Chap4

22. 设平稳过程 $\{X(t)\}$ 的协方差函数 $R(\tau)=\frac{a^2}{2}\cos\omega\tau+b^2e^{-a|\tau|}$, 求功率谱密度函数 $S(\omega)$.

$$\cos\omega_0\tau=\frac{1}{2}(e^{j\omega_0\tau}+e^{-j\omega_0\tau})$$

$$\int\cos\omega_0\tau\, e^{-j\omega\tau}d\tau=\frac{1}{2}\int(e^{j\omega_0\tau}+e^{-j\omega_0\tau})e^{-j\omega\tau}d\tau$$
$$=\frac{1}{2}\int(e^{j\tau(\omega_0-\omega)}+e^{-j\tau(\omega_0+\omega)})d\tau$$
$$=\pi(\delta(\omega+\omega_0)+\delta(\omega-\omega_0))$$

$$\int e^{-a|\tau|}e^{-j\omega\tau}d\tau=\int_{-\infty}^{0}e^{a\tau}e^{-j\omega\tau}d\tau+\int_0^{+\infty}e^{-a\tau}e^{-j\omega\tau}d\tau$$
$$=\int_{-\infty}^{0}e^{(a-j\omega)\tau}d\tau+\int_0^{+\infty}e^{-(a+j\omega)\tau}d\tau$$
$$=\frac{2a}{\omega^2+a^2}$$

$$\therefore S(\omega)=\int R(\tau)e^{-j\omega\tau}d\tau$$
$$=\frac{a^2}{2}\int\cos\omega_0\tau\, e^{-j\omega\tau}d\tau+b^2\int e^{-a|\tau|}e^{-j\omega\tau}d\tau$$
$$=\frac{\pi a^2}{2}(\delta(\omega+\omega_0)+\delta(\omega-\omega_0))+\frac{2ab^2}{\omega^2+a^2}$$

23. 设 $\{X(t)\}$ 为 Gauss 平稳过程, 均值为零, $R_X(\tau)=Ae^{-a|\tau|}\cos\beta\tau$. 令 $Y(t)=X^2(t)$, 验证 $R_Y(\tau)=A^2e^{-2a\tau}(1+\cos2\beta\tau)$, 并求出 $S_Y(\omega)$.

由 4.3.2 中的平方检波可知.

$$R_Y(\tau)=2R_X^2(\tau)=2A^2e^{-2a|\tau|}\cos^2\beta\tau$$
$$=A^2e^{-2a|\tau|}(1+\cos2\beta\tau)$$
$$R_Y(\tau)=A^2e^{-2a|\tau|}+A^2e^{-2a|\tau|}\cos2\beta\tau$$

由 $e^{-a|\tau|}\leftrightarrow\frac{2a}{a^2+\omega^2}$

$$e^{-a|\tau|}\cos\omega_0\tau\leftrightarrow\frac{1}{2\pi}\cdot\frac{2a}{a^2+\omega^2}*\pi[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)]=\frac{a}{a^2+(\omega+\omega_0)^2}+\frac{a}{a^2+(\omega-\omega_0)^2}$$

可知 $S_Y(\omega)=\int R_Y(\tau)e^{-j\omega\tau}d\tau$

$$=\int A^2e^{-2a|\tau|}e^{-j\omega\tau}d\tau+\int A^2e^{-2a|\tau|}\cos2\beta\tau\, e^{-j\omega\tau}d\tau$$
$$=A^2\left(\frac{4a}{\omega^2+4a^2}+\frac{2a}{4a^2+(\omega+2\beta)^2}+\frac{2a}{4a^2+(\omega-2\beta)^2}\right)$$
$$=2aA^2\left(\frac{2}{\omega^2+4a^2}+\frac{1}{4a^2+(\omega+2\beta)^2}+\frac{1}{4a^2+(\omega-2\beta)^2}\right)$$

24. 设 $\{X(t)\}$ 为 Gauss 平稳过程, 均值为零, 功率谱密度 $S(\omega)=\frac{1}{1+\omega^2}$. 求 $X(t)$ 落在区间 [0.5, 1] 中的概率.

$$R(\tau)=\frac{1}{2\pi}\int\frac{1}{1+\omega^2}e^{j\omega\tau}d\omega$$

设 $f(z)=\frac{1}{1+z^2}e^{jz|\tau|}=\frac{1}{(z+j)(z-j)}e^{jz|\tau|}$ 则 $z=j$ 为 $f(z)$ 的一级极点,

$$\therefore Res[f(z), z]=(z-j)\frac{1}{(z+j)(z-j)}e^{jz|\tau|}=\frac{1}{2j}e^{-|\tau|}$$

则由留数定理

$$R(\tau)=\frac{1}{2\pi}\cdot2\pi j\, Res[f(z), z]=\frac{1}{2}e^{-|\tau|}$$

则 $\sigma^2=R(0)=\frac{1}{2}$ $\mu=0$.

由 $\{X(t)\}$ 为 Gauss 平稳过程, 服从 $\sigma^2=\frac{1}{2}$, $\mu=0$ 的正态分布.

$$\therefore P(0.5\le X(t)\le1)=P\left(\frac{0.5}{\frac{\sqrt2}{2}}\le\frac{X(t)}{\frac{\sqrt2}{2}}\le\frac{1}{\frac{\sqrt2}{2}}\right)=0.16$$

28. 求下列功率谱密度对应的协方差函数:

(1) $S(\omega)=\frac{\omega^2+64}{\omega^4+29\omega^2+100}$;

(2) $S(\omega)=\frac{1}{(1+\omega^2)^2}$;

(1) $R(\tau)=\frac{1}{2\pi}\int\frac{\omega^2+64}{\omega^4+29\omega^2+100}e^{j\omega\tau}d\omega$

设 $f(z)=\frac{z^2+64}{(z^2+4)(z^2+25)}e^{jz|\tau|}$ 则 $z_1=2j$ $z_2=5j$ 分别是 $f(z)$ 的一级极点,

$$\therefore Res[f(z), z_1]=(z+z_1)\frac{z^2+64}{(z^2+4)(z^2+25)}\Big|_{z=z_1}=\frac{5}{7j}e^{-2|\tau|}$$
$$Res[f(z), z_2]=(z+z_2)\frac{z^2+64}{(z^2+4)(z^2+25)}\Big|_{z=z_1}=-\frac{13}{70j}e^{-5|\tau|}$$

则由留数定理.

$$R(\tau)=\frac{1}{2\pi}\cdot2\pi j\{Res[f(z), z_1]+Res[f(z), z_2]\}$$
$$=j\left\{\frac{5}{7j}e^{-2|\tau|}-\frac{13}{70j}e^{-5|\tau|}\right\}$$
$$=\frac{5}{7}e^{-2|\tau|}-\frac{13}{70}e^{-5|\tau|}$$

(2) $R(\tau) = \frac{1}{2\pi}\int \frac{1}{(1+\omega^2)^2} e^{j\omega\tau} d\omega$

设 $f(z) = \frac{e^{jz|\tau|}}{(1+z^2)^2} = \frac{e^{jz|\tau|}}{(z+j)^2(z-j)^2}$ 则 $z=j$ 为 $f(z)$ 的2级极点

$\therefore \mathrm{Res}[f(z), j] = \frac{1}{(2-1)!}\lim\limits_{z\to j}\frac{d}{dz}((z-j)^2 \cdot f(z))$

$= \lim\limits_{z\to j}\left[\frac{e^{jz|\tau|}}{(z+j)^2}\right]'$

$= \left.\frac{e^{jz|\tau|}\cdot j|\tau|(z+j)^2 - e^{jz|\tau|}2(z+j)}{(z+j)^4}\right|_{z=j}$

$= \frac{1}{4j}\cdot e^{-|\tau|}(1+|\tau|)$

由留数定理:

$R(\tau) = \frac{1}{2\pi}\cdot 2\pi j\ \mathrm{Res}[f(z), j] = \frac{1}{4}e^{-|\tau|}(1+|\tau|)$