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Lecture 4 Notes

These notes correspond to Section 1.3 in the text.

Classification of Differential Equations

There are many types of differential equations, and a wide variety of solution techniques, even for equations of the same type, let alone different types. We now introduce some terminology that aids in classification of equations and, by extension, selection of solution techniques.

- An *ordinary differential equation*, or ODE, is an equation that depends on one or more derivatives of functions of a single variable. Differential equations given in the preceding examples are all ordinary dfiferential equations, and we will consider these equations exclusively in this course.

- A *partial differential equation*, or PDE, is an equation that depends on one or more *partial* derivatives of functions of several variables. In many cases, PDE are solved by reducing to multiple ODE.

  Example The *heat equation*
  $$\frac{\partial u}{\partial t} = k^2 \frac{\partial^2 u}{\partial x^2},$$
  where k is a constant, is an example of a partial differential equation, as its solution $u(x,t)$ is a function of two independent variables, and the equation includes partial derivatives with respect to both variables. □

- The *order* of a differential equation is the order of the highest derivative of any unknown function in the equation.

  Example The differential equation
  $$\frac{dy}{dt} = ay - b,$$
  where a and b are constants, is a first-order differential equation, as only the first derivative of the solution $y(t)$ appears in the equation. On the other hand, the ODE
  $$y'' + 3y' + 2y = 0$$
  is a second-order differential equation, whereas the PDE known as the *beam equation*
  $$u_t = u_{xxxx}$$
  is a fourth-order differential equation. □

- A differential equation is *linear* if any linear combination of solutions of the equation is also a solution of the equation. A differential equation that is not linear is said to be *nonlinear*.

Nonlinear equations are, in general, very difficult to solve, so in many cases one approximates a nonlinear equation by a linear equation, called a *linearization*, that is more readily solved.

Example The ODE

$$y' + 3t^2 y = e^t, \quad y'' + (\sin t)y' + ty = 0$$

are examples of linear differential equations. Note that coefficients of these equations may be functions of the independent variable t, but not of the dependent variable y. On the other hand, the PDE known as *Burger's equation*,

$$u_t + uu_x = 0,$$

is a nonlinear differential equation. A linearization may be obtained by replacing the coefficient u of u_x with a constant or a function of only x or t. □

A differential equation of any type, in conjunction with any other information such as an initial condition, is said to describe a *well-posed problem* if it satisfies three conditions, known as *Hadamard's conditions* for well-posedness:

- A solution of the problem exists.
- A solution of the problem is unique.
- The unique solution *depends continuously* on the problem data, which may include initial values or coefficients of the differential equation. That is, a small change in the data corresponds to a small change in the solution.

Unfortunately, problems can easily fail to be well-posed by not satisfying any of these conditions. However, in this course we will learn to solve a variety of initial value problems that are well-posed.