## References

[1] K. Schwarzschild, *Math Ann.*, vol. 55, p. 177, 1902. See also references [2] and [17].

[2] S. N. Karp and A. Russek, "Diffraction by a wide slit," *J. Appl. Phys.*, vol. 27, pp. 886–894, 1956.

[3] J. E. Burke, D. Censor, and V. Twersky, "Exact inverse-separation series for multiple scattering in two dimensions," *J. Acoust. Soc. Am.*, vol. 37, no. 1, pp. 5–13, 1965.

[4] J. B. Keller, "Geometrical theory of diffraction," *J. Opt. Soc. Am.*, vol. 52-2, pp. 116–130, 1962.

[5] V. Twersky, "Multiple scattering of waves and optical phenomena," *ibid.*, pp. 145–171.

[6] M. S. Macrakis, "Theoretical and experimental study of the backscattering cross section of an infinite ribbon," *J. Appl. Phys.*, vol. 31, pp. 2261–2266, 1960.

[7] B. J. Morse, "Diffraction by polygonal cylinders," *J. Math. Phys.*, vol. 5, no. 2, pp. 199–214, 1964.

[8] P. M. Russo, R. C. Rudduck, and L. Peters, Jr., "A method for computing *E*-plane pattern of horn antennas," *IEEE Trans. Antennas and Propagation*, vol. AP-13, pp. 219–224, March 1965.

[9] J. S. Yu, R. C. Rudduck, and L. Peters, Jr., "Comprehensive analysis for *E*-plane of horn antennas by edge diffraction theory," *IEEE Trans. Antennas and Propagation*, vol. AP-14, pp. 138–149, March 1966.

[10] R. C. Rudduck, "Application of wedge diffraction to antenna theory," NASA Contractor Rept., NASA-CR-372, 1966.

[11] R. B. Dybdal, R. C. Rudduck, and L. L. Tsai, "Mutual coupling between TEM and $TE_{01}$ parallel-plate waveguide apertures," *IEEE Trans. Antennas and Propagation*, vol. AP-14, pp. 574–580, September 1966.

[12] R. A. Ross, "Radar cross section of rectangular flat plates as a function of aspect angle," *IEEE Trans. Antennas and Propagation*, vol. AP-14, pp. 329–335, May 1966.

[13] E. V. Jull, "Diffraction by a wide aperture in an anisotropic medium," *Can. J. Phys.*, vol. 44, pp. 313–729, 1966.

[14] W. Pauli, "On asymptotic series for functions in the theory of diffraction of light," *Phys. Rev.*, vol. 54, pp. 924–931, December 1, 1938.

[15] P. M. Morse and P. J. Rubenstein, "The diffraction of waves by ribbons and by slits," *ibid.*, pp. 895–898.

[16] P. Morse and H. Feshbach, *Methods of Theoretical Physics*. New York: McGraw-Hill, 1958, pp. 1428–1432.

[17] A. Sommerfeld, *Optics*. New York: Academic Press, 1954, pp. 245–289.

[18] J. B. Keller, "Diffraction by an aperture," *J. Appl. Phys.*, vol. 28, no. 4, pp. 426–444, 1957.


# Geometrical Shadowing of a Random Rough Surface

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*Abstract*—**In the context of the backscattering of waves from a random rough surface, a theoretical model is used to investigate the geometrical self-shadowing of a surface described by Gaussian statistics. Expressions are derived for various shadowing probabilities as functions of the parameter characterizing surface roughness and of the angle of incidence of the illuminating beam. The theoretical shadowing functions compare closely with those obtained experimentally from a recent computer simulation of a Gaussian surface.**

## Introduction

Conventional theories of the scattering of electromagnetic radiation from random rough surfaces [1], [2] assume that every element of the surface contributes to the scattered wave. This assumption neglects the shadowing of the surface by itself, an effect that may be expected to be important at large angles of incidence. Beckmann [3] has recently suggested an improvement in the theory to include geometrical shadowing, and has used it in the context of lunar and planetary radar studies to explain the functional form of the angular spectrum of power backscattered from the limbs of the moon. Beckmann argues that the expression for the angular power spectrum calculated in the absence of shadowing should, to a first approximation, be multiplied by a shadowing function $S$, the probability that a point on the rough surface is illuminated by the radar beam.

Brockelman and Hagfors [4] have argued that this shadowing function is, in principle, not appropriate. Hagfors [2] has shown that for a random rough surface which imposes deep phase modulation in the scattered wave, those surface elements which are oriented perpendicular to the incoming beam are the prime contributors to the backscattered wave. The shadowing function $R$ which should be used in this case is the probability that a point on the rough surface with favorably oriented slope is illuminated by the radar beam. $R$ is in contrast with the function $S$ which accords equal weight to all slopes. In addition, Shaw [5] and Brockelman and Hagfors [4] have criticized Beckmann's derivation of the expression for $S$ [3]. This paper represents an approximate method for calculating $R$ and $S$ free of these criticisms.

## Shadowing Theory

The model to be used describes the rough surface as a mean smooth surface upon which are superimposed positive and negative undulations of height generated by a stationary random process. Locally the underlying surface may be

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considered plane, coinciding with the $z=0$ plane of a Cartesian coordinate system. The density of surface height deviations, $\xi$, from the mean plane in the $z$ direction is described by a continuous probability function $P_1(\xi)$, of zero mean, chosen to be Gaussian for computational ease, where the probability of finding a height deviation within the range $\Delta\xi$ about $\xi$ is

$$P_1(\xi)\Delta\xi = \frac{1}{(2\pi)^{1/2}\sigma} e^{-\xi^2/2\sigma^2}\cdot\Delta\xi \tag{1}$$

and $\sigma$ is the root-mean-square height deviation. The horizontal scale of the relief is contained within an autocorrelation function $\rho(r)$, defined by

$$\rho(r) = \langle \xi(\boldsymbol{R} + \boldsymbol{r})\cdot\xi(\boldsymbol{R})\rangle \tag{2}$$

where $\boldsymbol{R}$ and $\boldsymbol{r}$ are vectors lying in the mean plane and the average is taken over all $\boldsymbol{R}$. $\rho(r)$ is independent of the direction of $\boldsymbol{r}$ for an isotropic surface. Higher dimensional density functions and their appropriate correlation matrices may be derived from the autocorrelation function $\rho(r)$ [6]. In particular, the joint density function of surface slopes $p(=\partial z/\partial x)$ and $q(=\partial z/\partial y)$ for the Gaussian surface described by (1) is

$$P_{22}(p, q) = \frac{1}{2\pi w^2}\exp\left[-\left(\frac{p^2+q^2}{2w^2}\right)\right] = P_2(p)\cdot P_2(q) \tag{3}$$

where $w^2$, the mean-square surface slope, is $[-\rho''(0)]$, the primes denoting double differentiation with respect to $r$. Based upon this model the shadowing functions will be calculated with an approach essentially similar to that used in a recent paper by Wagner [7] and suggested by Beckmann [3].

The problem is the following: what is the probability $S(\xi_0, p_0, q_0, \theta)$ that a point $F$ on a random rough surface, of given height $\xi_0$ above the mean plane, and with local slopes $p_0$, $q_0$, will not lie in shadow when the surface is illuminated with a parallel beam of radiation at an angle of incidence $\theta$ to the mean plane? Fig. 1 illustrates a section through the surface. The origin of coordinates is taken in the mean plane below $F$ and the axes are oriented with the incoming beam lying in the $x=0$ plane. Only parts of the surface in this plane to the right of $F$ can shadow $F$, and $S(\xi_0, p_0, q_0, \theta)$, or $S(F, \theta)$ for short, is equivalent to the probability that no part of the surface to the right of $F$ will intersect the ray $FS$. This, in turn, may be written as the limit

$$S(F, \theta) = \lim_{\tau\to\infty} S(F, \theta, \tau) \tag{4}$$

where $S(F, \theta, \tau)$ is the probability that no part of the surface between $y=0$ and $y=\tau$ will intersect the ray $FS$. A differential equation for $S(F, \theta, \tau)$ may be developed thus:

$$S(F, \theta, \tau + \Delta\tau) = S(F, \theta, \tau)\cdot Q(\Delta\tau \mid F, \theta, \tau) \tag{5}$$

where now $Q(\Delta\tau \mid F, \theta, \tau)$ is the conditional probability that the surface will not intersect $FS$ in the interval $\Delta\tau$, given that it does not in the interval $\tau$. Turning this around and suppressing the functional dependence upon $F$, $\theta$,

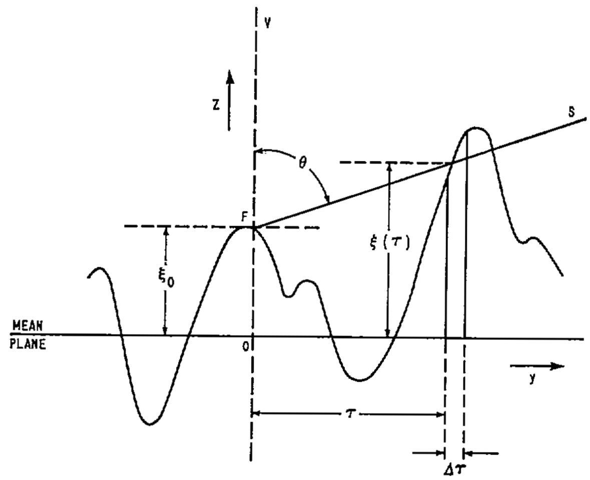


Fig. 1. Section of a random rough surface illuminated from $S$.

$$Q(\Delta\tau \mid F, \theta, \tau) = 1 - g(\tau)\Delta\tau \tag{6}$$

where $g(\tau)\Delta\tau$ is the conditional probability that the surface in $\Delta\tau$ *will* intersect the ray $FS$ given that it does not in the interval $\tau$. Equation (5), again suppressing explicit $F$ and $\theta$ dependence now becomes,

$$S(\tau + \Delta\tau) = S(\tau)\cdot\{1 - g(\tau)\Delta\tau\}. \tag{7}$$

Expanding $S(\tau+\Delta\tau)$ about $\tau$ in a Taylor series to first order in $\Delta\tau$ leads to the differential equation

$$\frac{dS(\tau)}{d\tau} = -g(\tau)\cdot S(\tau) \tag{8}$$

which may be integrated to yield

$$S(\tau) = S(0)\exp\left\{-\int_0^\tau g(\tau)d\tau\right\}. \tag{9}$$

$S(0)$ will clearly be unity if $q_0$ is less than $\cot\theta$ and zero otherwise, so $S(0)=h(\mu-q_0)$, where $h$ is the unit step function and $\mu=\cot\theta$. Equation (4) now becomes

$$S(F, \theta) = h(\mu - q_0)\exp\left\{-\int_0^\infty g(\tau)d\tau\right\}. \tag{10}$$

The heart of the task lies in the evaluation of $g(\tau)$ and the subsequent integration over $\tau$. Instead of an attempt at a complete analysis, $g(\tau)$ will be approximated by replacing $g(\tau)\Delta\tau$ with the conditional probability that $F$ will be shadowed by the surface in $\Delta\tau$ given that it is not shadowed by the surface at $y=\tau$. (This avoids the difficulty of including the effects of correlation between points on the surface in $\Delta\tau$ and the infinity of points in $\tau$.)

If the surface at $\tau$ does not shadow $F$,

$$\xi(\tau) < \xi_0 + \mu\tau. \tag{11}$$

This is symbolically denoted as circumstance $\alpha$.

If the surface in $\Delta\tau$ does shadow $F$, and $q\equiv q(\tau)$,

$$\xi(\tau) < \xi_0 + \mu\tau;\ \xi(\tau + \Delta\tau) > \xi_0 + \mu(\tau + \Delta\tau);\ q \geq \mu \tag{12}$$

that is, $\xi(\tau)$ must lie in the interval $(q-\mu)\Delta\tau$ below $\xi_0+\mu\tau$, and $q \geq \mu$. This is denoted as circumstance $\beta$. $g(\tau)\Delta\tau$ is just the conditional probability that $\beta$ will occur given $\alpha$, or

$$g(\tau)\Delta\tau = P(\beta \mid \alpha). \tag{13}$$

A well-known relationship in probability theory [already used implicity in the derivation of (5)] links $P(\beta|\alpha)$ with $P(\alpha, \beta)$ and $P(\alpha)$—respectively the joint probability of $\alpha$ and $\beta$ occurring and the probability of $\alpha$ occurring by itself:

$$P(\alpha, \beta) = P(\beta \mid \alpha) \cdot P(\alpha). \tag{14}$$

Therefore,

$$g(\tau)\Delta\tau = \frac{P(\alpha, \beta)}{P(\alpha)}\,. \tag{15}$$

If $P_3(\xi, q|F, \tau)$ is the joint probability density function of $\xi$ and $q$ at $y=\tau$, conditional upon given height and slopes at $F$, then from the meaning of circumstances $\alpha$ and $\beta$ [(11) and (12)],

$$P(\alpha, \beta) = \Delta\tau \int_\mu^\infty dq(q-\mu)[P_3(\xi, q \mid F, \tau)]_{\xi=\xi_0+\mu\tau} \tag{16}$$

and

$$P(\alpha) = \int_{-\infty}^{+\infty} dq \int_{-\infty}^{\xi_0+\mu\tau} d\xi P_3(\xi, q \mid F, \tau) \tag{17}$$

and so

$$g(\tau)\Delta\tau = \frac{\Delta\tau \displaystyle\int_\mu^\infty dq(q-\mu)[P_3(\xi, q \mid F, \tau)]_{\xi=\xi_0+\mu\tau}}{\displaystyle\int_{-\infty}^{+\infty} dq \int_{-\infty}^{\xi_0+\mu\tau} d\xi P_3(\xi, q \mid F, \tau)} \tag{18}$$

where the renormalization effected by the denominator allows for the condition that $\xi(\tau)$ is known to be $\leq \xi_0+\mu\tau$.

With known distribution and autocorrelation functions the integrals in (18) may be evaluated in full, but this is a tedious process and not illuminating. Great simplicity is gained by neglecting correlation between the height and slopes at $F$ and those at $y=\tau$. The conditional density function in this case reduces to a product of simple Gaussian functions:

$$P_3(\xi, q \mid F, \tau) = P_1(\xi)P_2(q) = \frac{1}{2\pi\sigma w} e^{-\xi^2/2\sigma^2 - q^2/2w} \tag{19}$$

and within this approximation (18) becomes

$$g(\tau) = \left(\frac{2}{\pi}\right)^{1/2} \frac{\mu}{\sigma} \cdot \frac{\Lambda(\mu) \cdot e^{-(\xi_0+\mu\tau)^2/2\sigma^2}}{\left[2 - \operatorname{erfc}\left(\dfrac{\xi_0+\mu\tau}{\sqrt{2}\sigma}\right)\right]} \tag{20}$$

where erfc is the error function complement, and

$$2\Lambda(\mu) = \left(\left(\frac{2}{\pi}\right)^{1/2} \cdot \frac{w}{\mu} e^{-\mu^2/2w^2} - \operatorname{erfc}(\mu/\sqrt{2}w)\right). \tag{21}$$

The integration over $\tau$ [from (10)] is now simple and leads to

$$\begin{aligned} S(F, \theta) &= S(\xi_0, p_0, q_0, \theta) \\ &= h(\mu - q_0)[1 - \tfrac{1}{2}\operatorname{erfc}(\xi_0/\sqrt{2}\sigma)]^\Lambda. \end{aligned} \tag{22}$$

Some deductions may be drawn from (22). A very high part of the surface ($\xi_0 \to +\infty$) is certain to be illuminated ($S \to 1$), and a very low part ($\xi_0 \to -\infty$) is certain to be in shadow ($S \to 0$). In addition, there is no shadow cast if the surface is illuminated from directly above ($\mu \to \infty$), and the whole surface is in shadow at grazing illumination ($\mu \to 0$). All these conclusions satisfy one's intuition about the shadowing of a rough surface.

Two further distributions may be deduced from $S(F, \theta)$: the probability of $F$ not being shadowed, independent of $\xi_0$, which is

$$\begin{aligned} S(p_0, q_0, \theta) &= \int_{-\infty}^{+\infty} S(\xi_0, p_0, q_0, \theta)P_1(\xi_0)d\xi_0 \\ &= \frac{h(\mu - q_0)}{[\Lambda(\mu) + 1]} \end{aligned} \tag{23}$$

and the probability that a point on the surface will not be shadowed, independent of height and slope, $S(\theta)$:

$$\begin{aligned} S(\theta) &= \int_{-\infty}^{+\infty} d\xi_0 \int_{-\infty}^{+\infty} dp_0 \int_{-\infty}^{+\infty} \\ &\quad \cdot dq_0 P_1(\xi_0)P_2(p_0)P_2(q_0)S(\xi_0, p_0, q_0, \theta) \\ &= \frac{[1 - \frac{1}{2}\operatorname{erfc}(\mu/\sqrt{2}w)]}{[\Lambda(\mu) + 1]}\,. \end{aligned} \tag{24}$$

The function $R(\theta)$, defined as the conditional probability that a point on the surface will not lie in shadow given that its local slope is perpendicular to the incident beam, is just

$$R(\theta) = [S(p_0, -q_0, \theta)]_{\substack{p_0=0 \\ q_0=\frac{1}{\mu}}} = \frac{1}{\Lambda(\mu) + 1} \tag{25}$$

The expressions for both $S(\theta)$ and $R(\theta)$ [(24) and (25)] are compared in Fig. 2 with those derived by Brockelman and Hagfors [4] from a computer simulation of an illuminated Gaussian random rough surface, for various values of the rms slope parameter $w$. Good agreement is achieved between present theory and "experiment."

The theory may be shown to be self-consistent in the following respect. The sum of the areas of the illuminated surface elements projected onto the plane normal to the direction of the incident beam is independent of the roughness of the surface and equal to the projected area of the underlying mean plane, as indeed it should be. Fig. 3 illustrates a portion of the surface illuminated from the right. $\delta A$ is a surface element at $F$ with local normal $FN$. $FV$ is normal to the mean plane, and $FS$ is the illuminating ray. The illuminated area $T$ of the whole surface projected onto the plane perpendicular to $FS$ is

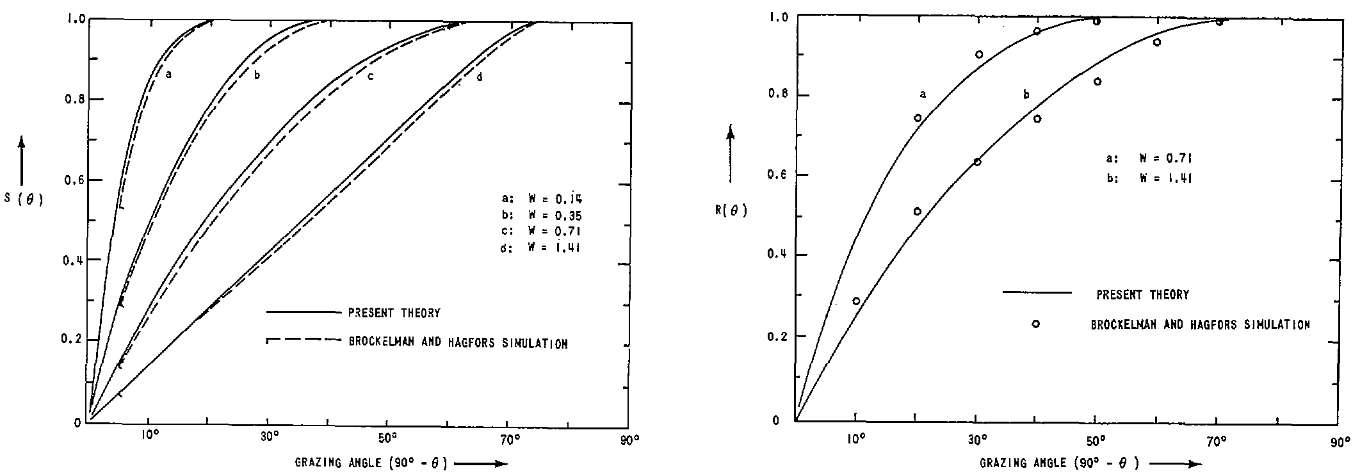


Fig. 2. Comparison of the theoretical shadowing functions $S(\theta)$ (*left*) and $R(\theta)$ (*right*) with the simulation of Brockelman and Hagfors.

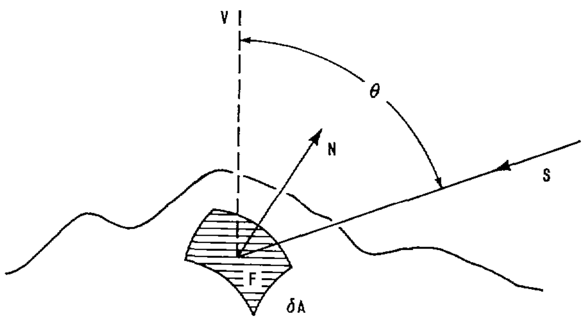


Fig. 3. Part of a random rough surface illuminated from $S$.

$$T = \iint_{\text{surface illuminated}} dA \cos N\hat{F}S. \tag{26}$$

Integration over a large area $A_0$ in the mean plane is equivalent to taking an average over the illuminated facets (invoking the ergodic theorem) and so

$$T = A_0 \int_{-\infty}^{+\infty} dp \int_{-\infty}^{+\infty} dq \cdot S(p, q, \theta) P_2(p) P_2(q) \cdot \frac{\cos N\hat{F}S}{\cos N\hat{F}V} = A_0 \cos\theta \tag{27}$$

where $p$ and $q$ are the local slopes at $F$.

## Conclusions

The expressions (24) and (25) are of simple analytical form and as such may be useful approximations to the true shadowing functions.

It is worth comparing the present relatively simple theory with that of Wagner [7]. Wagner does not use the device of renormalization [in (18)], and instead includes correlation explicitly. He is forced to approximate the integral over $\tau$ [in (10)], and at the expense of analytical complexity gains a slightly closer agreement with the simulation of Brockelman and Hagfors. An advantage of the present method is that the self-consistency condition (27) is satisfied identically, not approximately.

With the basic approach described above, other shadowing probabilities may be deduced. In particular, the shadowing function relevant to a bistatic scattering experiment, in which source and receiver are not collinear, can be calculated. The details have been presented elsewhere by the author [8] in connection with optical shadowing on the moon.

## References

[1] P. Beckmann and A. Spizzichino, *The Scattering of Electromagnetic Waves from Rough Surfaces.* New York: Pergamon, 1963.

[2] T. Hagfors, "Backscattering from an undulating surface with applications to radar returns from the moon," *J. Geophys. Research,* vol. 69, pp. 3779–3784, September 1964.

[3] P. Beckmann, "Shadowing of random rough surfaces," *IEEE Trans. Antennas and Propagation,* vol. AP-13, pp. 384–388, May 1965.

[4] R. A. Brockelman and T. Hagfors, "Note on the effect of shadowing on the backscattering of waves from a random rough surface," *IEEE Trans. Antennas and Propagation,* vol. AP-14, pp. 621–626, September 1966.

[5] L. Shaw, "Comments on 'Shadowing of random surfaces,'" *IEEE Trans. Antennas and Propagation,* vol. AP-14, p. 253, March 1966.

[6] D. Middleton, *Introduction to Statistical Communication Theory.* New York: McGraw-Hill, 1960, chs. 7 and 8.

[7] R. J. Wagner, "Shadowing of randomly rough surfaces," TRW Systems Rept. 7401-6012-R0000, 1966.

[8] B. G. Smith, "Lunar surface roughness, shadowing and thermal emission," *J. Geophys. Research* (to be published).