1、用Doolittle分解法解线性方程组

$$\begin{cases} 5x_1 + x_2 + 2x_3 = 2 \\ x_1 + 3x_2 - x_3 = 4 \\ 2x_1 + 2x_2 + 5x_3 = 6 \end{cases}$$

由题可得. $Ax = b$.

$$A = \begin{pmatrix} 5 & 1 & 2 \\ 1 & 3 & -1 \\ 2 & 2 & 5 \end{pmatrix} \quad b = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix}$$

则令 $A = LU$ 原方程组化为 $LUx = b$

设 $Ux = y$ 则 $\begin{cases} Ly = b \\ Ux = y \end{cases}$

由

$$A = \begin{pmatrix} 5 & 1 & 2 \\ 1 & 3 & -1 \\ 2 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & & \\ \overset{\frac{1}{5}}{l_{21}} & 1 & \\ \overset{\frac{2}{5}}{l_{31}} & \overset{\frac{4}{7}}{l_{32}} & 1 \end{pmatrix} \begin{pmatrix} \overset{5}{u_{11}} & \overset{1}{u_{12}} & \overset{2}{u_{13}} \\ & \overset{\frac{14}{5}}{u_{22}} & \overset{-\frac{7}{5}}{u_{23}} \\ & & u_{33} \end{pmatrix}$$

则 $u_{11} = a_{11} = 5$

$u_{12} = a_{12} = 1$

$u_{13} = a_{13} = 2$.

$5l_{21} = a_{21} = 1 \Rightarrow l_{21} = \frac{1}{5}$

$\frac{1}{5} + u_{22} = a_{22} = 3 \Rightarrow u_{22} = \frac{14}{5}$

$\frac{2}{5} + u_{23} = a_{23} = -1 \Rightarrow u_{23} = -\frac{7}{5}$

$5l_{31} = a_{31} = 2 \Rightarrow l_{31} = \frac{2}{5}$

$\frac{2}{5} + \frac{14}{5}l_{32} = a_{32} = 2 \Rightarrow l_{32} = \frac{4}{7}$

$\frac{4}{5} - \frac{4}{5} + u_{33} = a_{33} = 5 \Rightarrow u_{33} = 5$

则 $L = \begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{5} & 1 & 0 \\ \frac{2}{5} & \frac{4}{7} & 1 \end{pmatrix} \quad U = \begin{pmatrix} 5 & 1 & 2 \\ & \frac{14}{5} & -\frac{7}{5} \\ & & 5 \end{pmatrix}$

则对 $Ly = b$ 有

$$\begin{pmatrix} 1 & 0 & 0 \\ \frac{1}{5} & 1 & 0 \\ \frac{2}{5} & \frac{4}{7} & 1 \end{pmatrix} y = \begin{pmatrix} 2 \\ 4 \\ 6 \end{pmatrix} \Rightarrow y = (2, \frac{18}{5}, \frac{22}{7})^T$$

则对 $Ux = y$ 有

$$\begin{pmatrix} 5 & 1 & 2 \\ & \frac{14}{5} & -\frac{7}{5} \\ & & 5 \end{pmatrix} x = \begin{pmatrix} 2 \\ \frac{18}{5} \\ \frac{22}{7} \end{pmatrix} \Rightarrow x = (-\frac{6}{35}, \frac{8}{5}, \frac{22}{35})^T$$

2、求如下矩阵的Crout分解

$$A = \begin{pmatrix} 4 & 1 & 0 & 0 \\ -1 & 4 & 2 & 0 \\ 0 & -1 & 4 & 3 \\ 0 & 0 & -1 & 4 \end{pmatrix}$$

由题可得.

$$A = \begin{pmatrix} 4 & 1 & 0 & 0 \\ -1 & 4 & 2 & 0 \\ 0 & -1 & 4 & 3 \\ 0 & 0 & -1 & 4 \end{pmatrix} = \begin{pmatrix} l_{11} & & & \\ l_{21} & l_{22} & & \\ l_{31} & l_{32} & l_{33} & \\ l_{41} & l_{42} & l_{43} & l_{44} \end{pmatrix} \cdot \begin{pmatrix} 1 & u_{12} & u_{13} & u_{14} \\ & 1 & u_{23} & u_{24} \\ & & 1 & u_{34} \\ & & & 1 \end{pmatrix}$$

则 $l_{11} = a_{11} = 4$

$l_{21} = a_{21} = -1$

$l_{31} = a_{31} = 0$

$l_{41} = a_{41} = 0$

$4u_{12} = a_{12} = 1 \Rightarrow u_{12} = \frac{1}{4}$

$$4u_{13} = a_{13} = 0 \Rightarrow u_{13} = 0$$

$$4u_{14} = a_{14} = 0 \Rightarrow u_{14} = 0$$

$$-\frac{1}{4} + l_{22} = a_{22} = 4 \Rightarrow l_{22} = \frac{17}{4}$$

$$l_{32} = a_{32} = -1 \Rightarrow l_{32} = -1$$

$$l_{42} = a_{42} = 0 \Rightarrow l_{42} = 0$$

$$\frac{17}{4}u_{23} = a_{23} = 2 \Rightarrow u_{23} = \frac{8}{17}$$

$$\frac{17}{4}u_{24} = a_{24} = 0 \Rightarrow u_{24} = 0$$

$$-\frac{8}{17} + l_{33} = a_{33} = 4 \Rightarrow l_{33} = \frac{76}{17}$$

$$l_{43} = a_{43} = -1 \Rightarrow l_{43} = -1$$

$$\frac{76}{17}u_{34} = a_{34} = 3 \Rightarrow u_{34} = \frac{51}{76}$$

$$-\frac{51}{76} + l_{44} = a_{44} = 4 \Rightarrow l_{44} = \frac{355}{76}$$

则

$$A = \begin{pmatrix} 4 & 1 & 0 & 0 \\ -1 & 4 & 2 & 0 \\ 0 & -1 & 4 & 3 \\ 0 & 0 & -1 & 4 \end{pmatrix} = \begin{pmatrix} 4 & 0 & 0 & 0 \\ -1 & \frac{17}{4} & 0 & 0 \\ 0 & -1 & \frac{76}{17} & 0 \\ 0 & 0 & -1 & \frac{355}{76} \end{pmatrix} \begin{pmatrix} 1 & \frac{1}{4} & 0 & 0 \\ 0 & 1 & \frac{8}{17} & 0 \\ 0 & 0 & 1 & \frac{51}{76} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$