第四章

插值法(3)—Newton 插值

第3节 Newton 插值

Lagrange 插值虽然易算，但若要增加一个节点时，全部基函数$l_i(x)$都需要重新计算。也就是说，Lagrange 插值不具有继承性。

1. 能否重新在P_n中寻找新的函数？

2. 希望每加一个节点时，只在原有插值的基础上附加部分计算量（或者说添加一项）即可。

➤ 3.1 Newton插值函数

问题1 求n次多项式$N_n(x)$

$$N_n(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + \cdots$$
$$+c_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}) \qquad (1)$$

使其满足条件 $N_n(x_i) = f(x_i), i = 1, 2, \cdots, n$ (2)

为使$N_n(x)$表达式简单，记

$$\varphi_i(x) = (x - x_0)\cdots(x - x_{i-1}), i = 0, 1, 2, \cdots, n \qquad (3)$$

其中(1). $\varphi_0(x) = 1$； (2). $\varphi_{i+1}(x) = (x - x_i)\varphi_i(x)$

定义1. 由式(3)定义的$n + 1$个函数$\varphi_0(x), \varphi_1(x), \cdots, \varphi_n(x)$称为以$x_0, x_1, \cdots, x_n$为节点的Newton插值函数.

问题2 n次多项式$N_n(x)$的系数？

$$N_n(x) = c_0 + c_1(x - x_0) + c_2(x - x_0)(x - x_1) + \cdots$$
$$+ c_n(x - x_0)(x - x_1)\cdots(x - x_{n-1}) \qquad (1)$$

使其满足条件 $N_n(x_i) = f(x_i), i = 0, 1, 2, \cdots, n$ (2)

求系数$c_i, i = 0, 1, \cdots, n.$

$$N_n(x_0) = f(x_0) = c_0,$$

$$N_n(x_1) = f(x_1) = f(x_0) + c_1(x_1 - x_0)$$

解得 $c_1 = \dfrac{f(x_1) - f(x_0)}{x_1 - x_0} \triangleq f[x_0, x_1].$

$$N_n(x_2) = f(x_2) = f(x_0) + f[x_0, x_1](x_2 - x_0) + c_2(x_2 - x_0)(x_2 - x_1)$$

$$c_2 = \frac{f(x_2)-f(x_0)-f[x_0,x_1](x_2-x_0)}{(x_2-x_0)(x_2-x_1)} = \frac{f(x_2)-f(x_0)-\frac{f(x_1)-f(x_0)}{x_1-x_0}(x_2-x_0)}{(x_2-x_0)(x_2-x_1)}$$

$$= \frac{f(x_2)-f(x_1)+\frac{f(x_1)-f(x_0)}{x_1-x_0}(x_1-x_0)-\frac{f(x_1)-f(x_0)}{x_1-x_0}(x_2-x_0)}{(x_2-x_0)(x_2-x_1)}$$

$$= \frac{f(x_2)-f(x_1)-\frac{f(x_1)-f(x_0)}{x_1-x_0}(x_2-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1}-\frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2-x_0} = \frac{f[x_1,x_2]-f[x_0,x_1]}{x_2-x_0}$$

$$\triangleq f[x_0, x_1, x_2]$$

➢ 3.2 差商 /* divided difference */

定义2. 给定区间$[a,b]$上的不同点$x_0, x_1, \cdots, x_n$，以及函数值$f(x_0), f(x_1), \cdots, f(x_n)$, 则称

$$f[x_i, x_j] = \frac{f(x_j)-f(x_i)}{x_j-x_i}$$为$f(x)$在x_i, x_j处的一阶差商；

$$f[x_i, x_j, x_k] = \frac{f[x_j,x_k]-f[x_i,x_j]}{x_k-x_i}$$为$f(x)$在x_i, x_j, x_k处的二阶差商；

$$f[x_0, x_1, \cdots, x_n] = \frac{f[x_1,x_2,\cdots,x_n]-f[x_0,x_1,\cdots,x_{n-1}]}{x_n-x_0}$$为n阶差商.

注1. 差商又称为均值.

例1. 已知如下数据，求各阶差商.

x_i	1	2	3	4
$f(x_i)$	0	-5	-6	3

解　各阶差商，如下表

x_i	$f(x_i)$	一阶差商	二阶差商	三阶差商
1	0			
2	-5	$f[1,2]=-5$		
3	-6	$f[2,3]=-1$	$f[1,2,3]=2$	
4	3	$f[3,4]=9$	$f[2,3,4]=5$	$f[1,2,3,4]=1$

$$f[x_i,x_j]=\frac{f(x_j)-f(x_i)}{x_j-x_i}\qquad f[x_i,x_j,x_k]=\frac{f[x_j,x_k]-f[x_i,x_j]}{x_k-x_i}$$

差商的性质

$\omega'_{n+1}(x_i)=(x_i-x_0)\cdots(x_i-x_{i-1})(x_i-x_{i+1})\cdots(x_i-x_n)$

性质1. 差商与函数值的关系 (由归纳法可证)

$$f[x_0,x_1,\cdots,x_n]=\sum_{i=0}^{n}\frac{f(x_i)}{\omega'_{n+1}(x_i)},\text{ 其中}\omega_{n+1}(x)=(x-x_0)\cdots(x-x_n).$$

证明：$f[x_0,x_1]=\dfrac{f(x_1)-f(x_0)}{x_1-x_0}=\dfrac{f(x_0)}{x_0-x_1}+\dfrac{f(x_1)}{x_1-x_0}.$

$$f[x_0,x_1,x_2]=\frac{\frac{f(x_2)-f(x_1)}{x_2-x_1}-\frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2-x_0}=\frac{f(x_0)}{(x_0-x_1)(x_0-x_2)}+\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}$$

$$+\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)}.$$

$$f[x_0,x_1,x_2,x_3]=\frac{f[x_0,x_1,x_2]-f[x_1,x_2,x_3]}{x_0-x_3}=\sum_{j=0}^{3}\frac{f(x_j)}{\omega'_{m+1}(x_j)}$$

$$=\frac{\frac{f(x_0)}{(x_0-x_1)(x_0-x_2)}+\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)}+\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}-\frac{f(x_1)}{(x_1-x_2)(x_1-x_3)}-\frac{f(x_2)}{(x_2-x_1)(x_2-x_3)}-\frac{f(x_3)}{(x_3-x_1)(x_3-x_2)}}{x_0-x_3}$$

$$=\frac{f(x_0)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}+\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}+\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}+\frac{f(x_3)}{(x_3-x_1)(x_3-x_2)(x_3-x_0)}$$

$$\frac{\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)}+\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)}-\frac{f(x_1)}{(x_1-x_2)(x_1-x_3)}-\frac{f(x_2)}{(x_2-x_1)(x_2-x_3)}}{(x_0-x_3)}$$

$$=\frac{f(x_1)}{(x_1-x_2)}\left(\frac{1}{x_1-x_0}-\frac{1}{x_1-x_3}\right)\left(\frac{1}{x_0-x_3}\right)+\frac{f(x_2)}{(x_2-x_1)}\left(\frac{1}{x_2-x_0}-\frac{1}{x_2-x_3}\right)\left(\frac{1}{x_0-x_3}\right)$$

$$=\frac{f(x_1)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}+\frac{f(x_2)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}$$

差商的性质

性质1. 差商与函数值的关系 (由归纳法可证)

$$f[x_0,x_1,\cdots,x_n]=\sum_{i=0}^{n}\frac{f(x_i)}{\omega'_{n+1}(x_i)},\text{ 其中}\omega_{n+1}(x)=(x-x_0)\cdots(x-x_n).$$

性质2. 对称性，差商大小与节点排序无关. (由性质1可得)

$$f[x_0,\cdots,x_i,\cdots,x_j,\cdots,x_n]=f[x_0,\cdots,x_i,\cdots,x_j,\cdots,x_n].$$

性质3. 线性性质，若$F(x)=\alpha f(x)+\beta g(x)$，则 (由性质1可得)

$$F[x_0,x_1\cdots,x_n]=\alpha f[x_0,x_1\cdots,x_n]+\beta g[x_0,x_1\cdots,x_n].$$

性质4. 差商与导数的关系，若$f(x)$有n阶导数，则

$$f[x_0,x_1,\cdots,x_n]=\frac{f^{(n)}(\xi)}{n!},$$ 其中ξ介于$x_0,x_1,\cdots,x_n$之间.

证：$N_n(x)=f(x_0)+f[x_0,x_1](x-x_0)+\cdots+f[x_0,x_1,\cdots,x_n](x-x_0)\cdots(x-x_{n-1})$

令$g(x)=f(x)-N_n(x)$，则$g(x_0)=g(x_1)=\cdots=g(x_n)=0$

重复使用$Rolle$定理可得。

推论. 重节点的差商，若$f(x)$有n阶导数，则

$$f[\underbrace{x_i,x_i,\cdots,x_i}_{n+1}]=\frac{f^{(n)}(x_i)}{n!}.$$

证：对性质4取极限，$x_k\to x_i, k=0,1\cdots,i-1,i+1,\cdots,n.$

➢ 3.3 Newton插值多项式

定义3. 设$x_0, x_1, \cdots, x_n$为区间$[a, b]$上的不同点，插值函数为

$$\varphi_0(x) = 1,\quad \varphi_i(x) = (x - x_0)(x - x_1)\cdots(x - x_{i-1}),$$

$f[x_0] = f(x_0)$, $f[x_0, x_1, \cdots, x_i]$为i阶差商，$i = 1, 2, \cdots n$.

称$N_n(x) = \sum_{i=0}^{n} f[x_0, x_1, \cdots x_i]\varphi_i(x)$为**Newton插值多项式.**

即
$$N_n(x) = f(x_0) + f[x_0, x_1](x - x_0) + f[x_0, x_1, x_2](x - x_0)(x - x_1)$$
$$+ \cdots + f[x_0, x_1, \cdots, x_n](x - x_0)(x - x_1)\cdots(x - x_{n-1})$$

注1. Newton插值多项式满足插值条件

$$N_n(x_i) = f(x_i), \quad i = 0, 1, 2, \cdots, n.$$

注2. 由Newton插值公式可以看出, 每当增加一个结点时, Newton插值多项式只在原有插值多项式的基础上增加一项即可，即

$$N_{n+1}(x) = N_n(x) + f[x_0, \cdots, x_n, x_{n+1}](x - x_0)\cdots(x - x_{n-1})(x - x_n)$$

例1. 已知如下数据，求Newton插值多项式.

x_i	1	2	3	4
$f(x_i)$	0	-5	-6	3

解　各阶差商，如下表

x_i	$f(x_i)$	一阶差商	二阶差商	三阶差商
1	0			
2	-5	$f[1,2]=-5$		
3	-6	$f[2,3]=-1$	$f[1,2,3]=2$	
4	3	$f[3,4]=9$	$f[2,3,4]=5$	$f[1,2,3,4]=1$

$$N_n(x)=f(1)+f[1,2](x-1)+f[1,2,3](x-1)(x-2)$$

$$+f[1,2,3,4](x-1)(x-2)(x-3)=x^3-4x^2+3.$$

定理1. 记$R_n(x)$为Newton插值多项式的余项，则

$$R_n(x) = f(x) - N_n(x) = f[x_0, x_1, \cdots, x_n, x](x - x_0)\cdots(x - x_n).$$

证明见教材P147.

注3. 因为多项式插值余项为$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x - x_0)\cdots(x - x_n)$.

所以$f[x_0, x_1, \cdots, x_n, x] = \frac{f^{(n+1)}(\xi)}{(n+1)!}$.

例1. 已知$f(x)=-2x^7+3x^5-5x^4+7x-6$，求

(1). $f[2^0,2^1,\cdots,2^6]$，(2). $f[2^0,2^1,\cdots,2^6,x]$，(3). $f[2^0,2^1,\cdots,2^7,x]$.

解：由性质4可知，

$$f[2^0,2^1,\cdots,2^6]=\frac{f^{(6)}(\xi)}{6!}=\left.\frac{-2\cdot 7!\,x}{6!}\right|_{x=\xi}=-14\xi,\xi\in(2^0,2^6).$$

由定理1可知，

$$f[2^0,2^1,\cdots,2^6,x]=\frac{f^{(7)}(\xi)}{7!}=\left.\frac{-2\cdot 7!}{7!}\right|_{x=\xi}=-2.$$

$$f[2^0,2^1,\cdots,2^7,x]=\frac{f^{(8)}(\xi)}{8!}=\left.\frac{-2\cdot 0}{8!}\right|_{x=\xi}=0.$$