

xSPDE user's guide

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1 Introduction

This xSPDE user's guide can be used as the basis for a 3-4 hour self-taught tutorial in using xSPDE for solving practical stochastic problems. It includes exercises for the reader. A full-length manual is available online with more details.

1.1 Stochastic equations

Stochastic equations are equations with random noise terms [1, 2]. They occur in many fields of biology, chemistry, engineering, economics, physics, meteorology and other disciplines. An ordinary stochastic equation, which is readily solved numerically [3, 4] for a real or complex vector \mathbf{a} is:

$$\frac{\partial \mathbf{a}}{\partial t} = \dot{\mathbf{a}} = \mathbf{A}(\mathbf{a}) + \mathbf{B}(\mathbf{a}) \boldsymbol{\zeta}(t). \quad (1.1)$$

Here, \mathbf{A} is a vector, \mathbf{B} a matrix, and $\boldsymbol{\zeta}$ is a real noise vector, delta-correlated in time:

$$\langle \zeta_i(t) \zeta_j(t') \rangle = \delta(t - t') \delta_{ij} \quad (1.2)$$

Stochastic partial differential equations including space dependence are also soluble with xSPDE. These are described in Chapter 4.

1.2 xSPDE

xSPDE is a Matlab based software toolbox. It has functions that can numerically solve both ordinary and partial differential stochastic equations, and graph results of correlations and averages. The equations can include partial spatial derivatives, like the Maxwell or Schrödinger equations. They are often found in physics, where they are used to model quantum noise and thermal or technical noise, all of which is measurable.

There are many equations of this type, and the xSPDE toolbox can treat a wide range. The general structure permits drop-in replacements of the functions provided. Different simulations can be carried out sequentially, to simulate the various stages in an experiment or other process.

The code supports parallelism at both the vector instruction level and at the thread level, using Matlab matrix instructions and the parallel toolbox. It calculates averages of arbitrary functions of any number of complex or real fields, as well as Fourier transforms in time or space. It provides error estimates for both step-size and sampling error.

xSPDE is distributed without any guarantee, under the MIT open-source license. Contributions and bug reports always welcome.

See: www.github.com/peterddrummond/xspde_matlab.

2 xSPDE simulations

The general purpose xSPDE toolbox function is:

- **xspde(in)**, the combined simulation/graphics function.

2.1 Running xSPDE interactively

To simulate a stochastic equation interactively, first make sure Matlab path is pointing to the xSPDE folder. If not, proceed as follows:

- Click on the Matlab HOME tab (top left), then Set Path
- Click on Add with Subfolders
- Find the xSPDE folder in the drop-down menu, and select it
- Click on close to save the path.

Next, type *clear* to clear old data, and enter the xSPDE inputs and functions into the command window. For the simplest of examples, do this by cutting and pasting from an electronic file of this manual. You can also use the Matlab built-in editor.

A typical script is as follows:

```
in.[label1] = [parameter1]
in.[label2] = ...
in.da = @(a,z,r) [derivative]
xspde(in)
```

Note the following points to remember:

- The notation `in.[label1] = [parameter1]` inputs a value to the `in.` data structure
- There are many input possibilities, and they all have default values.
- You don't have to save the data if you just want an immediate plot
- The notation `@(a,z,r) [derivative]` defines an inline function, namely da/dt .
- In xSPDE, `a` is the stochastic variable, `z` the random noise, and `r` the parameters.
- For example, `r.t` is the current time, `r.x` the grid of x-coordinates.

2.2 Inputs

All xSPDE simulations use a structure *in.* for parameters. All xSPDE functions use a structure *r.* that combines inputs and generated parameters. Any name will do for either structure, as long as you are consistent.

All inputs have default values, which can be changed by typing parameters into the *in* structure. If you only need the first value of a vector or array, just input the value that is required.

2.2.1 Common xSPDE inputs

The most common xSPDE parameters are:

Label	Type	Default value	Description
<i>in.fields</i>	integer	1	Stochastic field variables
<i>in.noises</i>	integer	1	Number of noises
<i>in.dimension</i>	integer	1	Space-time dimensions
<i>in.name</i>	string	' '	Simulation name
<i>in.olabels</i>	string	'a_1'	Observable labels
<i>in.da</i>	function	0	The stochastic derivative
<i>in.initial</i>	function	0	Function to initialize variables
<i>in.ensembles</i>	integer vector	[1,1,1]	Hierarchy of stochastic trajectories
<i>in.observe</i>	function	$a(1,:)$	Observable function
<i>in.linear</i>	function	0	Linear interaction picture response
<i>in.ranges</i>	real vector	10	Ranges in time and space
<i>in.steps</i>	integer vector	1	Time-steps per plot point
<i>in.points</i>	integer vector	51	Output lattice points in [t,x,y,z]
<i>in.images</i>	integer cell array	{0,..}	Movie images in time per observable
<i>in.transforms</i>	vector cells	{[0 0 0 0],...}	Output Fourier transforms in [t,x,y,z]

A complete list of user inputs is given in Chapter (6).

2.2.2 Use the dot

Matlab formulae often require a 'dot' or a 'colon'. To multiply vectors element-wise, like $a_i = b_i c_i$, the notation $a = b.*c$ indicates all elements are multiplied. This is used in xspde to speed up calculations in parallel.

A formula for a stochastic field may require you to address the first index - which is the field component - but treat all the other elements in parallel. To do this, use the notation $a(1,:)$.

Whenever a formula combines operations over spatial lattices or ensembles, remember to **USE THE DOT**.

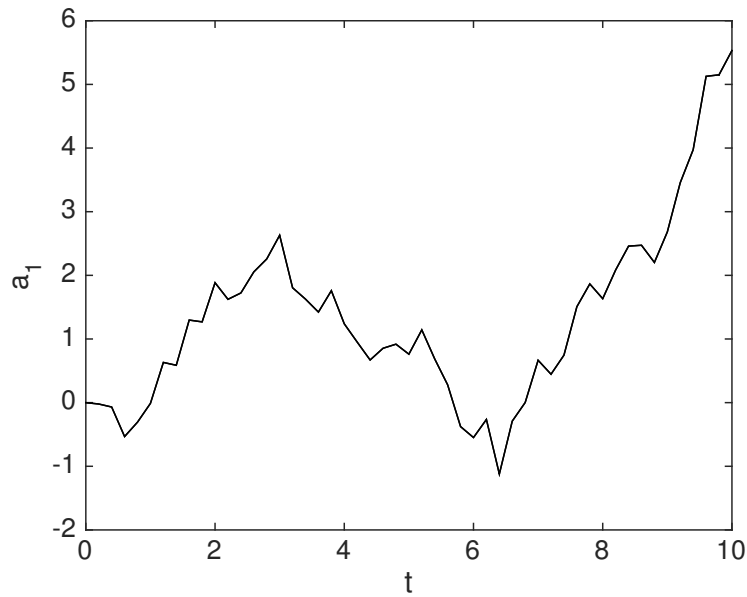


Figure 2.1: The simplest case: a random walk.

2.3 The random walk

The first example is the simplest possible stochastic equation:

$$\dot{a} = \zeta(t), \quad (2.1)$$

Exercises

Remember that, unless you type *clear* first, any changes to the input structure are additive; so you will get the sum of all the previous inputs as well as your new input.

- **Run the complete xSPDE script given below in Matlab.**

It is simplest to cut and paste from an electronic file into the Matlab command window. Be careful; pasting can cause subtle changes that may require correction.

You should get the output in Fig (2.1).

```
in.da=@(a,z,r) z; xspde(in);
```

Here *in.da* defines the derivative function, with *z* the noise.

- **What do you see if you average over 10000 trajectories ?**

```
in.ensembles = 10000;
xspde(in);
```

- What do you see if you plot the mean square distance? Note that variances should increase linearly with t .

```
in.observe = @(a,r) a.^2;
in.olabels = '<a^2>';
xspde(in);
```

- What if you add a force that takes the particle back to the origin?

$$\dot{a} = -a + \zeta(t), \quad (2.2)$$

```
in.da=@(a,z,r) -a+z
xspde(in);
```

Mathematical exercise:

- If you like mathematics; as an offline exercise, try to solve the equivalent diffusion equation for the probability P to find the variance $\langle a^2 \rangle$! The equation is:

$$\frac{\partial P(a)}{\partial t} = \left[\frac{\partial}{\partial a} + \frac{1}{2} \frac{\partial^2}{\partial a^2} \right] P(a)$$

2.4 Laser quantum noise

Next we treat a model for the quantum noise of a single mode laser as it turns on, near threshold:

$$\dot{a} = ga + b\zeta(t) \quad (2.3)$$

where the noise is complex, $\zeta = (\zeta_1 + i\zeta_2)$, so that:

$$\langle \zeta(t)\zeta^*(t') \rangle = 2\delta(t-t'). \quad (2.4)$$

Here the coefficient b describes the quantum noise of the laser, and is inversely proportional to the equilibrium photon number.

Exercises

You should type *clear* first when starting new simulations.

- **Solve for the case of $g = 0.1$, $b = 0.01$**

Most lasers have more than 100 photons, and hence less noise!

```
clear
in.noises=2;
in.observe = @(a,r) abs(a)^2;
in.olabels = '|a|^2';
in.da=@(a,z,r) a+0.01*(z(1)+i*z(2));
xspde(in);
```

Consider the case where the laser saturates to a steady state:

$$\dot{a} = \left(1 - |a|^2\right) a + b\zeta(t) \quad (2.5)$$

- **Solve for the saturated laser case**

You should get the output graph in Fig (2.2).

```
in.da=@(a,z,r) (1-abs(a)^2)*a+0.01*(z(1)+i*z(2));
xspde(in);
```

2.5 Spectra

Frequency spectra have many uses, especially for understanding the steady-state fluctuations of any physical system. Take the random walk equation in a complex space,

$$\frac{da}{dt} = -a + \zeta(t)$$

where $\zeta(t) = \zeta_1(t) + i\zeta_2(t)$. Use an initial condition of $a = (v_1 + iv_2)/\sqrt{2}$, with $\langle v_i^2 \rangle = 1$, to start near the steady state.

For sufficiently long time-intervals, the solution in frequency space given by:

$$\tilde{a}(\omega) = \frac{\tilde{\zeta}(\omega)}{1 - i\omega}$$

The expectation value of the noise spectrum, in the large T limit, is therefore:

$$\begin{aligned} \langle |\tilde{a}(\omega)|^2 \rangle &= \frac{1}{2\pi(1+\omega^2)} \int \int e^{i\omega(t-t')} \langle \zeta(t)\zeta^*(t') \rangle dt dt' \\ &= \frac{T}{\pi(1+\omega^2)} \end{aligned}$$

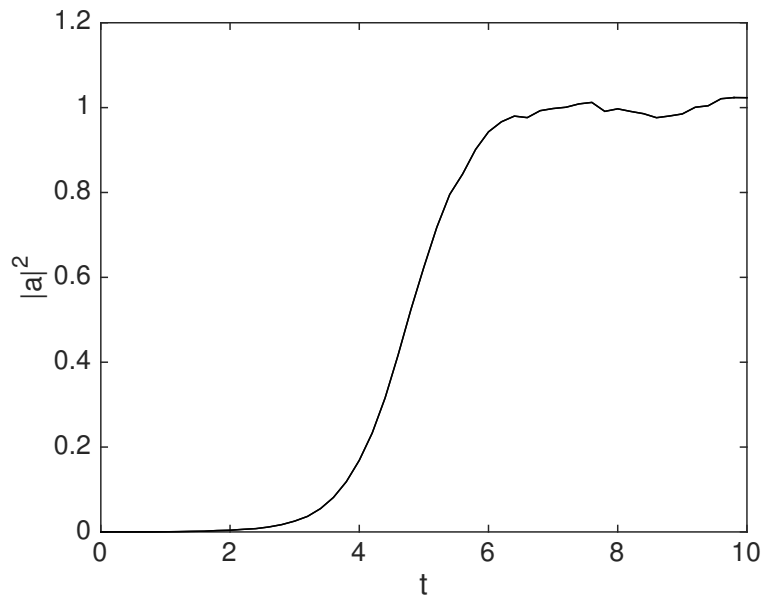


Figure 2.2: Simulation of the stochastic equation describing a laser turning on.

Exercises

- Plot the spectrum over a range of $t = 100$, with 640 points and a random initial equation near the equilibrium value.

The input parameters are given below. There are parallel operations here, for ensemble averaging, so **USE THE DOT**.

```
clear
in.points = 640;
in.ranges = 100;
in.noises = 2;
in.ensembles = 10000;
in.initial = @(v,r) (v(1,:)+1i*v(2,:))/sqrt(2);
in.da = @(a,z,r) -a + z(1,:)+1i*z(2,:);
in.observe = @(a,r) a.*conj(a);
in.transforms = 1;
in.olabels = '|a(w)|^2';
xspde(in);
```

Note that `in.transforms = 1` tells xSPDE to Fourier transform the field over the time coordinate, to give a spectrum. The first argument v of the initial function is a random field. You can transform any field in any dimension, see Chapter 6 for details.

To define as many observables as you like, use a Matlab cell array;

```
in.observe{1} = ..;  
in.observe{2} = ..;
```

- **Simulate over a range of $t = 200$. What changes do you see? Why?**
- **Change the equation to the laser noise equations. Why is the spectrum much narrower?**

2.6 Ito and Stratonovich equations

The xSPDE toolbox is mainly designed to treat Stratonovich equations [1], which are the broad-band limit of a finite band-width random noise equation. Another type of stochastic equation is the Ito form, which is a limit where derivatives are evaluated at the start of each step.

We can write an Ito equation as a difference equation:

$$d\mathbf{a} = \mathbf{A}^I [\mathbf{a}] + \underline{\mathbf{B}} [\mathbf{a}] \cdot d\mathbf{w}(t). \quad (2.6)$$

Here $\langle dw_i(\mathbf{x}) dw_j(\mathbf{x}') \rangle = \delta_{ij} dt$.

When \mathbf{B} is not constant, the Ito drift term \mathbf{A}^I is different to the corresponding Stratonovich one. The relationship is:

$$A_i = A_i^I - \frac{1}{2} \sum_{j,m} \frac{\partial B_{ij}}{\partial a_m} B_{mj}. \quad (2.7)$$

2.7 Financial calculus

A well-known Ito stochastic equation is the Black-Scholes equation, used to price financial options. It describes the fluctuations in a stock value:

$$da = \mu a dt + a \sigma dw, \quad (2.8)$$

where $\langle dw^2 \rangle = dt$. Since the noise is multiplicative, the equation is different in Ito and Stratonovich forms of stochastic calculus.

The corresponding Stratonovich equation, as used in xSPDE is:

$$\dot{a} = \left(\mu - \sigma^2/2 \right) a + a \sigma \zeta(t). \quad (2.9)$$

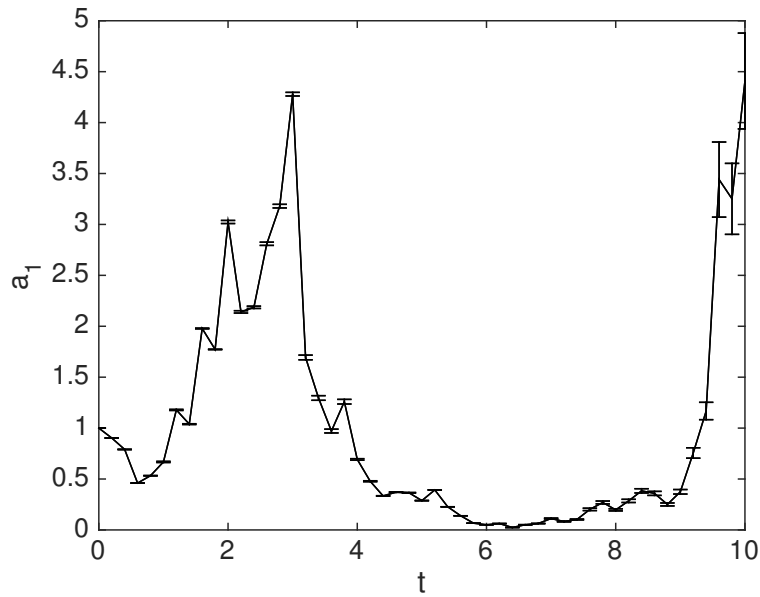


Figure 2.3: Simulation of the Black-Scholes equation describing stock prices.

Exercises

- Solve for a startup with a volatile stock having $\mu = 0.1$, $\sigma = 1$.

An interactive xSPDE script in Matlab is given below with an output graph in Fig (2.3), Note the spiky behaviour, typical of multiplicative noise, and also of the risky stocks in the small capitalization portions of the stock market.

```
clear
in.initial=@(v,r) 1
in.da=@(a,z,r) -0.4*a+a*z
xspde(in)
```

Note that *in.initial* describes the initialization function. The first argument of $\text{@}(v, r)$ is v , an initial random variable with unit variance. The error-bars are estimates of step-size error. Errors can be reduced by using more time-steps.

- Solve for a more mature stock having $\mu = 0.1$, $\sigma = 0.1$.

3 xSPDE Projects

Projects are larger scale investigations, where files are used to record inputs and data. This requires a new input parameter:

Label	Type	Default value	Description
<i>in.file</i>	string	"	Matlab or HDF5 data file

3.1 Ensembles and sampling errors

Averages over stochastic ensembles are the specialty of xSPDE, which requires specification of the ensemble size. A sophisticated hierarchy of ensemble specifications in three levels is possible, to allow maximum resource utilization, so that:

$$in.ensembles = [ensembles(1), ensembles(2), ensembles(3)].$$

The first, *ensembles*(1), gives within-thread parallelism, allowing vector instruction use for single-core efficiency. The second, *ensembles*(2), gives a number of independent trajectories calculated serially.

The third, *ensembles*(3), gives multi-core parallelism, and requires the Matlab parallel toolbox. This improves speed when there are multiple cores, and one should optimally put *ensembles*(3) equal to the available number of CPU cores.

The *total* number of stochastic trajectories or samples is

$$ensembles(1) \times ensembles(2) \times ensembles(3).$$

Either *ensembles*(2) or *ensembles*(3) are required if sampling error-bars are to be calculated, owing to the sub-ensemble averaging method used in xSPDE to calculate sampling errors accurately.

3.2 xSPDE batch jobs

An XPDE session can either run simulations interactively, described in Chapter 2, or else using a function file called a project file. This allows xSPDE to run in a batch mode, as needed for longer projects which involve large ensembles. In either case, the Matlab path must include the xSPDE folder.

A minimal xSPDE project function is as follows:

3 xSPDE Projects

```
function = project.m
in.[label1] = [parameter1]
in.[label2] = ...
    in.da = @(a,z,r) [my derivative]
xspde(in)
end
```

For standard graph generation, the script input or project function should end with the combined function **xspde(in)**. Alternatively, to generate simulation data and graphs separately, the function *xsim* runs the simulation, and *xgraph* makes the graphs. The two-stage option is better for running batch jobs, which you can graph at a later time.

After preparing a project file using the editor, click on the Run arrow above the editor window. To create a data file, you must enter the filename before running the simulation, using the *in.file = filename* input. A simple interactive workflow is as follows:

- Create the metadata *in*, and include a file name
- Change the Matlab directory path to home using *cd ~*.
- Run the simulation with **xsim(in)**.
- Run **xgraph(in)**, and the data will be accessed and graphed.

3.3 Kubo project

To get started on more complex programs, we next simulate the Kubo oscillator, which is an oscillator with a random frequency:

$$\dot{a} = ia\zeta \quad (3.1)$$

Exercises

- **Simulate the Kubo oscillator using a file, *Kubo.m*, with two layers of ensembles to allow sampling error estimates. The error vector *error* returned by the function gives the time-step, then sampling error estimates.**

```
function [error] = Kubo()
in.name = 'Kubo oscillator';
in.ensembles = [400,16];
in.initial = @(v,r) 1+0*v;
in.da = @(a,z,r) i*a.*z;
in.olabels = '<a_1>';
in.file = 'kubo.mat';
error=xsim(in);
xgraph(in);
end
```

If you run this more than once with the same file-name, you will get a warning and a new file-name. Under these circumstance, xgraph will graph the data in the existing file. This is because xSPDE will not overwrite files, to protect your previous data.

3.4 Challenge problem #1

This is a much harder example, involving a full nonlinear quantum phase-space simulation. The method can also be used to investigate quantum non-equilibrium phase transitions, tunneling in open systems, quantum entanglement, Einstein-Podolsky-Rosen paradoxes, Bell violations, and many other problems treated in the literature[1, 2].

The following new ideas are introduced:

1. **in.ranges is the space-time integration range.**
2. **in.steps gives integration steps per plot-point, for accuracy.**

A simple case is the nonlinear driven quantum oscillator - for example, an optomechanical, atomic or nonlinear optical medium in a driven cavity.

The phase-space Fokker-Planck equation is equivalent to an SDE with the following form

$$\begin{aligned}\frac{d\alpha}{dt} &= -\tilde{\gamma}\alpha - i\kappa\alpha^2\beta + \mathcal{E} + i\sqrt{i\kappa}\alpha\xi_1(t), \\ \frac{d\beta}{dt} &= -\tilde{\gamma}^*\beta + i\kappa\beta^2\alpha + \mathcal{E}^* + \sqrt{i\kappa}\beta\xi_2(t),\end{aligned}\tag{3.2}$$

where $\xi_i(t)$ are independent Gaussian noise terms with zero means and the following nonzero correlations

$$\langle \xi_i(t)\xi_j(t') \rangle = \delta_{ij}\delta(t-t').\tag{3.3}$$

Note that these equations require that $\kappa \ll \gamma$, which is typical experimentally, otherwise one must use additional stochastic gauge stabilization to improve sampling error.

Defining the dimensionless parameters ε and λ as

$$\varepsilon = \frac{2\mathcal{E}}{\kappa}, \quad \lambda = \frac{2\tilde{\gamma}}{\kappa}.\tag{3.4}$$

3 xSPDE Projects

Provided $\gamma + \Delta\omega < 0$ and $(\text{Re}\lambda)^2 > 3|\lambda|^2$ there is a bistable region. This means that for $|\varepsilon|^2$ in a certain range, there are two physically possible values of n_0 . For $\tilde{\gamma} = 2.5 - 10i$, $\kappa = 1$, the bistable region is around $\mathcal{E} = 8$. The observed quantity is the particle number, $\langle \hat{a}^\dagger \hat{a} \rangle = \langle \alpha\beta \rangle$, with stable values of $n \approx 1$ and $n \approx 6$.

Exercises

- **Simulate the nonlinear oscillator by creating a file, *Nonlinear.m***
- **Can you observe quantum tunneling of $n = \alpha\beta$ in the bistable regime?**
- **Do you see evidence of a transient Schroedinger ‘cat state’ with a negative n value?**

Since tunneling is random, for real experimental comparisons, one would have to measure correlation functions and spectra. However, a tunneling event in a simulation indicates that it is likely in experiment too. These calculations require long time scales, **in.ranges**, to observe tunneling, and a large number of time steps per plotted point, **in.steps**, to maintain accuracy in the quantum simulations.

4 Stochastic partial differential equations

A stochastic partial differential equation for a complex vector field defined in space-time dimension $d = 1 - 4$, that is, including time and up to three space dimensions, can be written in differential form as

$$\frac{\partial \mathbf{a}}{\partial t} = \mathbf{A}[\mathbf{a}] + \mathbf{B}[\mathbf{a}] \cdot \boldsymbol{\zeta}(t) + \mathbf{L}[\nabla] \cdot \mathbf{a}. \quad (4.1)$$

Here \mathbf{a} is a real or complex vector or vector field. The initial conditions are arbitrary functions. $\mathbf{A}[\mathbf{a}]$ and $\mathbf{B}[\mathbf{a}]$ are vector and matrix functions of \mathbf{a} , $\mathbf{L}[\nabla]$ is a matrix of linear terms and derivatives, diagonal in the vector indices, and $\boldsymbol{\zeta} = [\zeta^x, \zeta^k]$ are real delta-correlated noise fields such that:

$$\begin{aligned} \langle \zeta_i^x(t, \mathbf{x}) \zeta_j^x(t, \mathbf{x}') \rangle &= \delta(\mathbf{x} - \mathbf{x}') \delta(t - t') \delta_{ij} \\ \langle \zeta_i^k(t, \mathbf{k}) \zeta_j^k(t, \mathbf{k}') \rangle &= f(\mathbf{k}) \delta(\mathbf{k} - \mathbf{k}') \delta(t - t') \delta_{ij}. \end{aligned} \quad (4.2)$$

Transverse boundary conditions are assumed periodic. The term $\mathbf{L}[\nabla]$ may be omitted if $d = 1$, as there are no space dimensions. Here $f(\mathbf{k})$ is an arbitrary momentum filter, for correlated noise. The linear function L is input as a separate function in xSPDE, to allow for greatest efficiency and use of an interaction picture.

4.1 Stochastic Ginzburg-Landau equation

Including space-time dimensions with $d = 3$, an example of a SPDE is the stochastic Ginzburg-Landau equation. This describes symmetry breaking: the system develops a spontaneous phase which can vary spatially as well. The model is widely used to describe lasers, magnetism, superconductivity, superfluidity and even particle physics:

$$\dot{a} = (1 - |a|^2) a + b\zeta(t) + c\nabla^2 a \quad (4.3)$$

where

$$\langle \zeta(x) \zeta^*(x') \rangle = 2\delta(t - t') \delta(x - x'). \quad (4.4)$$

The following new ideas are introduced for this problem:

1. **in.dimension is the space-time dimension.**
2. **The 'dot' notation used for parallel operations over lattices.**
3. **in.linear is the linear operator - a laplacian in these cases.**

4. `in.images` produces movie-style images at discrete time slices.
5. `D.x` indicates a derivative operation, $\partial/\partial x$.
6. $-5 < x < 5$ is the default `xSPDE` coordinate range in space.

Exercises

- Solve the stochastic G-L equation for $b = 0.001$ and $c = 0.01i$.
- Change to a real diffusion so that $c = 0.1$.

In the first case, you should get the output graphed in Fig (4.1) .

```
in.noises=2;
in.dimension=3;
in.steps=10;
in.linear = @(D,r) i*0.01*(D.x.^2+D.y.^2);
in.observe = @(a,~) abs(a).^2;
in.images =6;
in.olabels = '|a|^2';
in.da=@(a,z,~) (1-abs(a(1,:)).^2).*a+0.001*(z(1,:)+i*z(2,:));
xspde(in)
```

Here the notation $z(1,:)$ is like the dot: the colon means that the operation is repeated over all values of the second index, which is the spatial lattice index.

4.2 Soliton

The famous nonlinear Schrödinger equation (NLSE) is:

$$\frac{da}{dt} = \frac{i}{2} \left[\nabla^2 a - a \right] + ia |a|^2$$

Together with the initial condition that $a(0, x) = \text{sech}(x)$, this has a soliton, an exact solution that doesn't change in time:

$$a(t, x) = \text{sech}(x)$$

The Fourier transform at $k = 0$ is simply:

$$\tilde{a}(t, 0) = \frac{1}{\sqrt{2\pi}} \int \text{sech}(x) dx = \sqrt{\frac{\pi}{2}}$$

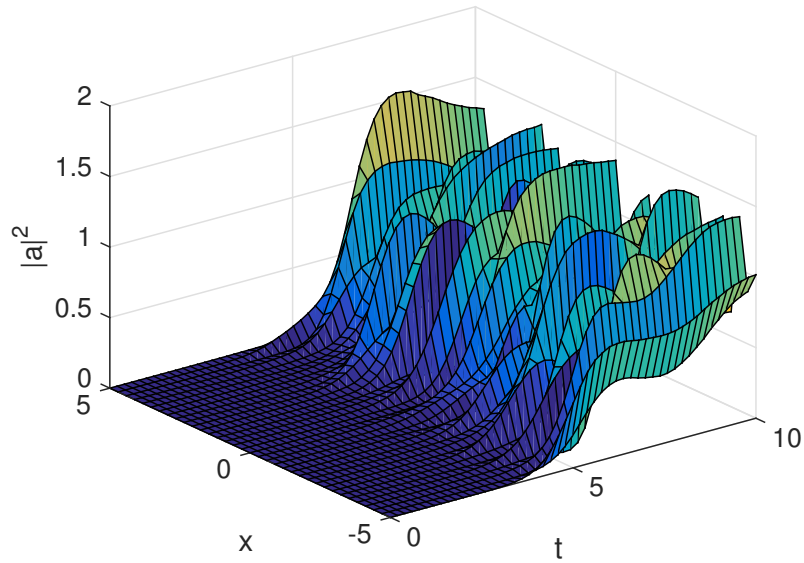


Figure 4.1: Simulation of the stochastic equation describing symmetry breaking in two dimensions. Spatial fluctuations are caused by the different phase-domains that interfere. The graph obtained here is projected onto the $y = 0$ plane.

Exercises

- Solve the NLSE for a soliton

```
function [e] = Soliton()
in.name = 'NLS soliton';
in.dimension = 2;
in.initial = @(v,r) sech(r.x);
in.da = @(a,z,r) i*a.*(conj(a).*a);
in.linear = @(D,r) 0.5*i*(D.x.^2-1.0);
in.olabels = {'a_1(x)'};
in.compare{1} = @(t,~) 1;
e = xspde(in);
end
```

- Add an additive complex noise of $0.01(dw_1 + idw_2)$ to the differential equation, then replot with an average over 1000 samples.

4.3 Gaussian diffraction

Free diffraction of a Gaussian wave-function in three dimensions, is given by

$$\frac{da}{dt} = \frac{i}{2} \nabla^2 a$$

Together with the initial condition that $a(0, x) = \exp(-|\mathbf{x}|^2/2)$, this has an exact solution for the diffracted intensity in either ordinary space or momentum space:

$$\begin{aligned} |a(t, \mathbf{x})|^2 &= \frac{1}{(1+t^2)^{3/2}} \exp(-|\mathbf{x}|^2 / (1+t^2)) \\ |\tilde{a}(t, \mathbf{k})|^2 &= \exp(-|\mathbf{k}|^2) \end{aligned}$$

Exercises

- Solve the diffraction equation without noise

```
function [e] = Gaussian()
in.dimension = 4;
in.initial = @(v,r) exp(-0.5*(r.x.^2+r.y.^2+r.z.^2));
in.linear = @(D,r) 1i*0.05*(D.x.^2+D.y.^2+D.z.^2);
in.observe = @(a,r) a.*conj(a);
in.olabels = '|a(x)|^2';
in.file = 'Gaussian.mat';
in.images = 4;
e = xsim(in);
e = xgraph(in);
end
```

- Add an additive complex noise of $0.01(dw_1 + idw_2)$ to the Gaussian differential equation, then replot with an average over 10 samples.

Note that for this, you'll need to define $in.da = @(a, z, r) 0.01 * (z(1, :) + i * z(2, :))$

4.4 Planar noise

The next example is growth of thermal noise of a two-component complex field in a plane, given by the equation

$$\frac{d\mathbf{a}}{dt} = \frac{i}{2}\nabla^2\mathbf{a} + \boldsymbol{\zeta}(t, \mathbf{x})$$

where $\boldsymbol{\zeta}$ is a delta-correlated complex noise vector field:

$$\zeta_j(t, \mathbf{x}) = [\zeta_j^{re}(t, \mathbf{x}) + i\zeta_j^{im}(t, \mathbf{x})] / \sqrt{2},$$

with the initial condition that the initial noise is delta-correlated in position space

$$a(0, \mathbf{x}) = \boldsymbol{\zeta}^{(in)}(\mathbf{x})$$

where:

$$\boldsymbol{\zeta}^{(in)}(\mathbf{x}) = [\boldsymbol{\zeta}^{re(in)}(\mathbf{x}) + i\boldsymbol{\zeta}^{im(in)}(\mathbf{x})] / \sqrt{2}$$

This has an exact solution for the noise intensity in either ordinary space or momentum space:

$$\begin{aligned} \langle |a_j(t, \mathbf{x})|^2 \rangle &= (1+t)/\Delta V \\ \langle |\tilde{a}_j(t, \mathbf{k})|^2 \rangle &= (1+t)/\Delta V_k \\ \langle \tilde{a}_1(t, \mathbf{k}) \tilde{a}_2^*(t, \mathbf{k}) \rangle &= 0 \end{aligned}$$

Here, the noise is delta-correlated, and ΔV , ΔV_k are the cartesian space and momentum space lattice cell volumes respectively. Suppose that $N = N_x N_y$ is the total number of spatial points, and $V = R_x R_y$, where there are $N_{x(y)}$ points in the $x(y)$ -direction, with a total range of $R_{x(y)}$. Then, $\Delta x = R_x/N_x$, $\Delta k_x = 2\pi/R_x$, so that:

$$\begin{aligned} \Delta V &= \Delta x \Delta y = \frac{V}{N} \\ \Delta V_k &= \Delta k_x \Delta k_y = \frac{(2\pi)^2}{V}. \end{aligned}$$

In the simulations, two planar noise fields are propagated, one using noise generated in position space, the other with noise generated in momentum space. This allows use of finite correlation lengths when needed.

Exercises

- Solve the planar noise growth equation

```

function [e] = Planar()
in.name = 'Planar noise growth';
in.dimension = 3;
in.fields = 2;
in.ranges = [1,5,5];
in.steps = 2;
in.noises = [2,2];
in.ensembles = [10,4,4];
in.initial = @Initial;
in.da = @Da;
in.linear = @Linear;
in.observe = @(a,r) a(1,:).*conj(a(1,:));
in.olabels= '<|a_1(x)|^2>';
in.compare = @(t,in) [1+t]/in.dV;
in.images = 4;
e = xspde(in);
end

function a0 = Initial(v,r)
a0(1,:) = (v(1:)+1i*v(2:))/sqrt(2);
a0(2,:) = (v(3:)+1i*v(4:))/sqrt(2);
end

function da = Da(a,z,r)
da(1,:) = (z(1:)+1i*z(2:))/sqrt(2);
da(2,:) = (z(3:)+1i*z(4:))/sqrt(2);
end

function L = Linear(D,r)
lap = D.x.^2+D.y.^2;
L(1,:) = 1i*0.5*lap(:);
L(2,:) = 1i*0.5*lap(:);
end

```

- Add a decay rate of $-a$ to the differential equation, then replot
- Add growth and nonlinear saturation terms

4.5 Characteristic equation

The next example is the characteristic equation for a traveling wave at constant velocity. It is included to illustrate what happens at periodic boundaries, when Fourier-transform methods are used for propagation. There are a number of methods known to prevent this effect, including addition of absorbers - often called apodization - at the boundaries. The equation is:

$$\frac{da}{dt} + \frac{da}{dx} = 0$$

Together with the initial condition that $a(0, x) = \text{sech}(2x + 5)$, this has an exact solution that propagates at a constant velocity:

$$a(t, x) = \text{sech}(2(x - t) + 5)$$

The time evolution at $x = 0$ is simply:

$$a(t, 0) = \text{sech}(2(t - 5/2))$$

Exercises

- Solve the characteristic equation given above, noting the effects of periodic boundaries.

```
function [e] = Characteristic()
in.name = 'Characteristic'
in.dimension = 2;
in.initial = @(v,r) sech(2.*(r.x+2.5));
in.da = @(a,z,r) 0*a;
in.linear = @(D,r) -D.x;
in.olabels = 'a_1(x)';
in.compare = @(t,in) sech(2.*(t-2.5));
xspde(in);
end
```

- Recalculate with the opposite velocity, and a new exact solution.

4.6 Challenge problem #2

Counter-intuitively, a random potential prevents normal wave-packet spreading in quantum-mechanics. This is Anderson localization: a famous property of quantum mechanics in a random potential. A typical experimental method is to confine an ultra-cold Bose-Einstein condensate (BEC) in a trap, then release the BEC in a random external potential produced by a laser. The expansion rate of the BEC is reduced by the Anderson

localization due to the random potential. Physically, the observable quantity is the particle density $n = |\psi|^2$.

This can be treated either using a Schroedinger equation with a random potential, at low density, or using the Gross-Pitaevskii (GP) equation to include atom-atom interactions at the mean field level. In this example of a problem where strong localization occurs, the general equations are:

$$\frac{\partial \psi}{\partial t} = \frac{1}{i\hbar} \left[-\frac{\hbar^2}{2m} \nabla^2 + V(\mathbf{r}) + g|\psi|^2 \right] \psi$$

In calculations, it is best to use a dimensionless form by rescaling coordinates and fields. A simple way to simulate this with xSPDE is to treat ψ as a scaled field $a(1)$, and to assume the random potential field $V(\mathbf{r})$ as caused by interactions with second random field $|a(2)|^2$. This has the advantage that it is similar to the actual experiment, and allows one to treat time-dependent potentials as well, if desired.

With the rescaling, this simplifies to:

$$\frac{\partial a_1}{\partial \tau} = i \left[\frac{\partial^2}{\partial \zeta^2} - |a_2|^2 - |a_1|^2 \right] a_1$$

A convenient initial condition is to use:

$$\begin{aligned} a_1 &= a_0 \exp(-\zeta^2) \\ \langle a_2(\zeta) a_2(\zeta') \rangle &= v \delta(\zeta - \zeta') \end{aligned}$$

Exercise

- **Solve Schroedinger's equation without a random potential, to observe expansion.**
- **Include a random potential v , to observe localization.**
- **Experiment with nonlinear terms and higher dimensions.**

Note that the GP equation is a mean field approximation; this is still not a full solution of the many-body problem! Also, the experiments are somewhat more complicated than this, and actually observe the momentum distribution.

5 Logic and data

The simulation program logic is straightforward. It is a very compact function called **xspde**. This calls **xsim**, for the simulation, then **xgraph** for the graphics. Most of the work is done by other specialized functions. Input parameters come from an **input** array, output is saved either in a **data** array, or else in a specified file. During the simulation, global averages and error-bars are calculated for both time-step and sampling errors. When completed, timing and error results are printed.

5.1 Simulation function, **xsim**

5.1.1 Input and data arrays

To explain xSPDE in full detail,

- Simulation inputs are stored in the **input** cell array.
- This describes a *sequence* of simulations, so that **input**={**in1,in2,...**}.
- Each structure **in** describes a simulation, whose output is the input of the next.
- The main simulation function is called using **xsim(input)**.
- Averages are recorded sequentially in the **data** cell array.
- Raw trajectory data is stored in the **raw** cell array if required.

The sequence **input** has a number of individual simulation objects **in**. Each includes parameters that specify the simulation, with functions that give the equations and observables. If there is only one simulation, just one individual specification **in** is needed. In addition, xSPDE generates graphs with its own graphics program.

5.1.2 User functions

initial is used to initialize each integration in time. This is a user-defined function, which can involve random numbers if there is an initial probability distribution. This creates a stochastic field on the lattice, called **a**. The default is **xinitialise**, which sets fields to zero.

transfer is used to initialize sequential simulations, where previous field values are used to initialize the next integration phase.

observe is the observation function whose output is averaged over the ensembles, called from **xpath**. The default, **xobserve**, returns the real amplitudes.

linear is the linear response, including transverse derivatives in space. The default, **xlinear**, sets this to zero.

da is called by **step** to calculate derivatives at every step in the process, including the stochastic terms.

step is the algorithm that computes each step in time. This also generates the random numbers at each time-step. The default is robust and widely applicable: **xMP** uses a midpoint or central-difference technique in the interaction picture. Other options are **xEuler**, **xRK2**, **xRK4**, for Euler and Runge-Kutta integration methods. This can be user-modified by initializing the handle `in.step`, and adding a new integration function.

5.2 Graphics function, **xgraph**

The main functions involved for graphics are:

xgraph is called by **xSPDE** when the ensemble loops finished. The results are graphed and output if required.

xgrpreferences is called by **xgraph** to set the graphics defaults that are not already entered.

Comparison results are calculated if available from the user-specified **in.compare**, an error summary is printed, and the results plotted using the **xgraph** routine, which is a function that graphs the observables. It is prewritten to cover a range of useful graphs, but can be modified to suit the user. The code is intended to cascade down from higher to lower dimension, generating different types of user-defined graphs. Each type of graph is generated once for each specified graphics function.

Graphics results depend on the dimension:

dimension=4 For the highest space dimension, only a slice through $z = 0$ is available. This is then graphed as if it was in three dimensions.

dimension=3 For two dimensions, distinct graphic images of observable *vs* x, y are plotted at *images* time slices. Otherwise, only a slice through $y = 0$ is available. This is then treated as if it was in two dimensions.

dimension=2 For two dimensions, one three-dimensional image of observable *vs* x, t is plotted. Otherwise, only a slice through $x = 0$ is available. This is otherwise treated as in one dimension.

dimension=1 For one dimensions, one image of observable *vs* t is plotted, with data at each lattice point in time. Exact results, error bars and sampling error bounds are included if available.

In addition to time-dependent graphs, the **xgraph** function can generate *images* (3D) and *transverse* (2D) plots at specified points in time, up to a maximum given by the number of time points specified. The number of these can be individually specified for each graphics output. The images available are specified in *imagetype*: 3D perspective plots, grey-scale colour plots and contour plots.

5.3 Errors

5.3.1 Step-size errors

Errors due to a finite step-size are estimated by comparing two simulations with different step-sizes and the same random sequence, to make sure the final results are accurate. The final 2D output graphs will have error-bars if *in.errorchecks* = 2 was specified, which is also the default parameter setting. Error-bars below a minimum relative size compared to the vertical range of the plot, specified by the graphics variable *minbar*, are not plotted.

There is a clear strategy if the errors are too large. Either increase the *points*, which gives more plotted points and lower errors, or increase the *steps*, which reduces the step size without changing the graphical resolution. The default algorithm and extrapolation order can also be changed, please read the full xSPDE manual when doing this. Error bars on the graphs can be removed by setting *errorchecks* = 1 or increasing *minbar* in final graphs.

5.3.2 Sampling errors

Sampling error estimation in xSPDE uses sub-ensemble averaging. This generally leads to more reliable sampling error estimates, and makes efficient use of the vector instruction sets that are used by Matlab. Ensembles are specified in three levels. The first, *ensemble(1)*, is called the number of samples for brevity. All computed quantities returned by the **observe** functions are first averaged over the samples, which are calculated efficiently using a parallel vector of trajectories. By the central limit theorem, these low-level sample averages are distributed as a normal distribution at large sample number.

Next, the sample averages are averaged **again** over the two higher level ensembles, if specified. This time, the variance is accumulated. The variance of these distributions is used to estimate a standard deviation in the mean, since each computed quantity is now a normally distributed result. This method is applied to all the *graphs* observables. The two lines generated represent $\bar{o} \pm \sigma$, where *o* is the observe function output, and σ is the standard deviation in the mean.

The highest level ensemble, *ensemble(3)*, is used for parallel simulations. This requires the Matlab parallel toolbox. Either type of high-level ensemble, or both together, can be used to calculate sampling errors.

Note that one standard deviation is not a strong bound; errors are expected to exceed this value in 32% of observed measurements. Another point to remember is that stochas-

tic errors are often correlated, so that a group of points may all have similar errors due to statistical sampling.

If `ensembles(2) > 1` or `ensembles(3) > 1`, which allows xSPDE to calculate sampling errors, it will plot upper and lower limits of one standard deviation. If the sampling errors are too large, try increasing `ensembles(1)`, which increases the trajectories in a single thread. An alternative is to increase `ensembles(2)`. This is slower, but is only limited by the compute time, or else to increase `ensembles(3)`, which gives higher level parallelization. Each is limited in different ways; the first by memory, and the second by time, the third by the number of available cores. Sampling error control helps ensures accuracy.

Note that error bars and sampling errors are only graphed for 2D graphs of results vs time. The error-bars are not plotted when they are below a user-specified size, to improve graphics quality. Higher dimensional graphs do not include this, for visibility reasons, but they are still recorded in the data files. Errors caused by the spatial lattice are not checked automatically in the xSPDE code. They must be checked by manually, by comparing results with different transverse lattice ranges and step-size.

5.4 Object architecture and user hints

5.4.1 xSPDE data objects

The xSPDE data objects include parameters and methods, but with a very open type of object-oriented architecture to allow extensibility. All xSPDE functions are modular and easily replacable. In many cases this is as easy as just defining a new function handle to replace the default value. To define your own integration function, include in the xSPDE input the line:

```
in.step=@Mystep;
```

Next, include anywhere on your Matlab path the function definition, for example:

```
function a = Mystep(a,xi,dt,r)
% a = Mystep(a,xi,dt,r) propagates a step my way.
..
a = ...;
end
```

5.4.2 xSPDE hints

- When using xSPDE, it is a good idea to run the batch test script, `Batchtest.m`.
- `Batchtest.m` tests your parallel toolbox installation. If you have no license for this, omit the third ensemble setting.

5 *Logic and data*

- To create a project file, it is often easiest to start with an existing project function with a similar equation: see Examples folder.
- Graphics parameters can also be included in the structure **in**, to modify graphs.
- Comparison functions can be included if you want to compare with analytic results.
- A full list of functionality is listed next.

6 Table of xSPDE metadata

The input cell array, *input*, includes input data in structures *in*. This data is also passed to the *xgraph* function. The inputs are numbers, vectors, strings, functions and cell arrays. All xSPDE metadata has preferred values, so only changes from the preferences need to be input. The resulting data is stored internally as a sequence of structures in a cell array, to describe the simulation sequence.

Simulation metadata, including all preferred default values that were used in a particular simulation, is also stored for reference in any xSPDE output files. This is done in both the *.mat* and the *.h5* output files, so the entire simulation can be easily reconstructed or changed.

Inputs can be numbers, vectors, strings or cells arrays. Some conventions are used, as follows:

- All input data has default values
- Vector inputs of numbers are enclosed in square brackets, [...].
- Where multiple inputs of strings, functions or vectors are needed they should be enclosed in curly brackets, {...}, to create a cell array.
- Vector or cell array inputs with only one member don't require brackets.
- Incomplete or partial vector or cell array inputs are filled in with the last applicable default value.
- New function definitions can be handles pointing elsewhere.

6.1 Simulation data: the *in* structure

The simulation data of the *in* structure is passed to the *xsim* program, to define the simulation details. User application constants can be included, but must not be reserved names. However, names starting with '*in.c*' are always available to users of xSPDE. Note that use of globals is **not** recommended, as it is incompatible with the Matlab parallel toolbox.

6.1.1 Simulation parameters

Label	Type	Typical value	Description
<i>version</i>	string	'xSPDEv1.04'	Current version number (added automatically)
<i>name</i>	string	'Kubo'	Simulation name
<i>dimension</i>	integer	1	Space-time dimension
<i>fields</i>	integer	1	Number of stochastic fields
<i>ranges</i>	vector	10	Range of coordinates in [t,x,y,z]
<i>origin</i>	vector	0	Origin of coordinates in [t,x,y,z]
<i>points</i>	vector	51	Output lattice points in [t,x,y,z]
<i>noises</i>	vector	[1 0]	Number of noise fields in [x,k]
<i>randoms</i>	vector	[1 0]	Initial random fields in [x,k]
<i>ensembles</i>	vector	[1 1 1]	Ensembles used for averaging
<i>steps</i>	integer	1	Integration steps per output point
<i>iterations</i>	integer	4	Maximum iterations
<i>order</i>	integer	1	Extrapolation order
<i>errorchecks</i>	integer	2	Number of error-checking cycles
<i>seed</i>	integer	0	Seed for random number generator
<i>file</i>	string	'f.mat' or 'f.h5'	File-name for HDF5 or Matlab data file
<i>ipsteps</i>	integer	2	IP transforms per time-step
<i>graphs</i>	integer	1	Number of observables
<i>print</i>	integer	1	Print switch, 1 for medium output
<i>c</i>	any	-	User specified static constants or structures
<i>olabels</i>	string cells	{ 'a_1', ... }	Observable labels
<i>transforms</i>	vector cells	{ [0 0 0 0], ... }	Fourier transforms in [t,x,y,z]
<i>raw</i>	integer	0	Raw trajectory switch, 1 for output

6.1.2 User defined functions

The user-definable functions, output field dimensions and calling arguments are:

Label	Output dimension	Arguments	Purpose
<i>*da</i>	d.a	(a,z,r)	Stochastic derivative
<i>*initial</i>	d.a	(v,r)	Function to initialize fields
<i>*linear</i>	d.a	(D,r)	Linear interaction picture function
<i>rfilter</i>	d.random2	(r)	Random input filter function in k-space
<i>nfilter</i>	d.noise2	(r)	Noise filter function in k-space
<i>transfer</i>	d.a	(v,r,a0,r0)	Function to initialize fields in sequences
<i>step</i>	d.a	(a,z,dt,r)	Function to calculate one time-step
<i>grid</i>	d.r	(r)	Grid calculator for lattice
<i>prop</i>	d.a	(a,r)	Interaction picture (IP) propagation
<i>propfactor</i>	d.a	(nc,r)	IP propagator array calculation
<i>observe</i>	d.int	(a,r)	Observable function cell array
<i>compare</i>	points(1)	(t,in)	Comparison function cell array

6.2 Internal data: the *r* structure

Internally, xSPDE data is stored in a cell array, *latt*, of the structures *r* passed to functions. This includes the data given above from the *in* structures. In addition, it includes the computed parameters given below, which includes various internal array dimensions. These are internally flattened where possible to reduce dimensionality and simplify parallel operations.

Label	Type	Typical value	Description
<i>t</i>	real	5	Current value of time, <i>t</i>
<i>w</i>	real	0.6	Frequency passed to in.observe : ω
<i>x, y, z</i>	array	-	Grid of <i>x, y, z</i>
<i>kx, ky, kz</i>	array	-	Grid of $[k_x, k_y, k_z]$
<i>dx</i>	vector	0.2	Steps in $[t, x, y, z]$
<i>dk</i>	vector	0.6160	Steps in $[\omega, k_x, k_y, k_z]$
<i>dt</i>	double	0.2000	Output time-step
<i>V</i>	real	1	Spatial lattice volume
<i>K</i>	real	1	Momentum lattice volume
<i>dV</i>	real	1	Spatial cell volume
<i>dK</i>	real	1	Momentum cell volume
<i>xc</i>	vector cells	{[0,... 10]}	Space-time coordinate axes in $[t, x, y, z]$
<i>kc</i>	vector cells	{[0,..15,-15...]}	Computational axes in $[\omega, k_x, k_y, k_z]$
<i>gk</i>	vector cells	{[-15,... 15]}	Graphics axes in $[\omega, k_x, k_y, k_z]$
<i>kr</i>	vector	30	Range in $[\omega, k_x, k_y, k_z]$
<i>wtp</i>	vector	-	Frequency phase-factors
<i>s.dx</i>	double	1	Initial stochastic normalization
<i>s.dxt</i>	double	3.1623	Propagating stochastic normalization
<i>s.dk</i>	double	1	Initial k stochastic normalization
<i>s.dkt</i>	double	3.1623	Propagating k stochastic normalization
<i>n.space</i>	integer	35	Number of spatial lattice points
<i>n.lattice</i>	integer	3500	Total lattice: ensembles(1) x n.space
<i>n.ensemble</i>	integer	20	ensembles(2) x ensembles(3)
<i>n.random</i>	integer	2	Number of initial random fields
<i>n.noise</i>	integer	2	Number of noise fields
<i>d.int</i>	vector	[1 1]	Dimensions for lattice integration
<i>d.a</i>	vector	[1 1]	Dimensions for <i>a</i> field (flattened)
<i>d.r</i>	vector	[1 1]	Dimensions for coordinates (flattened)
<i>d.ft</i>	vector	[1 1 1]	Dimensions for field transforms
<i>d.k</i>	vector	[0 1 1]	Dimensions for noise transforms
<i>d.obs</i>	vector	[1 1 1 1]	Dimensions for observations
<i>d.data</i>	vector	[1 3 51 1]	Dimensions for average data (flattened)
<i>d.raw</i>	vector	[1 1 51 1]	Dimensions for raw data (flattened)

6.3 Graphics data: the extended in structure

The simulation data of the *in* structure is also passed to the *xgraph* program, along with optional graphics parameters to create an extended in structure, as follows.

Label	Type	Typical value	Description
<i>minbar</i>	cell	$\{0.01, \dots\}$	Minimum relative error-bar size
<i>font</i>	cell	$\{18, \dots\}$	Font size for graph labels
<i>xlabels</i>	cell	$\{ 't' 'x' 'y' 'z' \}$	Strings for x-axis labels
<i>klabels</i>	cell	$\{ '\omega' 'k_x' 'k_y' 'k_z' \}$	Transformed axis labels
<i>headers</i>	cell	$\{0, \dots\}$	Switch for graph headers
<i>images</i>	cell	$\{0, \dots\}$	Number of movie style images
<i>imagetype</i>	cell	$\{0, \dots\}$	Type of movie style images
<i>transverse</i>	cell	$\{0, \dots\}$	Number of transverse plots
<i>pdimension</i>	cell	$\{2, \dots\}$	Maximum plot dimensions
<i>*compare</i>	function cell	$\{ @(t, in) t^2 \}$	Comparison functions

Bibliography

- [1] C. W. Gardiner, *Handbook of Stochastic Methods: for Physics, Chemistry and the Natural Sciences* (Springer 2004).
- [2] P. D. Drummond and M. Hillery, *The Quantum Theory of Nonlinear Optics* (Cambridge University Press, Cambridge, 2014).
- [3] P. D. Drummond and I. K. Mortimer, *Computer simulations of multiplicative stochastic differential equations*. J. Comp. Phys. **93**, 144-170 (1991).
- [4] P.E. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations* (Springer, 1995).
- [5] M. J. Werner and P. D. Drummond, *Robust algorithms for solving stochastic partial differential equations*. J. Comp. Phys. **132**, 312-326 (1997).
- [6] B. M. Caradoc-Davies, *Vortex dynamics in Bose-Einstein condensates* (Doctoral dissertation, PhD thesis, University of Otago (NZ), 2000).
- [7] G. R. Collecutt, P. D. Drummond, *Xmds: eXtensible multi-dimensional simulator*. Comput. Phys. Commun. **142**, 219-223 (2001).
- [8] Graham R. Dennis, Joseph J. Hope, Mattias T. Johnsson, *XMDS2: Fast, scalable simulation of coupled stochastic partial differential equations*, Computer Physics Communications **184**, 201–208 (2013).
- [9] <http://sourceforge.net/projects/xmds/>