\alpha=0}dx$$

而 $\frac{\partial y}{\partial \alpha} = \eta(x)$ 对任意 $\eta(x)$ 上式成立

$$\therefore \frac{\partial f}{\partial y} - \frac{d}{dx}\left(\frac{\partial f}{\partial \dot{y}}\right) = 0$$

电磁场 $L = \frac{1}{2}mv^2 - q\phi + q\vec{A}\cdot\vec{v}$

则 $P_x = m\dot{x} + qA_x$

能量函数 $h = \sum_\alpha \frac{\partial L}{\partial \dot{q}_\alpha}\dot{q}_\alpha - L$

$\frac{dh}{dt} = -\frac{\partial L}{\partial t}$

若L不显含时间则能量函数守恒

$h \overset{?}{=} T+V$

→ ① T中 $\vec{r}_i$ 与 q_j 不含时
② V中不含速度

3+1 维中有7个守恒量

3动量 能量

利用哈密顿原理 → 拉格朗日方程

$$J=\int_{t_1}^{t_2} L(q,\dot{q},t)\,dt$$

$$\therefore \delta J=\int_{t_1}^{t_2} L(q+\delta q,\dot{q}+\delta\dot{q},t)\,dt-\int_{t_1}^{t_2} L(q,\dot{q},t)\,dt$$

而 $L(q+\delta q,\dot{q}+\delta\dot{q},t)\approx L(q,\dot{q},t)+\frac{\partial L}{\partial q}\delta q+\frac{\partial L}{\partial \dot{q}}\delta\dot{q}$

$$\therefore \delta J=\int_{t_1}^{t_2}\left(\frac{\partial L}{\partial q}\delta q+\frac{\partial L}{\partial \dot{q}}\delta\dot{q}\right)dt$$

而 $\dot{q}=\frac{d}{dt}$ $\therefore \delta\dot{q}=\frac{d}{dt}(\delta q)$ $\delta q(t_1)=\delta q(t_2)=0$

$$\therefore \delta J=\int_{t_1}^{t_2}\frac{\partial L}{\partial q}\delta q+\int_{t_1}^{t_2}\frac{\partial L}{\partial \dot{q}}\frac{d}{dt}(\delta q)\,dt=\int_{t_1}^{t_2}\frac{\partial L}{\partial q}\delta q\,dt+\frac{\partial L}{\partial \dot{q}}\delta q\Big|_{t_1}^{t_2}-\int_{t_1}^{t_2}\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\delta q\,dt$$

$$=\int_{t_1}^{t_2}\left[\frac{\partial L}{\partial q}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)\right]\delta q\,dt=0$$

δq 在 $t_1\sim t_2$ 间任意变化 则 $\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}}\right)-\frac{\partial L}{\partial q}=0$

哈密顿原理 → 哈密顿正则方程

$$\delta I\equiv\delta\int_{t_1}^{t_2}L\,dt=\int_{t_1}^{t_2}\left(\sum_\alpha\frac{\partial L}{\partial \dot{q}_\alpha}\dot{q}_\alpha-H(q,p,t)\right)dt$$

记 $\frac{\partial L}{\partial \dot{q}_\alpha}=p_\alpha$ 故 $\delta I=\int_{t_1}^{t_2}\delta\left(\sum_\alpha p_\alpha\dot{q}_\alpha-H(q,p,t)\right)dt$

$$=\int_{t_1}^{t_2}\left(\sum_\alpha(\delta p_\alpha\dot{q}_\alpha+p_\alpha\delta\dot{q}_\alpha)-\delta H\right)dt$$

而 $\delta H(p,q,t)=\frac{\partial H}{\partial p_\alpha}\delta p_\alpha+\frac{\partial H}{\partial q_\alpha}\delta q_\alpha$

$$\text{上式}=\sum_\alpha\int_{t_1}^{t_2}\left(\dot{q}_\alpha\delta p_\alpha-\frac{\partial H}{\partial p_\alpha}\delta p_\alpha+p_\alpha\frac{d\delta q_\alpha}{dt}-\frac{\partial H}{\partial q_\alpha}\delta q_\alpha\right)dt$$

而 $\int_{t_1}^{t_2} p_\alpha \frac{d\delta q_\alpha}{dt} dt = p_\alpha \delta q_\alpha \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \dot{p}_\alpha \delta q_\alpha dt$

故上式 $= \sum_\alpha \int_{t_1}^{t_2} ((\dot{q}_\alpha - \frac{\partial H}{\partial p_\alpha}) \delta p_\alpha - (\dot{p}_\alpha + \frac{\partial H}{\partial q_\alpha}) \delta q_\alpha) dt$

而由勒让德变换 $H = p\dot{q} - L$

$\therefore dH = \frac{\partial H}{\partial p} dp + \frac{\partial H}{\partial q} dq + \frac{\partial H}{\partial t} dt = \cancel{p d\dot{q}} + \dot{q} dp - \frac{\partial L}{\partial q} dq - \cancel{\frac{\partial L}{\partial \dot{q}} d\dot{q}} - \frac{\partial L}{\partial t} dt$

$\therefore \frac{\partial H}{\partial p_\alpha} = \dot{q}_\alpha$ 数学上恒成立

故上式 $\sum_\alpha - \int_{t_1}^{t_2} (\dot{p}_\alpha + \frac{\partial H}{\partial q_\alpha}) \delta q_\alpha dt$

δq_α 任意变化则得到 $\dot{p}_\alpha = - \frac{\partial H}{\partial q_\alpha}$

若 $L' = L + \frac{dF}{dt}$ 则 L' 也满足拉格朗日方程 证明如下

$F(q, t)$ $\frac{dF}{dt} = \sum_\alpha \frac{\partial F}{\partial q_\alpha} \dot{q}_\alpha + \frac{\partial F}{\partial t}$

$\frac{\partial \dot{F}}{\partial q_i} = \sum_\alpha \frac{\partial F}{\partial q_\alpha \partial q_i} \dot{q}_\alpha + \frac{\partial F}{\partial q_i \partial t}$

而 $\frac{\partial \dot{F}}{\partial \dot{q}_i} = \frac{\partial F}{\partial q_i}$ $\therefore \frac{d}{dt}(\frac{\partial F}{\partial q_i}) = \sum_\alpha \frac{\partial^2 F}{\partial q_i \partial q_\alpha} \dot{q}_\alpha + \frac{\partial^2 F}{\partial q_i \partial t} = \frac{\partial \dot{F}}{\partial q_i}$

$\therefore \frac{d}{dt}(\frac{\partial \dot{F}}{\partial \dot{q}_i}) - \frac{\partial \dot{F}}{\partial q_i} = 0$ 故成立

循环坐标其对应动量守恒

Chapter 3 有心力

二维问题 Azimuth angle (方位角) θ 天顶角 $\psi = \frac{1}{2}\pi$

$$\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = const \qquad l = mr^2\dot{\theta} \text{ 守恒}$$

维力定理 记 $G = \sum_i \vec{p_i}\cdot\vec{r_i}$

$$\therefore \frac{dG}{dt} = \sum_i \dot{\vec{r_i}}\cdot\vec{p_i} + \sum_i \dot{\vec{p_i}}\cdot\vec{r_i}$$

$$= \sum_i \dot{\vec{r_i}}\, m\dot{\vec{r_i}} + \sum_i \vec{F_i}\cdot\vec{r_i} = mv^2 + \sum_i \vec{F_i}\cdot\vec{v_i}$$

$$\therefore \frac{1}{\tau}\int_0^\tau \frac{dG}{dt}dt = \overline{\frac{dG}{dt}} = 2\overline{T} + \overline{\sum_i \vec{F_i}\cdot\vec{v_i}} = \frac{1}{\tau}(G(\tau)-G(0)) \quad (\tau\to\infty)$$

$$\therefore \overline{T} = -\frac{1}{2}\overline{\sum_i \vec{F_i}\cdot\vec{r_i}}$$ 维里

推理想气体玻意耳定律

$$\overline{T} = \frac{3}{2}Nk_BT$$

$$P_n = -\frac{d\vec{F_i}}{dA_i} \qquad \therefore \frac{1}{2}\sum \vec{F_i}\cdot\vec{r_i} = -\frac{P}{2}\int \vec{n}\cdot\vec{r}\,dA = -\frac{P}{2}\int \nabla\cdot\vec{r}\,dV = -\frac{3}{2}PV = -\frac{3}{2}Nk_BT$$

$$\therefore PV = Nk_BT = nRT$$

若为保守力 $\vec{F_i} = -\nabla V_i$

$$\therefore \overline{T} = -\frac{1}{2}\overline{\sum_i \vec{F_i}\cdot\vec{r_i}} = \frac{1}{2}\overline{\sum_i \nabla V_i\cdot\vec{r_i}}$$

对于某一个粒子 $\overline{T} = \frac{1}{2}\overline{\frac{\partial V}{\partial r}r}$

而 $V = ar^{n+1}$ $\therefore \frac{\partial V}{\partial r}r = (n+1)V$

$$\therefore \overline{T} = \frac{1}{2}(n+1)\overline{V}$$

万有引力

$n = -2$

$$\therefore \overline{T} = -\frac{1}{2}\overline{V}$$

比耐公式

$$L = T - V = \frac{1}{2}m(\dot{r}^2 + (r\dot{\theta})^2) - V(r)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) - \frac{\partial L}{\partial r} = m\ddot{r} - mr(\dot{\theta})^2 + \frac{\partial V(r)}{\partial r} = 0$$

$$m\ddot{r} - \frac{l^2}{mr^3} + \frac{\partial V(r)}{\partial r} = 0$$

$$l = mr^2\dot{\theta} \Rightarrow \frac{d}{dt} = \frac{l}{mr^2}\frac{d}{d\theta}$$

∴ $m\ddot{r} - \frac{l^2}{mr^3} + \frac{dV}{dr} = 0 \Rightarrow \frac{l}{r^2}\frac{d}{d\theta}\left(\frac{l}{mr^2}\frac{dr}{d\theta}\right) - \frac{l^2}{mr^3} + \frac{dV}{dr} = 0$

∴ 令 $u = \frac{1}{r}$ $\quad dr = -\frac{1}{u^2}du$

∴ $u^2\frac{d}{d\theta}\left(\frac{l^2}{m}u^2 - \frac{1}{u^2}\frac{du}{d\theta}\right) - \frac{l^2}{m}u^3 + \frac{dV}{dr} = 0$

∴ $-u^2\frac{l^2}{m}\frac{d^2u}{d\theta^2} - \frac{l^2}{m}u^3 + \frac{d(V(\frac{1}{u}))}{dr} = 0$ $\qquad dr = -\frac{1}{u^2}du$

∴ $\frac{d^2u}{d\theta^2} + u + \frac{m}{l^2}\frac{d(V(\frac{1}{u}))}{du} = 0$

$E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + V(r)$ $\qquad l = mr^2\dot{\theta}$

$l = mr^2\dot{\theta}$ $\qquad \frac{dr}{dt} = \frac{du}{dt}\cdot -r^2 = -\frac{du}{d\theta}r^2\dot{\theta} = -\frac{du}{d\theta}\frac{l}{m}$

有心力结论 $e = \sqrt{1 + \frac{2El^2}{mk^2}}$ $\qquad a = -\frac{k}{2E}$

$b = a\sqrt{1-e^2} = \sqrt{-\frac{2El^2}{mk^2}} \cdot -\frac{k}{2E} = -\sqrt{-\frac{l^2}{2Em}}$

∴ $A = \pi ab = \pi\sqrt{-\frac{l^2k^2}{8mE^3}}$

又∵ $\frac{dA}{dt} = \frac{1}{2}r^2\dot{\theta} = \frac{l}{2m} \Rightarrow \overset{\text{周期}}{T} = \frac{A}{\frac{dA}{dt}} = \pi\sqrt{-\frac{mk^2}{2E^3}} = 2\pi\sqrt{\frac{m}{k}}\,a^{\frac{3}{2}}$

而 m 有效质量 $\frac{1}{\mu} = \frac{1}{M} + \frac{1}{m}$

$T = 2\pi\sqrt{\frac{\mu}{k}}\,a^{\frac{3}{2}} = 2\pi\sqrt{\frac{1}{G(M+m)}}\,a^{\frac{3}{2}}$ （开普勒第三定律）

$$V'(r) \equiv V(r) + \frac{l^2}{2mr^2}$$

$$E = \frac{1}{2}m\dot{r}^2 + \frac{l^2}{2mr^2} + V(r) = \frac{1}{2}m(\dot{r}_0 + \dot{x})^2 + V'(r_0) + \frac{dV'}{dr}\Big|_{r_0}x + \frac{1}{2}\frac{d^2V'}{dr^2}x^2\Big|_{r_0} + \cdots$$

∴ $E - V'(r_0) - \frac{1}{2}m\dot{r}_0^2 \approx \frac{1}{2}m\dot{x}^2 + \frac{1}{2}m\omega^2x^2$ $\qquad \omega = \sqrt{\frac{\frac{d^2V'}{dr^2}\big|_{r_0}}{m}}$

Kepler 问题 五个量守恒 $p_x \ p_y \ p_z \ E$ $\vec{A} = \vec{p}\times\vec{L} - mk\frac{\vec{r}}{r}$

证明龙格-楞次矢量守恒

$$\dot{\vec{p}}\times\vec{L} = \frac{d}{dt}(\vec{p}\times\vec{L}) = \frac{mf(r)}{r}[\vec{r}\times(\vec{r}\times\dot{\vec{r}})]$$

$$A\times(B\times C) = B(AC) - C(AB) = \vec{r}(\vec{r}\cdot\dot{\vec{r}}) - \dot{\vec{r}}r^2$$

$$\dot{\vec{p}} = \frac{f(r)\vec{r}}{r} \quad \vec{L} = \vec{r}\times m\dot{\vec{r}}$$

$$\therefore \dot{\vec{p}}\times\vec{L} = \frac{mf(r)}{r}[\vec{r}r\dot{r} - \dot{\vec{r}}r^2] = -mf(r)r^2\left[\frac{\dot{\vec{r}}}{r} - \frac{\vec{r}\dot{r}}{r^2}\right] = -mf(r)r^2\frac{d}{dt}\left(\frac{\vec{r}}{r}\right)$$

而Kepler问题 $f(r) = -\frac{k}{r^2}$

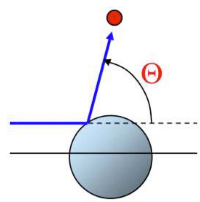
$$\therefore \frac{d}{dt}(\vec{p}\times\vec{L}) = \frac{d}{dt}\left(-\frac{mk\vec{r}}{r}\right)$$ 故 $\vec{p}\times\vec{L} + \frac{mk\vec{r}}{r}$ 守恒

散射

散射截面 $\sigma(\Omega)d\Omega = \frac{\text{单位时间进入立体角}d\Omega\text{散射点数目}}{\text{入射强度}}$

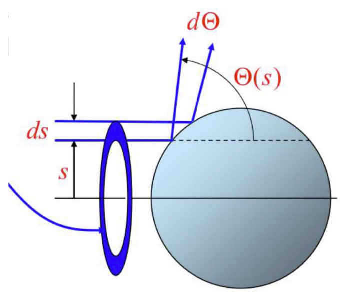
↓

微分散射截面



$$\sigma(\Omega)d\Omega = \sigma(\Theta)\sin\Theta\, d\Theta\, [d\phi]$$

$$\therefore N = \int_0^{2\pi} d\phi\, I\sigma(\Theta) = I\sigma(\Theta)2\pi\sin\Theta\, d\Theta$$



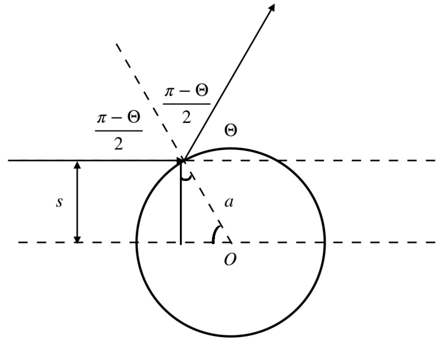
$$2\pi s\,ds = \sigma(\Theta)2\pi\sin\Theta\, d\Theta$$

$$\therefore \sigma(\Theta) = \frac{s}{\sin\Theta}\Big/\left|\frac{ds}{d\Theta}\right|$$

$$\therefore \text{总散射截面 } \sigma_T = \int_0^{\pi} 2\pi\sin\Theta\,\sigma(\Theta)\,d\Theta$$

$$= \int_0^{a} 2\pi s\,ds = \pi a^2$$

$$S = a\sin\frac{\pi-\Theta}{2} = a\cos\frac{\Theta}{2}$$



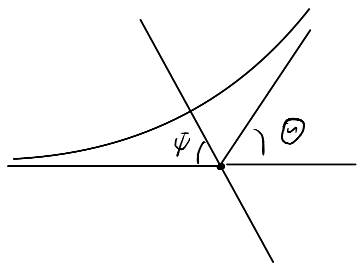
$$\therefore \frac{ds}{d\Theta} = -\frac{a}{2}\sin\frac{\Theta}{2}$$

$$\therefore \sigma(\Theta) = \frac{s}{\sin\Theta}\cdot\frac{a}{2}\sin\frac{\Theta}{2} = \frac{a\sin\frac{\Theta}{2}\,a\cos\frac{\Theta}{2}}{4\sin\frac{\Theta}{2}\cos\frac{\Theta}{2}} = \frac{a^2}{4}$$

$$\therefore \sigma_T = \int_0^{\pi}\int_0^{2\pi}\frac{a^2}{4}\sin\Theta\, d\Theta\, d\varphi = \pi a^2$$

硬球散射

$$l = |r\times p_0| = sp_0 = s\sqrt{2mE}$$



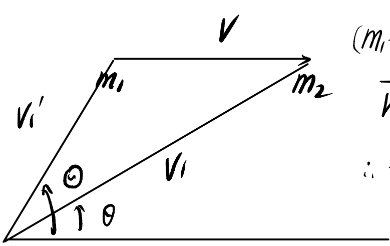
$$\Theta = \pi - 2\psi \qquad \cos\psi = \frac{1}{e}$$

$$\therefore \cot\frac{\Theta}{2} = \sqrt{e^2-1} = \sqrt{\frac{2El^2}{mk^2}} = \frac{2sE}{k}$$

$$\therefore d\cot\frac{\Theta}{2} = \frac{2E}{k}ds = \frac{-\sin\frac{\Theta}{2}\frac{1}{2}\sin\frac{\Theta}{2} - \cos\frac{\Theta}{2}\frac{1}{2}\cos\frac{\Theta}{2}}{\sin^2\frac{\Theta}{2}} = -\frac{1}{2\sin^2\frac{\Theta}{2}}d\Theta$$

$$\therefore \sigma(\Theta) = \frac{k}{\sin\Theta}\frac{1}{2E}\cdot\frac{k}{2E}\frac{\cot\frac{\Theta}{2}}{2\sin^2\frac{\Theta}{2}} = \frac{k^2}{4E^2}\frac{1}{4\sin^4\frac{\Theta}{2}}$$

实验室坐标



动量定理

$$(m_1+m_2)\vec{V} = m_1\vec{V}_0 \qquad \vec{V} = \frac{\mu}{m_2}\vec{V}_0$$

$$\vec{V}_1\sin\theta = \vec{V}_1'\sin\Theta \qquad \vec{V}_1\cos\theta = \vec{V}_1'\cos\Theta + V$$

$$\therefore \tan\theta = \frac{\sin\Theta}{\cos\Theta+\rho} \qquad \rho = \frac{V}{V_1'} = \frac{\mu}{m_2}\frac{V_0}{V_1'}$$

$$\therefore V_1^2 = V_1'^2 + V^2 + 2V_1'V\cos\Theta$$

$$\Rightarrow \cos\theta = \frac{\cos\Theta+\rho}{\sqrt{\rho^2+1+2\rho\cos\Theta}}$$

$\because \int 2\pi I \sigma(\Theta) \sin\Theta |d\Theta| = \int 2\pi I \sigma'(\theta) \sin\theta |d\theta|$

$\therefore \sigma'(\theta) = \sigma(\Theta) \frac{\sin\Theta}{\sin\theta} \left|\frac{d\Theta}{d\theta}\right| = \sigma(\Theta) \left|\frac{d\cos\Theta}{d\cos\theta}\right|$ 代入

而若 $m_1 = m_2$

则 $\cos\theta = \sqrt{\frac{1+\cos\Theta}{2}}$

$\therefore \sigma'(\theta) = 4\cos\theta \cdot \sigma(\Theta) \quad (\theta \leq \frac{\pi}{2}) \quad (\rho = 1)$

当 $m_2 \gg m_1$ 时 $\rho \to 0 \quad \sigma(\theta) \sim \sigma(\Theta)$

Chapter 4 刚体转动

欧拉角

$$D = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{（进动）} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \text{（章动）} \quad B = \begin{bmatrix} \cos\psi & \sin\psi & 0 \\ -\sin\psi & \cos\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{（自转）}$$

$\therefore A = DCB$

轴矢量 镜面对称相同

赝矢量 矢量的叉积总为赝矢量 镜面对称反号

力矩 $\vec{r} \times \vec{F}$

角动量 $\vec{r} \times \vec{p}$

角速度 $\vec{\omega}$，加速度 $\vec{\alpha}$

空间系与刚体系

$$\left(\frac{d\vec{G}}{dt}\right)_{space} = \left(\frac{d\vec{G}}{dt}\right)_{body} + \omega \times \vec{G}$$

$$\omega = n_z\dot{\phi} + n_\xi\dot{\theta} + n_z\dot{\psi} = \begin{bmatrix} \dot{\phi}\sin\psi\sin\theta + \dot{\theta}\cos\psi \\ \dot{\phi}\cos\psi\sin\theta - \dot{\theta}\cos\psi \\ \dot{\phi}\cos\theta + \dot{\psi} \end{bmatrix}$$

故 $\omega_\phi = \dot{\phi}\sin\psi\sin\theta + \dot{\theta}\cos\psi$

$\omega_\theta = \dot{\phi}\cos\psi\sin\theta - \dot{\theta}\cos\psi$

$\omega_\psi = \dot{\phi}\cos\theta + \dot{\psi}$

$$\vec{L} = \vec{I}\cdot\vec{\omega}$$

$$T = \frac{\vec{\omega}\cdot I\cdot\vec{\omega}^T}{2}$$

$$I_{ij} = \int \rho(\vec{r})(r^2\delta_{ij} - x_i x_j)\,d\vec{r}$$

惯量主轴

$$\frac{d\vec{J}}{dt} = \dot{J}_1\vec{e}_1 + \dot{J}_2\vec{e}_2 + \dot{J}_3\vec{e}_3 + J_1\dot{\vec{e}}_1 + J_2\dot{\vec{e}}_2 + J_3\dot{\vec{e}}_3$$

$$= \dot{J}_1\vec{e}_1 + \dot{J}_2\vec{e}_2 + \dot{J}_3\vec{e}_3 + J_1(\vec{\omega}_1\times\vec{e}_1) + J_2(\vec{\omega}_2\times\vec{e}_2) + J_3(\vec{\omega}_3\times\vec{e}_3)$$

$$= I_1\dot{\omega}_1^{\vec{e}_1} + I_2\dot{\omega}_2^{\vec{e}_2} + I_3\dot{\omega}_3^{\vec{e}_3} + \vec{\omega}\times\vec{J} = \vec{N}$$

$$\therefore \begin{cases} I_1\dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) = N_1 \\ I_2\dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) = N_2 \\ I_3\dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) = N_3 \end{cases}$$

欧拉运动方程

$\vec{N} = 0$ 时

$$I = I_1 n_1^2 + I_2 n_2^2 + I_3 n_3^2$$

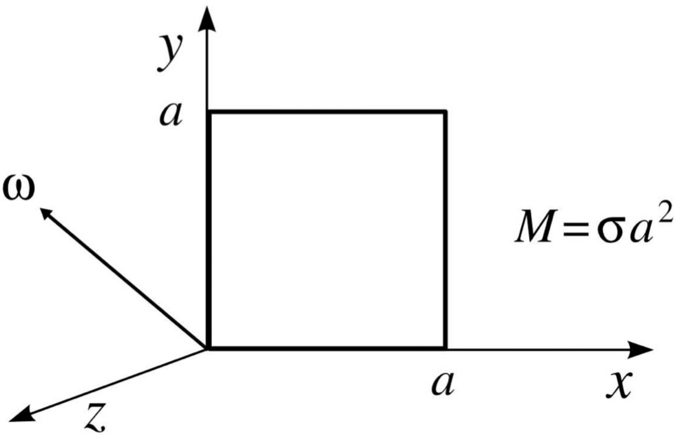
考虑 $\rho = \frac{n}{\sqrt{I}}$

$$\therefore 1 = I_i\rho_i^2 = I_1\rho_1^2 + I_2\rho_2^2 + I_3\rho_3^2$$

转动惯量 $I = \int r^2\,dm$

平行轴定理 $I_{zz} = I_{xx} + I_{yy}$

Problem. Calculate the tensor of inertia and the principal axes of inertia of a square covered with mass for a corner of the square.



$$r^2=x^2+y^2$$

解：$I_{xx}=\int_0^a\int_0^a \sigma(r^2-x^2)\,dxdy = a\sigma\int_0^a y^2dy = a\sigma\frac{1}{3}a^3=\frac{1}{3}Ma^2$

$I_{xy}=-\int_0^a\int_0^a \sigma xy\,dxdy = -\frac{1}{2}a^2\frac{1}{2}a^2\sigma = -\frac{1}{4}a^2 M$

$I_{yy}=\int_0^a\int_0^a\sigma(r^2-y^2)\,dxdy=\frac{1}{3}Ma^2$

$I_{zz}=\int_0^a\int_0^a\sigma(r^2-z^2)\,dxdy=\int_0^a\int_0^a\sigma(x^2+y^2)\,dxdy=\frac{2}{3}a^4\sigma=\frac{2}{3}Ma^2$

$I_{xz}=I_{yz}=0$

∴ $\vec{I}=\begin{pmatrix}\frac{1}{3} & -\frac{1}{4} & 0\\ -\frac{1}{4} & \frac{1}{3} & 0\\ 0 & 0 & \frac{2}{3}\end{pmatrix}Ma^2$

$\vec{I}_0-IE=0$

∴ $\begin{pmatrix}\frac{1}{3}I_0-I & -\frac{1}{4}I_0 & 0\\ -\frac{1}{4}I_0 & \frac{1}{3}I_0-I & 0\\ 0 & 0 & \frac{2}{3}I_0-I\end{pmatrix}=\left[\left(\frac{1}{3}I_0-I\right)^2-\frac{1}{16}I_0^2\right]\left(\frac{2}{3}I_0-I\right)$

$=\left(\frac{1}{12}I_0-I\right)\left(\frac{7}{12}I_0-I\right)\left(\frac{2}{3}I_0-I\right)=0$

则 $I_1=\frac{1}{12}I_0$ $I_2=\frac{7}{12}I_0$ $I_3=\frac{2}{3}I_0$ 对应的特征向量

$I_1=\frac{1}{12}I_0$ 时

$$\begin{pmatrix} \frac{1}{4}I_0 & -\frac{1}{4}I_0 & 0 \\ -\frac{1}{4}I_0 & \frac{1}{4}I_0 & 0 \\ 0 & 0 & \frac{7}{12}I_0 \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}=0 \Rightarrow \vec{n}_1=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

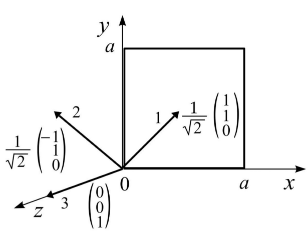
$I_2=\frac{7}{12}I_0$ 时

$$\begin{pmatrix} -\frac{1}{4}I_0 & -\frac{1}{4}I_0 & 0 \\ -\frac{1}{4}I_0 & -\frac{1}{4}I_0 & 0 \\ 0 & 0 & \frac{1}{12}I_0 \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}=0 \Rightarrow \vec{n}_2=\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$I_3=\frac{2}{3}I_0$ 时

$$\begin{pmatrix} -\frac{1}{3}I_0 & -\frac{1}{4}I_0 & 0 \\ -\frac{1}{4}I_0 & -\frac{1}{3}I_0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}=0 \Rightarrow \vec{n}_3=\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

惯量椭球



$$\vec{I}=\begin{pmatrix} \frac{1}{3} & -\frac{1}{4} & 0 \\ -\frac{1}{4} & \frac{1}{3} & 0 \\ 0 & 0 & \frac{2}{3} \end{pmatrix}Ma^2$$

$\therefore \frac{I_0}{3}\rho_x^2-\frac{I_0}{2}\rho_x\rho_y+\frac{I_0}{3}\rho_y^2+\frac{2I_0}{3}\rho_z^2=1$ (求任一方向上的惯量

x轴 $\vec{n}=(1,0,0)$ $\rho_x=\frac{n_1}{\sqrt{I_x}}$

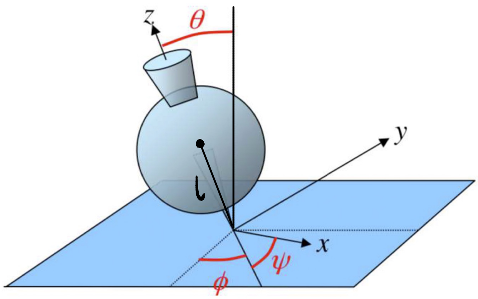
$\therefore I_x=\frac{1}{3}I_0 \quad I_y=\frac{1}{3}I_0 \quad I_z=\frac{2}{3}I_0$

椭球原点与P点处的切平面之间距为常数 $\frac{\sqrt{2T}}{L}$

$I_1 = I_2$ 时

对称陀螺

$$I_1\dot{\omega}_1 = \omega_2\omega_3(I_2 - I_3)$$
$$I_2\dot{\omega}_2 = \omega_3\omega_1(I_3 - I_1)$$
$$\dot{\omega}_3 = 0$$
$$\therefore \dot{\omega}_1 = -\Omega\omega_2 \quad \dot{\omega}_2 = \Omega\omega_1 \quad \Omega \equiv \frac{I_3 - I_1}{I_1}\omega_3$$



$$\omega = \begin{bmatrix} \dot\phi\sin\psi\sin\theta + \dot\theta\cos\psi \\ \dot\phi\cos\psi\sin\theta - \dot\theta\cos\psi \\ \dot\phi\cos\theta + \dot\psi \end{bmatrix}$$

$$T = \frac{1}{2}I_1(\omega_1^2 + \omega_2^2) + \frac{1}{2}I_3\omega_3^2$$
$$= \frac{1}{2}I_1(\dot\phi^2\sin^2\theta + \dot\theta^2) + \frac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2$$

ϕ 与 ψ 是循环坐标

$$V = Mgl\cos\theta \qquad \therefore L = T - V = \frac{1}{2}I_1(\dot\phi^2\sin^2\theta + \dot\theta^2) + \frac{1}{2}I_3(\dot\phi\cos\theta + \dot\psi)^2 - Mgl\cos\theta$$

$$\therefore P_\phi = I_1\dot\phi\sin^2\theta + I_3(\dot\phi\cos\theta + \psi)\cos\theta = const = I_1 a$$
$$P_\psi = I_3(\dot\phi\cos\theta + \dot\psi) = I_3\omega_3 = const = I_1 b$$

$$\therefore \dot\phi = \frac{b - a\cos\theta}{\sin^2\theta} \qquad \dot\psi = \frac{I_1 a}{I_3} - \cos\theta\,\frac{b - a\cos\theta}{\sin^2\theta}$$

$E = T + V$ 而 $\frac{1}{2}I_3\omega_3^2$ 守恒

$$\therefore E' = E - \frac{1}{2}I_3\omega_3^2 = \frac{1}{2}I_1\dot\theta^2 + \frac{I_1}{2}\frac{(b - a\cos\theta)^2}{\sin^2\theta} + Mgl\cos\theta$$

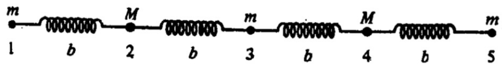
Chapter 6 微扰　　书上不同的概念

$$|V-\omega^2 T|=0$$

5．(a) 在线性三原子分子中，假设初始条件是中心原子保持静止，但已从平衡位置移过了一段距离 a_0，而其他两个原子则仍在它们的平衡位置处．求质心附近微幅纵振动的振幅．给出简正模式的振幅．

(b) 重复 (a) 中的问题，不过现在的中心原子最初位于平衡位置并有一初速度 v_0．

6．把一些质点和理想弹簧排列起来模拟一个线性五原子分子，其形状如下图所示：



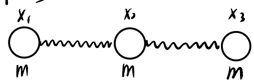
所有弹簧的力常数都相等．求纵振动的本征频率和简正模式．[提示：把坐标 η_i 变换为 ζ_i，它们的定义为

$$\eta_3=\zeta_3,\quad \eta_1=\frac{\zeta_1+\zeta_5}{\sqrt{2}},\quad \eta_5=\frac{\zeta_1-\zeta_5}{\sqrt{2}}$$

加上 η_2 和 η_4 的对称表达式．这样，特征行列式将分解成较低阶的行列式．]

$$\eta_i = a_{i1}(A_1\cos\omega_1 t + B_1\sin\omega_1 t) + a_{i2}(A_2\cos\omega_2 t + B_2\sin\omega_2 t) + a_{i3}(A_3\cos\omega_3 t + B_3\sin\omega_3 t)$$

最基本三原子



弹簧原长为 b

$$\therefore V=\frac{k}{2}(x_2-x_1-b)^2+\frac{k}{2}(x_3-x_2-b)^2$$

$$T=\frac{1}{2}m(\dot{x}_1^2+\dot{x}_2^2+\dot{x}_3^2)$$

记 $b=x_{20}-x_{10}=x_{30}-x_{20}$　　$\dot{x}_1=x_1-x_{10}=\eta_1$

$$\therefore V=\frac{k}{2}(\eta_2-\eta_1)^2+\frac{k}{2}(\eta_3-\eta_2)^2=\frac{1}{2}k(\eta_1^2+\eta_3^2)+k\eta_2^2-k\eta_1\eta_2-k\eta_2\eta_3$$

$$=\begin{pmatrix}\frac{1}{2}k & -\frac{1}{2}k & 0\\ -\frac{1}{2}k & k & -\frac{1}{2}k\\ 0 & -\frac{1}{2}k & \frac{1}{2}k\end{pmatrix}\begin{pmatrix}\eta_1\\ \eta_2\\ \eta_3\end{pmatrix}$$

$$T=\frac{1}{2}m(\dot{\eta}_1^2+\dot{\eta}_2^2+\dot{\eta}_3^2)$$

$$\therefore |V-\omega^2 T| = \begin{vmatrix} k-m\omega^2 & -k & 0\\ -k & 2k-m\omega^2 & -k\\ 0 & -k & k-m\omega^2\end{vmatrix}=0$$

$$=(k-m\omega^2)^2(2k-m\omega^2)-2k^2(k-m\omega^2)$$

$$=(k-m\omega^2)(2k^2-3km\omega^2+m^2\omega^4-2k^2)$$

$$=(k-m\omega^2)m\omega^2(m\omega^2-3k)$$

$$\therefore \omega_1=0\quad \omega_2=\sqrt{\frac{k}{m}}\quad \omega_3=\sqrt{\frac{3k}{m}}$$

则对应的特征向量 $a_1=\frac{1}{\sqrt{3m}}\begin{pmatrix}1\\1\\1\end{pmatrix}$　$a_2=\frac{1}{\sqrt{2m}}\begin{pmatrix}1\\0\\-1\end{pmatrix}$　$a_3=\frac{1}{\sqrt{6m}}\begin{pmatrix}1\\-2\\1\end{pmatrix}$

Chapter 7

勒让德变换

$\frac{\partial L}{\partial \dot{q}_i} = p_i$ 则 $\frac{\partial H}{\partial p_i} = \dot{q}_i$ ⟶ 勒让德变换得到的结果

$L = L(q_1 \cdots q_n, \dot{q}_1 \cdots \dot{q}_n, t)$

故 $G = G(q_1 \cdots q_n, p_1 \cdots p_n, t) = \sum_i p_i \dot{q}_i - L \qquad p_i = \frac{\partial L}{\partial \dot{q}_i}$

$\therefore \frac{d}{dt}(p_i) = \frac{\partial L}{\partial q_i}, \quad \frac{\partial G}{\partial q_i} = -\frac{\partial L}{\partial q_i}$

$\therefore \dot{p}_i = -\frac{\partial G}{\partial q_i}$ (拉格朗日得到的结果)

$$dH = p\,d\dot{q}_i + \dot{q}_i\,dp - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial \dot{q}}d\dot{q} - \frac{\partial L}{\partial t}dt$$

$$= \frac{\partial H}{\partial q}dq + \frac{\partial H}{\partial p}dp + \frac{\partial H}{\partial t}dt$$

而 $p = \frac{\partial L}{\partial \dot{q}}$

$$= \dot{q}_i\,dp - \frac{\partial L}{\partial q}dq - \frac{\partial L}{\partial t}dt$$

达朗贝尔
↓
拉格朗日方程

$\therefore \frac{\partial H}{\partial p} = \dot{q}_i$ 数学恒等变形

$\frac{\partial H}{\partial q} = -\frac{\partial L}{\partial q} = -\dot{p}$ 拉格朗日方程

Chapter 8 正则变换

条件

⟶ 相位空间做变换

$$\delta \int_{t_1}^{t_2} \left(\sum_i p_i \dot{q}_i - H(q,p,t)\right) dt = 0$$

与 $\delta \int_{t_1}^{t_2} \left(\sum_i P_i \dot{Q}_i - K(Q,P,t)\right) dt = 0$

$\therefore$ 有 $\left(\sum_i p_i \dot{q}_i - H\right)\lambda = \sum_i P_i \dot{Q}_i - K + \frac{dF}{dt}$ ⟶ 生成函数

q, Q, t

$q \quad P \quad t$

$p \quad Q \quad t$

$p \quad P \quad t$

$F = F(q, Q, t)$

$$\therefore \frac{dF}{dt} = \frac{\partial F}{\partial q}\dot{q} + \frac{\partial F}{\partial Q}\dot{Q} + \frac{\partial F}{\partial t}dt$$

$$\therefore p_i\dot{Q}_i - K + \frac{\partial F_1}{\partial q}\dot{q}_i + \frac{\partial F_1}{\partial Q}\dot{Q}_i + \frac{\partial F_1}{\partial t}dt = p_i\dot{q}_i - H$$

$$\therefore \left(P_i + \frac{\partial F_1}{\partial Q}\right)\dot{Q}_i - \left(p_i - \frac{\partial F_1}{\partial q}\right)\dot{q}_i - \left(K - H - \frac{\partial F_1}{\partial t}\right)dt = 0$$

$$\therefore P_i = -\frac{\partial F_1(q, Q, t)}{\partial Q} \qquad p_i = \frac{\partial F_1(q, Q, t)}{\partial q} \qquad K = H + \frac{\partial F_1}{\partial t}$$

例 $H(q, p) = \frac{p^2}{2m} + \frac{1}{2}kq^2 = \frac{1}{2m}(p^2 + m^2\omega^2 q^2) \qquad \omega \equiv \sqrt{\frac{k}{m}}$

我们做一个正则变换 $p = f(P)\cos Q \quad q = \frac{f(P)}{m\omega}\sin Q$

第一类正则变换

$$\therefore P = -\frac{\partial F_1}{\partial Q} \qquad p = \frac{\partial F_1}{\partial q}$$

$$p = m\omega q \cot Q$$

$$\therefore m\omega q \cot Q = \frac{\partial F_1}{\partial q}$$

$$\therefore F_1 = \frac{1}{2}m\omega q^2 \cot Q$$

$$\therefore P = \frac{1}{2}m\omega q^2 \frac{1}{\sin^2 Q} \rightarrow q = \sqrt{\frac{2P\sin^2 Q}{m\omega}} \qquad p = \sqrt{2m\omega P}\cos^2 Q$$

$$K = H$$

$$\therefore K(Q, P) = \frac{1}{2m}(2m\omega P\cos^2 Q + 2m\omega P\sin^2 Q) = \omega P = E$$

$$\therefore \dot{Q} = \frac{\partial K}{\partial P} = \omega \Rightarrow Q = \omega t + \varphi$$

$$\dot{P} = -\frac{\partial K}{\partial \dot{Q}} = 0 \Rightarrow P = C \qquad \therefore q = \sqrt{\frac{2C}{m\omega}}\sin(\omega t + \varphi)$$

是否满足正则变换

$$\dot{Q}_i = \frac{\partial K}{\partial P_i} = \frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial P_i} + \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial P_i}$$

$$\dot{P}_i = -\frac{\partial K}{\partial Q} = \frac{\partial H}{\partial q_j}\frac{\partial q_j}{\partial Q} + \frac{\partial H}{\partial p_j}\frac{\partial p_j}{\partial Q}$$

$$[Q_i, H] = \frac{\partial Q_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial Q_i}{\partial p_j}\frac{\partial H}{\partial q_j}$$

$$[P_i, H] = \frac{\partial P_i}{\partial q_j}\frac{\partial H}{\partial p_j} - \frac{\partial P_i}{\partial p_j}\frac{\partial H}{\partial q_j}$$

泊松括号

$[u, u] = 0$

$[u, v] = -[v, u]$

$[au+bv, w] = a[u, w] + b[v, w]$

$[uv, w] = [u, w]v + [v, w]u$

雅可比恒等式 $[u, [v, w]] + [v, [w, u]] + [w, [u, v]] = 0$

$[q_j, p_k] = \delta_{jk}$

$\frac{du}{dt} = [u, H] + \frac{\partial u}{\partial t} = 0$ 守恒