插值函数：设 $f(x)$ 为定义在区间 $[a.b]$ 上的函数. $a\le x_0<x_1<\cdots<x_n\le b$. 设 Φ 为给定的某一函数类 $\{\varphi_0(x).\ \varphi_1(x)\cdots\varphi_m(x)\}$ 为它的一组基. 在 Φ 上找一个插值函数 $g(x)$ 使其满足 $g(x_i)=f(x_i)\quad i=0\ldots n$.

定理：$\{x_i\}_{i=0}^{n}$ 为 $n+1$ 个节点，$\Phi=\mathrm{span}\{\varphi_0.\ \varphi_1\ldots\ \varphi_n\}$ $n+1$ 维空间. 则插值函数存在唯一. 当且仅当

$$\begin{vmatrix} \varphi_0(x_0) & \cdots & \varphi_n(x_0) \\ \vdots & \ddots & \vdots \\ \varphi_n(x_0) & \cdots & \varphi_n(x_n) \end{vmatrix} \neq 0$$

特点：1. 与基函数的选择无关

2. 与 $f(x)$ 的表达形式无关

3. 基函数个数与节点个数相同

1.1. Lagrange 插值多项式

(1) $g(x)=\sum_{i=0}^{n} l_i(x)f(x_i)$ 其中 $l_i(x_j)=\delta_{ij}=\begin{cases}0 & i\neq j\\ 1 & i=j\end{cases}$

$$l_i(x)=\frac{(x-x_0)\cdots(x-x_{i-1})(x-x_{i+1})\cdots(x-x_n)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$$

例：线性插值：$(x_0.f(x_0))$. $(x_1.f(x_1))$

$l_0(x)=\frac{x-x_1}{x_0-x_1}\quad l_1(x)=\frac{x-x_0}{x_1-x_0}\quad\Rightarrow\ g(x)=f(x_0)l_0(x)+f(x_1)l_1(x)$

(2) 插值误差：$R_n(x)=f(x)-L_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\cdots(x-x_n)$

$$R_2(x)=\frac{f^{(3)}(\xi)}{3!}(x-x_0)(x-x_1)(x-x_2),$$
$$\xi\in[\min\{x_0,x_1,x_2,x\},\max\{x_0,x_1,x_2,x\}]$$

(3) 事后误差估计 (未知 $f(x)$ 的情况)

给定 $\{x_i\}_{i=0}^{n+1}$ 任取 $n+1$ 个构造 $L_n(x)$ 如 $i=0\ldots n\Rightarrow L_n(x)$ 另取 $i=1\ldots n+1\Rightarrow\tilde{L}_n(x)$

则 $f(x)-L_n(x)=\frac{x-x_0}{x_0-x_{n+1}}\left(L_n(x)-\tilde{L}_n(x)\right)$

(4) Lagrange 插值的缺点：无袭承性. 增加一个节点，所有的基函数都要重新计算

1.2. Newton 插值多项式

(1) $N_n(x)=a_0+a_1(x-x_0)+\cdots+a_n(x-x_0)\cdots(x-x_{n-1})$

$a_0=f(x_0)\quad a_1=f[x_0.x_1]\quad a_2=f[x_2.x_1.x_0]\ \cdots\ a_n=f[x_0\ldots x_n]$

差商：$f[x_0.x_1]=\frac{f(x_1)-f(x_0)}{x_1-x_0}$

$f[x_0.x_1.x_2]=\frac{f[x_1.x_2]-f[x_0.x_1]}{x_2-x_0}$

$\vdots$

$f[x_0.x_1..x_n]=\frac{f[x_1\ldots x_n]-f[x_0.\ldots x_{n-1}]}{x_n-x_0}$

(2) $R_n(x)=\frac{f^{(n+1)}(\xi)}{(n+1)!}(x-x_0)\cdots(x-x_n)$

差商性质总结：

$f[x_0.\ldots x_n]=f[x_{i0}\ldots x_{in}]$

$f[x_0.\ldots x_n]=\sum_{i=0}^{n}\frac{f(x_i)}{(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)}$

$f[x_0\ldots x_n.x]=\frac{f^{(n+1)}(\xi_x)}{(n+1)!}$

1.3. Hermite 插值

定义：满足 $\{x_i.f(x_i)\}_{i=0}^{n}$ 和 $\{(x_i.f^{(k)}(x_i)),\ k=1,\ldots,k_i)\}_{i=0}^{n}$ 条件的插值称为 Hermite 插值.

给定一阶导数值的差商表

$$\begin{array}{llll}
x_0.\ f(x_0) & & & \\
x_0.\ f(x_0) & f[x_0.x_0] & & \\
x_1.\ f(x_1) & f[x_0.x_1] & f[x_0.x_0.x_1] & \\
x_1.\ f(x_1) & f[x_1.x_1] & f[x_0.x_1.x_1] & \\
x_2.\ f(x_2) & f[x_1.x_2] & \vdots & f[x_0.x_0.x_1.x_1] \\
\vdots & \vdots & & \cdots\quad\vdots
\end{array}$$

$x_n \quad f(x_n) \quad f[x_{n-1}, x_n] \quad f[x_{n-1}, x_{n-1}, x_n] \quad \cdots\cdots$

$x_n \quad f(x_n) \quad f[x_n, x_n] \quad f[x_{n-1}, x_n, x_n] \quad \cdots\cdots \quad f[x_0, x_0, \ldots, x_n, x_n]$

差商型二重密切 Hermite 插值公式: $H_{2n+1}(x) = f(x_0) + f[x_0, x_0](x-x_0) + f[x_0, x_0, x_1](x-x_0)^2 + \cdots + f[x_0, x_0, \ldots, x_n, x_n]\prod_{i=0}^{n-1}(x-x_i)^2(x-x_n)$

误差分析: $R(x) = \frac{f^{(k_0+k_1+\cdots+k_n+n+1)}(\xi)}{(k_0+k_1+\cdots+k_n+n+1)!}(x-x_0)^{k_0+1}\cdots(x-x_n)^{k_n+1}$

二重密切 Hermite 插值误差: $R(x) = \frac{f^{(2n+2)}(\xi)}{(2n+2)!}(x-x_0)^2\cdots(x-x_n)^2$

1.4. 分段插值

1. Runge 现象　等距高次插值 数值稳定性差
2. 分段线性插值. $a = x_0 < x_1 < \cdots < x_n = b$

每个小区间上作线性插值

$$S_i(x) = \frac{x - x_{i+1}}{x_i - x_{i+1}} f(x_i) + \frac{x - x_i}{x_{i+1} - x_i} f(x_{i+1}) \qquad x \in [x_i, x_{i+1}]$$

$$P_n(x) = \{S_i(x) : x \in [x_i, x_{i+1}] \quad i = 0.1.2\ldots n-1\}$$

误差: $|f(x) - P_n(x)| = \left|\frac{f^{(2)}(\xi)}{2!}(x-x_i)(x-x_{i+1})\right|$

$$\leq \frac{M_2}{2}\left(\frac{x_{i+1}-x_i}{2}\right)^2 \qquad M_2 = \max_{a\leq x\leq b}|f''(x)|$$

1.5. 三次样条插值.

定义: 给定区间 $[a.b]$ 上 $n+1$ 个节点, $a = x_0 < x_1 < \cdots < x_n = b$ 和这些点上的函数值 $\{f(x_i) = y_i \quad i = 0.1\ldots n\}$ 若 $S(x)$ 满足 $\{S(x_i) = y_i \quad i = 0.1\ldots n\}$ $S(x)$ 在每个小区间 $[x_i, x_{i+1}]$ 上至多是一个三次多项式 $S(x)$ 在 $[a.b]$ 上有连续的二阶导数

则称 $S(x)$ 为 $f(x)$ 关于剖分 $a = x_0 < x_1 < \cdots < x_n = b$ 的三次样条插值函数 称 $\{x_0 .. x_n\}$ 为样条结点,

记 $S''(x_i) = M_i \quad (i = 0 \ldots n) \qquad S_i(x) \equiv S(x) \quad x \in [x_i, x_{i+1}] \quad i = 0.1\ldots n-1$

$$\begin{cases} S_i(x_i) = f(x_i), & S_i(x_{i+1}) = f(x_{i+1}) \\ S_i''(x_i) = M_i, & S_i''(x_{i+1}) = M_{i+1} \end{cases}$$

记 $h_i = x_{i+1} - x_i$

由于 3 次多项式求导 2 次后. 为线性函数

$$S_i''(x) = M_i \frac{x - x_{i+1}}{-h_i} + M_{i+1}\frac{x - x_i}{h_i}$$

积分 2 次: $S_i(x) = \frac{(x_{i+1}-x)^3}{6h_i} M_i + \frac{(x-x_i)^3}{6h_i} M_{i+1} + C(x_{i+1} - x) + d(x - x_i)$

$$C = \frac{y_i}{h_i} - \frac{h_i M_i}{6} \qquad d = \frac{y_{i+1}}{h_i} - \frac{h_i M_{i+1}}{6}$$

M关系式: $\mu_i M_{i-1} + 2M_i + \lambda_i M_{i+1} = d_i \quad i = 1\ldots n-1$

$$\mu_i = \frac{h_{i-1}}{h_i + h_{i-1}} \qquad \lambda_i = \frac{h_i}{h_i + h_{i-1}} \qquad d_i = 6f[x_{i-1}, x_i, x_{i+1}]$$

添加 2 个附加条件可得 $n+1$ 阶方程组.

$$\begin{pmatrix} 2 & 1 & & & & \\ \mu_1 & 2 & \lambda_1 & & & \\ & \mu_2 & 2 & \lambda_2 & & \\ & & \ddots & \ddots & \ddots & \\ & & & \mu_{n-1} & 2 & \lambda_{n-1} \\ & & & & 1 & 2 \end{pmatrix} \begin{pmatrix} M_0 \\ M_1 \\ M_2 \\ \vdots \\ M_{n-1} \\ M_n \end{pmatrix} = \begin{pmatrix} d_0 \\ d_1 \\ d_2 \\ \vdots \\ d_{n-1} \\ d_n \end{pmatrix}$$

周期边界条件: $M_0 \equiv M_n$

m关系式: $\lambda_i m_{i-1} + 2m_i + \mu_i m_{i+1} = C_i \quad i = 1.2\ldots n-1$

$$\lambda_i = \frac{h_i}{h_i + h_{i-1}} \qquad \mu_i = \frac{h_{i-1}}{h_i + h_{i-1}} \qquad C_i = 3(\lambda_i f[x_{i-1}, x_i] + \mu_i f[x_i, x_{i+1}])$$

$n-1$ 个方程 $n+1$ 个未知数. 需要附加条件

(1) 固支边界条件: $S'(x_0) = m_0 \quad S'(x_n) = m_n$

(2) 给定 $S''(x_0) = M_0 \quad S''(x_n) = M_n$

$$2m_0 + m_1 = 3f[x_0, x_1] - \frac{h_1}{2}M_0$$

$$m_{n-1} + 2m_n = 3f[x_{n-1}, x_n] + \frac{h_n}{2}M_n$$

(3) 周期边界条件 $m_0 = m_n$ $M_0 = M_n$

计算过程如下：

			$\frac{h_i}{h_i+h_{i-1}}$	$1-\lambda_i$		$3(\lambda_i f[x_{i-1},x_i]+\mu_i f[x_i,x_{i+1}])$	
x_0	$f(x_0)$						
x_1	$f(x_1)$	h_0	λ_1	μ_1	$f[x_0,x_1]$	c_1	d_1
x_2	$f(x_2)$	h_1	λ_2	μ_2	$f[x_1,x_2]$	c_2	d_2
			$\vdots$	$\vdots$	$f[x_2,x_3]$	$\vdots$	$\vdots$
x_{n-1}	$f(x_{n-1})$	h_{n-2}	λ_{n-1}	μ_{n-1}	$\vdots$	c_{n-1}	d_{n-1}
x_n	$f(x_n)$	h_{n-1}			$f[x_{n-1},x_n]$		
		$h_i = x_{i+1}-x_i$					$6f[x_{i-1},x_i,x_{i+1}]$