Chap4

31. 设 $\{X_n, n=\cdots,-1,0,1,\cdots\}$ 为平稳序列, 协方差函数为 $R(\tau)$,

(1) 求 X_{n+1} 的形如 $\widehat{X}_{n+1}^{(1)} = aX_n$ 的最小误差方差预报, 这里 a 是待定常数,

(2) 求 X_{n+1} 的形如 $\widehat{X}_{n+1}^{(2)} = aX_n + bX_{n-1}$ 的最小均方误差预报, 这里 a 和 b 是待定常数.

(1) 设 $\hat{X}^* = aX_n$ 为 X_{n+1} 的最佳预报. 则由投影定理可知. $E[(X_{n+1}-\hat{X}^*)X_n]=0$

$$\begin{aligned} E[(X_{n+1}-\hat{X}^*)X_n] &= E(X_{n+1}X_n - aX_n^2) \\ &= E(X_{n+1}-\mu)(X_n-\mu) - aE(X_n-\mu)(X_n-\mu) \quad (\mu=0) \\ &= R(1) - aR(0) = 0 \end{aligned}$$

$\Rightarrow a = \dfrac{R(1)}{R(0)}$

(2) 设 $\hat{X}^* = aX_n + bX_{n-1}$ 为 X_{n+1} 的最佳预报. 则由投影定理可知. $E[(X_{n+1}-\hat{X}^*)X_n]=0$. $E[(X_{n+1}-\hat{X}^*)X_{n-1}]=0$

$$\begin{aligned} E[(X_{n+1}-\hat{X}^*)X_n] &= E(X_{n+1}X_n - aX_n^2 - bX_nX_{n-1}) \\ &= R(1) - aR(0) + bR(1) = 0 \end{aligned}$$

$\Rightarrow R(0)a + R(1)b = R(1)$

$$\begin{aligned} E[(X_{n+1}-\hat{X}^*)X_{n-1}] &= E(X_{n+1}X_{n-1} - aX_nX_{n-1} - bX_{n-1}^2) \\ &= R(2) - aR(1) - bR(0) \end{aligned}$$

$\Rightarrow R(1)a + R(0)b = R(2)$

$\Rightarrow a = \dfrac{R(1)(R(0)-R(2))}{R^2(0)-R^2(1)} \qquad b = \dfrac{R(2)R(0)-R^2(1)}{R^2(0)-R^2(1)}$

34. 证明如下两个滑动平均序列

$$X_n = \varepsilon_n + \alpha\varepsilon_{n-1},$$
$$Y_n = \varepsilon_n + \frac{1}{\alpha}\varepsilon_{n-1}$$

有相同的自相关函数.

由题可知 自相关函数 $r(h) = E[X(t)X(t+h)]$ $t\in T$

对于 $X_n = \varepsilon_n + \alpha\varepsilon_{n-1}$

$h=0$ 时 $R_X(0) = E[(\varepsilon_n+\alpha\varepsilon_{n-1})(\varepsilon_n+\alpha\varepsilon_{n-1})] = E[\varepsilon_n^2] + \alpha^2E[\varepsilon_{n-1}^2] = (1+\alpha^2)\sigma^2$

$h=1$ 时 $R_X(1) = E[(\varepsilon_n+\alpha\varepsilon_{n-1})(\varepsilon_{n+1}+\alpha\varepsilon_n)] = \alpha E[\varepsilon_n^2] = \alpha\sigma^2$

$h\geq 2$ 时 $R_X(h) = E[(\varepsilon_n+\alpha\varepsilon_{n-1})(\varepsilon_{n+\tau}+\alpha\varepsilon_{n+\tau-1})] = 0$

$$\Rightarrow \rho_X(h) = \begin{cases} 1 & h=0 \\ \frac{\alpha}{1+\alpha^2} & h=1 \\ 0 & h\geq 2 \end{cases}$$

对于 $Y_n = \varepsilon_n + \frac{1}{\alpha}\varepsilon_{n-1}$

$h=0$ 时 $R_X(0) = E[(\varepsilon_n+\frac{1}{\alpha}\varepsilon_{n-1})(\varepsilon_n+\frac{1}{\alpha}\varepsilon_{n-1})] = E[\varepsilon_n^2] + \frac{1}{\alpha^2}E[\varepsilon_{n-1}^2] = (1+\frac{1}{\alpha^2})\sigma^2$

$h=1$ 时 $R_X(1) = E[(\varepsilon_n+\alpha\varepsilon_{n-1})(\varepsilon_{n+1}+\alpha\varepsilon_n)] = \frac{1}{\alpha}E[\varepsilon_n^2] = \frac{1}{\alpha}\sigma^2$

$h\geq 2$ 时 $R_X(h) = E[(\varepsilon_n+\alpha\varepsilon_{n-1})(\varepsilon_{n+\tau}+\alpha\varepsilon_{n+\tau-1})] = 0$

$$\Rightarrow \rho_Y(h) = \begin{cases} 1 & h=0 \\ \frac{\alpha}{1+\alpha^2} & h=1 \\ 0 & h\geq 2 \end{cases}$$

$\therefore$ X 与 Y 具有相同的自相关函数

35. 设 $\{X_n, n=0,\pm1,\cdots\}$ 为 AR(p) 模型:

$$X_n = \alpha_1X_{n-1} + \cdots + \alpha_pX_{n-p} + \varepsilon_n, \quad n=\cdots,-1,0,1,\cdots.$$

试导出 Yule-Walker 方程:

$$R(h) = \alpha_1R(h-1) + \cdots + \alpha_pR(h-p), \quad h>0.$$

提示: AR(p) 模型两边同乘以 X_{n-k}, 然后取期望.

由 $X_n = \alpha_1X_{n-1} + \cdots + \alpha_pX_{n-p} + \varepsilon_n$

可知 $X_nX_{n-k} = \alpha_1X_{n-1}X_{n-k} + \cdots + \alpha_pX_{n-p}X_{n-k} + \varepsilon_nX_{n-k}$

两边取期望. $EX_nX_{n-k} = \alpha_1EX_{n-1}X_{n-k} + \cdots + \alpha_pEX_{n-p}X_{n-k} + E\varepsilon_nX_{n-k}$.

记 $R(\tau)$ 是 X_n 的协方差函数. 由 X 是平稳过程. 则 $R(\tau) = EX_nX_{n+\tau}$.

$X_{n-k} = \sum_{i=0}^{\infty}\beta_i\varepsilon_{n-k-i} \Rightarrow E\varepsilon_nX_{n-k} = 0$

$\therefore R(k) = \alpha_1R(k-1) + \cdots + \alpha_pR(k-p)$ 即 $R(h) = \alpha_1R(h-1) + \cdots + \alpha_pR(h-p)$

37. 考虑如下 AR(2) 模型：

(1) $X_n = 0.5X_{n-1} + 0.3X_{n-2} + \varepsilon_n$,

试用 Yule-Walker 方程导出协方差函数，证明它们的谱密度函数 $S(\omega)$ 是周期函数，并作出 $S(\omega)$ 在 $(-\pi,\pi)$ 上的图形.

由 Yule - Walker 方程可得

$R(h) = 0.5R(h-1) + 0.3R(h-2)$

$\because R(0) = E[X_n^2] = E[(0.5X_{n-1} + 0.3X_{n-2} + \varepsilon_n)^2] = 0.34R(0) + 0.3R(1) + \sigma^2$

$R(1) = E[X_nX_{n-1}] = E[0.5X_{n-1}^2 + 0.3X_{n-2}X_{n-1} + \varepsilon_nX_{n-1}] = 0.5R(0) + 0.3R(1)$

$\Rightarrow R(0) = \frac{175}{78}\sigma^2 \quad R(1) = \frac{125}{78}\sigma^2$

$S_1(\omega) = \sum_{\tau=-\infty}^{+\infty} e^{-j\omega\tau}R(\tau) = 0.5\sum_{\tau=-\infty}^{+\infty} e^{-j\omega(\tau+1)}R(\tau) + 0.3\sum_{\tau=-\infty}^{+\infty} e^{-j\omega(\tau+2)}R(\tau)$

$S_1(2\pi+\omega) = \sum_{\tau=-\infty}^{+\infty} e^{-j(2\pi+\omega)\tau}R(\tau) = \sum_{\tau=-\infty}^{+\infty} e^{-j2\pi}e^{-j\omega\tau}R(\tau) = \sum_{\tau=-\infty}^{+\infty} e^{-j\omega\tau}R(\tau) = S_1(\omega)$

其中 $e^{-j2\pi} = \cos 2\pi - j\sin 2\pi = 1$ 故 $S_1(\omega)$ 具有周期性.

2. 设 $\{W(t), t \geqslant 0\}$ 为 Brown 运动, 验证下列过程仍为 $[0,\infty)$ 上的 Brown 运动.

(1) $T(t) = tW(1/t)$,

(2) $W(t) = W(a^2t)/a$, $a > 0$.

(1) $T(t) = tW(\frac{1}{t})$

① $\lim_{t\to 0} T(0) = 0$

② $T(t+s) - T(s) = (t+s)W(\frac{1}{t+s}) - sW(\frac{1}{s})$

$= tW(\frac{1}{t+s}) + s[W(\frac{1}{t+s}) - W(\frac{1}{s})]$

$tW(\frac{1}{t+s}) \sim N(0, \frac{t^2}{t+s})$

$s[W(\frac{1}{t+s}) - W(\frac{1}{s})] \sim N(0, \frac{ts}{t+s})$

$\therefore T(t+s) - T(s) \sim N(0, t)$

③ $T(t)$ 具有独立增量.

(2) $Y(t) = \frac{1}{a}W(a^2t)$

① $Y(0) = 0$.

② $Y(t+s) - Y(s) = \frac{1}{a}W(a^2(t+s)) - \frac{1}{a}W(a^2s)$

$= \frac{1}{a}(W(a^2(t+s)) - W(a^2s))$

$\sim \frac{1}{a}N(0, a^2t) \sim N(0, t)$

③ $Y(t)$ 具有独立增量.

则 $Y(t) = \frac{1}{a}W(a^2t)$ 为 Brown 运动

5. 设 $\{Z(t), t \geqslant 0\}$ 为 Brown 桥, 验证 $W(t) = (t+1)Z\left(\frac{t}{t+1}\right)$ 是 Brown 运动.

$EZ(t) = 0$.

$EW(t) = (t+1)E[Z(\frac{t}{t+1})] = 0$

$\therefore E[W(t)W(s)] = (t+1)(s+1)E[Z(\frac{t}{t+1})Z(\frac{s}{s+1})]$

$= (t+1)(s+1)(1-\frac{t}{t+1})\frac{s}{s+1}$

$= s$

$\therefore W(t)$ 是 Brown 运动