第四章

插值法(4)—Hermite 插值

第4节 Hermite 插值

Lagrange插值，Newton插值虽然易算，但若要提高插值曲线的光滑性，即可导性，则需要增加插值条件，提高曲线光滑性。

不仅要求函数值重合，而且要求若干阶导数也重合。

即：要求插值函数$H(x)$满足$H(x_i)=f(x_i), H'(x_i)=f'(x_i),$ $\cdots$， $H^{(m)}(x_i)=f^{(m)}(x_i), i=0,1,2,\cdots,n.$

➢ 4.1 三次Hermite插值

问题1 已知两个节点的情况

设x_0, x_1为插值节点，$x_0 < x_1$，且已知

$$y_0 = f(x_0), y_1 = f(x_1), y_0' = f'(x_0), y_1' = f'(x_1),$$

在区间$[x_0, x_1]$上求多项式$H(x)$，使得满足插值条件

$$H(x_0) = y_0, H(x_1) = y_1, H'(x_0) = y_0', H'(x_1) = y_1'.$$

由于有4个条件，所以$H(x)$应为次数不超过3次的多项式，称为**Hermite三次插值，记作**

$$H_3(x) = y_0\alpha_0(x) + y_1\alpha_1(x) + y_0'\beta_0(x) + y_1'\beta_1(x).$$

特例分析 设$x_0=0, x_1=1$为插值节点，则原条件化为：

$$H(0)=y_0, H(1)=y_1, H'(0)=y_0', H'(1)=y_1'.$$

$$H_3(x)=y_0\alpha_0(x)+y_1\alpha_1(x)+y_0'\beta_0(x)+y_1'\beta_1(x).$$

因为$\alpha_0(x), \alpha_1(x), \beta_0(x), \beta_1(x)$均为3次多项式，代入条件，可得

$$\alpha_0(x)=(x-1)^2(1+2x) \qquad \beta_0(x)=x(x-1)^2$$

$$\alpha_1(x)=x^2(1-2(x-1)) \qquad \beta_1(x)=x^2(x-1)$$

故

$$H_3(x)=y_0(2x^3-3x^2+1)+y_1(3x^2-2x^3)+y_0'x(x-1)^2+y_1'x^2(x-1).$$

一般情况

设$t=\frac{x-x_0}{x_1-x_0}$，则$x_0\leftrightarrow t_0=0, x_1\leftrightarrow t_1=1$,

$$\alpha_0(t)=(t-1)^2(2t+1)=\left(\frac{x-x_1}{x_1-x_0}\right)^2\left(2\frac{x-x_0}{x_1-x_0}+1\right)=H_{0,0}(x)$$

$$\beta_0(t)=t(t-1)^2=\left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_1}{x_1-x_0}\right)^2=\widehat{H}_{1,0}(x)$$

$$\alpha_1(t)=t^2(1-2(t-1))=\left(\frac{x-x_0}{x_1-x_0}\right)^2\left(1-2\frac{x-x_1}{x_1-x_0}\right)=H_{0,1}(x)$$

$$\beta_1(t)=t^2(t-1)=\left(\frac{x-x_0}{x_1-x_0}\right)^2\left(\frac{x-x_1}{x_1-x_0}\right)=\widehat{H}_{1,1}(x)$$

两点的Hermite三次插值多项式：

$$H_3(x)=y_0H_{0,0}(x)+y_1H_{0,1}(x)+y_0'\widehat{H}_{1,0}(x)+y_1'\widehat{H}_{1,1}(x).$$

4.2 带重节点的差商

$$f[\underbrace{x_0, x_0, \cdots, x_0}_{n+1}] = \frac{f^{(n)}(x_0)}{n!} \Rightarrow f[x_0, x_0] = f'(x_0), f[x_1, x_1] = f'(x_1)$$

$$f[x_0, x_0, x_1] = \frac{f[x_0, x_1] - f[x_0, x_0]}{x_1 - x_0} = \frac{f[x_0, x_1] - f'(x_0)}{x_1 - x_0}$$

$$f[x_0, x_0, x_1, x_1] = \frac{f[x_0, x_1, x_1] - f[x_0, x_0, x_1]}{x_1 - x_0}$$

$$= \frac{\frac{f'(x_1) - f[x_0, x_1]}{x_1 - x_0} - \frac{f[x_0, x_1] - f'(x_0)}{x_1 - x_0}}{x_1 - x_0}$$

$$= \frac{f'(x_1) - 2f[x_0, x_1] + f'(x_0)}{(x_1 - x_0)^2}$$

两点的Hermite三次插值的重节点Newton插值格式

$$H_3(x) = f(x_0) + f[x_0, x_0](x - x_0) + f[x_0, x_0, x_1](x - x_0)^2$$

$$+f[x_0, x_0, x_1, x_1](x - x_0)^2(x - x_1)$$

证明：代入重节点差商公式，易验证满足插值条件。

$$H(x_0) = y_0, H(x_1) = y_1, H'(x_0) = y_0', H'(x_1) = y_1'.$$

定理2. 设$f(x) \in C^4[a, b]$, $x, x_0, x_1 \in [a, b]$，则存在$\xi \in [a, b]$，使得

$$R_n(x) = f(x) - H_3(x) = \frac{f^{(4)}(\xi)}{4!}(x - x_0)^2(x - x_1)^2.$$

证明：利用Newton插值余项可证。

例1：求不超过3次的多项式$H_3(x)$，使之满足插值条件：

$$H(-1)=-9,\quad H'(-1)=15,\quad H(1)=1,\quad H'(1)=-1$$

Hermite插值多项式的求法（两点的Hermite插值）

(1) 直接利用公式；　(2) 待定系数法；　(3) 利用插商表。

解法一　$H(-1)=-9,\ H(1)=1,\ H'(-1)=15,\ H'(1)=-1$

令$H(x)=a+bx+cx^2+dx^3$代入条件，则$\begin{cases} a-b+c-d=-9, \\ a+b+c+d=1, \\ b-2c+3d=15, \\ b+2c+3d=-1. \end{cases}$

解得：$a=0, b=4, c=-4, d=1$，故$H_3(x)=x^3-4x^2+4x.$

解法二　$H(-1)=-9$，$H'(-1)=15$，$H(1)=1$，$H'(1)=-1$

$$H_{1,0}(x)=\left(\frac{x-x_1}{x_1-x_0}\right)^2\left(2\frac{x-x_0}{x_1-x_0}+1\right)=\left(\frac{x-1}{2}\right)^2\left(2\frac{x+1}{2}+1\right)=\frac{1}{4}(x-1)^2(2+x)$$

$$H_{1,1}(x)=\left(\frac{x-x_0}{x_1-x_0}\right)^2\left(1-2\frac{x-x_1}{x_1-x_0}\right)=\left(\frac{x+1}{2}\right)^2\left(1-2\frac{x-1}{2}\right)=\frac{1}{4}(x+1)^2(2-x)$$

$$\widehat{H}_{1,0}(x)=\left(\frac{x-x_0}{x_1-x_0}\right)\left(\frac{x-x_1}{x_1-x_0}\right)^2=\left(\frac{x+1}{2}\right)\left(\frac{x-1}{2}\right)^2=\frac{1}{8}(x+1)(x-1)^2$$

$$\widehat{H}_{1,1}(x)=\left(\frac{x-x_0}{x_1-x_0}\right)^2\left(\frac{x-x_1}{x_1-x_0}\right)=\left(\frac{x+1}{2}\right)^2\left(\frac{x-1}{2}\right)=\frac{1}{8}(x-1)(x+1)^2$$

$$\therefore\quad H_3(x)=H_{0,0}(x)y_0+H_{0,1}(x)y_1+\widehat{H}_{1,0}(x)y_0'+\widehat{H}_{1,1}(x)y_1'$$

$$=x^3-4x^2+4x$$

解法三 $H(-1) = -9,\ \ H'(-1) = 15,\ \ H(1) = 1,\ \ H'(1) = -1$

x_i	$f(x_i)$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
-1	-9			
-1	-9	$f[-1,-1] = f'(-1) = 15$		
1	1	$f[-1,1] = 5$	$f[-1,-1,1] = -5$	
1	1	$f[1,1] = f'(1) = -1$	$f[-1,1,1] = -3$	$f[-1,-1,1,1] = 1$

$$H_3(x) = -9 + 15(x+1) - 5(x+1)^2 + (x+1)^2(x-1)) = x^3 - 4x^2 + 4x.$$

例2: P192.EX14 提示

x_i	$f(x_i)$	$f[x_i,x_{i+1}]$	$f[x_i,x_{i+1},x_{i+2}]$	$f[x_i,x_{i+1},x_{i+2},x_{i+3}]$
1	2			
2	4	$f[1,2]=2$		
2	4	$f[2,2]=f'(2)=3$	$f[1,2,2]=1$	
3	12	$f[2,3]=8$	$f[2,2,3]=5$	$f[1,2,2,3]=2$

$$H_3(x)=2+2(x-1)+1(x-1)(x-2)+2(x-1)(x-2)(x-2)$$

$$=2x^3-9x^2+15x-6$$

注：已知某点出现一阶导数，则该点作两次重节点，如果出现二阶导，则该点作三次重节点。

例3: P192.EX16 提示

x_i	$f(x_i)$	$f[x_i,x_{i+1}]$	$f[x_i,x_{i+1},x_{i+2}]$	$f[x_i,x_{i+1},x_{i+2},x_{i+3}]$
-1	2			
-1	2	$f[-1,-1]=f'(-1)=-1$		
0	-1	$f[-1,0]=-3$	$f[-1,-1,0]=-2$	
1	4	$f[0,1]=5$	$f[-1,0,1]=4$	$f[-1,-1,0,1]=3$
1	4	$f[1,1]=f'(1)=1$	$f[0,1,1]=-4$	$f[-1,0,1,1]=-4$
1	4	$f[1,1]=f'(1)=1$	$f[1,1,1]=f''(2)/2=0$	$f[0,1,1,1]=4$

$f[x_i,x_{i+1},x_{i+2},x_{i+3},x_{i+4}]$	$f[x_i,x_{i+1},x_{i+2},x_{i+3},x_{i+4},x_{i+5}]$
$f[-1,-1,0,1,1]=-7/2$	
$f[-1,0,1,1,1]=4$	$f[-1,-1,0,1,1,1]=15/4$

$$H_5(x) = 3.75x^5-3.5x^4-8x^3+7.5x^2+5.25x-1$$

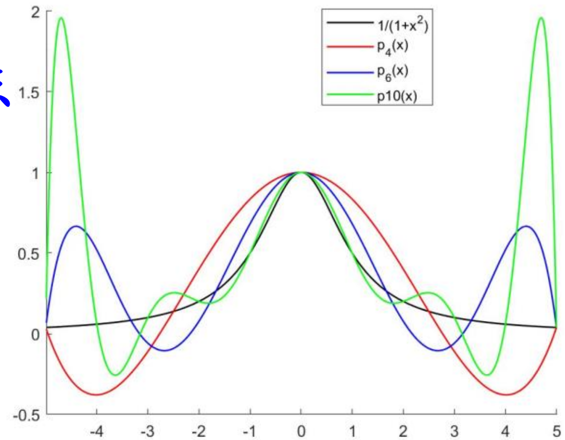
第5节 分段多项式插值

5.1 龙格Runge现象

利用插值多项式逼近连续函数$y=f(x)$时，并非插值多项式的次数越高越好．因为当插值多项式的次数较高时，给自变量一个小的扰动，就可能引起函数值较大的变化，从而使得截断误差很大．这种现象称为龙格现象．

例如：$f(x)=\frac{1}{1+x^2}$在区间$[-5,5]$上取等距插值节点

$$x_i=-5+10\cdot\frac{i}{n},\quad i=0,1,2,\cdots,n$$



其插值多项式$p_n(x)$在-1和1点附近产生震荡。

5.2 分段线性插值

分段线性插值就是通过插值点用折线段连接起来逼近$f(x)$.

设已知节点$a = x_0 < x_1 < \cdots < x_n = b$，且对应的函数值为

$f(x_0), f(x_1), \cdots, f(x_n)$，记$h_k = x_{k+1} - x_k$，$h = \max\limits_{0 \le k \le n-1} \{h_k\}$,

求一折线函数$I_h(x)$满足

(1). $I_h(x) \in C[a,b]$，　(2). $I_h(x_k) = f(x_k)$，

(3). $I_h(x_k)$在每个区间(x_k, x_{k+1})上是线性函数；

则称函数$I_h(x)$为分段线性插值函数.

$I_h(x)$满足在小区间$[x_{k-1},x_k]$的表达式为

$$y-y_{k-1}=\frac{y_k-y_{k-1}}{x_k-x_{k-1}}(x-x_{k-1})\Leftrightarrow y=y_{k-1}\frac{x-x_k}{x_{k-1}-x_k}+y_k\frac{x-x_{k-1}}{x_k-x_{k-1}}$$

$I_h(x)$满足在小区间$[x_k,x_{k+1}]$的表达式为

$$y=y_k\frac{x-x_{k+1}}{x_k-x_{k+1}}+y_{k+1}\frac{x-x_k}{x_{k+1}-x_k}$$

如果记$I_h(x)=\sum\limits_{k=0}^{n}y_kl_k(x)$，则

$$l_k(x)=\begin{cases}\dfrac{x-x_{k-1}}{x_k-x_{k-1}},x_{k-1}\le x\le x_k(k\ne 0)\\ \dfrac{x-x_{k+1}}{x_k-x_{k+1}},x_k\le x\le x_{k+1}(k\ne n)\\ 0,\ x\in[a,b],x\notin[x_{k-1},x_{k+1}].\end{cases}$$

➢ 分段线性插值的误差

定理1. 设$f \in C^2[a,b]$，$I_h(x)$为$[a,b]$上的分段线性插值函数，

则对$\forall x \in [a,b]$有：

$$|f(x) - I_h(x)| \le \frac{h^2}{8} M,$$

其中$h = \underset{0 \le k \le n-1}{Max}\{x_{k+1} - x_k\}$, $M = \underset{x \in [a,b]}{Max}|f''(x)|$.

并且$h \to 0$时，$I_h(x) \rightrightarrows f(x)$.

证明：见教材P159.

5.3 分段三次Hermite插值

设已知节点$a = x_0 < x_1 < \cdots < x_n = b$，且对应的函数值为

$f(x_0), f(x_1), \cdots, f(x_n)$，记$h_k = x_{k+1} - x_k$，$h = \underset{0 \le k \le n-1}{Max}\{h_k\}$,

构造一个导函数连续的分段多项式函数$I_h(x)$满足

(1). $I_h(x) \in C[a, b]$， (2). $I_h(x_k) = f(x_k)$， $I'_h(x_k) = f'(x_k)$

(3). $I_h(x_k)$在每个区间(x_k, x_{k+1})上是三次多项式函数；

则称函数$I_h(x)$为分段三次Hermite插值函数.

$I_h(x)$满足在小区间$[x_{k-1},x_k]$和$[x_k,x_{k+1}]$的表达式分别为

$$H_{3,k-1}(x)=f_{k-1}H_{k-1}(x)+\boxed{f_kH_k(x)}+f'_{k-1}\widehat{H}_{k-1}(x)+\boxed{f'_k\widehat{H}_k(x)}.$$

$$H_{k-1}(x)=\left(\frac{x-x_k}{x_{k-1}-x_k}\right)^2\left(1+2\frac{x-x_{k-1}}{x_k-x_{k-1}}\right),\quad \widehat{H}_{k-1}(x)=\left(\frac{x-x_{k-1}}{x_k-x_{k-1}}\right)\left(\frac{x-x_k}{x_{k-1}-x_k}\right)^2,$$

$$H_k(x)=\left(\frac{x-x_{k-1}}{x_k-x_{k-1}}\right)^2\left(1+2\frac{x-x_k}{x_{k-1}-x_k}\right),\quad \widehat{H}_k(x)=\left(\frac{x-x_{k-1}}{x_k-x_{k-1}}\right)^2\left(\frac{x-x_k}{x_{k-1}-x_k}\right).$$

$$H_{3,k}(x)=\boxed{f_kH_k(x)}+f_{k+1}H_{k+1}(x)+\boxed{f'_k\widehat{H}_k(x)}+f'_{k+1}\widehat{H}_{k+1}(x).$$

$$H_k(x)=\left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2\left(1+2\frac{x-x_k}{x_{k+1}-x_k}\right),\quad \widehat{H}_k(x)=\left(\frac{x-x_k}{x_{k+1}-x_k}\right)\left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2,$$

设$I_h(x)=\sum_{k=0}^{n}[f(x_k)\alpha_k(x)+f'(x_k)\beta_k(x)]$

于是 $$\alpha_k(x)=\begin{cases}\left(\frac{x-x_{k-1}}{x_k-x_{k-1}}\right)^2\left(1+2\frac{x-x_k}{x_{k-1}-x_k}\right), & x\in[x_{k-1},x_k]\\ \left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2\left(1+2\frac{x-x_k}{x_{k+1}-x_k}\right), & x\in[x_k,x_{k+1}]\\ 0, & else\end{cases}$$

$$\beta_k(x)=\begin{cases}\left(\frac{x-x_{k-1}}{x_k-x_{k-1}}\right)^2(x-x_k), & x\in[x_{k-1},x_k]\\ \left(\frac{x-x_{k+1}}{x_k-x_{k+1}}\right)^2(x-x_k), & x\in[x_k,x_{k+1}]\\ 0, & else\end{cases}$$

➢ 分段三次Hermite插值的误差

定理2. 设$f \in C^4[a,b]$，$I_h(x)$为分段三次Hermite插值函数，

则对$\forall x \in [a,b]$有：

$$|f(x) - I_h(x)| \le \frac{h^2}{384}M,$$

其中$h = \max\limits_{0\le k\le n-1}\{x_{k+1} - x_k\}$, $M = \max\limits_{x\in[a,b]}\left|f^{(4)}(x)\right|$.

并且$h \to 0$时，$I_h(x) \rightrightarrows f(x)$.

证明：见教材P161.