1、 $f(x)=\sqrt{x}$ 在离散点, $f(81)=9$. $f(100)=10$. $f(121)=11$ 利用2次 Lagrange 插值计算 $\sqrt{115}$ 的近似值.
并根据误差公式给出合适的误差界

$(x_0.\ f(x_0))=(81.\ 9)$ $(x_1.\ f(x_1))=(100.\ 10)$ $(x_2.\ f(x_2))=(121.\ 11)$

$$l_0(x)=\frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)}=\frac{(x-100)(x-121)}{(81-100)(81-121)}=\frac{1}{760}(x-100)(x-121)$$

$$l_1(x)=\frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)}=\frac{(x-81)(x-121)}{(100-81)(100-121)}=-\frac{1}{399}(x-81)(x-121)$$

$$l_2(x)=\frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)}=\frac{(x-81)(x-100)}{(121-81)(121-100)}=\frac{1}{840}(x-81)(x-100)$$

$$\therefore L_2(x)=\frac{9}{760}(x-100)(x-121)-\frac{10}{399}(x-81)(x-121)+\frac{11}{840}(x-81)(x-100)$$

$$\sqrt{115}=f(115)\approx L_2(115)\approx 10.725564$$

误差界: $R_n(x)=f(x)-L_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\cdots(x-x_n)$

$$f^{(n+1)}(x)=f^{(3)}(x)=\frac{3}{8}x^{-5/2}$$

$\because\ 81\le x\le 121 \Rightarrow 2.32845\times10^{-6}\le f^{(3)}(x)\le 6.35066\times10^{-6}$

$\therefore\ -3.2388366\times10^{-3}\le R_2(115)\le -1.1875\times10^{-3}$

实际误差为 -1.7587×10^{-3}

2. 设有插值节点, $a\le x_0<x_1<\cdots<x_n\le b$. 证明与这些节点相应的 Lagrange 插值基函数 $\{l_i(x)\ i=0.1..n\}$ 是线性无关的

证: 若存在一组系数 $\lambda_0\ldots\lambda_n$ 使得 $P=\lambda_0l_0+\lambda_1l_1+\cdots+\lambda_nl_n=0$

由 $l_i(x)$ 的函数性质可知

一方面多项式 P 满足 $P(x_0)=\lambda_0$. $P(x_1)=\lambda_1$ … $P(x_n)=\lambda_n$

另一方面 P 是零多项式 取值永远是0

故 $\lambda_0=\lambda_1=\cdots=\lambda_n=0$

$\therefore$ $l_0\ldots l_n$ 线性无关

3、利用插值数据 $(-1.0,\ 1.0)$, $(1.0.\ 3.0)$ $(3.0.\ 2.0)$ $(4.0,\ -1.0)$ 构造出三次 Lagrange 插值多项式 $L_3(x)$
并计算 $L_3(0)$. $L_3(2)$

$(x_0.\ f(x_0))=(-1.\ 1)$ $(x_1.\ f(x_1))=(1.\ 3)$ $(x_2.\ f(x_2))=(3.\ 2)$ $(x_3.\ f(x_3))=(4.\ -1)$

$$l_0(x)=\frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}=-\frac{1}{40}(x-1)(x-3)(x-4)$$

$$l_1(x)=\frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}=\frac{1}{12}(x+1)(x-3)(x-4)$$

$$l_2(x)=\frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}=-\frac{1}{8}(x+1)(x-1)(x-4)$$

$$l_3(x)=\frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}=\frac{1}{15}(x+1)(x-1)(x-3)$$

$$\therefore L_3(x)=-\frac{1}{40}(x-1)(x-3)(x-4)+\frac{1}{4}(x+1)(x-3)(x-4)-\frac{1}{4}(x+1)(x-1)(x-4)-\frac{1}{15}(x+1)(x-1)(x-3)$$

$\therefore L_3(2)=3.15$

$L_3(0)=2.1$

4. 设 $\{l_i(x)\}_{i=0}^{6}$ 是以 $\{x_i=2i\}_{i=0}^{6}$ 为节点的6次 Lagrange 插值函数
试求 $\sum_{i=0}^{6}(x_i^3+x_i^2+1)l_i(x)$ 和 $\sum_{i=0}^{6}(x_i^3+x_i^2+1)l_i'(x)$

由 Lagrange 插值法可知. 插值函数 $L_n(x)=\sum_{i=0}^{6}f(x_i)l_i(x)=\sum_{i=0}^{6}(x_i^3+x_i^2+1)l_i(x)$

则被插函数 $f(x)=x^3+x^2+1$

插值误差 $R_n(x)=f(x)-L_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\cdots(x-x_n)$

$\because f^{(7)}(x)=0 \Rightarrow R_n(x)=0$

$\therefore L_6(x)=\sum_{i=0}^{6}(x_i^3+x_i^2+1)l_i(x)=f(x)=x^3+x^2+1$

$\because \sum_{i=0}^{6}(x_i^3+x_i^2+1)l_i'(x)=[\sum_{i=0}^{6}(x_i^3+x_i^2+1)l_i(x)]'=L_6'(x)$

$\therefore \sum_{i=0}^{6}(x_i^3+x_i^2+1)l_i'(x)=3x^2+2x.$

综上 $\sum_{i=0}^{6}(x_i^3+x_i^2+1)l_i(x)=x^3+x^2+1$

$\sum_{i=0}^{6}(x_i^3+x_i^2+1)l_i'(x)=3x^2+2x.$