The Note of Quantum Information and Quantum Computation

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§1 Lecture 1 :Introduction

§1.1 About the Course

§1.1.1 Class:

Web. Class 1 - 4 - 201

§1.1.2 Email:

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§1.1.3 References:

(1)YongZhang Online lecture notes on QIC Version 5

(2)Nielsen and Chuang : QC and QI

§1.2 Information and Computation

§1.2.1 About

David Deutsch,1985

What computers can or cannot compute is determined by the law of physics and not mathematics

Information is physical (Rolf Landauer 1961) and is encoded in the state of a physical system

Classical Information ->encoded Classical System

quantum information ->encoded quantum System

Computation is a physical process (David 1985) and is performed is an physical realizable process

Universe is quantum Information

§1.2.2 Definition of quantum Information and quantum Computation

1. QIC is the study of using fundamental principles of quantum mechanics to perform information processing and computational tasks

2. QIC is the study of performing information processing and computation tasks in quantum mechanical system

3. QIC is the study of combing quantum systems and classical systems to perform information process and computational tasks

4. QIC represents a further development of quantum mechanics and understands fundamental principles of quantum mechanics from the point of information and computation

Think style of QI and QC

- think about information and computation physically namely detise physical system to represent and process information
- think about physics computationally and informationally namely describe physics in terms of information and computation

§2 Lecture 2 :QUBITS

§2.1 Basic concepts of qubit

Qubit Quantum Binary Digit

Unit Smallest unit of quantum information

Range set Two dimensional Hilbert space(linear space)

State:

$$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle \tag{1}$$

Remark 2.1. Vectors as an element of linear space[$|0\rangle, |1\rangle$]
$|0\rangle, |1\rangle$ are orthogonal basis of $\mathscr{H}_2 = space\{|0\rangle, |1\rangle\}$
and $|\alpha|^2 + |\beta|^2 = 1$
Hidden information is infinity

COPY:No-cloning theorem(No perfect quantum copy)
Observed Information:2 state
QIC is a kind of ART
$\infty \to 2$
$|\psi\rangle = \alpha |0\rangle + \beta |1\rangle$
Quantum measurements (Wave function collerse) NOT unitary evolution NOT Schrodinger equation
Probability $|\alpha|^2$[After measurement]
$|\psi\rangle \to |\alpha|^2 |0\rangle$
Information loss
Irreversible process
Un-unitary process

QIC is powerful because its hidden information is infinitely large due to linear superposition principle

But it is difficult to manipulate such hidden information

Because of quantum measurement the weirdness

Classical bits : 0 or 1[2 Choice]

Quantum bits : $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$ has [∞ choice]

$\hat{A}$ operator and $|\psi_+\rangle$ state : $\hat{A}|\psi_+\rangle = +1|\psi_+\rangle$

$\hat{A}|\psi_-\rangle = -1|\psi_-\rangle$

§2.2 The state of vector formalism of a qubit

Start

$\mathscr{H}_2 = span\{|0\rangle, |1\rangle\}$

Computation basis

$|0\rangle = (1, 0)$ And $|1\rangle = (0, 1)$

A normalized state vector modules global(phase factor)

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle \tag{2}$$

$|\alpha|^2 + |\beta|^2 = 1, \alpha$ and $\beta \in$ Complex Number

$\alpha, \beta \rightarrow 4$ real numbers

$|\alpha|^2 + |\beta|^2 = 1 \rightarrow$ one real constraints

Irrelevant global phase $\rightarrow$ one real constrains

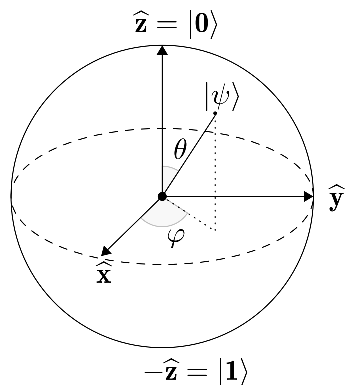
Two indepent real parameters to characterize a qubit

$$|\psi_+\rangle = cos\frac{\theta}{2}|0\rangle + e^{i\varphi}sin\frac{\theta}{2}|1\rangle \tag{3}$$

When $\vec{n} = (1, 0, 0) = \vec{e}_x|_{\theta=\frac{\pi}{2},\varphi=\pi}$

we get $|\psi(\vec{e}_x)\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) = |+\rangle$

Bloch vector $\vec{n} = (sin\theta cos\varphi, sin\theta sin\varphi, cos\theta)$



$\vec{e}_y = (0, 1, 0)$so we can get:

$|\psi_+(\vec{e_y})\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle) {=} |+\rangle^{'}, |\psi(-\vec{e_y})\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle) = |-\rangle^{'}$
as the same way $\vec{e_z} = (0,0,1), -\vec{e_z} = (0,0,-1)$
so$|\psi(\vec{e_z})\rangle = |0\rangle, |\psi(-\vec{e_z})\rangle = |1\rangle$
$\left|\psi_+\vec{(n)}\right\rangle$depends on the value of θ and φ

Bloch Sphere is a qubit

§2.3 The Stabilzer formalism of a qubit

Group Theory(concepts such as Pauli Group)

$|\psi_+(\theta,\varphi)\rangle$ is completely fixed as the eigenstate of $\sigma_n = \vec{\sigma}\cdot\vec{n}$ with eigenvalue
And $|\psi_+(\theta,\varphi)\rangle, 0 \le \theta \le \pi, 0 \le \varphi \le 2\pi$ in the bolch sphere

§2.3.1 Spin - 1/2 operator

$$\vec{J} = \frac{1}{2}\hbar\vec{\sigma} \tag{4}$$

$\vec{\sigma} = (\sigma_x, \sigma_y, \sigma_z)$[Pauli matrix],And $\{|\psi\rangle\ \ |\sigma_n|\psi\rangle = |\psi\rangle\}$

$$\sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \sigma_y = \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \sigma_z = \sigma_3 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Theorem 2.2

If we define $\sigma_n = \vec{\sigma}\cdot\vec{n}$

so $\vec{n} = (n_x, n_y, n_z) = (sin\theta cos\varphi, sin\theta sin\varphi, cos\theta)$ is the Bloch Vector

and $\sigma_n\,|\psi_+(\theta,\phi)\rangle = |\psi_+(\theta,\phi)\rangle$

Proof.

$$\vec{\sigma_n} = \vec{\sigma}\cdot\vec{n} = \sigma_x n_x + \sigma_y n_y + \sigma_z n_z = sin\theta cos\varphi\sigma_x + sin\theta sin\varphi\sigma_y + cos\theta\sigma_z$$

$$= sin\theta cos\varphi \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + sin\theta sin\varphi \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + cos\theta \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} cos\theta & sin\theta e^{-i\varphi} \\ sin\theta e^{i\varphi} & -cos\theta \end{pmatrix}$$

□

namely

$$\sigma_n = \begin{pmatrix} cos\theta & sin\theta e^{-i\varphi} \\ sin\theta e^{i\varphi} & -cos\theta \end{pmatrix} \tag{5}$$

$|\psi_+(\theta,\varphi)\rangle = \begin{pmatrix} cos\frac{\theta}{2} \\ e^{i\varphi}sin\frac{\theta}{2} \end{pmatrix}$ and we can easily prove $\sigma_n\,|\psi_+(\theta,\varphi)\rangle = |\psi_+(\theta,\varphi)\rangle$
Stabilizer formalism of qubit
expectation value

Theorem 2.3

$$\langle\psi_+(\theta,\varphi)|\vec{\sigma}\cdot\vec{m}|\psi_+(\theta,\varphi)\rangle = \vec{m}\cdot\vec{n}$$

Remark 2.4. Note:

$(\vec{\sigma}\cdot\vec{n})(\vec{\sigma}\cdot\vec{n}) = (\sigma_i n_i)(\sigma_j n_j) = (\sigma_i\sigma_j)(n_i n_j) = (\delta_{ij} + i\varepsilon_{ijk}\sigma_k)(n_i n_j) = I$

$\vec{n} = (n_x, n_y, n_z) = (sin\theta cos\varphi, sin\theta sin\varphi, cos\theta), \vec{m} = (m_x, m_y, m_z) = (sin\theta' cos\varphi', sin\theta' sin\varphi', cos\theta')$

σ_n has ± 1 eigenvalue

ψ_+:eigenvalue = 1 ψ_-:eigenvalue = -1

$$|\psi_+(\theta,\varphi)\rangle = \begin{pmatrix} cos\frac{\theta}{2} \\ e^{i\varphi}sin\frac{\theta}{2} \end{pmatrix} \tag{6}$$

$$|\psi_-(\theta,\varphi)\rangle = \begin{pmatrix} -e^{-i\varphi}sin\frac{\theta}{2} \\ cos\frac{\theta}{2} \end{pmatrix} \tag{7}$$

Example 2.5

$n = \vec{e}_x, \sigma_n = \sigma_x = \sigma_1 = X$[Pauli-X gate]

$X\,|\psi_+(\pm\vec{e}_x)\rangle = \pm\,|\psi_+(\pm\vec{e}_x)\rangle \to \sigma_x\,|\pm\rangle = \pm\,|\pm\rangle$

Example 2.6

$n = \vec{e}_y, \sigma_n = \sigma_y = \sigma_2 = X$[Puali-Y gate]

$\sigma_y\,|\pm\rangle' = \pm\,|\pm\rangle'$

Example 2.7

$\sigma_z\,|\psi_+(\pm\vec{e}_z)\rangle = \pm\,|\psi_+(\pm\vec{e}_z)\rangle$

quantum System Open System

Schrodinger eq. $\to$ Closed System

(State Vector)

We consider the density matrix of a qubit

§2.3.2 ρ:density matrix

Remark 2.8. For the expression in terms of density matrix we have:

$$tr(\rho(\vec{\sigma}\cdot\vec{m})) = \vec{n}\cdot\vec{m}$$

where$\rho = |\psi_+(\theta,\varphi)\rangle\langle\psi_+(\theta,\varphi)|$

and ρ has some property

$$\rho \geq 0$$

$$tr(\rho) = 1$$

$$\rho^+ = \rho$$

Density matrix of a quantum statistic mechanics

$\vec{P}$ = polarization vector,$|\vec{P}| \leq 1$

$$\rho(\vec{P}) = \frac{1}{2}(I_2 + \vec{P}\cdot\vec{\sigma})$$

so how to realize the qubit

- Electron
  - Spin:$\frac{1}{2}$
  - Mass:0.5Mev
  - Qubit:Spin-State
- Photon
  - Spin:1
  - Mass:0
  - Qubit:Photon Polarization

§2.4 Two bit system

The corpotational basis of orthonormal=

$$span|x_1x_2\rangle, x_1, x_2 = 0, 1 = span|00\rangle, |01\rangle, |10\rangle, |11\rangle$$

$$|\psi\rangle = \sum_{x_1,x_2}\alpha_{x_1,x_2}|x_1x_2\rangle$$

Genetri picture of 2-qubit-system

and we define: $|x_1x_2\rangle = |x_1\rangle\otimes|x_2\rangle$[Tensor Product]

Maximalled entanglend state

For examples : Bell States

$$|\beta_{x,y}\rangle$$

Bell states are maximally entangled two-qubit pure states,also named as EPR pair states

Example 2.9

$$\beta_{x_1,x_2} = \frac{1}{\sqrt{2}}(|0x_2\rangle + (-1)^{x_1}|1\bar{x_2}\rangle)[\bar{x_2} = 1 \otimes x_2]$$

(NOT Gate is the quantum analog of classical logical gate X_{OR} Gate)

Remark 2.10. In classical world states must be orthogonal to each other due to non-superposition

Therefore the classical copy machines allowed to exist

§2.5 Two qubit pure states and Bell states

Definition 2.11. A two- qubit is described by $\mathscr{H}_4$ with the computational basis

$$\mathscr{H}_4 = span\{|x_1x_2\rangle, x_1, x_2 = 0, 1\} = span\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$$

$$|\psi\rangle \in \mathscr{H}_4, |\psi\rangle = \sum_{x_1,x_2} \alpha_{x_1,x_2} |x_1x_2\rangle$$

Remark 2.12. If a two-qubit pure state can not be expressed as $|\psi\rangle = |\alpha\rangle \otimes |\beta\rangle$

$$|\alpha\rangle, |\beta\rangle \in \mathscr{H}_4$$

then it is called an to entangled pure state

otherwise if a two qubit pure state:

$$|\psi = |\alpha\rangle \otimes |\beta\rangle\rangle$$

called separable state or product state

$$|\beta_{x_1,x_2}\rangle = \frac{1}{\sqrt{2}}(|0x_2\rangle + (-1)^{x_1} |1\bar{x}_2\rangle)$$

$\bar{x_2} = 1 \oplus x_2$ and $|\psi(i,j)\rangle$ can be expressed as $(I_2 \otimes X^i Z^j) |\psi(0,0)\rangle$ while $i, j = 0, 1$

Lemma 2.13

$$|\psi(i,j)\rangle = (I_2 \otimes X^i Z^j)\,|\psi(0,0)\rangle = \frac{1}{\sqrt{2}}(I_2 \otimes X^i Z^j)(|00\rangle + |11\rangle)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle \otimes X^i Z^j\,|0\rangle + |1\rangle \otimes X^i Z^j\,|1\rangle) = \frac{1}{\sqrt{2}}(|0\rangle \otimes X^i\,|0\rangle + (-1)^j\,|1\rangle \otimes X^i\,|1\rangle)$$

$$= \frac{1}{\sqrt{2}}(|0\rangle \otimes |i\rangle + (-1)^j\,|1\rangle \otimes |\bar{i}\rangle)$$

Remark 2.14. $\langle\psi(i,j)|\psi(i,j)\rangle = 1$

(1) For the case of $i = j = 0$ we have $|\psi(0,0)\rangle = |\psi(0,0)\rangle$

(2)For the case of $i = 0, j = 1$

$$RHS = (I_2 \otimes Z)\,|\psi(0,0)\rangle = (I_2 \otimes Z)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) = |\psi(0,1)\rangle$$

(3)For the case of $i = 1, j = 0$

$$RHS = (I_2 \otimes X)\,|\psi(0,0)\rangle = (I_2 \otimes X)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) = |\psi(1,0)\rangle$$

(4)For the case of $i = j = 1$

$$RHS = (I_2 \otimes XZ)\,|\psi(0,0)\rangle = (I_2 \otimes XZ)\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

$$= (I_2 \otimes X)\frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)\frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) = |\psi(1,1)\rangle$$

Example 2.15

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$$

and so on[relative phase +,-]

Note1:$|\beta_{x_1,x_2}\rangle$ are called Bell states or the EPR states
EPR is Einstein-Pdolsky-Rossen
Note2:The Bell states are maximally entangled bipartile pure state

Orthognal Relation:

$$\langle \beta_{x_1,x_2}\rangle \, \beta'_{x_1,x_2} = \delta_{x_1 x_1'} \delta_{x_2 x_2'}$$

Completness Relation:

$$\sum_{x_1,x_2=0} \left|\beta_{x_1,x_2}\right\rangle \left\langle \beta_{x_1,x_2}\right| = I_4 = I_2 \otimes I_2$$

Definition 2.16. Bell Transform is a unitary basis transformation matrix from the computational basis to the bell basis

Remark 2.17.
- $|\psi(i,j)\rangle \, |i,j = 0,1$ is the Bell basis of $\mathscr{H}_2 \otimes \mathscr{H}_2$
- $|i,j\rangle \, |i,j = 0,1$ is the product basis of $\mathscr{H}_2 \otimes \mathscr{H}_2$

§2.6 Three Qubit Pure State and GHZ states

Def : A three-qubit is a composite system of three qubit

$$\mathscr{H}_{2^3} = \mathscr{H}_8 = \mathscr{H}_2 \otimes \mathscr{H}_2 \otimes \mathscr{H}_2$$

$\mathscr{H}_8 = span\{|x_1x_2x_3\rangle\,, x_1, x_2, x_3 = 0, 1\}$

$$|\psi\rangle \in \mathscr{H}_8, |\psi\rangle = \sum_{x_1,x_2,x_3} \alpha_{x_1x_2x_3} \left|x_1x_2x_3\right\rangle$$

and $\sum_{x_1,x_2,x_3} |\alpha_{x_1x_2x_3}|^2 = 1$

Example 2.18

For example :
$|GHZ(0,0,0)\rangle_{\pm} = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$
The GHZ states forms an orthonormal basis called the GHZ basis

The GHZ transform is a unitary basis transform matrix from the computation basis of GHZ basis

§2.7 n-qubit pure state and n-qubit GHZ states

$$\mathscr{H}_{2^n} = span\{|x_1x_2\cdots x_n\rangle \, x_1, x_2 \cdots x_n = 0, 1\}$$

$$|\psi\rangle \in \mathscr{H}_{2^n}, |\psi\rangle = \sum_{x_1x_2x_3\cdots} \alpha_{x_1x_2\cdots x_n} \left|x_1x_2\cdots x_n\right\rangle$$

And $\sum_{x_1x_2\cdots x_n} |\alpha_{x_1x_2\cdots x_n}|^2 = 1$

The n-qubit GHZ states

$$|GHZ(x_1,x_2,\cdots,x_n)\rangle_{\pm} = \frac{1}{\sqrt{2}}(|x_1x_2\cdots x_n\rangle \pm |\bar{x_1}\cdots\bar{x_n}\rangle)$$

Maximalled entangled n-qubit pure state

§3 Lecture 3:Quantum Circuit Model

Definition 3.1. Quantum Gates:

A quantum gate U

describes a change of a quantum state vector

satisfy

$$U(\alpha|\psi\rangle + \beta|\varphi\rangle) = \alpha U|\psi\rangle + \beta U|\varphi\rangle$$

Gate U satisfy $U^{\dagger}U = I$

> **Remark 3.2.** The linearty is associaated with linearly superposition principle
>
> The unitary is a associated with:
>
> The probaility conservation
>
> $$\langle\psi\rangle\,\varphi = \langle\psi|\,U^{\dagger}U\,|\varphi\rangle = 1$$
>
> The Schrodinger equation give rise to unitary time evolution
>
> $$|\psi(t)\rangle = U(t)\,|\psi(t)\rangle$$

Definition 3.3. Single qubit Gates :U are 2×2 unitary matrices denoted by U(2) group

Definition 3.4. n-qubit gate U are $2^n\times2^n$ unitary matrix defined by U(2^n) group

Definition 3.5. Quantum Circuit Model is circuit model of quantum computation and its describes a squance of a finite number of quantum gates actigon a finite number of qubits

> **Remark 3.6.** Circuit Model is very special type of computation model in computer science
>
> **Turing Machine**
>
> Digrantial represetation of a circuit model consist of wires wires and gates

§3.1 single qubit gate

Some basic gate:

§3.1.1 The identity gate

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

§3.1.2 The rotational gate

(gate around the x-axis about angle α)

$$R_x(\alpha) = e^{-i\frac{\sigma_x}{2}\alpha} = \begin{pmatrix} cos\frac{\alpha}{2} & -isin\frac{\alpha}{2} \\ -isin\frac{\alpha}{2} & cos\frac{\alpha}{2} \end{pmatrix} = cos\frac{\alpha}{2}I - i\sigma_x sin\frac{\alpha}{2}$$

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, R_x(\pi) = \sigma_x$$

$$x\left|i\right\rangle = \left|\bar{i}\right\rangle$$

[Quantum NOT gate]

§3.1.3 The rotational Gate around Z-axis angle δ

$$R_z(\delta) = e^{-i\frac{\sigma_z}{2}\delta} = cos\frac{\delta}{2}I - i\sigma_z sin\frac{\sigma}{2} = \begin{pmatrix} e^{-i\frac{\delta}{2}} & 0 \\ 0 & e^{i\frac{\delta}{2}} \end{pmatrix}$$

Example 3.7

Pauli Z gate

$$z = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = iR_z(\pi)$$

$$z\left|i\right\rangle = (-1)^2\left|i\right\rangle, i = 0, 1$$

$$z\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \begin{pmatrix} \alpha \\ -\beta \end{pmatrix}$$

Example 3.8

The phase gate

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = e^{i\frac{\pi}{4}}\begin{pmatrix} e^{-i\frac{\pi}{4}} & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix} = e^{i\frac{\pi}{4}}R_z(\frac{\pi}{2})$$

Example 3.9

The T-gate or $\frac{\pi}{8}$ gate

$$T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\frac{\pi}{4}} \end{pmatrix} = e^{i\frac{\pi}{8}} \begin{pmatrix} e^{-i\frac{\pi}{8}} & 0 \\ 0 & e^{i\frac{\pi}{8}} \end{pmatrix}$$

Example 3.10

The phase-shift gate:

$$R_\theta = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{pmatrix} = e^{i\frac{\theta}{2}} \begin{pmatrix} e^{-i\frac{\theta}{2}} & 0 \\ 0 & e^{i\frac{\theta}{2}} \end{pmatrix} = e^{i\frac{\theta}{2}} R_z(\theta)$$

It satisify

$$T^2 = S, S^2 = Z$$

§3.1.4 The rotational gate around the y-axis about angle γ

$$R_y(\gamma) = e^{-i\frac{\sigma_x}{2}\gamma} = cos\frac{\gamma}{2} I - i\sigma_y sin\frac{\gamma}{2} = \begin{pmatrix} cos\frac{\gamma}{2} & -isin\frac{\gamma}{2} \\ isin\frac{\gamma}{2} & cos\frac{\gamma}{2} \end{pmatrix}$$

$$\sigma_y = iR_y(\pi) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

rotation y axis can be expressed by rotation R_x, R_z

$$R_y(\alpha) = R_z(\frac{\pi}{2}) R_x(\alpha) R_z(-\frac{\pi}{2})$$

$$R_z(\alpha) = R_y(-\frac{\pi}{2}) R_x(\alpha) R_y(\frac{\pi}{2})$$

§3.1.5 Hadamard Gate is a basis transformation from the eigenstate of σ_z to the σ_x

$$H|0\rangle = |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), H|1\rangle = |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$$

basis transform gate

$$H|i\rangle = \frac{1}{\sqrt{2}}((-1)^i |i\rangle + |i\rangle)$$

while $i = 0, 1$

and Hadamard gate can be expressed as $H = \frac{1}{\sqrt{2}}(X + Z) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$

Namely H is a similarity transformation of diagonalization $\sigma_x \Leftrightarrow \sigma_z$

$$HXH = Z, HZH = X, HYH = -Y, H^\dagger = H, H^2 = I$$

Or if we in the point of rotation operator:

We have:

$$H = \frac{1}{\sqrt{2}}(\sigma_x + \sigma_z) = \frac{1}{\sqrt{2}}(\vec{e}_x + \vec{e}_z) \cdot \vec{\sigma} \tag{8}$$

so the $H = \hat{n} \cdot \vec{\sigma}$

while $\hat{n} = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$

Thus

$$iH = exp(-i\frac{\theta}{2}\hat{n} \cdot \vec{\sigma})$$

Definition 3.11.

$$U(\hat{n}, 0) = exp(-i\frac{\hat{n} \cdot \vec{\sigma}}{2}\theta)$$

We have $e^{iA\theta} = cos\theta I + sin\theta A$(Use the Taylor Series)

So $exp(-i\frac{\hat{n}\cdot\vec{\sigma}}{2}\theta) = (cos\theta - isin\theta)^{\frac{\hat{n}\cdot\sigma}{2}}$

When $\theta = \pi$ we have:

$$U(\hat{n}, \pi) = -i\sigma_n sin\frac{\pi}{2} = -i\sigma_n = -iH \tag{9}$$

so $H = iU(\hat{n}, \pi)$

Any single-qubit gate $A \in SU(2)$

Can be expressed as $A = e^{i\alpha}R_z(\beta)R_y(r)R_z(\delta)$ with real parameters $\alpha, \beta, \gamma, \delta$

$$A = e^{i\alpha}\begin{pmatrix} e^{i\frac{\beta+\delta}{2}}cos\frac{\gamma}{2} & -e^{-i\frac{\beta-\delta}{2}}sin\frac{\gamma}{2} \\ e^{i\frac{\beta-\delta}{2}}sin\frac{\gamma}{2} & e^{i\frac{\beta+\delta}{2}}cos\frac{\gamma}{2} \end{pmatrix}$$

Remark 3.12.

$$A = e^{i\alpha}R_\alpha(\beta + \frac{\pi}{2})R_x(r)R_\alpha(\delta - \frac{\pi}{2})$$

$$H = iR_x(\pi)R_y(\frac{\pi}{2}) = iR_y(\frac{\pi}{2})R_z(\pi)$$

§4 Lecture 4 Quantum Gate 1

Definition 4.1. n -qubit gates are $2^n \times 2^n$ mitary matrices $U^\dagger U = I_2$

A set of n-qubit gates form a represent of group $U(2^n)$

eg: Two-qubit gates are 4×4unitary matrices gives a represention of $U(4)$ group

§4.1 SWAP Gate

Definition 4.2. The SWAP gate care also called permatation

The classical $SWAP$gate $SWAP(x, y) = (y, x)x, y \in z_2$

The quantum SWAP $SWAP\left|x\right\rangle\left|y\right\rangle = \left|y\right\rangle\left|x\right\rangle$

Example 4.3

$$SWAP\left|00\right\rangle = \left|00\right\rangle, SWAP\left|01\right\rangle = \left|10\right\rangle, SWAP\left|10\right\rangle = \left|01\right\rangle, SWAP\left|11\right\rangle = \left|11\right\rangle$$

$$SWAP = \left\langle 00\right|\left|00\right\rangle + \left\langle 01\right|\left|10\right\rangle + \left\langle 10\right|\left|01\right\rangle + \left\langle 11\right|\left|11\right\rangle = \sum_{i,j=0}\left|ij\right\rangle\left\langle ji\right|$$

we can express it as matrix

$$\left|00\right\rangle = \begin{bmatrix}1\\0\\0\\0\end{bmatrix}, \left|01\right\rangle = \begin{bmatrix}0\\1\\0\\0\end{bmatrix}, \left|10\right\rangle = \begin{bmatrix}0\\0\\1\\0\end{bmatrix}, \left|11\right\rangle = \begin{bmatrix}0\\0\\0\\1\end{bmatrix}$$

so

$$SWAP = \begin{bmatrix}1 & 0 & 0 & 0\\0 & 0 & 1 & 0\\0 & 1 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

Definition 4.4. Control Unitary Gate:

$$CU = \left|c\right\rangle[control - qubit]\left|t\right\rangle[target - qubit]$$

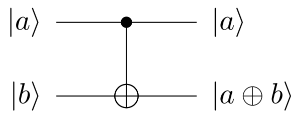
$$c = 0, \left|0\right\rangle\left|t\right\rangle \rightarrow \left|c\right\rangle\left|t\right\rangle$$

$$c = 1, \left|1\right\rangle\left|t\right\rangle \rightarrow \left|c\right\rangle U\left|t\right\rangle$$

§4.2 CNOT Gate

The CNOT-gate is the quantum analogue of the classical XOR gate

$$XOR(x, y) = (x, x \oplus y)$$



The quantum NOT gate is the pauli X gate

$$X = \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$CNOT_{12}$:1 is control qubit,2 is target qubit

$$CNOT_{12} = |0\rangle \langle 0| \otimes I_2 + |1\rangle \langle 1| \otimes X$$

so

$$CNOT = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

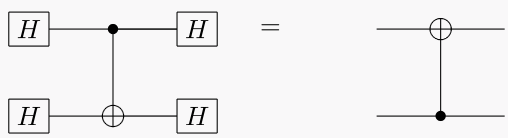
and the determine($CNOT_{12}$) = -1 and $CNOT_{12} \notin SU(4)$

Example 4.5

The CNOT gate is a perfect copy machine for the copy of the computational basis state $|0\rangle$and $|1\rangle$

$$CNOT\,|00\rangle = |00\rangle\,, CNOT\,|10\rangle = |11\rangle$$

Remark 4.6.



§4.2.1 The SWAP gate can be construted as a prouduct of three CNOT gates

$$SWAP = CNOT_{12}CNOT_{21}CNOT_{12}$$

Proof:

Step1:

$$|\psi_1\rangle = CNOT_{12}\,|\psi_0\rangle = CNOT_{12}\,|x\rangle\,|y\rangle = |x\rangle\,|x \oplus y\rangle$$

Step2:

$$|\psi_2\rangle = CNOT_{21}\,|\psi_1\rangle = CNOT_{21}\,|x\rangle\,|x \oplus y\rangle = |x \oplus y \oplus x\rangle\,|x \oplus y\rangle = |y\rangle\,|x \oplus y\rangle$$

Step3:

$$|\psi_3\rangle = CNOT_{12}\,|\psi_2\rangle = CNOT_{12}\,|y\rangle\,|x\oplus y\rangle = |y\rangle\,|x\oplus y\oplus x\rangle = |y\rangle\,|x\rangle$$

so SWAP gate three CNOT generation

§4.2.2 The Constrution of the Bell basis states using the CNOT gate and the Hadamarid gate

$$|\beta\rangle_{ij} = \frac{1}{\sqrt{2}}(I_2\otimes X^j Z^i)(|00\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|0j\rangle + (-1)^i\,|1j\rangle)$$

Phase bit $\langle x|x\rangle\,|\beta_{ij}\rangle = (-1)^i\,|\beta_{ij}\rangle$

Phase bit $\langle z|z\rangle\,|\beta_{ij}\rangle = (-1)^i\,|\beta_{ij}\rangle$

Note:

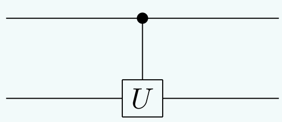
$$Z^i\,|0\rangle = |0\rangle\,, Z^i\,|1\rangle = (-1)^i\,|1\rangle$$

$$X^j\,|0\rangle = |j\rangle\,, X^j\,|1\rangle = |\bar{j}\rangle$$

§4.3 Decomposition of a two-qubit control unitary

Lemma 4.7

A control unitary two qubit gate can be decomposed as product of CNOT gate and single-qubit gates



$U = e^{i\alpha}A\times B\times C$,$ABC = I_2$

Proof:

$$CNOT = \begin{pmatrix} I_2 & 0 \\ 0 & X \end{pmatrix}$$

From the above figure the Controlled-U gate $= |0\rangle\langle 0|\otimes I_2 + |1\rangle\langle 1|\otimes U$

So the matrix form of it is :

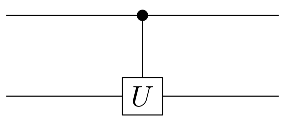
$$\begin{pmatrix} I_2 & 0 \\ 0 & e^{i\alpha}A\times B\times C \end{pmatrix} = \begin{pmatrix} ABC & 0 \\ 0 & e^{i\alpha}A\times B\times C \end{pmatrix}$$

$$= \begin{pmatrix} I_2 & 0 \\ 0 & e^{i\alpha}I_2 \end{pmatrix}\begin{pmatrix} A & 0 \\ 0 & A \end{pmatrix}\begin{pmatrix} I_2 & 0 \\ 0 & X \end{pmatrix}\begin{pmatrix} B & 0 \\ 0 & B \end{pmatrix}\begin{pmatrix} I_2 & 0 \\ 0 & X \end{pmatrix}\begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}$$

and

$$R_\alpha = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix}$$

so the



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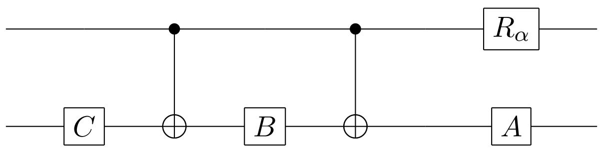


The composition is done!!

Theorem 4.8

Any control Unitary two-qubit gate can be discomposed as a product of CNOT gates and single-qubit gates

Proof:

$$U = e^{i\alpha} R_z(\beta) R_y(\gamma) R_z(\delta) \tag{10}$$

can be reformulated as :

$$U = e^{i\alpha} A \times B \times C \tag{11}$$

$$\boxed{A = R_z(\beta) R_y(\frac{\gamma}{2}), B = R_y(-\frac{\gamma}{2}) R_z(\frac{-\delta+\beta}{2}), C = R_z(\frac{\delta-\beta}{2})}$$

Ref:The **Barenco** gate (1985) is the controlled - $e^{-\frac{\pi}{4}} = R_x(\frac{\theta}{2})$ gate with irration $\theta/\pi = \alpha$

so corollary : The Barenco gate can be expressed as a product of CNOT gates and single qubit gate

§5 Lecture 5:Quantum Gate 2

§5.1 Toffoli gate

Quantum Toffoli gate and Fredkin gate: Def the quantum toffoli gate = Controlled -CNOT gate = Controlled -Controlled -NOT - gate

Toffoli gate:

$\text{Toffoli}(|x\rangle |y\rangle |z\rangle) \to |x\rangle |y\rangle |z \oplus xy\rangle$

Toffoli gate (123):

1,2 is control qubit

3 is target qubit

$\text{Toffoli123} = |0\rangle \langle 0| \otimes I_4 + |1\rangle \langle 1| \otimes CNOT_{23}$

The matrix form of Toffoli

Example 5.1

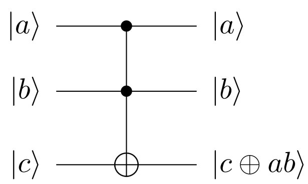
$x, y = 0$,$\text{Toffoli}(|x\rangle |y\rangle |z\rangle = |x\rangle |y\rangle |z\rangle)$

$x, y = 1$,$\text{Toffoli}(|x\rangle |y\rangle |z\rangle = |x\rangle |y\rangle |\bar{z}\rangle)$

The toffoli gate function:

Input: $|a\rangle$ and $|b\rangle$ and $|c\rangle$ the $|a\rangle$ and $|b\rangle$ is the control qubit and $|c\rangle$ is the target qubit

Output: $|a\rangle$ and $|b\rangle$ and $|c \oplus ab\rangle$



which is a classcial reversible gate for reversible computational

Quantum Fredkin gate is the Controlled-SWAP gate

(Intro SWAP gate)

SWAP gate can change the place of qubit.
For example:$\text{SWAP}(x, y) = (y, x)$

and its matrix form is

$$SWAP = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

so the SWAP form can be expressed below:

$$SWAP^x |y\rangle |z\rangle = |xz \oplus \bar{x}y\rangle |xy \oplus \bar{x}z\rangle$$

Proof. when $x = 0$,

$$Id |y\rangle |z\rangle = |1z \oplus 0y\rangle |1y \oplus 0z\rangle = |z\rangle |y\rangle$$

when $x = 1$,

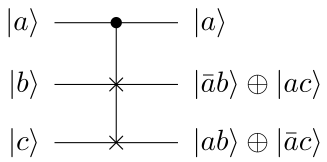
$$SWAP\,|y\rangle\,|z\rangle = |y\rangle\,|z\rangle = |0z \oplus 1y\rangle\,|0y \oplus 1z\rangle = |y\rangle\,|z\rangle$$

□

§5.2 Fredkin gate

$$SWAP\,|y\rangle\,|z\rangle = |1z \oplus 0y\rangle\,|1y \oplus 0z\rangle = |z\rangle\,|y\rangle$$

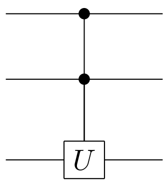
The quantum Fredkin Gate is the quantum analogue of the classical reversible gate



The quantum Fredkin gate is constructed as a product of three Toffoli gate

Proof omission

§5.3 Decomposition of a control unitary three qubit gate



Theorem 5.2

Any control unitary theorem three -qubit gates can be expressed as a product of control unitary two qubit gate

§6 Lecture 6:Quantum Gate 3

§6.1 Decompostion of any control unitary three qubit gate as can be pressed as a product of control unitary two qubit gate

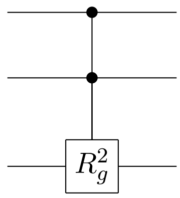
*

Corollary 1: A quantum Toffoli gate can be decomposed as a product of control unitary two qubit gate

Corollary 2: A quantum Fredkin gate can be decomposed as a product of control unitary two qubit gate

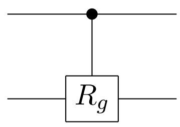
Corollary 3: The Deutsch gate(1985)

Dg =



$$\boxed{R_g = e^{-i\frac{\pi}{4}} R_x(\tfrac{\theta}{2}), R_g^2 = -iR_x(\theta)}$$

Bg[Barenco gate[1995]] =



The Deutsch gate can be decomposed as a product of the Barence gate

§6.2 Universal Quantum Gate Set

Definition 6.1. A universal Quantum gate set is a set of elementry gates so that any unitary matrix can be expressed as a product of elementary gates from this set

Any quantum gate:

Theorem 6.2

The Dentsch Gate[1985] is a universal quantum gate set

Theorem 6.3

The Barenco Gate[1995] with the swap gate forms a universal quantum gate set

Proof:

Step1 :The Dentsch Gate Dg can be expressed as a product of Bg,Bg$^+$ and CNOT

Step2 :The Bg$^+$ can be expressed as Bg SWAP and CNOT

Step3 :The CNOT is a product of Bg and SWAP

Step4 :The Dg is a product of Bg and SWAP

Definition 6.4. Genertic two qubit gate are defined as two qubit gate with eigenvalues $e^{i\theta_1}, e^{i\theta_2}, e^{i\theta_3}, e^{i\theta_4}$

while$\frac{\theta_i}{\pi}$ are irrational[无理数]

Theorem 6.5

Any generic[通用] two qubit gates with the swap gates forms a universal quantum gate set

Note:The Barenco Gate is not a generic two qubit gate

$\text{Bg} = |0\rangle\langle 0| \otimes I_2 + |1\rangle\langle 1| \otimes R_g$

Theorem 6.6

The CNOT gate with all single-qubits gates defines a universal quantum gate set

Proof:

Step1:Barenco gate with SWAP gives a universal quantum gate set

Step2:The Barenco gate can be expressed as CNOT with single - qubit gates

Step3:The SWAP gate can be expressed as a product CNOT

Therefore : [Barenco , SWAP] $\approx$ [CNOT,Single qubit gate]

§6.3 Finite number of single qubit gates Approxmately universal quantum computation

Definition 6.7. An approximately universal quantum gate set is a finiate set of elementary gate if any unitary matrix can be approximately composed as a product of elementary quanutum gates from this set to arbitrary precision

Theorem 6.8

All single - qubit gate can be approximately expressed as a product of the Hadmard gate $T = e^{i\frac{\pi}{4}} R_{\frac{\pi}{4}}(\theta)$

Theorem 6.9

The CNOT gate the Hadmard gate and T gate (the $\frac{\pi}{8}$) form an approximately universal quantum gate set.

Theorem 6.10

The Toffoli gate the Hadmard gate and the phase gate S form and approximately universal quantum gate set

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix} = e^{i\frac{\pi}{4}} R_{\frac{\pi}{2}}(\theta)$$

§6.4 Design of a unitary matrix is ART

Design unitary matrix as a product of elementary gates may be not efficient

It may require exponentially many gates

The design is a NP-Hard problem in computer science

§6.5 The Physical Implementation of Quantum Computer

Quantum computer

Definition 6.11. Quantum decoherence means the loss of quantum properties

Definition 6.12. Quantum noise leads to the loss of quantum information

The source of quantum noise against quantum computation include the computes of quantum computers the environment its operator and so on

Fact 1 Quantum Computer is a composite system of many subsystems

Fact 2 Quantum Computer is an open system

Interaction with the environment and its operator

In principle : Any quantum process can be regards as quantum computation but whether such quantum computation survives quantum noise or not

Is not an easy question to be answered

Definition 6.13. τ_Q is the minimal time that system remains its supposed quantum properties against quantum noise

$\tau_Q = \tau_1, \tau_2$

$\tau 1$: The relation time of an excited state $|e\rangle$ before returning to the ground state$|g\rangle$

$\tau 2$: The dephasing time of the relevant phase between $|e\rangle$ and $|g\rangle$

$$|\psi\rangle = \alpha |e\rangle + \beta |g\rangle \, [\alpha^2 + \beta^2 = 1]$$

density : $\rho = |\psi\rangle \langle\psi|$

so $\rho = |\alpha|^2 |e\rangle \langle e| + |\beta|^2 |g\rangle \langle g|$

coherence measurement

§7 Lecture 7:Physical Implementation of QC

§7.1 Fundamental principles of selecting a quantum system for quantum computation

Definition 7.1. τ_{op} is the time scale of performing a typical quantum operation such as a quantum gate

Definition 7.2. τ_Q is the time scale and $n_{op} = \frac{\tau_Q}{\tau_{op}}$ is the maximum number of operations that can be performed on a quantum computer before quantum noise kills required quantum properties of the quantum computer

Example 7.3
- Nuclear Spins:$n_{op} = \frac{\tau_Q}{\tau_{op}} \approx 10 - 10^{14}$
- Electron Spins:$n_{op} = 10^4$
- Optical Cavity $n_{op} = 10^9$

Fundamental Principle

n_{op} is required to be as large as possible [enough] to perform all neccessary quantum operations before quantum properties are destroyed by quantum noise

§7.2 Guding principle of building a QC

§7.2.1 Principle 1: Robust representation of a qubit against quantum noise

Note 1:Physcial representation of a qubit A natural qubit is an electron spin $\mathscr{H}_2 = \text{span}\{|0\rangle_L, |1\rangle_L\}$

logical qubit

$\mathscr{H}_2 = \text{span}\{|\uparrow\rangle, |\downarrow\rangle\}$

Note 2:A natural two qubits :two electronic spin states

$$|00\rangle_L = |\uparrow\uparrow\rangle, |01\rangle_L = |\uparrow\downarrow\rangle, |10\rangle_L = |\downarrow\uparrow\rangle, |11\rangle_L = |\downarrow\downarrow\rangle$$

A natural two - qubit a spin - $\frac{3}{2}$ particles spin states

$$|00\rangle_L = |\frac{3}{2}\rangle, |01\rangle_L = |\frac{1}{2}\rangle, |10\rangle_L = |-\frac{1}{2}\rangle, |11\rangle_L = |-\frac{3}{2}\rangle$$

Note 3:A well controlled qubit can be specified and controlled in the Rabi Oscillation

Note 4:A robust representaion of a qubit is discrete (finite) and protected by symmetry or topology or Integrability or else

§7.2.2 Principle 2:Able to prepare an initial qubit state

Note1:The computation basis state$|0\rangle^{\otimes n} = |0\rangle \otimes |0\rangle \otimes |0\rangle \cdots |0\rangle$ is a natural condida to as an inital state

It is not easy to keep all qubits on the same state such as $|0\rangle$ due to time evolution and quantum decoherence

Note2:The initial qubit state is a pure state instead of a mixed state

Note3:The initial state is able to survive for a sufficianty long time against noise

Other quantum states can be obtrived? as by the action of quantum gates on the initial state

§7.2.3 Principle 3:Able to perform a quantum measurement

Quantum measurement is indeed by the interaction of a quantum corpfer? with a classical device

Note 1:Strong measurement means strong couples and the wave function collapse

Note 2:Weak measurement means couplings is week

§7.2.4 Principle 4:Able to perform a universal quantum gate state

CONTROL SYSTEM

$$H_{total} = H_{QC} + H_{control} + H_{interaction}$$

Quantum gates are induced by the dymacial evoluation of the interaction Hamiltonian in Dirac picture [or in the interaction picture]

$U(t) = e^{\frac{iHt}{\hbar}}$

Note1:The Rabi oscillation yields a cell controlled qubit with cell defined quantum gates

Note2:The control Hamltonian induces quantum decoherence

Exact universal quantum gate set

CNOT ,All single -qubit gates

Approximately unniversal quantum gate set [CNOT,H,T]

§7.2.5 Principle 5:Able to perform classical computation

For example How to decompose an arbitary unitary matrix as a product of elementary gates

§7.2.6 Principle 6:Able to quantum communication

Between Different qubit

Note1:Transportation of flying qubit [such as photos]

Note2:Interaction between fixed qubits and flying qubit

§7.2.7 Principle 7:Able to perform faulty tolerant quantum corpatation

Able to perform guarantee error correcting codes to correct quantum errors

$$|\psi(t)_L\rangle = e^{-\frac{iwt}{2}}(\alpha|0\rangle + \beta e^{-iwt}|1\rangle) = e^{-\frac{iwt}{2}}\begin{pmatrix}1 & 0\\ 0 & e^{-iwt}\end{pmatrix}\begin{pmatrix}\alpha\\ \beta\end{pmatrix}$$

$$|\psi(t)_L\rangle = e^{-\frac{iwt}{2}}U(t)\,|\psi(0)\rangle_L$$

A logical single qubit gate

for example

Example 7.4

S gate eg1 : $t = \frac{3\pi}{2w}$

$$U(t)_{|t=\frac{3\pi}{2w}} = \begin{pmatrix}1 & 0\\ 0 & i\end{pmatrix} = S$$

Phase gate

eg2:$t = \frac{7\pi}{4w}$

$$U(t)_{|t=\frac{7\pi}{4w}} = \begin{pmatrix}1 & 0\\ 0 & e^{i\frac{\pi}{4}}\end{pmatrix} = T$$

$\frac{\pi}{8}$**gate**

eg3:$t = \frac{3\pi}{w}$

$$U(t)_{|t=\frac{7\pi}{4w}} = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix} = Z$$

About physical realization of the pauli-x gate

$$|0\rangle_L = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = |+\rangle$$

$$|1\rangle_L = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = |-\rangle$$

§8 Lecture 8:Harmonic Oscillator Quantum Computer

§8.1 A quantum simple harmonic oscillator

Dirac

$$H_0 = \hbar w(a^\dagger a + \frac{1}{2}) \tag{12}$$

a: energy-level lowering operator,w: frequency

In quantum field theory In quantum optics

a : annihilation operator $a^\dagger$: creation operator

$$a|n\rangle = \sqrt{n}|n-1\rangle \tag{13}$$

$|n\rangle$ is the excited state

$a^\dagger$: energy-level raising operator

so

$$a^\dagger|n\rangle = \sqrt{n+1}|n+1\rangle \tag{14}$$

so the $H_0|n\rangle = E_n|n\rangle$

energy eigenstate:$|n\rangle$

energy eigenvalue:$E_n = (n+\frac{1}{2})\hbar w$

Time evolution of the energy eigenstate

$|n(t)\rangle = e^{-iH_0t/\hbar}|n\rangle = exp(-iwt(n+\frac{1}{2}))|n\rangle$

the global phase can be neglected in QM:

$|n(t)\rangle = e^{-iwt/2}e^{-inwt}|n\rangle$

initial state

$\psi(t_0) = \sum_n c_n|n\rangle$

the time evolution of quantum state we have

$|\psi(t)\rangle = \sum_n c_n|n(t_0)\rangle$

we obtain the global phase factor we get :

$|\psi(t)\rangle = e^{-iwt/2}\sum_n c_n e^{-iwnt}|n\rangle$

A quantum state is automatically change due to the time evolution

§8.2 Physical construction of logical qubits

- Physical Realization of a logical qubit $\mathscr{H}_2 = span\{|0\rangle_L, |1\rangle_L\}$

  L:Logical,eg:$|0\rangle_L = |0\rangle$,$|1\rangle_L = |1\rangle$

- $|0\rangle_L = |n\rangle, |1\rangle_L = |n+1\rangle$

  and $E_{n+1} = (n+\frac{3}{2})\hbar w$

- $|0\rangle_L = |n\rangle, |1\rangle_L = |m\rangle, m > n+1$

  $E_m = (m+\frac{1}{2})\hbar w > E_n$

  Hence $|0\rangle_L = |0\rangle, |1\rangle = 1$ is the best choice for a physical implementation of a logical qubit because $E_1 = \frac{3}{2}\hbar w$

Example 8.1

$|00\rangle_L = |0\rangle, |01\rangle_L = |2\rangle\, |10\rangle_L = |4\rangle\, |11\rangle_L = |1\rangle$

Example 8.2

$|00\rangle_L = |0\rangle, |01\rangle_L = |2\rangle, |10\rangle_L = \frac{|4\rangle+|1\rangle}{\sqrt{2}}$

$$|11\rangle_L = \frac{|4\rangle-|1\rangle}{\sqrt{2}}$$

Example 8.3

$|00\rangle_L = |0\rangle, |01\rangle_L = \frac{|4\rangle+|1\rangle}{\sqrt{2}}$

$$|10\rangle_L = \frac{|4\rangle-|1\rangle}{\sqrt{2}}, |11\rangle_L = |2\rangle$$

with two simple harmonic oscillator

$$|00\rangle_L = |0\rangle_1 \otimes |0\rangle_2$$

$$|01\rangle_L = |0\rangle_1 \otimes |1\rangle_2$$

$$|10\rangle_L = |1\rangle_1 \otimes |0\rangle_2$$

$$|11\rangle_L = |1\rangle_1 \otimes |1\rangle_2$$

§8.3 Physical realization of n logical qubits with a single quantum harmonic oscillator

$$|000\cdots 0\rangle_L = |0\rangle$$

$$|000\cdots 1\rangle_L = |1\rangle$$

$$|000\cdots 10\rangle_L = |2\rangle$$

$$|1111\cdots 1\rangle_L = |2^n - 1\rangle$$

energy cost is $:E_n(2^n+\frac{1}{2})\hbar w$

with n quantum harmonic oscillator

energy cost $E = n \times E_1 = \frac{3n}{2}\hbar w$

obviously $(2^n + \frac{1}{2})\hbar w \gg \frac{3n}{2}\hbar w$

quantum digital cost

§9 Lecture 9:Harmonic Oscillator Quantum Computer

[Optical photo quantum computer]

§9.1 The physcial realization of the Pauli-X gate

A logical qubit

$$|0\rangle_L = |+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_L$$

$$|1\rangle_L = |-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

the basis σ_x

The pauli-x gate

Definition 9.1. $X|0\rangle_L = |1\rangle_L, X|1\rangle_L = |0\rangle_L$

Initial State : $|\psi(0)\rangle_L = \alpha|0\rangle_L + \beta|1\rangle_L = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$

and

$$|\psi(t)\rangle_L = U(t)|0\rangle_L = U(t)(\frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}})$$

$$= \frac{\alpha+\beta}{\sqrt{2}}|0\rangle + \frac{\alpha-\beta}{\sqrt{2}}e^{-iwt}|1\rangle$$

Gobal Phase $e^{-\frac{iwt}{2}}$ can be ignored

$$|\psi(t)\rangle_L = \frac{\alpha+\beta}{\sqrt{2}}(\frac{1}{\sqrt{2}}(|0\rangle_L + |1\rangle_L)) + \frac{\alpha-\beta}{\sqrt{2}}(\frac{1}{\sqrt{2}}(|0\rangle_L - |1\rangle_L))e^{-iwt}$$

$$= (\frac{1+e^{-iwt}}{2} + \frac{1-e^{-iwt}}{2}X)|\psi(0)\rangle_L$$

so the $U(t) = \begin{pmatrix} \frac{1+e^{-iwt}}{2} & \frac{1-e^{-iwt}}{2} \\ \frac{1-e^{-iwt}}{2} & \frac{1+e^{-iwt}}{2} \end{pmatrix}$

$$U(\frac{\pi}{w}) = X \tag{15}$$

§9.2 The physical realization of two qubit gates

$$|n(t)\rangle = U(t)|n\rangle = e^{-\frac{iwt}{2}}e^{-iwn}|n\rangle$$

when $t = \frac{\pi}{w}$

so$|n(\frac{\pi}{w})\rangle = -iX(-1)^n|n\rangle$

for example:The CNOT gate - controlled - X gate

Logical Two qubit

Definition 9.2.

$$|00\rangle_L = |0\rangle_L|0\rangle_L = |0\rangle$$

$$|01\rangle_L = |0\rangle_L|1\rangle_L = |2\rangle$$

$$|10\rangle_L = |1\rangle_L|0\rangle_L = \frac{|4\rangle + |1\rangle}{\sqrt{2}}$$

$$|11\rangle_L = |1\rangle_L|1\rangle_L = \frac{|4\rangle - |1\rangle}{\sqrt{2}}$$

that can achieve the CNOT gate

Note 1:SWAP gate and CZ gate can constructed by the same way

CZ:

$$|00\rangle_L = |0\rangle_L|0\rangle_L = |0\rangle$$

$$|01\rangle_L = |0\rangle_L|1\rangle_L = |2\rangle$$

$$|10\rangle_L = |1\rangle_L|0\rangle_L = |4\rangle$$

$$|11\rangle_L = |1\rangle_L|1\rangle_L = |1\rangle$$

Note 2:Obviously the construction of two-qubit gates on a harmonic oscillator quantum computer depends on the choice of logical qubits

Note 3:When the eigenvalue of a unitary matrix are known,such as CNOT gate CZ gate SWAP gate It is easy to construct them on the harmonic oslliator Quantum Computer

However when the eigenvalue of a quanutm gate are unknown,the construct becomes difficult.

§9.3 The control Hamiltonian into a harmonic oscillator system

$$H = H_0 + H_{int} + H_1$$

with the Rabi oscillator interaction between harmonic oscillator and control system are induces the physical realization of single - qubit gates and two qubit gates

§9.4 Optical Photon Quantum Computation

Note:Quantum Optics

will be talked on the fall semster about quantum filed theory

Quantum Optics + Quantum Electrodynamics[without special relativity]

A quantum of electromagnetic fields in vaccum or in cavity

Chargeless particle No - directing intercetion betweem photos

Long Distance stable transport [in vaccum or in optical files]

A lot of optical instruments of manipulation photos

Non-linear kerr medit Beam Spliter

§9.5 Model for photos generated by two indenpendent

Simple harmonic oscillators

$$H = \hbar w a^{+} a + \hbar w b^{+} b$$

a^{+}, b^{+} creation a, b annilation

and $a\,|vac\rangle = 0, b\,|vac\rangle = 0, a^{+}\,|vac\rangle = 1, b^{+}\,|vac\rangle = 1$

§10 Lecture 10:Optical photo quantum computers

§10.1 Quantum harmonic oscillator models for photos

Two independent cavities

$$[b, b^{+}] = [b^{+}, b] = 1, [b, b] = [b^{+}, b^{+}] = 0$$

Vaccum energy is not considered in the Hamiltonian

$H_0 = \hbar w a^{+} a + \hbar w b^{+} b, H_0\,|vac\rangle = 0$

Time evoluation

$$|mn(t)\rangle_{ba} = e^{-\frac{iH_0 t}{\hbar}}\,|mn\rangle_{ba} = e^{-i(m+n)t}\,|mn\rangle_{ba} \tag{16}$$

§10.2 A single - Rail representation of a qubit

Only use photos in one cavity to represent a qubit

$|0\rangle_L = |00\rangle_{ba} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, |1\rangle_L = |01\rangle_{ba} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

$$|\psi(0)\rangle_L = \alpha_0\,|0\rangle_L + \alpha_1\,|1\rangle_L = \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}_L$$

Time evolution of $|\psi\rangle_L$

$$|\psi(t)\rangle_L = \alpha_0 |00\rangle_{ba} + \alpha_1 e^{-iwt} |01\rangle_{ba} = \begin{pmatrix} 1 & 0 \\ 0 & e^{-iwt} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}_L$$
$$= e^{-\frac{-iwt}{2}} \begin{pmatrix} e^{\frac{iwt}{2}} & 0 \\ 0 & e^{-\frac{iwt}{2}} \end{pmatrix} = e^{-\frac{iwt}{2}} R_z(-wt) \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}_L$$

$R_z(\theta) = e^{-\frac{i}{2}\sigma_z \theta}$

Free time evoluation automatically changes the qubit so that the single - rail representation is not a good coodidate for a logical qubit

§10.3 The Dual-rail represetation of a qubit be use photos in two cavities to represent a qubit

$$|0\rangle_L = |01\rangle_{ba} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}_L, |1\rangle_L = |10\rangle_{ba} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}_L$$

so $|\psi(0)\rangle_L = \alpha_0 |01\rangle_{ba} + \alpha_1 |11\rangle_{ba}$

$$|\psi(t)\rangle_L = e^{-iwt}(\alpha_0 |01\rangle_{ba} + \alpha_1 |11\rangle_{ba}) = e^{-iwt} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}_L$$

Free time evoluation in the dual - rail represetation of a qubit - only gives a global phase factor irrelevant to physics result

§10.4 The single - qubit gate $R_z(\Delta)$ via the phase shifter

Definition 10.1. A phase shifter is a transporent medium with length L and index of refraction n , compared to the index of refraction of the vaccum n_0

$$\Delta = \frac{L}{v_c} - \frac{L}{v_0} = \frac{(n - n_0)L}{c} \tag{17}$$

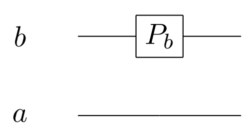
The phase shifter operator P

we have $P|0\rangle_a = |0\rangle_a, P|1\rangle_a = e^{i\Delta}|1\rangle_a$

The phase - shifter Hamiltonian $t = \Delta$

$$P_a = e^{-\frac{iH_a t}{\hbar}}, H_a = \hbar(n_0 - n)a^+ a$$

for the same way $P_b = e^{-\frac{iH_b t}{\hbar}}, H_b = \hbar(n_0 - n)b^+ b$



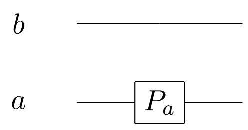
so

$$|\psi(t)\rangle_L = P_b \otimes Id |\psi_L\rangle = (P_b \otimes Id)(\alpha_0 |01\rangle_{ba} + \alpha_1 |10\rangle_{ba})$$

$$= \alpha_0 |01\rangle_{ba} + \alpha_1 e^{i\Delta t} |10\rangle_{ba}$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & e^{i\Delta} \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}_L = e^{-\frac{i\Delta}{2}} R_z(\Delta) \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}_L$$

For the same way



so

$$|\psi(t)\rangle_L = P_a \otimes Id |\psi_L\rangle = (P_a \otimes Id)(\alpha_0 |01\rangle_{ba} + \alpha_1 |10\rangle_{ba})$$

$$= \alpha_0 |01\rangle_{ba} + \alpha_1 e^{i\Delta t} |10\rangle_{ba}$$

$$= \begin{pmatrix} e^{i\Delta} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}_L = e^{\frac{i\Delta}{2}} R_z(-\Delta) \begin{pmatrix} \alpha_0 \\ \alpha_1 \end{pmatrix}_L$$

§10.5 Single qubit gate $R_y(2\theta)$ via the Beam Spliter

$H_{bs} = i(ab^+ - a^+ b), B = e^{-iH_{bs}Q} = e^{(a^+ b - ab^+)Q}$

Thm:

$$Ba^+ B^+ = a^+ cos\theta + b^+ sin\theta$$

$$Bb^+ B^+ = b^+ cos\theta - a^+ sin\theta$$

Proof:

Lemma :Baker–Campbell–Hausdorff formula BCH formula

$$e^{\theta G} A e^{-\theta G} = \sum_{n=0}^{\infty} \frac{\theta^n}{n!} C_n$$

C_n defined recursiuely as the squence of commutations $C_0 = A$

and $C_1 = [G, C_0], C_2 = [G, C_1] \cdots C_n = [G, C_{n-1}]$

and $B = e^{G\theta}, G = ab^+ - a^+ b$

we have some equation:

$$[G, a] = [ab^+ - a^+ b, a] = b, [G, a] = [ab^+ - a^+ b, b] = -a$$

And $A = a, C_0 = a, C_1 = b, C_{2k} = (-1)^k a, C_{2k+1} = (-1)^k b$

so

$$BaB^+ = \sum_0^{\infty} \frac{\theta^n}{n!} C_n = \sum_{n=0}^{\infty} \frac{\theta^{2k}}{(2k)!} [(-1)^k a] + \sum_{n=0}^{\infty} \frac{\theta^{2k+1}}{(2k+1)!} [(-1)^k b] = acos\theta + bsin\theta$$

Note:"a" and "b" are equally important so that exchange of a with b gives $B\left|00\right\rangle_{ba} = exp(ab^{+} - a^{+}b)\left|00\right\rangle_{ba}$

so

$$B\left|0\right\rangle_{L} = B\left|01\right\rangle_{ba} = Ba^{+}\left|00\right\rangle_{ba}$$

$$= (Ba^{+}B^{+})B\left|00\right\rangle_{ba} = Ba^{+}B^{+}\left|00\right\rangle_{ba}$$

$$= (a^{+}cos\theta + b^{+}sin\theta)\left|00\right\rangle_{ba}$$

$$= cos\theta\left|01\right\rangle_{ba} + sin\theta\left|10\right\rangle_{ba} = cos\theta\left|0\right\rangle_{L} + sin\theta\left|1\right\rangle_{L}$$

for the same method:

$$B\left|1\right\rangle_{L} = B\left|10\right\rangle_{ba}$$

$$= Bb^{+}B^{+}B\left|00\right\rangle_{ba}$$

$$= (b^{+}cos\theta - a^{+}sin\theta)\left|00\right\rangle_{ba}$$

$$= -sin\theta\left|0\right\rangle_{L} + cos\theta\left|1\right\rangle_{L}$$

$$\boxed{\left|0\right\rangle_{L} = \begin{pmatrix}1\\0\end{pmatrix}, \left|1\right\rangle_{L} = \begin{pmatrix}0\\1\end{pmatrix}, B = \begin{pmatrix}cos\theta & -sin\theta\\ sin\theta & cos\theta\end{pmatrix}}$$

and $B = \begin{pmatrix}cos\theta & -sin\theta\\ sin\theta & cos\theta\end{pmatrix} = cos\theta I_2 - isin\theta\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix} = cos\theta I_2 - isin\theta\sigma_y$

and $R_y(\theta) = e^{-\frac{i\sigma_y\theta}{2}}$

Thm: $R_z(-2\Delta)$ from Phase-shifter and $R_y(2\theta)$ from Beam Spliter form a universal gate set for gerenating all single qubit gates

The control - Z gate or the control -X gate via the non-inear kerr media

No direct interaction between photons

An indirect interaction between photons which is mediated by atoms in the non-linear kerr media with length L

$$H_{xpm} = -Xa^{+}ab^{+}b = -Xb^{+}ba^{+}a$$

coupling coefficied

so the $K_{opeator} = e^{iXa^{+}ab^{+}bL}$

the number operator $b^{+}b$ and $a^{+}a$

Example 10.2 (the number operator $b^{+}b$ and $a^{+}a$)

$$b^{+}ba^{+}a\left|01\right\rangle_{ba} = 0, b^{+}ba^{+}a\left|00\right\rangle_{ba} = 0$$

$$b^{+}ba^{+}a\left|10\right\rangle_{ba} = 0, b^{+}ba^{+}a\left|11\right\rangle_{ba} = \left|11\right\rangle_{ba}$$

so we have

$$K\left|00\right\rangle_{ba} = \left|00\right\rangle_{ba}, K\left|10\right\rangle_{ba} = \left|10\right\rangle_{ba}, K\left|01\right\rangle_{ba} = \left|01\right\rangle_{ba}, K\left|11\right\rangle_{ba} = e^{iXL}\left|11\right\rangle_{ba}$$

The natural logical qubits:

$$\left|e_{00}\right\rangle = \left|0_L 0_L\right\rangle = \left|01\right\rangle_{ba} \otimes \left|01\right\rangle_{ba}$$

$$\left|e_{01}\right\rangle = \left|0_L 1_L\right\rangle = \left|01\right\rangle_{ba} \otimes \left|10\right\rangle_{ba}$$

$$\left|e_{11}\right\rangle = \left|1_L 1_L\right\rangle = \left|10\right\rangle_{ba} \otimes \left|10\right\rangle_{ba}$$

$$\left|e_{10}\right\rangle = \left|1_L 0_L\right\rangle = \left|10\right\rangle_{ba} \otimes \left|01\right\rangle_{ba}$$

Apply the SWAP gate on the first logical qubit
When $XL = \pi$ the $e^{iXL} = e^{\pi} = -1$

§11 Lecture 11:Coding and quantum teleportation

§11.1 Super Dense Coding

The Holevo bound in quantum information theory shows that transmitting a qubit without an entangled resource is capable of sendius a classical qubit at most

Definition 11.1. (Super)Dense coding is a quantum information protocol in which Alice sends two qubits of classical information to Bob

Only by transmitting a qubit to Bob when they share an entangled resource such as the Bell States
such as

$$Alice \rightarrow \left|\beta\right\rangle_{AB} Bob$$

Task:Send Two Classical Bits

Step 1: Experiment setup
Expermental Set up Charlie prepares the Bell State $\left|\beta_{00}\right\rangle = \frac{1}{\sqrt{2}}(\left|00\right\rangle + \left|11\right\rangle)$ and

sends and sends one of two qubits to Alice and the other to Bob

$|\beta_{00}\rangle_{cc} \to |\beta_{00}\rangle_{AB}$[Alice has one and Bob has the another one]

Step 2:Local unitary transformation

Alice encodes two classical bits in the logical unitary transforms such as Z^iX^j or that can be also expressed as I_2, X, Z, ZX and performs the local procedure : Transmit a qubit

Example 11.2

when I_2:

$|\psi(0,0)\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

Example 11.3

when Z:

$$|\psi(0,1)\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$$

$$= I_2 \otimes Z\,|\psi(0,0)\rangle\,[Bob]$$

$$= Z \otimes I_2\,|\psi(0,0)\rangle\,[Alice]$$

Example 11.4

when X:

$$|\psi(1,0)\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$= I_2 \otimes X\,|\psi(0,0)\rangle\,[Bob]$$

$$= X \otimes I_2\,|\psi(0,0)\rangle\,[Alice]$$

Example 11.5

when ZX:

$$|\psi(1,0)\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$$

$$= I_2 \otimes XZ\,|\psi(0,0)\rangle\,[Bob]$$

$$= ZX \otimes I_2\,|\psi(0,0)\rangle\,[Alice]$$

Step 3: The local unitary transform $Z^j X^i$ on her qubit and $|\beta_{ij}\rangle_{ab} = (Z^j X^i \otimes Id)\,|\beta_{00}\rangle_{ab}$

and that can be write like : $A[Z^i X^j], B$

Two classical bits (i, j) local unitary transform $Z^i X^j$ The Bell states $|\beta_{ij}\rangle$

so we have the

00	01	10	11
Id	X	Z	XZ
$\|\beta_{00}\rangle$	$\|\beta_{01}\rangle$	$\|\beta_{10}\rangle$	$\|\beta_{11}\rangle$

Step 4: Alice sends her qubit to Bob so that Bob process two qubits

Note

Note 1 : The Two bits of classical information (i, j) is encoded in the correlations of two qubit

Note 2 : A single - qubit in $|\beta_{AB}\rangle$ carries information of (i,j) because of reduced Density Matrix $\rho_A = \rho_B = \frac{1}{2} I_2$

Note 3 : Alice must send her qubit to Bob so that two qubit (i, j) can be transmitted

Step 5: Bob performs the Bell measurement on his two qubit to determine two classical bits $(i, j)$by Alice

$$(X \otimes X)\,|\beta_{ij}\rangle = (-1)^i\,|\beta_{ij}\rangle$$

$$(Z \otimes Z)\,|\beta_{ij}\rangle = (-1)^j\,|\beta_{ij}\rangle$$

Remark 11.6. In fact there are two qubits which have been transmitted the one by Charli to Bob and the other by Alice to Bob

Remark 11.7. Alice and Bob share an n -fold tensor produce of Bell states

$\otimes_{k=1}^{n} |\beta_{ij}\rangle^{(1)} = |\beta_{ij}\rangle^{(k)} \otimes |\beta_{ij}\rangle^{(n)}$

sending n qubits is to transmit 2n classical bits

Remark 11.8. On the one hand, the word ”dense” in dense coding means that sending one qubit is to transmit two classical bits

> **Remark 11.9.** On the other hand, we still have that sending two qubits is to transmit two classical bits,if we think about it the following way:Alice prepares the entangled stateβ_{00} and then sends one qubit to Bob, so Alice sends two qubits to Bob in the entire procedure

Local Unitary Transform	Final State	Two bits
Alice	Bob	Bob
I_2	$\|\psi(0,0)\rangle$	(0,0)
X	$\|\psi(1,0)\rangle$	(1,0)
Z	$\|\psi(0,1)\rangle$	(0,1)
ZX	$\|\psi(1,1)\rangle$	(1,1)

§11.2 Quantum Teleportation

Teleportation is a word used in the movie like "Star Trek"

Definition 11.10. Quantum teleportation is a kind of inverse of dense coding in the sense

	Resolve	Sending	Transmission
Dense	entanglement	1 qubit	2 bits
Teleportation	entanglement	2 qubit	1 qubit

Definition 11.11. Quantum Teleportation is an information protocl in which Alice transmits an unknown qubit to Bob for away from her by sharing a maximally entangled state with Bob and sending two classical bits to Bob

Definition 11.12.

$$|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle) \tag{18}$$

$$|\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle) \tag{19}$$

$$|\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle) \tag{20}$$

$$|\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle) \tag{21}$$

Lemma 11.13

$$|\psi\rangle \otimes |\phi^+\rangle = \frac{1}{2}(|\phi^+\rangle \otimes |\psi\rangle + (X \otimes I_2 |\phi^+\rangle) \otimes X |\psi\rangle$$

$$+(\mathrm{Z}\otimes I_2 |\phi^+\rangle) \otimes Z |\psi\rangle + (ZX \otimes I_2 |\phi^+\rangle) \otimes XZ |\psi\rangle)(22)$$

The Standard description of QT

Task: Alice and Bob are in different locations and Alice wishes to transmitan unknown qubits $|\psi\rangle_A = \alpha |0\rangle + \beta |1\rangle$ to Bob

Alice : After quantum teleportation $|\psi_A\rangle$ is otherwise and $|\psi_A\rangle \otimes |\psi_B\rangle$ violates the no-cloning

Alice to Bob

Before QT Alice has $|\psi_A\rangle$

After QT Bob has $|\psi_B\rangle$

Step 1: State Preparation Alice and Bob share a maximally entangled state $|\beta_{01}\rangle_{ab}$ and Alice wishes to send an unknown qubit to Bob so that the prepared state is : $|\psi_A\rangle \otimes |\beta_{00}\rangle_{AB} = \frac{1}{2}\sum_{i,j=0}^{1} |\beta_{ij}\rangle_{AA} X^j Z^i |\psi_B\rangle$

Proof.

$$|\psi_A\rangle |\beta_{00}\rangle_{AB} = \frac{1}{\sqrt{2}}(\alpha |0\rangle + \beta |1\rangle)_A(|00\rangle + |11\rangle)_{AB}$$

$$= \frac{1}{\sqrt{2}}(\alpha |000\rangle + \alpha |011\rangle + \beta |100\rangle + \beta |111\rangle)_{AAB}$$

$$= \frac{1}{2} |\beta_{00}\rangle_{AA} (\alpha |0\rangle + \beta |1\rangle)_B + \frac{1}{2} |\beta_{10}\rangle_{AA} (\alpha |0\rangle - \beta |1\rangle)_B$$

$$+\frac{1}{2} |\beta_{01}\rangle_{AA} (\alpha |1\rangle + \beta |0\rangle)_B + \frac{1}{2} |\beta_{11}\rangle_{AA} (\alpha |1\rangle - \beta |0\rangle)_B$$

$$= \frac{1}{2} |\beta_{00}\rangle_{AA} |\psi_B\rangle + \frac{1}{2} |\beta_{10}\rangle_{AA} Z |\psi\rangle_B + \frac{1}{2} |\beta_{01}\rangle_{AA} |\psi\rangle_B + \frac{1}{2} |\beta_{11}\rangle_{AA} XZ |\psi\rangle_B$$

$$= \frac{1}{2} \sum_{i,j=0}^{1} |\beta_{ij}\rangle_{AA} X^j Z^i |\psi_B\rangle$$

□

Step 2: The Bell Measurement Performedly Alice on her two qubit Alice makes joint measurement for the observables $X \otimes X$ and $Z \otimes Z$,on the composite of the subsystem A and the unknown particle that Alice wants to send to Bob.The measurement results and two-bit information associated with the measurement datum,are listed in Table below

Alice on her two qubit:

$$(X \otimes X) |\beta_{ij}\rangle_{AB} = (-1)^i |\beta_{ij}\rangle_{AA} \tag{23}$$

$$(Z \otimes Z) |\beta_{ij}\rangle_{AB} = (-1)^j |\beta_{ij}\rangle_{AA} \tag{24}$$

so $|\psi\rangle_A \otimes |\beta_{00}\rangle_{AB}$ Wave function Collopse

Post-measurement	$Z \otimes Z$	$X \otimes X$	Two bits
$\|\beta_{00}\rangle$	1	1	(0,0)
$\|\beta_{01}\rangle$	1	-1	(1,0)
$\|\beta_{10}\rangle$	-1	1	(0,1)
$\|\beta_{11}\rangle$	-1	-1	(1,1)

Before measurement $\rho_{AAB} = |\psi\rangle_{AAB} \langle\psi|$

Where $|\psi\rangle_{AAB} = |\psi\rangle_A |\beta_{00}\rangle_{AB}$

After measurement $|\psi\rangle_{AAB} \to |\phi\rangle_{AAB}$

with the probability $\frac{1}{4}$

so $|\phi_{ij}\rangle_{AAB} = |\beta_{ij}\rangle_{AA} X^j Z^i |\psi\rangle_B$

$\rho'_B = \frac{1}{2} I_2$ means that after Alice's Bell measurement Bob knows nothing about his qubit

Step 3:Classical Communication

Alice informs her measurement result (i, j) to Bob $|\psi\rangle_{AAB} \to |\phi\rangle_{AAB}$ reduced density for Bob

$$\rho'_B = tr_{AA}\rho'_{AAB} = \sum_{k,l=0}^{1} \langle\beta_{kl}|_{AA} \rho'_{AAB} |\beta_{kl}\rangle_{AA}$$
$$= \frac{1}{4} \sum_{k,l=0}^{1} X^k Z^l |\psi\rangle_B \langle\psi| Z^l X^k\rangle = \frac{1}{2} I_2$$

Note

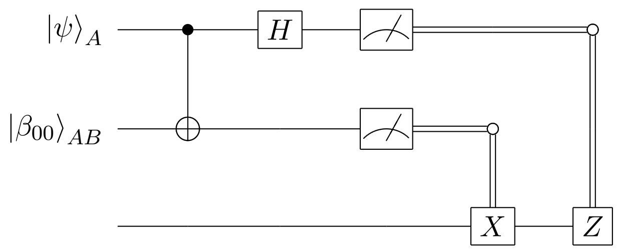
Note 1:

Classical Communication is a part of physics

Note 2:

The classical communication obeys the causoliey law proposed by the teleporation obeys the special relativity

§11.3 Quantum Circuit Model of Teleportation



Step 1:State Preparation

$$\begin{aligned}|\psi\rangle_{AAB} &= (|\psi\rangle)_A \otimes |\beta_{01}\rangle_{AB} \\ &= \frac{1}{\sqrt{2}}(\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|100\rangle + \beta\,|111\rangle)_{AAB}\end{aligned} \tag{25}$$

First qubit : Control qubit

Target qubit

Step 2:Product - Basis Measurement

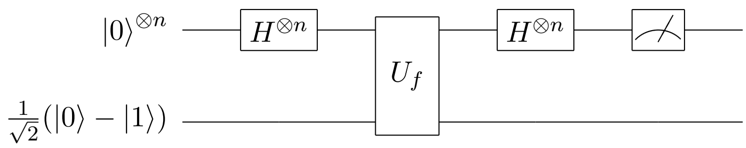
$$\begin{aligned}|\psi(t_2)\rangle &= (CNOT \otimes I_2)\,|\psi(t_1)\rangle_{AAB} \\ &= \frac{1}{\sqrt{2}}(\alpha\,|000\rangle + \alpha\,|011\rangle + \beta\,|110\rangle + \beta\,|101\rangle)_{AAB} \\ &= \frac{1}{\sqrt{2}}(\alpha\,|0\rangle\,(|00\rangle + |11\rangle) + \beta\,|1\rangle\,(|10\rangle + |01\rangle))_{AAB}\end{aligned} \tag{26}$$

Step 3: H operator

$$\begin{aligned}&|\psi(t_3)\rangle = (H \otimes I_2 \otimes I_2)\,|\psi(t_2)\rangle \\ &= \frac{1}{2}(\alpha(|0\rangle + |1\rangle)_A(|00\rangle + |11\rangle)_{AB} + \beta(|0\rangle - |1\rangle)_A(|10\rangle + |01\rangle))_{AB} \\ &= \frac{1}{2}\,|\beta_{00}\rangle_{AA}\,|\psi_B\rangle + \frac{1}{2}\,|\beta_{10}\rangle_{AA}\,Z\,|\psi\rangle_B + \frac{1}{2}\,|\beta_{01}\rangle_{AA}\,|\psi\rangle_B + \frac{1}{2}\,|\beta_{11}\rangle_{AA}\,XZ\,|\psi\rangle_B\end{aligned} \tag{27}$$

§12 Lecture 12:Quantum Algorithm

§12.1 Deutsch - Jozsa's Algorithm



and then I will introduce two lemma

Lemma 12.1

$$H^{\otimes n}|x\rangle = \frac{1}{2^{\frac{n}{2}}} \sum_{y\in\{0,1\}^n} (-1)^{x\cdot y}|y\rangle \tag{28}$$

Before proof I will introduce some notation:

State:

$$\begin{cases} |x\rangle = |x_1x_2\cdots y_n\rangle \\ |y\rangle = |y_1y_2\cdots y_n\rangle \end{cases} \tag{29}$$

Product:

$$x\cdot y = x_1y_1 \oplus x_2y_2\cdots\oplus x_ny_n \tag{30}$$

with

$$x_iy_i = x_i AND y_i \tag{31}$$

Proof.

$$\begin{aligned} &H^{\otimes n}|x\rangle = H|x_1\rangle\otimes H|x_2\rangle\otimes\cdots H|x_n\rangle \\ &= (\frac{1}{\sqrt{2}})^n \sum_{y_1=0}^{1}(-1)^{x_1\cdot y_1}|y_1\rangle \sum_{y_2=0}^{1}(-1)^{x_2\cdot y_2}|y_2\rangle\cdots\sum_{y_n=0}^{1}(-1)^{x_n\cdot y_n}|y_n\rangle \\ &= (\frac{1}{\sqrt{2}})^n \sum_{y\in\{0,1\}^n}(-1)^{x\cdot y}|y\rangle \end{aligned} \tag{32}$$

□

Lemma 12.2

Phase kick - back for n - qubit

Define:

$$U_f : |a\rangle |y\rangle \to |a\rangle |y \oplus f(a)\rangle \tag{33}$$

with

$$f : \forall a \in \{0,1\}^n \to f(a) \in \{0,1\} \tag{34}$$

And we call $|a\rangle$ as the first register,which is an n - qubit state,and $|y\rangle$ as the second register.which is a single qubit

Therefore:

$$U_f |x\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) = (-1)^{f(x)} |x\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{35}$$

Proof.

$$\begin{aligned} U_f |x\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) &= |x\rangle \frac{|f(x)\rangle - |1 + f(x)\rangle}{\sqrt{2}} \\ = |x\rangle \frac{|f(x)\rangle - |\bar{f}(x)\rangle}{\sqrt{2}} \\ = (-1)^{f(x)} |x\rangle \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \end{aligned} \tag{36}$$

□

Step 1:Initial State

$$|\psi(t_1)\rangle = H^{\otimes n} |0\rangle^{\otimes n} \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{37}$$

Step 2:Hadamard Gate Operator

$$|\psi(t_2)\rangle = H^{\otimes n} |0\rangle^{\otimes n} \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{38}$$

> **Remark 12.3.** Creat linearly superposition of all computational basis with Lemma 1
>
> $$H^{\otimes n} |0\rangle^{\otimes n} = \frac{1}{2^{\frac{n}{2}}} \sum |x\rangle$$

Step 3:

$$|\psi(t_3)\rangle = \frac{1}{2^{\frac{n}{2}}} \sum U_f(|x\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)) = \frac{1}{2^{\frac{n}{2}}} \sum (-1)^{f(x)} |x\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{39}$$

Remark 12.4. Phase kick - back technique with Lemma 2

Step 4: Apply $H^{\otimes n}$ again with Lemma 1

$$|\psi(t_4)\rangle = (H^{\otimes n} \otimes Id)\,|\psi(t_3)\rangle = \frac{1}{2^{\frac{n}{2}}}\sum(-1)^{f(x)} H^{\otimes n}\,|x\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{40}$$

$$= \frac{1}{2^{\frac{n}{2}}}\sum_{x\in\{0,1\}}\sum_{y\in\{0,1\}}(-1)^{f(x)\oplus x\cdot y}\,|y\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{41}$$

Step 5:Quantum measurement with the projector

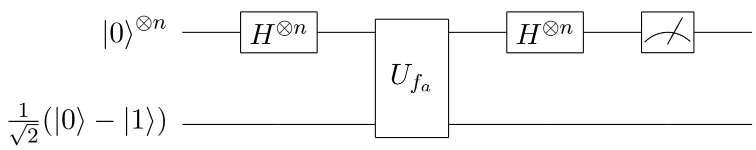
$\underbrace{|0\rangle^{\otimes n}}\,|0\rangle \otimes I_2$

$|\widetilde{\psi(t_4)}\rangle\ = f(x)$ constant $(-1)^{f(x)}\,|0\rangle^{\otimes n}\,\frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$

or $|\widetilde{\psi(t_4)}\rangle = f(x)$ balanced 0

Remark 12.5. The Dentsch Algrithm only makes one query to determine whether $f(x)$ constant or balanced ,whereas the classical deterministrate algrithm need at least $2^{n-1}+1$ queries This is an example for the expontial speed up of QA beyond

§12.2 Bernstein - Vazirant's Algorithm



INPUT : A black - box for computing an unknown function $f_a : \{0,1\}^n \to \{0,1\}$

PROMISE: The function f_a has the form $f(x) = ax = a_1x_1 \oplus a_2x_2 \oplus \cdots a_nx_n$

PROBLEM: Determine of the n-qubit string "a"

Step 1:Initial State

$$|\psi(t_1)\rangle = |0\rangle^{\otimes n} \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{42}$$

Step 2

$$|\psi(t_2)\rangle = \frac{1}{2^{\frac{n}{2}}}\sum_{x\in\{0,1\}}|x\rangle \otimes \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle) \tag{43}$$

Step 3

$$|\psi(t_3)\rangle = \frac{1}{2^{\frac{n}{2}}}\sum_{x\in\{0,1\}}(-1)^{a\cdot x}|x\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle) \tag{44}$$

Step 4

Lemma 12.6

$$\sum_{x\in\{0,1\}^n}(-1)^{(a\oplus y)\cdot x} = \delta_{a\oplus y,0}2^n \tag{45}$$

$$LHS = \sum_{x_1=0}^{1}(-1)^{(a_1\oplus y_1)\cdot x_1}\cdots\sum_{x_n=0}^{1}(-1)^{(a_n\oplus y_n)\cdot x_n} \tag{46}$$

and

$$\sum_{x_1=0}^{1}(-1)^{(a_1\oplus y_1)\cdot x_1} = (-1)^0+(-1)^{a_1\oplus y_1} = \begin{cases} 2 & \text{if } a_1\oplus y_1 = 0 \\ 0 & \text{if } a_1\oplus y_1 = 1 \end{cases} \tag{47}$$

so $\sum_{x_1=0}^{1}(-1)^{(a_1\oplus y_1)\cdot x_1} = \delta_{a_1\oplus y_1,0}2$

and we finally get:

$$\sum_{x\in\{0,1\}^n}(-1)^{(a\oplus y)\cdot x} = \delta_{a\oplus y,0}2^n \tag{48}$$

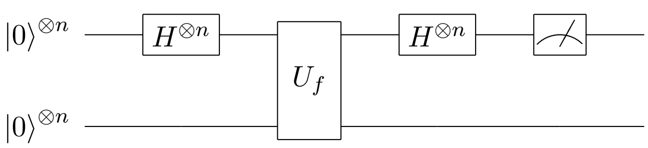
□

$$\begin{aligned} |\psi(t_4)\rangle &= \frac{1}{2^n}\sum_{x\in\{0,1\}}\sum_{y\in\{0,1\}}(-1)^{(a\oplus y)\cdot x}|y\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle) \\ &= \sum_{y\in\{0,1\}^n}\delta_{a\oplus y,0}|y\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle) \\ &= |a\rangle\otimes\frac{1}{\sqrt{2}}(|0\rangle-|1\rangle) \end{aligned} \tag{49}$$

Proof. **Step 5**

Quantum Measurement $|a\rangle\langle a|\otimes I_2$ yields the result of "a"

§12.3 Simon's Algorithm



INPUT : A black - box for compuating an unknown function $f(x):\{0,1\}^n\to\{0,1\}$

PROMISE: $f(x)$ is a period function with the period $a = (a_1, a_2, a_3 a_n)$ so that $f(x) = f(y)$ if and only if $x = y$ or $x = y \oplus a$

PROBLEM : Determine the period of the n -bit string "a" by making queries to U_f

ANSWER: With classical algorithms $2^{\frac{n}{2}}$ queries

with Simon's algorithms n queries Expontial speed up of QA beyond CA

Step 1:Initial State

$$|\psi(t_1)\rangle = |0\rangle^{\otimes n} \otimes |0\rangle^{\otimes n} \tag{50}$$

Step 2:

$$|\psi(t_2)\rangle = H^{\otimes n} |0\rangle^{\otimes n} \otimes |0\rangle^{\otimes n} = \frac{1}{2^{\frac{n}{2}}} \sum_{x \in \{0,1\}^n} |x\rangle \otimes |0\rangle^{\otimes n} \tag{51}$$

Step 3:

$$|\psi(t_3)\rangle = U_f |\psi(t_2)\rangle = \frac{1}{2^{\frac{n}{2}}} \sum_{x \in \{0,1\}^n} |x\rangle |f(x)\rangle \tag{52}$$

and now $f(x) = f(x \oplus a)$

so:

$$\widetilde{\psi(t_3)} = \frac{1}{2^{\frac{n}{2}}} (|x_0\rangle \oplus |x_0 \oplus a\rangle) |f(x_0)\rangle \tag{53}$$

Step 4:Apply $H^{\otimes n}$ to $\widetilde{\psi(t_3)}$

$$\begin{aligned} \widetilde{\psi(t_4)} &= H^{\otimes n} \frac{1}{2}^{\frac{n}{2}} (|x_0\rangle + |x_0 \oplus a\rangle) |f(x_0)\rangle \\ &= \frac{1}{2^n} \sum_{y \in \{0,1\}^n} ((-1)^{x_0 \cdot y} + (-1)^{(x_0 \oplus a) \cdot y}) |y\rangle |f(x_0)\rangle \\ &= \frac{1}{2^n} \sum_{y \in \{0,1\}^n} (-1)^{x_0 \cdot y} (1 + (-1)^{a \cdot y}) |y\rangle |f(x_0)\rangle \end{aligned} \tag{54}$$

so we get: $\begin{cases} |\widetilde{\psi(t_4)}\rangle = 0 & \textbf{if } a \cdot y = 0 \\ |\widetilde{\psi(t_4)}\rangle = \frac{1}{2^n} \sum_{y \in \{0,1\}^n} 2 |y\rangle |f(x_0)\rangle & \textbf{if } a \cdot y = 1 \end{cases}$

Step 5:

Measure the register qubit by $|y_1\rangle \langle y_1| \otimes I_2$ with probability 1 we have $a \cdot y = 0$ else we have$a \cdot y = 1$

§13 Density Matrix

§13.1 Density matrix as state of quantum open system

Density matrix[operator] describes the state of a quantum open system

measurement theory in terms of the density operator

$$\begin{aligned}\langle\psi| P_n |\psi\rangle &= \sum_{a_k} \langle\psi| P_n |a_k\rangle \langle a_k| |\psi\rangle \\ &= \sum_{a_k} \langle a_k| |\psi\rangle \langle\psi| P_n |a_k\rangle \\ &= tr(|\psi\rangle \langle\psi| P_n) = tr(\rho P_n)\end{aligned} \tag{55}$$

so the density operator can be defined as :

$$\rho = |\psi\rangle \langle\psi| \tag{56}$$

§13.2 Reduced density matrix(State for subsystem)

Let's assmue that the state of the composite physical system C consisted of subsystem A and subsystem B, is in state $|\psi\rangle_C$.Then,we can get the density matrix for the composite physical system C:

$$\rho_{AB} = |\psi\rangle_{AB} |\psi\rangle \tag{57}$$

Reduced density matrix ρ_A of the general bipartite system

$$|\psi\rangle_{AB} = \sum_i \sum_j a_{ij} |i\rangle_B \otimes |j\rangle_B \tag{58}$$

$$\rho_A = tr_B \rho_{AB} = \sum_i \langle i|\rho_{AB}|i\rangle_B \tag{59}$$

§14 Supplement