Modelling and Simulation

Lecture Bionics/Biomimetics MSc.

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Linear State Equations

Most systems are non-linear and the solution is difficult to find. Therefore, it is often helpful to try to approximate non-linear relationships by linear equations. Linear approximations reflect the behaviour of the system only approximately but are much easier to treat. Linear approximations describe the systems behaviour only in a certain region and the error grows along.

The principle of linearization can be explained best for a function of one variable x.

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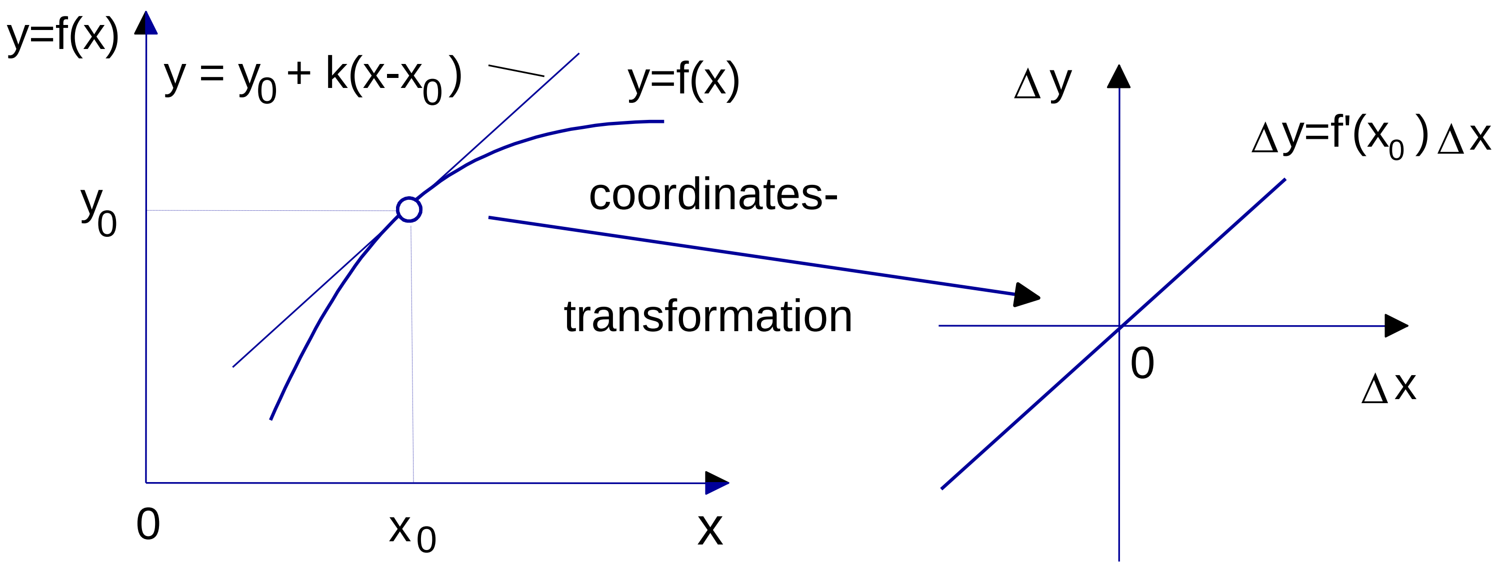
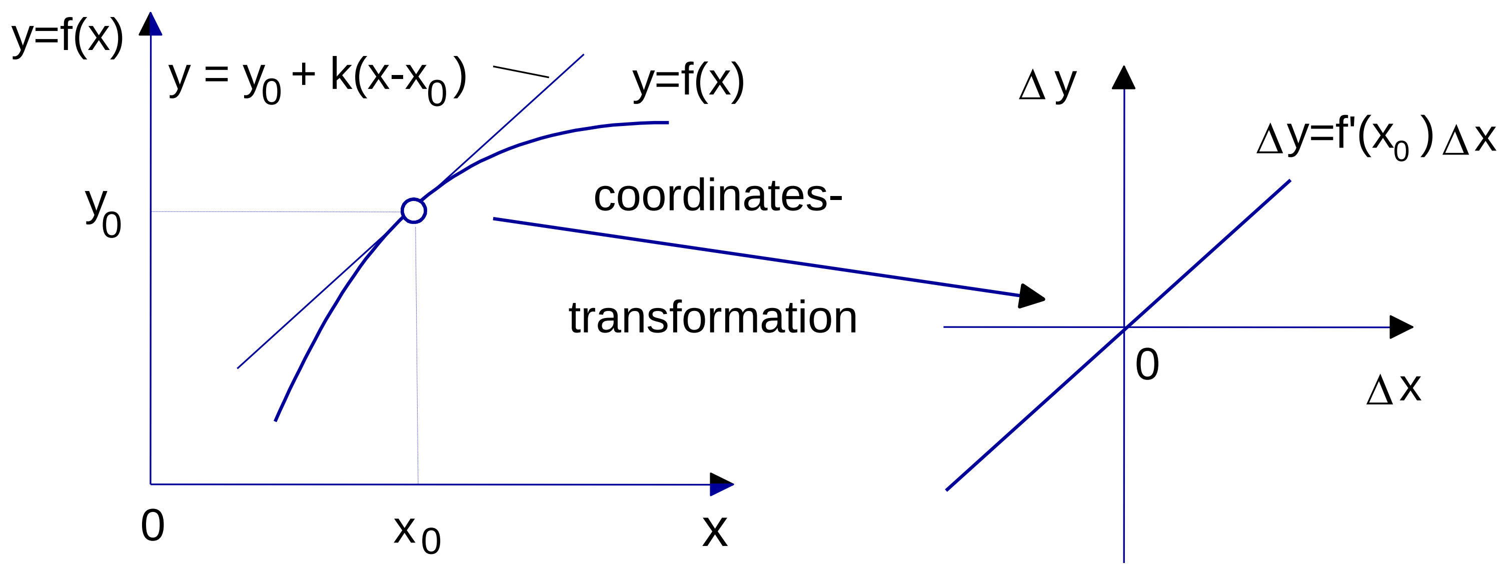


Linear State Equations



In the proximity of a working point (x_0, y_0) where the system behaviour is under investigation *f* can be approximated by the linear function: $y = y_0 + k(x - x_0)$
by applying the linear transformation $\Delta x = x - x_0 \quad \Delta y = y - y_0$
Finally, we arrive at the linear and homogeneous linear relation

$$\Delta y = \frac{\partial f(x)}{\partial x} \Delta x = k \Delta x$$

Linear State Equations

Example 1:
The function f(α) = sin(α) shall be linearized in the proximity of the point $\alpha = \alpha_0$ with $|\alpha-\alpha_0| \ll 1$.
By applying the method shown above one arrives at

$$f(\alpha) = \sin\alpha = f(\alpha_0 + \Delta\alpha) \approx f(\alpha_0) + \left[\frac{\partial f}{\partial \alpha}\right]_{\alpha=\alpha_0} (\alpha - \alpha_0) \approx \sin\alpha_0 + \underbrace{\left[\frac{\partial \sin\alpha}{\partial \alpha}\right]_{\alpha=\alpha_0}}_{\cos(\alpha_0)} \underbrace{(\alpha - \alpha_0)}_{\Delta\alpha}$$

and with the linear transformation $\Delta\alpha := \alpha - \alpha_0$

we get the linear relation $\Delta f = \hat{f}(\Delta\alpha) = \cos\alpha_0 \Delta\alpha$

Example 2: single mass pendulum
For small angles the linearized equation of motion for the single mass pendulum reads

$$\ddot{\varphi} = -\frac{g}{l}\varphi$$

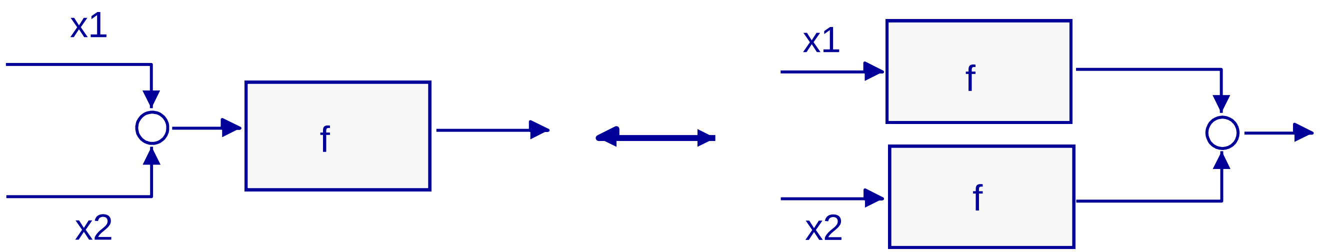
Linear State Equations

Definition: a function f(x) is called *linear*, if the relations below are valid:

$$f(x_1 + x_2) = f(x_1) + f(x_2)$$ Additivity

$$f(\lambda x) = \lambda f(x) \qquad \lambda \in \Re$$ Homogeneity

Shown as block diagram description:



Examples:
function:

$f(x) = a \cdot x$	linear
$f(x) = x^2$	non-linear
$f(x) = a \cdot \mathrm{sign}(x)$	non-linear

Linear State Equations

The transformation of the state equations into their linearized form is generally possible for arbitrary time-dependent functions

$x_0(t)$ and $u_0(t)$

However, normally it makes most sense to linearize with respect to points which are also stationary points or equilibrium positions of the system

$$\mathbf{x} = \mathbf{x}_0 + \Delta\mathbf{x} \qquad \mathbf{u} = \mathbf{u}_0 + \Delta\mathbf{u} \qquad |\Delta\mathbf{x}| << |\mathbf{a}_0| \qquad |\Delta\mathbf{u}| << |\mathbf{b}_0|$$

Here both $\mathbf{a}_0$ und $\mathbf{b}_0$

are typical values of the system. Applying this expressions in the non-linear state equations and by expanding the functions f and g around the points (t, x_0) and (t, u_0) up to the second term we arrive at

$$\frac{d}{dt}(\mathbf{x}_0 + \Delta\mathbf{x}) = \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0, t) + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{x=x_0} \Delta\mathbf{x} + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right]_{u=u_0} \Delta\mathbf{u}$$

$$\mathbf{y}_0 + \Delta\mathbf{y} = \mathbf{g}(\mathbf{x}_0, \mathbf{u}_0, t) + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right]_{x=x_0} \Delta\mathbf{x} + \left[\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right]_{u=u_0} \Delta\mathbf{u}$$

Linear State Equations

The prescribed solution must fulfil the state equations, therefore

$$\frac{d}{dt}(\mathbf{x}_0) = \mathbf{f}(\mathbf{x}_0, \mathbf{u}_0, t) \qquad \mathbf{y}_0 = \mathbf{g}(\mathbf{x}_0, \mathbf{u}_0, t)$$

Thus we arrive at the linear state equations

$$\frac{d}{dt}(\Delta\mathbf{x}) = \left[\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right]_{x=x_0} \Delta\mathbf{x} + \left[\frac{\partial\mathbf{f}}{\partial\mathbf{u}}\right]_{u=u_0} \Delta\mathbf{u} \qquad \Delta\mathbf{y} = \left[\frac{\partial\mathbf{g}}{\partial\mathbf{x}}\right]_{x=x_0} \Delta\mathbf{x} + \left[\frac{\partial\mathbf{g}}{\partial\mathbf{u}}\right]_{u=u_0} \Delta\mathbf{u}$$

By introducing the *Jacobi* - Matrices

$$\mathbf{A} = \left[\frac{\partial\mathbf{f}}{\partial\mathbf{x}}\right]_{x=x_0} \quad \mathbf{B} = \left[\frac{\partial\mathbf{f}}{\partial\mathbf{u}}\right]_{u=u_0} \quad \mathbf{C} = \left[\frac{\partial\mathbf{g}}{\partial\mathbf{x}}\right]_{x=x_0} \quad \mathbf{D} = \left[\frac{\partial\mathbf{g}}{\partial\mathbf{u}}\right]_{u=u_0}$$

The linear equations

$$\Delta\dot{\mathbf{x}} = \mathbf{A}(t)\Delta\mathbf{x}(t) + \mathbf{B}(t)\Delta\mathbf{u}(t)$$

$$\Delta\mathbf{y} = \mathbf{C}(t)\Delta\mathbf{x}(t) + D(t)\Delta\mathbf{u}(t)$$

By expanding the matrix elements we get the *Jacobi* - matrix

$$\mathbf{A} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \frac{\partial f_2}{\partial x_2} & & \vdots \\ \vdots & & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \cdots & \frac{\partial f_n}{\partial x_{1n}} \end{bmatrix}_{\substack{x_1 = x_{10} \\ \dots \\ x_n = x_{n0}}}$$

Linear State Equations

By omitting the Δ we get the
linear state equations

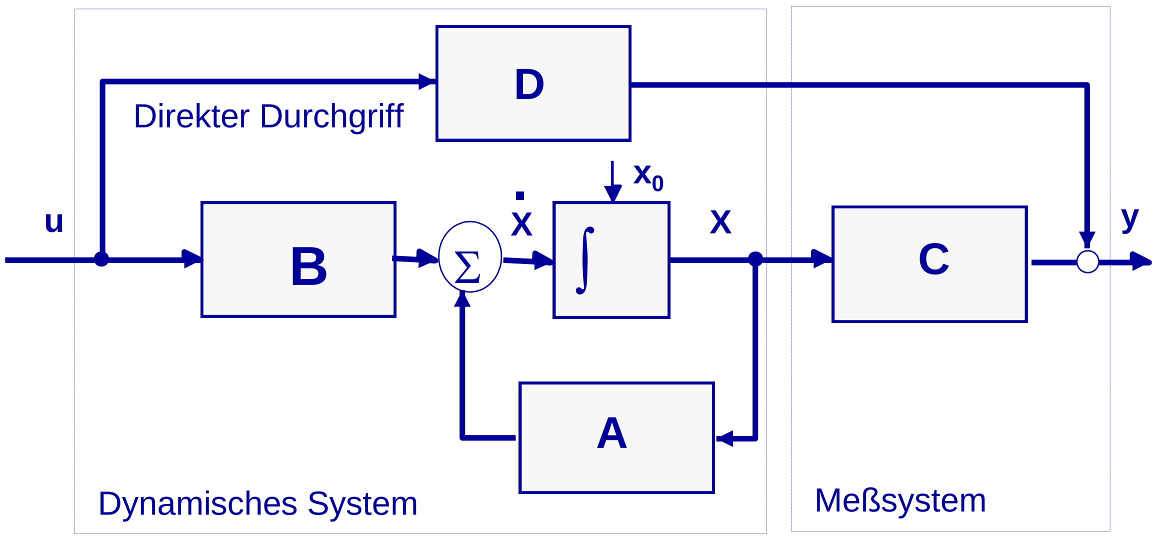
$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$

$$\mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) + \mathbf{D}(t)\mathbf{u}(t)$$

with

A: n x n - System matrix
B: n x r - input matrix
C: m x n – observation matrix
D: m x r - feed through matrix

Block diagram



Time variant system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t)$$

Time invariant system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t)$$

Linear State Equations

Example : Mass point pendulum

$$\dot{\mathbf{x}} = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 \end{bmatrix} \cong \begin{bmatrix} x_2 \\ -\frac{g}{l}x_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & 0 \end{bmatrix}}_{\mathbf{A}}\mathbf{x}$$