Modelling and Simulation

Bionics, Biomimetics MSc.

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HOCHSCHULE RHEIN-WAAL
Rhine-Waal University of Applied Sciences

Organisation

Lectures <u>weekly</u> (Room 06 01 14)

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password: ModSim1718

Class starts at 08:15

Organisation

Content

- Introduction
- Modelling of Dynamic Systems
  Introduction to MATLAB / Simulink
- Solution of State Equations
- Numerical Methods for Dynamic Systems

Introduction

Target:

To understand and influence the behaviour of complex dynamic systems.

For this purpose it is necessary to know and to understand the methods of modelling and analysis of this kind of systems along with their sub-systems and components.

In this lecture we will deal with

- The mathematical description of dynamic systems
- Modelling techiques (with a focus on mechanical, electrical and biological systems)
- Numerical methods

Mechanical Systems

Mechanical Systems

Modelling and Simulation?
What? Why?

Mechanical Systems

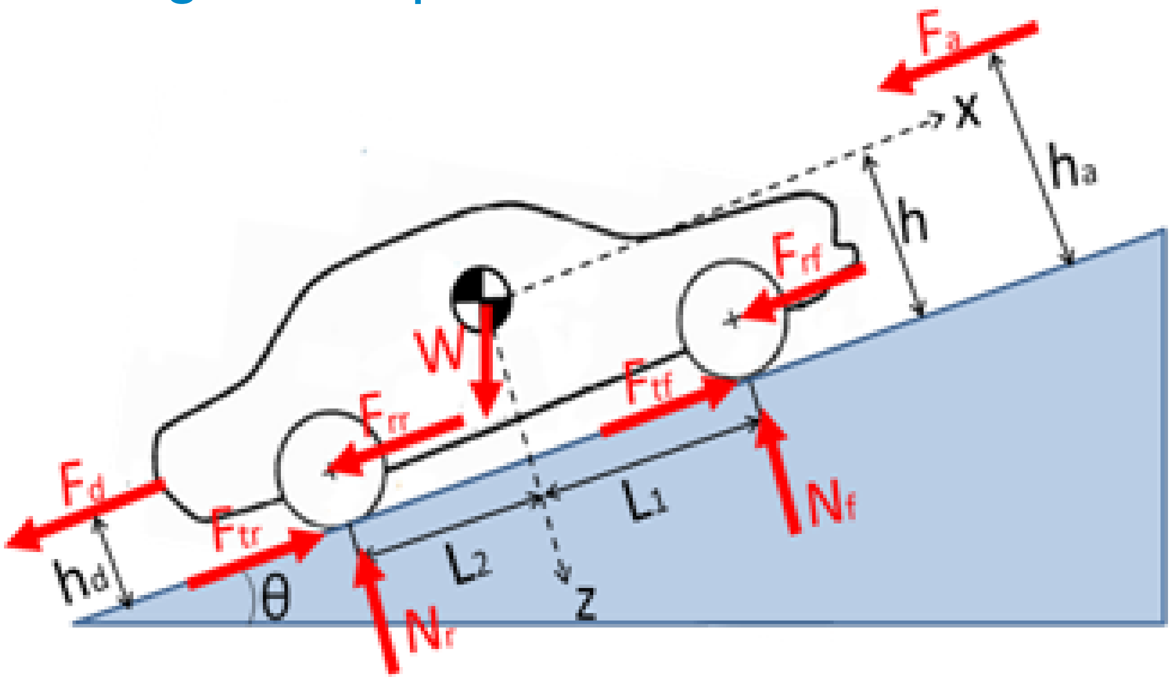
Modelling and Simulation?

1. Identify effects and quantities that we are interested in.
   (e.g. forces and accelerations in vertical and longitudinal vehicle dynamics)

Mechanical Systems

Modelling and Simulation?

2. Modelling: Abstract from the concrete object and use a „domain specific“ modelling technique



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Mechanical Systems

Modelling and Simulation?

3. Formulate the mathematical description of your model.
   (e.g. equations of motion)

$$\sum F_x = m \cdot \ddot{x}_c$$
$$m \cdot \ddot{x}_c = [K.(\xi_1 + \xi_2) + c.(\xi_1 + \xi_2)].\sin\alpha$$
$$+ (F_{r1} + F_{r2}).\cos\alpha$$

$$\sum F_y = m \cdot \ddot{y}_c$$
$$m \cdot \ddot{y}_c = -[K.(\xi_1 + \xi_2) + c.(\xi_1 + \xi_2)].\cos\alpha$$
$$+ (F_{r1} + F_{r2}).\sin\alpha - P$$

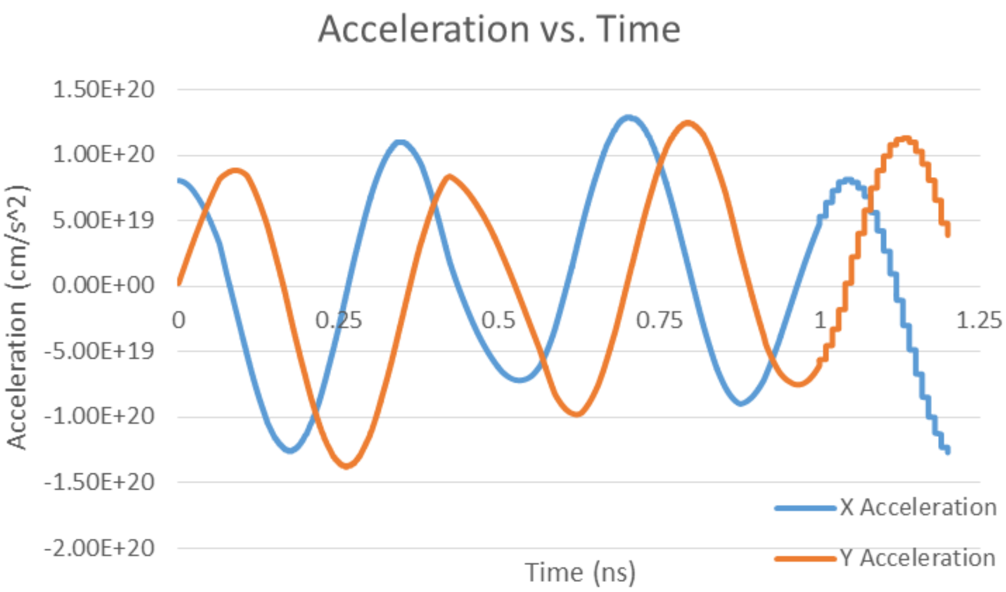
$$\sum M_{CM} = I \cdot \ddot{\alpha}$$
$$I \cdot \ddot{\alpha} = (K \cdot \xi_1 + c \cdot \dot{\xi}_1) \cdot L_1 - (K \cdot \xi_2 + c \cdot \dot{\xi}_2) \cdot L_2$$
$$+ F_{r_1} \cdot (h_1 + \xi_1) + F_{r_2} \cdot (h_2 + \xi_2)$$

Mechanical Systems

Modelling and Simulation?

4. Solve the equations of motion.

Numerics!



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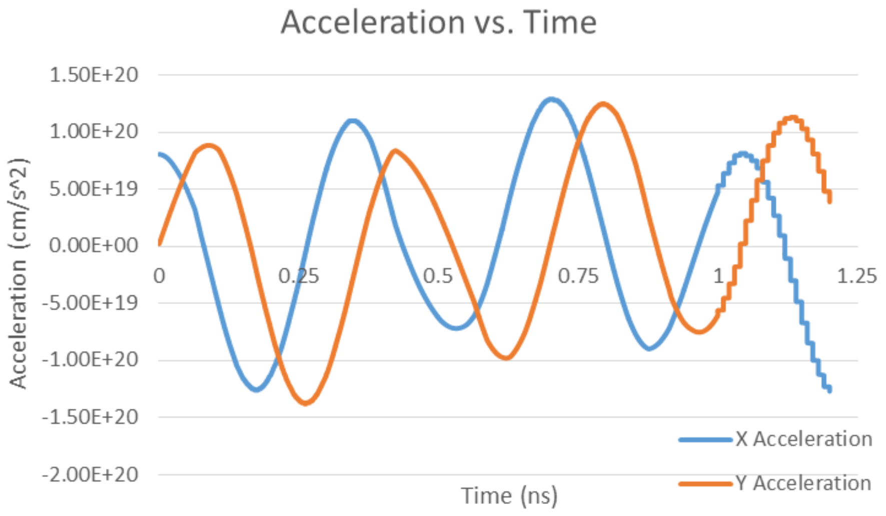
Mechanical Systems

Modelling and Simulation?

5. Interpretation and visualization of results, parameter studies, model optimization,…



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Systems

Definition (Hiller):

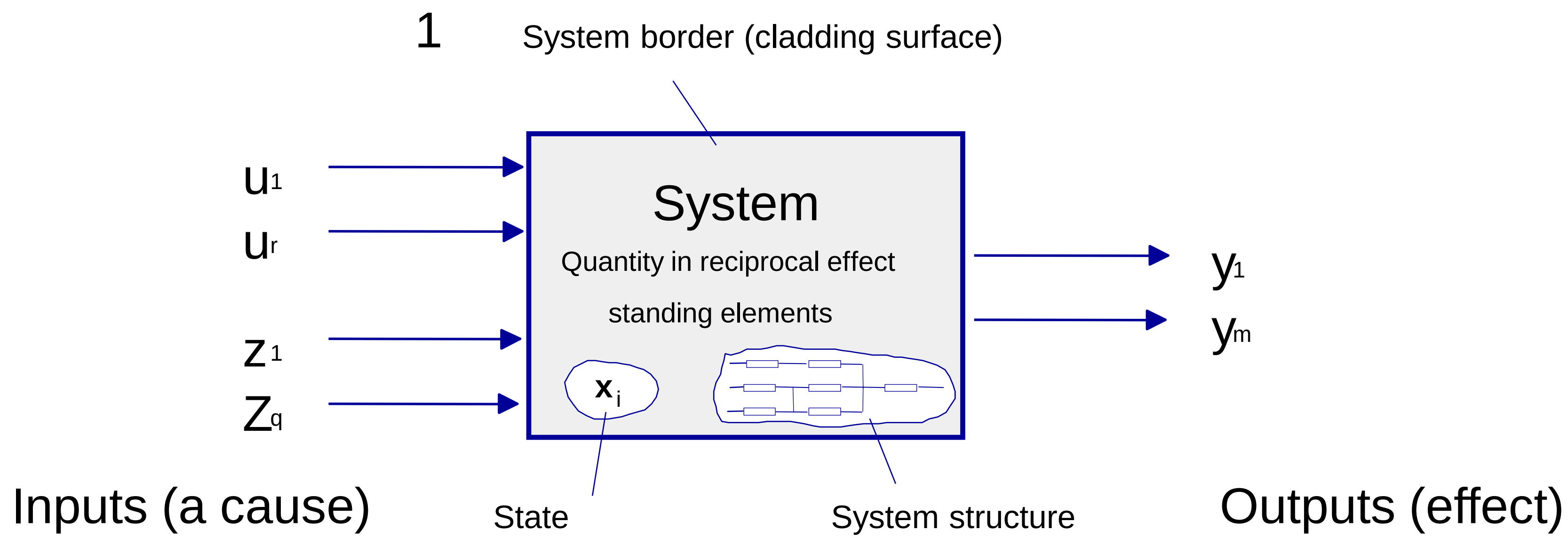
A system is a set of elements (parts, components, objects)

- which mutually influences each other (interaction)
- which are subject to external influence and affection (inputs)
- which effect to the external (outputs)

Alternative definition:
A ***system*** is a defined arrangement of things influencing each other. This arrangement is separated from the environment by a thought surface. Connections of the system with its environment are cut by the surface. The characteristics and conditions transferred by means of these connections are the inputs and outputs of the system.

Representation

Symbolically a system is represented by a box (block), whereby the cladding surface symbolizes the demarcation for environment.



The connections of the block with the environment are

- **r** ***Inputs*** **u_i**
- **m** ***Outputs*** **y_i**
- **q** ***Variable disturbances*** **z_i**

If no connections with the environment are present, then one speaks of a ***closed system.***

Systems

Definition: ***State variables*** are those quantities of a system, whose knowledge is sufficient, in order to describe the system performance completely.

- The state variables are time-dependent.
- The states are summarized in the n x 1state vector

Notes:

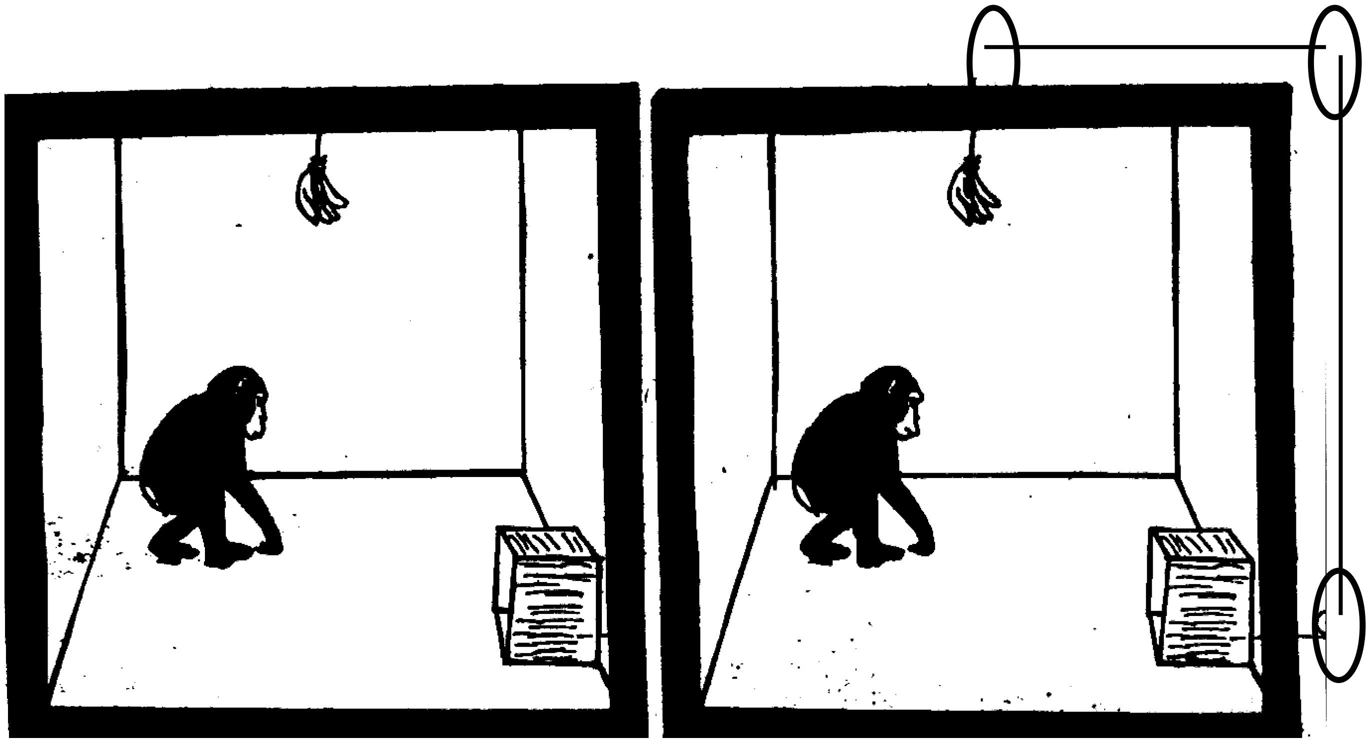
- The variables of state are to a certain extent "internal sizes" of the system.
- The choice of the variables of state is not unique.
  Variables of state must be independent and redundancy-free. From the requirement of being redundancy-free the number of state variables is fixed.
- Usually not all state variables are necessary to know for a particular application.
  Outputs of a system often follow from a combination of state variables.

Systems

System demarcation

The system must be defined against its environment. This demarcation results not from the physical borders, but from the question, which is the basis for the view of system.

How important a correct demarcation is, this cartoon shows (not completely seriously meant):



Systems

Examples of state variables for various disciplines

Electrical engineering	Mechanics	Chemical processes	Vital statistics
Current	Position	Temperature	Number of inhabitants
Voltage	Velocity	Mass proportion	State of environment
Electric charge	Acceleration		CO_2 Pollution
	Kinetic Energy		Ozon value
	Potential Energy		

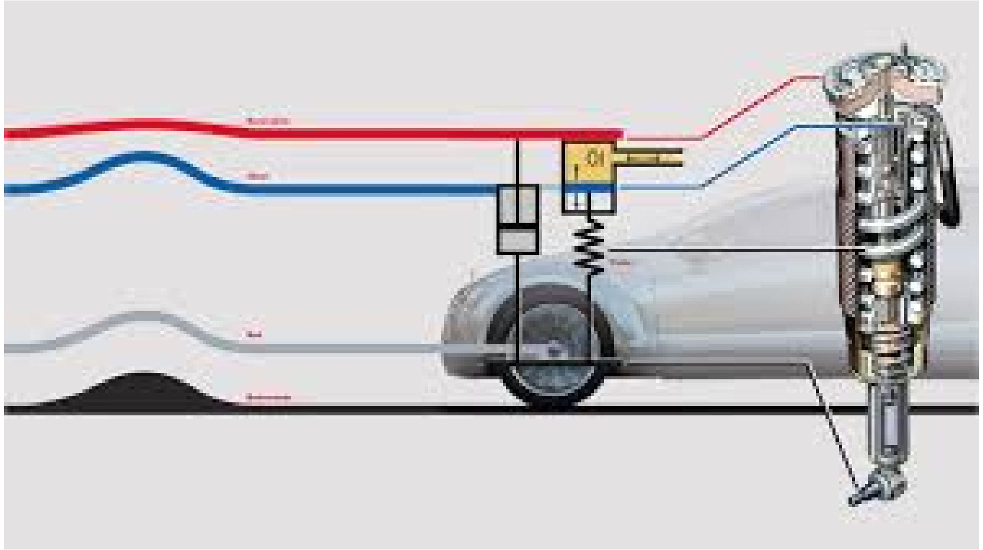
Systems

Examples for system elements from various disciplines

Electrical engineering	**Mechanics**	**Chemical processes**	**Vital statistics**
Resistance	Mass	Reservoir	Birth
Capacitor	Spring	Valve	Death
Coil	Damper	Pipe	Disease
Transistor	Beam	Filter	Consumption
Amplifier	Bearing	Reactor	
Filter	Guidance		
	Force actuator		

Systems

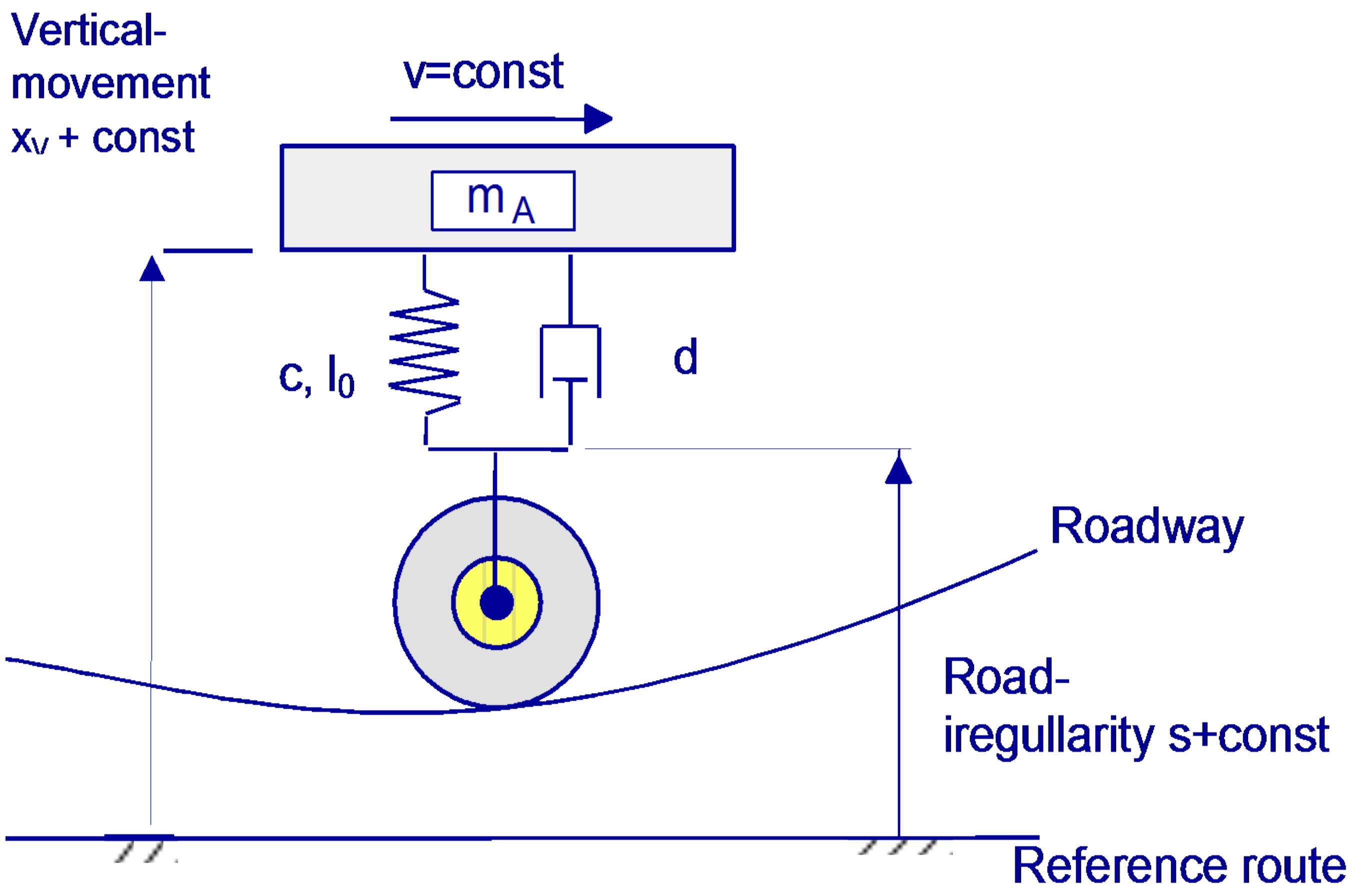
Example: Wheel suspension



Quelle: Auto Motor Sport

Systems

Example: Wheel suspension



Systems

Motion equations (mechanics):

$$m_A \ddot{x}_V = -c\left(x_V - s - l_0\right) - d\left(\dot{x}_V - \dot{s}\right) - m_A g$$

$$m_A \ddot{x}_V + d\dot{x}_V + cx_V = \underbrace{cs + d\dot{s}}_{h(t)} + cl_0 - m_A g$$

$$\Rightarrow \quad \ddot{x}_V = \frac{1}{m_A}\left(h - mg - d\dot{x}_V - cx_V + cl_0\right)$$

$$0 = -cx_V + cl_0 - m_A g \quad \Rightarrow \quad x_{V0} = l_0 - \frac{m_A}{c} g$$

$$x_{V_A} = x_V - x_{V_0} = x_V - l_0 + \frac{m_A}{c} g$$

$$\ddot{x}_{V_A} = \frac{1}{m_A}\left(h - d\dot{x}_{V_A} - cx_{V_A}\right)$$

Systems

Equations of state

Set:

$$x_1 = x_{V_A}$$
$$x_2 = \dot{x}_{V_A}$$

$$\Rightarrow$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = \ddot{x}_{V_A} = \frac{1}{m_A}\left(h - dx_2 - cx_1\right)$$

Input sizes: $u_1 = h(t)$ Road irregularity

Variables of state: x_1, x_2 Position and speed

Output values:

$y_1 = \dot{x}_2$ Structure acceleration

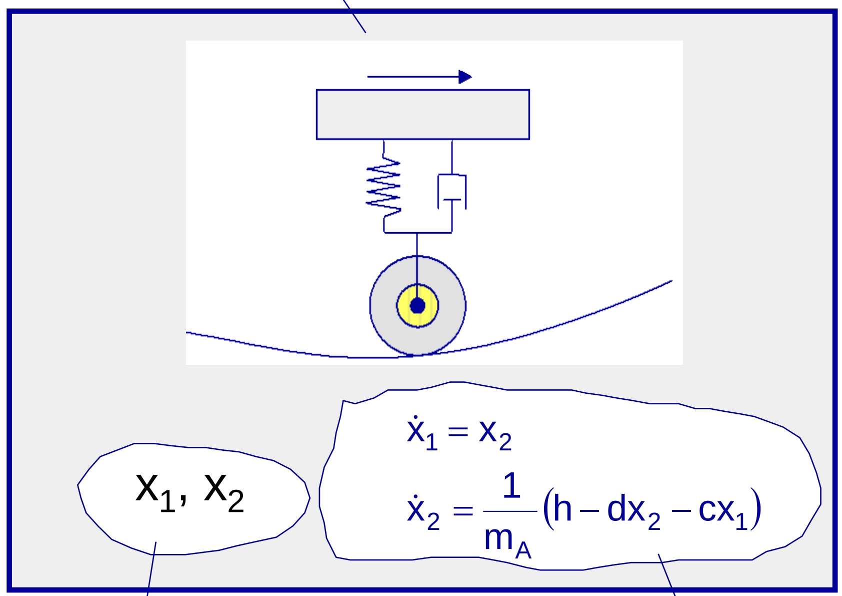
$y_2 = h - dx_2 - cx_1$ Wheel load fluctuation

Systems

System
Quarter vehicle

Input
- Road irregularity

$u_1 = h(t)$



Output
- Structure acceleration
- Wheel load fluctuations

y_1

y_2

States
- Position
- Speed

System structure
- State Equations

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Models

Dynamic System

- Mechanics
- Electrical Engineering
- Fluidics
- Thermodynamics
- Controls
- Computer Science

Verbal system description
"work statement"

Modelling
if necessary within the specification language of the respective technical discipline

Mathematical description of model

Generalization of model description

Models

For the understanding of complex systems a concept, which describes the dynamics behavior of the system sufficiently exact is often necessary. This concept takes place with domain-specific systems of a certain discipline predominantly within the common "model language".

Mechanics: mechanical components

Electro-technics: Connection diagrams, etc.

In case of multi-domain systems however a common modeling is necessary. Often simplification of the subsystems is required.

Models

Model concepts

Models are simplified images of the reality, in order to be able to analyze certain functions and behaviours. Models are e.g. the basis for simulation of the system. For different investigations, in particular with different investigation depth, different models are needed.

- **Scaled physical models**
- **Mathematical-physical models.**

In this lecture we will deal exclusively with mathematical-physical models.

Block diagrams

Block diagrams represent a descriptive representation of the model equations, i.e. block diagrams provide a representation of the interconnections between the single parts of a system.

Signal - Arrow
Operation - Block

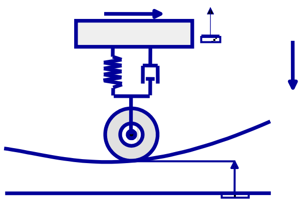
A block diagram thus always consists of

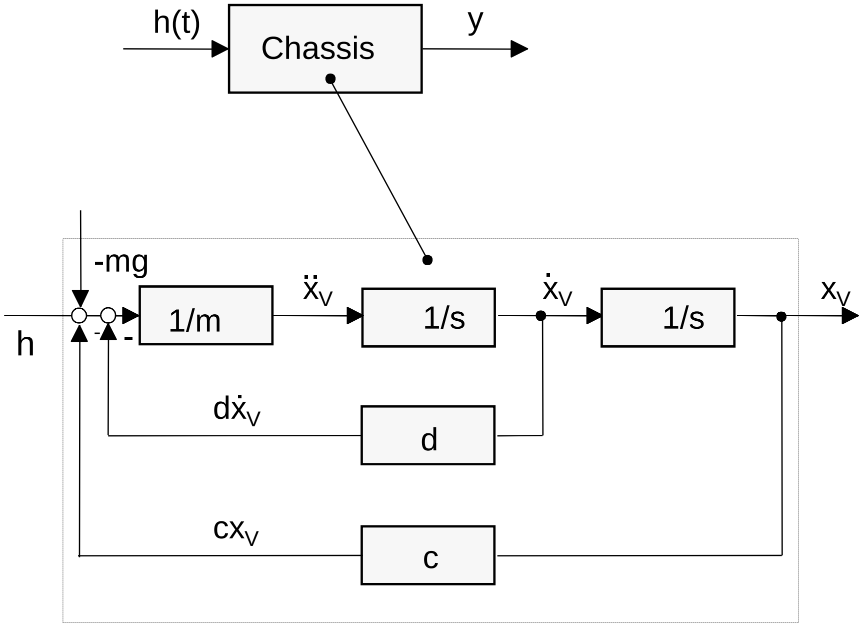
- The blocks
- Arrows for in and outgoing signals at the blocks, whereby by a head of the arrow in the respective direction of action it is indicated whether it concerns the output signal or one of the input signals.
- Definition of the input and output values
- Summations, represented as small circles

Further details to the structure and the elements of block diagrams are standardized in *DIN 19226.*

Examples of block diagrams

Wheel suspension



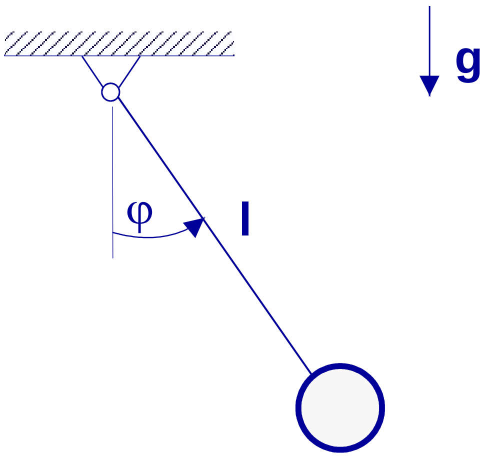


$$\ddot{x}_V = \frac{1}{m_A}(h - mg - d\dot{x}_V - cx_V)$$

Modelling

Definition: An equation is called *ordinary differential equation* (ODE), if it contains apart from the variable also at least one derivative of this variable. *Systems with concentrated parameters* (e.g. *multi-body systems*) are described by ODE.

Example: ODE (*equation of motion*) for a mathematical pendulum



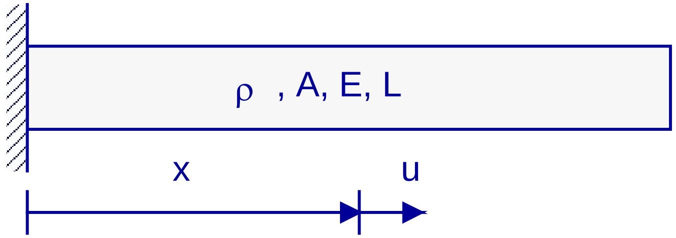
$$ml^2\ddot{\varphi} = -mgl\sin\varphi$$

Modelling

Definition: A function of several independent variable and their derivatives is called *partial differential equation* (PDE).
Systems with (locally) distributed parameters (e.g. continuous mechanical systems) are described by PDE.
Example: Longitudinal oscillations of a continuous bar

Designations:

- ρ: Density
- E: Elasticity modulus
- A: Bar cross section
- L: Bar length

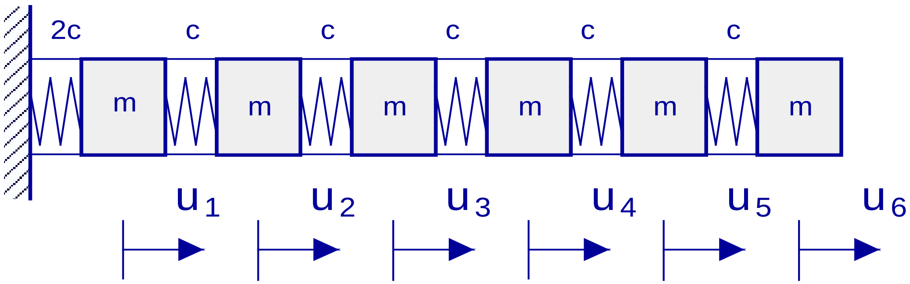


$$\rho A \ddot{u}(x,t) = \frac{EA}{L^2}\frac{\partial^2 u}{\partial x^2}(x,t)$$

Here derivatives with respect to the local coordinate x and time arise.

Modelling

Note: A system with distributed parameters can frequently be approximated by a spatial discretisation by a system with concentrated parameters. In this case it receives a system instead of ODE instead of one PDE:



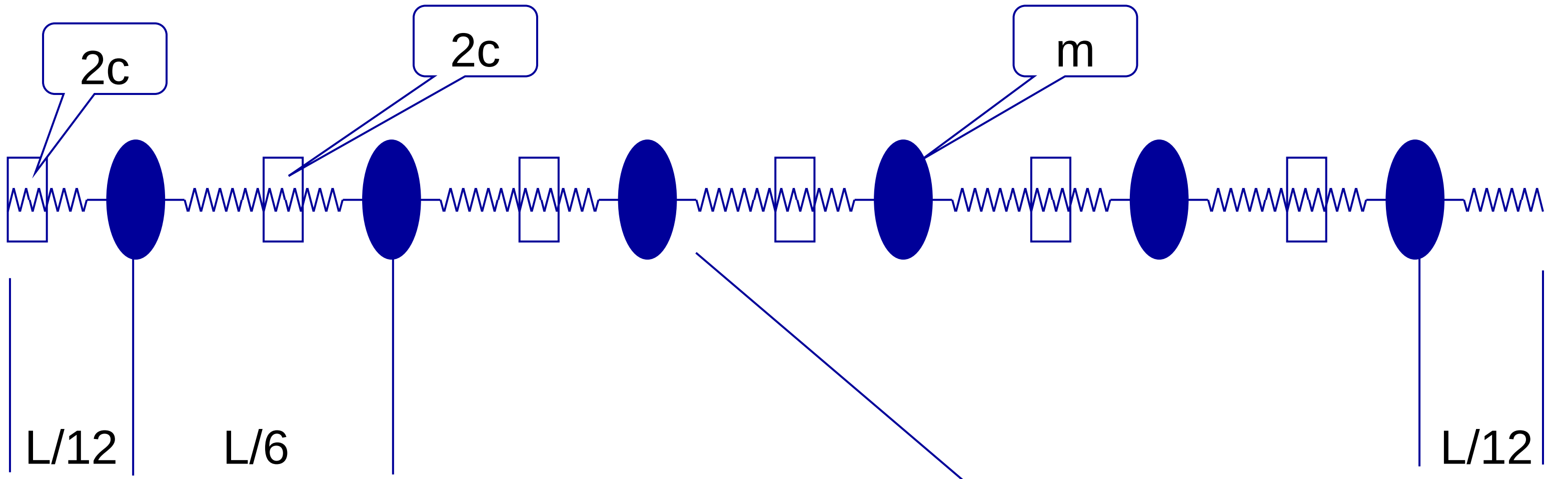
$$
\begin{aligned}
m\ddot{u}_1 &= -3cu_1 + cu_2 \\
m\ddot{u}_2 &= cu_1 - 2cu_2 + cu_3 \\
m\ddot{u}_3 &= cu_2 - 2cu_3 + cu_4 \\
m\ddot{u}_4 &= cu_3 - 2cu_4 + cu_5 \\
m\ddot{u}_5 &= cu_4 - 2cu_5 + cu_6 \\
m\ddot{u}_6 &= cu_5 - cu_6
\end{aligned}
$$

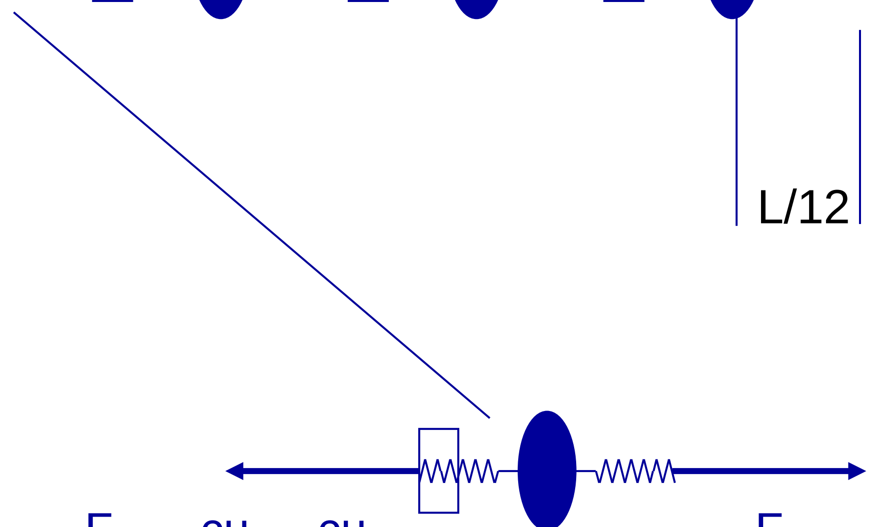
with $c = \frac{AE}{L/6}$

and $m = \frac{\rho AL}{6}$

$$
\ddot{\mathbf{u}} = \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \ddot{u}_3 \\ \ddot{u}_4 \\ \ddot{u}_5 \\ \ddot{u}_6 \end{bmatrix} = \frac{36E}{\rho L^2} \begin{bmatrix} -3 & 1 & 0 & 0 & 0 & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & -2 & 1 & 0 \\ 0 & 0 & 0 & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} = \mathbf{Au}
$$

Modelling





spring constant

$$\sigma = \frac{F}{A} = E\varepsilon = \frac{u_i}{L/6} \Rightarrow F = \underbrace{AE\frac{1}{L/6}}_{c} u_i$$

$$F_{23} = cu_2 - cu_3 \qquad F_{23} = -cu_3 + cu_4$$

mass

$$m = \frac{\rho AL}{6}$$

Newton´s Law

$$m\ddot{u}_3 = cu_2 - cu_3 - cu_3 + cu_4 = cu_2 - 2cu_3 + cu_4$$

Modelling

In systems with concentrated parameters the dynamic system can be described by ODE as follows:

implicit Form $\mathbf{F}^{i}\left(t, \mathbf{y}(t), \mathbf{y'}(t), \ldots, \mathbf{y}^{(\mathbf{n})}(t)\right) = \mathbf{0}$

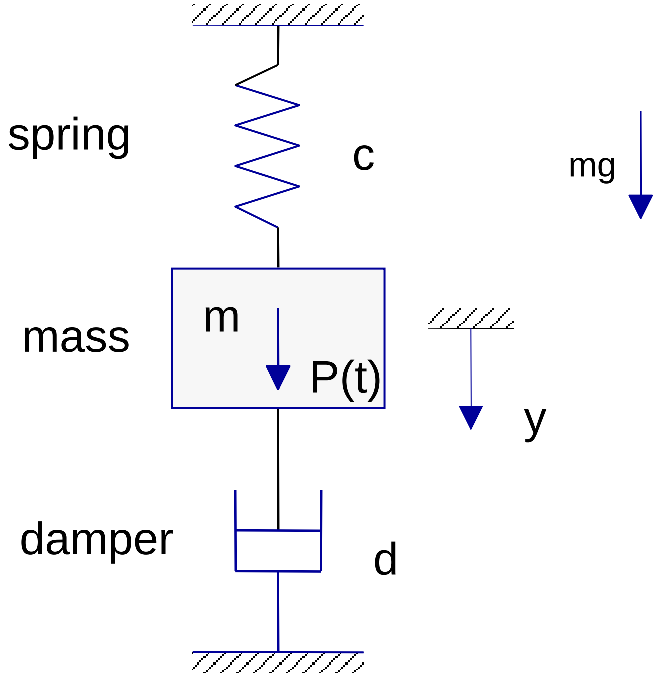
or *explicit Form* $\mathbf{y}^{(\mathbf{n})}(t) = \mathbf{F}^{\mathbf{e}}\left(t, \mathbf{y}(\mathbf{t}), \mathbf{y'}(\mathbf{t}), \ldots, \mathbf{y}^{(\mathbf{n-1})}(t)\right)$

Notes:

- In some cases the transfer of the implicit form into the explicit equation form is *analytically* not possible. In the system dynamics this is however usually not the case.
- In practice often systems arise, which can be described by coupled differential and algebraic equations (DAE = differential Algebraic Equations). An example of it are multi-body systems with kinematic loops.
- The number of the highest arising derivative is called order of the ODE.
- Since the function y in system dynamics usually depends on the time, we mark the derivatives of it by a dot on top of it.

Modelling

Example: oscillator with viscous damper



implicit equation:

$$F^{i}\left(t, y, \dot{y}, \ddot{y}\right) = m\ddot{y} + d\dot{y} + cy - P(t) = 0$$

explicit equation:

$$\ddot{y}(t) = F^{e}\left(t, y, \dot{y}\right) = \left[-\frac{1}{m}\left(d\dot{y} + cy - P(t)\right)\right]$$

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Modelling

By introduction the state variables $x_1,\ldots,x_n$

and the substitutions
$$x_1 = y \quad x_2 = y' \quad x_n = y^{(n-1)}$$
$$x_1 = y \quad x_2 = \dot{x} \quad x_n = \dot{x}_{n-1}$$

$$y^{(n)}(t) = F^e(t, y(t), y'(t), \ldots, y^{(n-1)}(t))$$
$$\dot{x}_n(t) = F^e(t, x(t), \dot{x}(t), \ldots, \dot{x}_{n-1}(t))$$

can be transferred into a system of 1th order differential equations.

Example

Linear oscillator $\ddot{y} + 2\delta\dot{y} + \nu_0{}^2 y = h(t)$

Substitution $x_1 = y \quad x_2 = \dot{y}$

Equations of state
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = h(t) - 2\delta x_2 - \nu_0{}^2 x_1$$

State Equations

State equations

$$\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t), t)$$

$$\mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t), t)$$

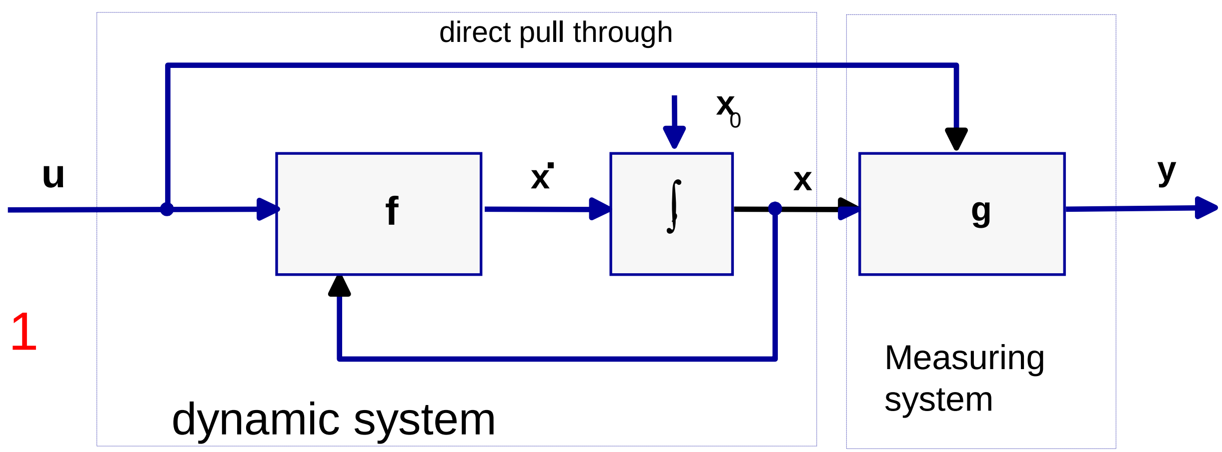
x : n x 1 – state vector

y : m x 1 – output vector

u : r x 1 – control vector

f,g : non-linear n x 1 - / m x 1 – vector-function

Block diagram

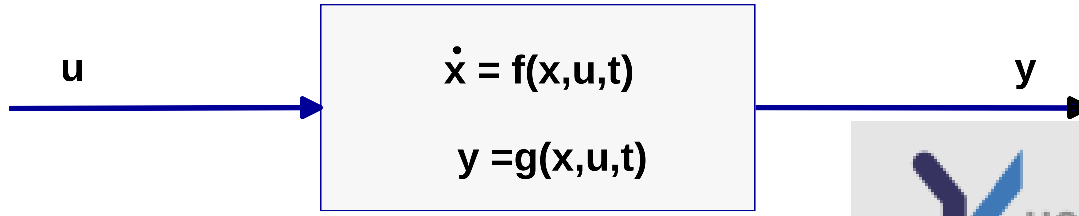


example : $\quad n = 2, m = 1, r = 1$

$\dot{x}_1 = f_1(x_1, x_2, u_1, t)$

$\dot{x}_2 = f_2(x_1, x_2, u_1, t)$

$y_1 = g_1(x_1, x_2, u_1, t)$



State Equations

Example: Mass Point

Newton's law: $m\ddot{y} = F$

Substitution: $x_1 = y \qquad x_2 = \dot{y}$

State equations: $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = f(x_1, x_2, u_1, t) = \begin{bmatrix} x_2 \\ \frac{F}{m} \end{bmatrix}$ with $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad u = \frac{F}{m}$

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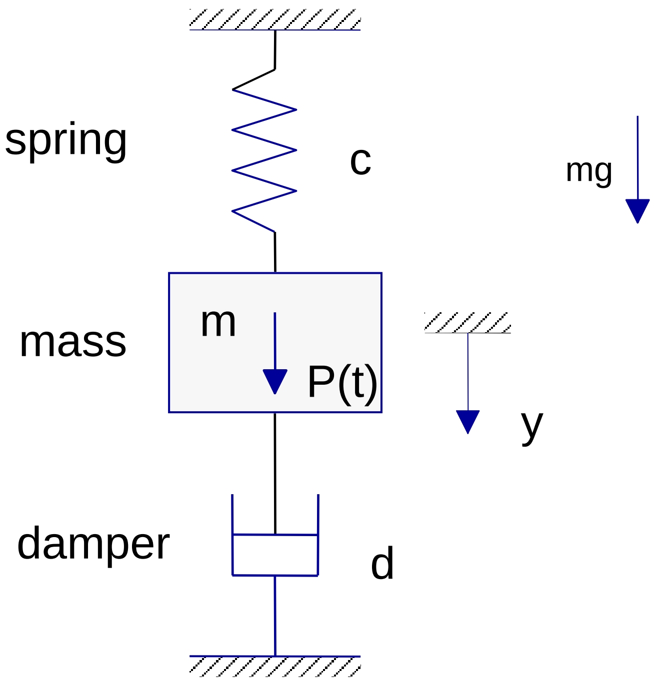
State Equations

Example: oscillator with viscous damper



Implicit equation:

$$F^{i}(t,y,\dot{y},\ddot{y}) = m\ddot{y} + d\dot{y} + cy - P(t) = 0$$

Explicit equation:

$$\ddot{y}(t) = F^{e}(t,y,\dot{y}) = \left[-\frac{1}{m_1}\left(d\dot{y} + cy - P(t)\right)\right]$$

State equations

$$x_1 = y \qquad \dot{x}_1 = \dot{y} = f_1(x_1, x_2, t) = x_2$$

$$x_2 = \dot{y} \qquad \dot{x}_2 = \ddot{y} = f_2(x_1, x_2, t) = -\frac{c}{m}x_1 - \frac{d}{m}x_2 + \frac{1}{m}P(t)$$

$$\Rightarrow \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{m} & -\frac{d}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} P(t)$$

$$u_1 = P(t) \qquad y_1 = g(x_1, x_2, u_1, t) = x_1$$

$$\Rightarrow [y_1] = [1 \quad 0] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

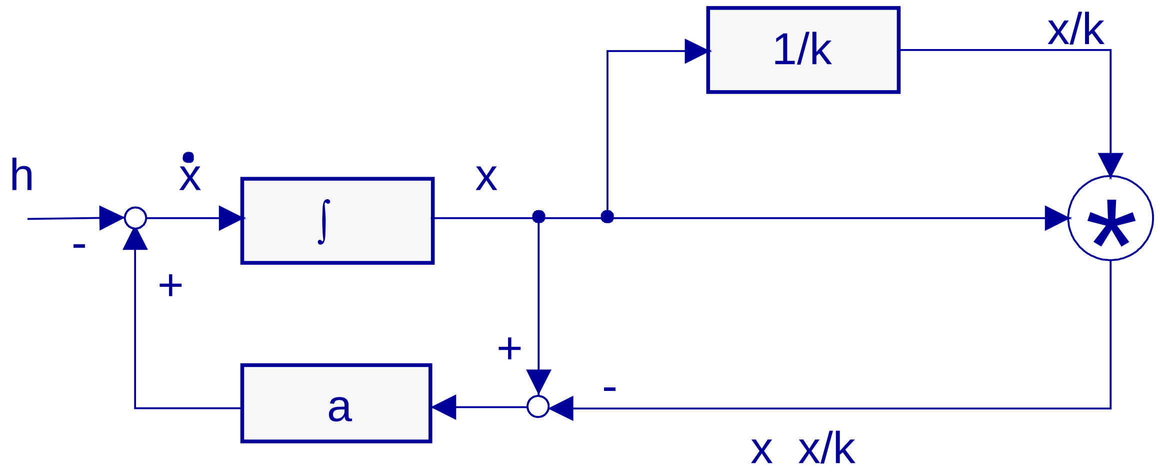
State Equations

Example logistic growth

state variable (asset e.g. fishes) : x

control variable (e.g. fishing): $u = h = \text{const}$

State equations: $\dot{x} = a\left(1 - \dfrac{x}{k}\right)x - h$



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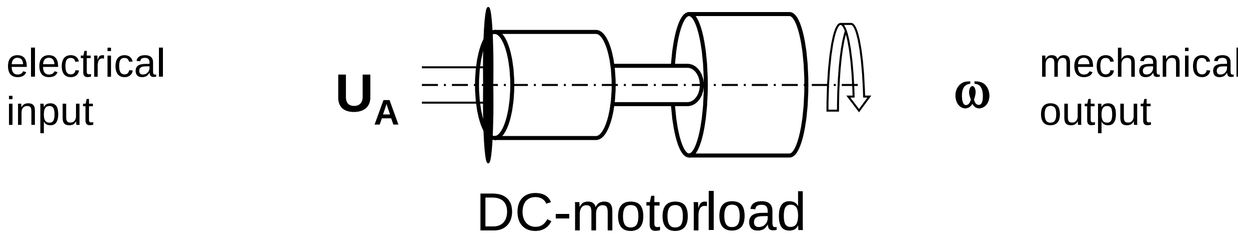
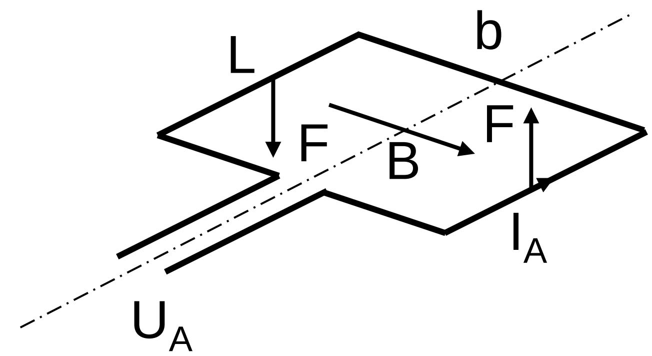
State Equations

Example: DC Motor





torque: $M = \alpha n L b\,(B I_A) = k_1\,(B I_A)$

where

- n: number of conductor loops
- B: magnetic field strength
- L, b: dimensions of conductor loop
- I_A: anchor current
- α: compensation factor

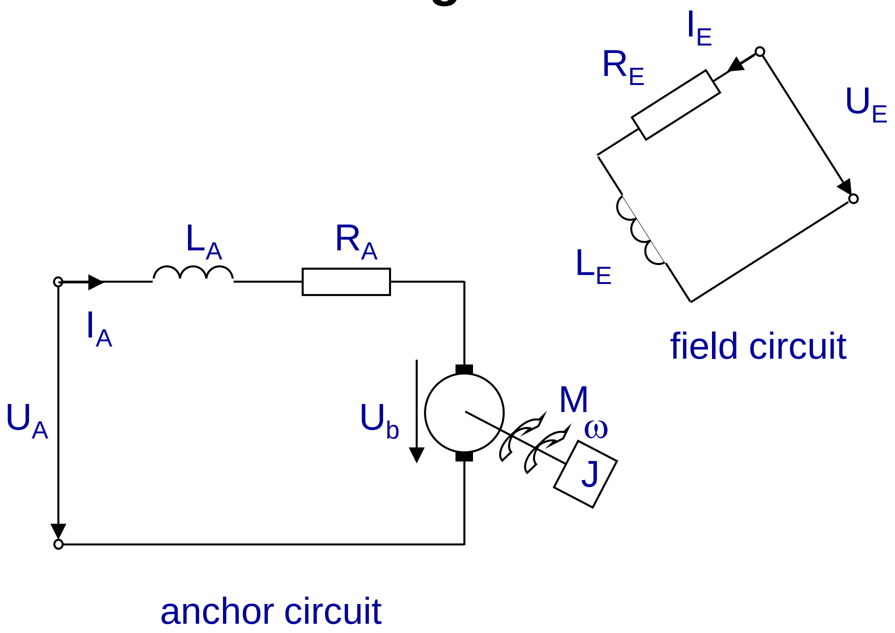
induced voltage (EMF) $U_b = k_2\, B\, \omega$

where

- k_2: constant
- ω: anchor rotation velocity

State Equations

model plan
circuit drawing



for a anchor controlled motor
I_E = const and hence B = const

$U_b = k_2 B\omega = k_3\omega$

anchor voltage

U_A

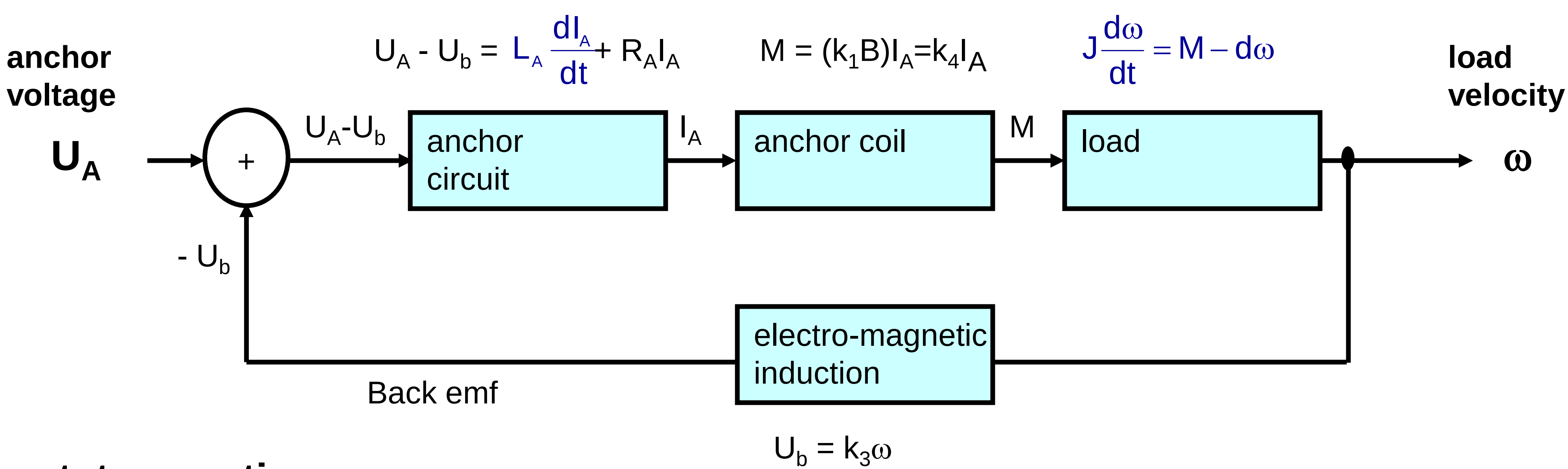
induction law for anchor circuit

$$U_A - U_b - R_A I_A - L_A \frac{dI_A}{dt} = 0$$

$$U_A - U_b = L_A \frac{dI_A}{dt} + R_A I_A$$

State Equations

anchor controlled motor



state equations

$$x_1 = I_A \qquad \dot{x}_1 = \dot{I}_A = f_1(x_1, x_2, t) = -\frac{R_A}{L_A} x_1 - \frac{k_3}{L_A} x_2 + \frac{1}{L_A} U_A$$

$$\underbrace{\frac{L_A}{R_A}}_{\substack{\text{time constant} \\ T_A}} \dot{x}_1 + x_1 = \frac{k_3}{R_A} x_2 + \frac{1}{R_A} U_A$$

$$x_2 = \omega \qquad \dot{x}_2 = \dot{\omega} = f_2(x_1, x_2, t) = \frac{k_4}{J} x_1 - \frac{d}{J} x_2$$

$$\underbrace{\frac{J}{d}}_{\substack{\text{time constant} \\ T_J}} \dot{x}_2 + x_2 = \frac{k_4}{d} x_1$$

$$u_1 = U_A \qquad y_1 = g(x_1, x_2, u_1, t) = \omega = x_2$$

Steady State Solutions

Definition:
The solution $x_0(t)$ of a dynamic system is called ***steady state***
if for a given load $u_0(t)$
$F(x_0(t),u_0(t),t) = 0$ and hence $\dot{x}(t) = 0$

If $u_0(t)=0$ then $x_0(t)$ is an equilibrium solution for the free not controlled system.

Remark: This equation is generally non-linear and therefore normally numerical methods have to be used to find a solution.

Steady State Solutions

Example 1: single mass point

$$\mathbf{f}(\mathbf{x},\mathbf{u},\mathbf{t}) = \begin{bmatrix} x_2 \\ \frac{F}{m} \end{bmatrix} = \mathbf{0} \Rightarrow x_2 = 0 \quad \text{and} \quad F = 0$$

Example 2: spring-mass-oscillator with viscous damper

$$\mathbf{f}(\mathbf{x},\mathbf{u},\mathbf{t}) = \begin{bmatrix} x_2 \\ -\frac{1}{m}\left(cx_1 + dx_2 - mg - P(t)\right) \end{bmatrix} = \mathbf{0} \qquad \Rightarrow x_2 = 0 \Rightarrow x_1 = \frac{mg}{c}$$

$\frac{mg}{c}$ is the displacement of the mass m due to gravity until equilibrium is reached

Steady State Solutions

Example 3: single mass pendulum

$$\mathbf{f}(\mathbf{x},\mathbf{u},\mathbf{t}) = \begin{bmatrix} x_2 \\ -\frac{g}{l}\sin x_1 \end{bmatrix} = \mathbf{0} \qquad \Rightarrow x_2 = 0 \qquad \sin x_1 = 0$$

$$\text{hence} \quad x_1 = 0, \pi, 2\pi, \ldots$$

Remark: these equilibrium points show completely different stability behaviour

Example 4: logistic growth with constant harvest

$$f(x,u,t) = a\left(1 - \frac{x}{k}\right)x - h = 0 \Rightarrow ax - \frac{a}{k}x^2 - h = 0$$

$$\Rightarrow x_{1/2} = \frac{1}{2}\left(k \pm \sqrt{k^2 - 4\frac{kh}{a}}\right)$$

For h=0 we get in particular $\quad x_1 = 0 \qquad x_2 = k$