1、设有常微分初值问题

$$\begin{cases} y'(x) = -y(x) \quad (0 \le x \le 1) \\ y(0) = 1 \end{cases}$$

假设求解区间[0.1]被n等分(n充分大) 令 $h=\frac{1}{n}$. $x_k=\frac{k}{n}$ $(k=0.1\dots n)$

(a) 分别写出向前Euler公式.向后Euler公式.梯形公式以及改进的Euler公式求上述微分方程数值解时的差分格式

(即 y_{k+1} 与 y_k 二者之间的递推关系)

(b) 设 $y_0=y(0)$ 分别求此四种公式(方法)下的近似值 y_n 的表达式;

(这里的 y_n 即是 $y(x_n)\equiv y(1)$ 的近似值)

(c) 当n充分大(即区间长度 $h\to 0$)时.分别判断四种方法下的近似值 y_n 是否收敛到原问题的真解 $y(x)$ 在 $x=1$ 处的值.(i.e. $y(1)$)

(a) 向前Euler公式: $\begin{cases} y_{k+1} = y_k - hy_k = (1-h)y_k \\ y_0 = y(0) = 1 \end{cases}$

向后Euler公式: $\begin{cases} y_{k+1} = y_k - hy_{k+1} \Rightarrow y_{k+1} = \frac{1}{1+h}y_k \\ y_0 = y(0) = 1 \end{cases}$

梯形公式: $\begin{cases} y_{k+1} = y_k - \frac{h}{2}(y_k + y_{k+1}) \Rightarrow y_{k+1} = \frac{2-h}{2+h}y_k \\ y_0 = y(0) = 1 \end{cases}$

改进的Euler公式: $\begin{cases} y_{k+1} = y_k + \frac{h}{2}[-y_k + (1-h)y_k] \Rightarrow y_{k+1} = (1-\frac{h^2}{2})y_k \\ y_0 = y(0) = 1 \end{cases}$

(b) 向前Euler公式. $y_n = (1-h)^n y_0$

向后Euler公式: $y_n = (\frac{1}{1+h})^n y_0$.

梯形公式: $y_n = (\frac{2-h}{2+h})^n y_0$

改进的Euler公式: $y_n = (1-\frac{h^2}{2})^n y_0$.

(c) 精确解为 $y=e^{-x}$

向前Euler公式: $y_n = (1-h)^{\frac{x}{h}} y_0$.

则 $\lim\limits_{n\to\infty} y_n = \lim\limits_{h\to 0} y_n = \lim\limits_{h\to 0}((1-h)^{-\frac{1}{h}})^{-x} = e^{-x}$ 故收敛

向后Euler公式: $y_n = (\frac{1}{1+h})^{\frac{x}{h}} y_0$.

则 $\lim\limits_{n\to\infty} y_n = \lim\limits_{h\to 0} y_n = \lim\limits_{h\to 0}((1+h)^{\frac{1}{h}})^{-x} = e^{-x}$ 故收敛

梯形公式: $y_n = (\frac{2-h}{2+h})^n y_0$

则 $\lim\limits_{n\to\infty} y_n = \lim\limits_{h\to 0} y_n = \lim\limits_{h\to 0}(\frac{2-h}{2+h})^n = \lim\limits_{h\to 0}[(1-\frac{2h}{2+h})^{-\frac{2+h}{2h}}]^{-\frac{2x}{2+h}} = e^{-x}$ 故收敛

改进的Euler公式: $y_n = (1-\frac{h^2}{2})^n y$

则 $\lim\limits_{n\to\infty} y_n = \lim\limits_{h\to 0} y_n = \lim\limits_{h\to 0}(1-\frac{h^2}{2})^{\frac{x}{h}} = \lim\limits_{h\to 0}[(1-\frac{h^2}{2})^{-\frac{2}{h^2}}]^{-\frac{hx}{2}} = 1$ 故不收敛

2. 试推导差分格式

$$y_{n+1} = y_{n-1} + \frac{h}{3}[7f(x_n.y_n) - 2f(x_{n-1}.y_{n-1}) + f(x_{n-2}.y_{n-2})]$$

的局部截断误差.即验证

$$T_{n+1} \equiv y(x_{n+1}) - y_{n+1} = \frac{1}{3}h^4 y^{(4)}(x_{n-1}) + O(h^5)$$

(提示:将差分格式右端的某些项在某点处同时作Taylor展开)

$y(x_{n+1})$ 在 x_{n-1} 处作Taylor展开:

$$y(x_{n+1}) = y(x_{n-1}) + (2h)y'(x_{n-1}) + \frac{(2h)^2}{2!}y''(x_{n-1}) + \frac{(2h)^3}{3!}y'''(x_{n-1}) + \frac{(2h)^4}{4!}y^{(4)}(x_{n-1}) + O(h^5)$$

$f(x_n.y_n)$ 在 x_{n-1} 处作Taylor展开:

$$f(x_n.y_n) = y'(x_n) = y'(x_{n-1}) + hy''(x_{n-1}) + \frac{h^2}{2!}y'''(x_{n-1}) + \frac{h^3}{3!}y^{(4)}(x_{n-1}) + O(h^4)$$

$$f(x_{n-1}.y_{n-1}) = y'(x_{n-1})$$

$f(x_{n-2}、y_{n-2})$ 在 x_{n-1} 处作Taylor展开:

$$f(x_{n-2}.y_{n-2}) = y'(x_{n-2}) = y'(x_{n-1}) - hy''(x_{n-1}) + \frac{h^2}{2!}y'''(x_{n-1}) - \frac{h^3}{3!}y^{(4)}(x_{n-1}) + O(h^4)$$

代入 $y_{n+1} = y_{n-1} + \frac{h}{3}[7f(x_n.y_n) - 2f(x_{n-1}.y_{n-1}) + f(x_{n-2}.y_{n-2})]$

$$
\begin{aligned}
y_{n+1} = {} & y(x_{n-1}) \\
& + \tfrac{7}{3}h\left[y'(x_{n-1}) + h y''(x_{n-1}) + \tfrac{h^2}{2!}y'''(x_{n-1}) + \tfrac{h^3}{3!}y^{(4)}(x_{n-1}) + O(h^4)\right] \\
& - \tfrac{2}{3}h\, y'(x_{n-1}) \\
& + \tfrac{1}{3}h\left[y'(x_{n-1}) - h y''(x_{n-1}) + \tfrac{h^2}{2!}y'''(x_{n-1}) - \tfrac{h^3}{3!}y^{(4)}(x_{n-1}) + O(h^4)\right] \\
= {} & y(x_{n-1}) + 2h y'(x_{n-1}) + 2h^2 y''(x_{n-1}) + \tfrac{4}{3}h^3 y'''(x_{n-1}) + \tfrac{1}{3}h^4 y^{(4)}(x_{n-1}) + O(h^5)
\end{aligned}
$$

又∵ $y(x_{n+1}) = y(x_{n-1}) + (2h)y'(x_{n-1}) + \frac{(2h)^2}{2!}y''(x_{n-1}) + \frac{(2h)^3}{3!}y'''(x_{n-1}) + \frac{(2h)^4}{4!}y^{(4)}(x_{n-1}) + O(h^5)$

$= y(x_{n-1}) + 2h y'(x_{n-1}) + 2h^2 y''(x_{n-1}) + \frac{4}{3}h^3 y'''(x_{n-1}) + \frac{2}{3}h^4 y^{(4)}(x_{n-1}) + O(h^5)$

∴ $T_{n+1} = y(x_{n-1}) - y_{n-1} = \frac{1}{3}h^4 y^{(4)}(x_{n-1}) + O(h^5)$

3. 试用线性多步法构造 $p=1$ $q=2$ 时的隐式差分格式. 求该格式局部截断误差的误差主项并判断它的阶. 最后为该隐式格式设计一种合适的预估－校正格式

先确定积分区间以及积分节点，积分区间为 $[x_{n-1}, x_{n+1}]$

$q+1=3$ 个积分节点分别为 $\{x_{n+1}, x_n, x_{n-1}\}$ 记步长 $h = x_n - x_{n-1} = x_{n+1} - x_n$

故差分格式为

$$y_{n+1} = y_{n-1} + [\alpha_0 f(x_{n+1}, y_{n+1}) + \alpha_1 f(x_n, y_n) + \alpha_2 f(x_{n-1}, y_{n-1})]$$

$$\alpha_0 = \int_{x_{n-1}}^{x_{n+1}} \frac{(x-x_n)(x-x_{n-1})}{(x_{n+1}-x_n)(x_{n+1}-x_{n-1})}dx = \frac{1}{3}h$$

$$\alpha_1 = \int_{x_{n-1}}^{x_{n+1}} \frac{(x-x_{n-1})(x-x_{n+1})}{(x_n-x_{n-1})(x_n-x_{n+1})}dx = \frac{4}{3}h.$$

$$\alpha_2 = \int_{x_{n-1}}^{x_{n+1}} \frac{(x-x_{n+1})(x-x_n)}{(x_{n-1}-x_{n+1})(x_{n-1}-x_n)}dx = \frac{1}{3}h.$$

故差分格式为:

$$y_{n+1} = y_{n-1} + \frac{h}{3}[f(x_{n+1}, y_{n+1}) + 4f(x_n, y_n) + f(x_{n-1}, y_{n-1})]$$

误差分析如下.

将 y_{n-1} 在 x_{n+1} 处作 Taylor 展开

$$y_{n-1} = y(x_{n-1}) = y(x_{n+1}) - (2h)y'(x_{n+1}) + \frac{(2h)^2}{2!}y''(x_{n+1}) - \frac{(2h)^3}{3!}y'''(x_{n+1}) + \frac{(2h)^4}{4!}y^{(4)}(x_{n+1}) - \frac{(2h)^5}{5!}y^{(5)}(x_{n+1}) + O(h^6)$$

将 $f(x_{n+1}, y_{n+1})$ 在 $y(x_{n+1})$ 处对变量 y 作 Taylor 展开

$$f(x_{n+1}, y_{n+1}) = f(x_{n+1}, y(x_{n+1})) + f_y(x_{n+1}, \xi)(y_{n+1} - y(x_{n+1})) = y'(x_{n+1}) - f_y(x_{n+1}, \xi)T_{n+1}$$

将 $f(x_n, y_n)$ 在 x_{n+1} 处作 Taylor 展开

$$f(x_n, y_n) = y'(x_n) = y'(x_{n+1}) - h y''(x_{n+1}) + \frac{h^2}{2!}y'''(x_{n+1}) - \frac{h^3}{3!}y^{(4)}(x_{n+1}) + \frac{h^4}{4!}y^{(5)}(x_{n+1}) + O(h^5)$$

将 $f(x_{n-1}, y_{n-1})$ 在 x_{n+1} 处作 Taylor 展开

$$f(x_{n-1}, y_{n-1}) = y'(x_{n-1}) = y'(x_{n+1}) - (2h)y''(x_{n+1}) + \frac{(2h)^2}{2!}y'''(x_{n+1}) - \frac{(2h)^3}{3!}y^{(4)}(x_{n+1}) + \frac{(2h)^4}{4!}y^{(5)}(x_{n+1}) + O(h^5)$$

将上式代入差分格式

$$
\begin{aligned}
y_{n+1} = {} & y(x_{n+1}) - (2h)y'(x_{n+1}) + \tfrac{(2h)^2}{2!}y''(x_{n+1}) - \tfrac{(2h)^3}{3!}y'''(x_{n+1}) + \tfrac{(2h)^4}{4!}y^{(4)}(x_{n+1}) - \tfrac{(2h)^5}{5!}y^{(5)}(x_{n+1}) + O(h^6) \\
& + \tfrac{1}{3}h\left[y'(x_{n+1}) - f_y(x_{n+1}, \xi)T_{n+1}\right] \\
& + \tfrac{4}{3}h\left[y'(x_{n+1}) - h y''(x_{n+1}) + \tfrac{h^2}{2!}y'''(x_{n+1}) - \tfrac{h^3}{3!}y^{(4)}(x_{n+1}) + \tfrac{h^4}{4!}y^{(5)}(x_{n+1}) + O(h^5)\right] \\
& + \tfrac{1}{3}h\left[y'(x_{n+1}) - (2h)y''(x_{n+1}) + \tfrac{(2h)^2}{2!}y'''(x_{n+1}) - \tfrac{(2h)^3}{3!}y^{(4)}(x_{n+1}) + \tfrac{(2h)^4}{4!}y^{(5)}(x_{n+1}) + O(h^5)\right]
\end{aligned}
$$

合并同类项得

$$0 = T_{n+1} - \frac{h}{3}f_y(x_{n+1}, \xi)T_{n+1} + \frac{1}{90}h^5 y^{(5)}(x_{n+1}) + O(h^6)$$

$$\Rightarrow 0 = \left(1 - \frac{1}{3}h f_y(x_{n+1}, \xi)\right)T_{n+1} + \frac{1}{90}h^5 y^{(5)}(x_{n+1}) + O(h^6)$$

$$\Rightarrow T_{n+1} = -\frac{1}{90}h^5 y^{(5)}(x_{n+1}) + O(h^6)$$

故该隐式差分格式的局部截断误差为 $T_{n+1} \approx -\frac{1}{90}h^5 y^{(5)}(x_{n+1})$ 精度阶数为 $5-1=4$

其预估－校正格式为

$$
\begin{cases}
\overline{y}_{n+1} = y_{n-1} + \frac{h}{3}[7f(x_n, y_n) - 2f(x_{n-1}, y_{n-1}) + f(x_{n-2}, y_{n-2})] \\
y_{n+1} = y_{n-1} + \frac{h}{3}[f(x_{n+1}, \overline{y}_{n+1}) + 4f(x_n, y_n) + f(x_{n-1}, y_{n-1})]
\end{cases}
$$