Journal of Computational and Applied Mathematics 255 (2014) 142–149

Contents lists available at ScienceDirect

Journal of Computational and Applied Mathematics

journal homepage: www.elsevier.com/locate/cam

A practical formula for computing optimal parameters in the HSS iteration methods

Yu-Mei Huang *

School of Mathematics and Statistics, Lanzhou University, Lanzhou 730000, Gansu, PR China

ARTICLE INFO

Article history:
Received 31 July 2011
Received in revised form 2 January 2013

Keywords:
The HSS iteration methods
Optimal parameter
Cubic equation
Matrix trace

ABSTRACT

In the HSS iteration methods proposed by Bai, Golub and Ng [Z.-Z. Bai, G.H. Golub, M.K. Ng, Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems, SIAM. J. Matrix Anal. Appl. 24 (2003) 603–626], the determination of the optimal parameter is a tough task when solving a non-Hermitian positive definite linear system. In this paper, a new and simple strategy for obtaining the optimal parameter is proposed, which computes the optimal parameter by solving a cubic polynomial equation. The coefficients of this polynomial are determined by several traces of some matrices related to the symmetric and skew-symmetric parts of the coefficient matrix of the real linear system. Numerical experiments show that our new strategy is very effective for approximating the optimal parameter in the HSS iteration methods as it leads to fast convergence of the method.

1. Introduction

The system of linear equations

$$Ax = b, \quad A \in \mathbb{C}^{n\times n} \text{ nonsingular, and } x, b \in \mathbb{C}^n, \tag{1}$$

often arises in scientific computing. When A is large sparse and non-Hermitian positive definite, Bai, Golub and Ng have proposed the Hermitian and skew-Hermitian splitting (HSS) iteration methods to solve the linear system (1), see [1]. Note that the matrix A possesses the Hermitian/skew-Hermitian splitting [2,3]

$$A = H + S,$$

where

$$H = \frac{1}{2}(A + A^*) \quad \text{and} \quad S = \frac{1}{2}(A - A^*),$$

with A^* being the conjugate transpose of the matrix A. The HSS iteration methods in [1] are designed as follows: Given an initial guess $x^{(0)} \in \mathbb{C}^n$, for $k = 0, 1, 2, \ldots$, compute the iteration $x^{(k+1)}$ according to the following formula until the prescribed stopping criterion is satisfied:

$$\begin{cases} (\alpha I + H)x^{(k+\frac{1}{2})} = (\alpha I - S)x^{(k)} + b, \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{(k+\frac{1}{2})} + b, \end{cases} \tag{2}$$

where α is a given positive constant, I is the identity matrix.

* Tel.: +86 09318912483.
E-mail address: huangym@lzu.edu.cn.

0377-0427/$ – see front matter
http://dx.doi.org/10.1016/j.cam.2013.01.023

The HSS iteration scheme (2) can be rewritten as the matrix–vector form

$$x^{(k+1)} = \mathcal{M}(\alpha)x^{(k)} + \mathcal{G}(\alpha)b, \quad k = 0, 1, 2, \ldots, \tag{3}$$

where

$$\mathcal{M}(\alpha) = (\alpha I + S)^{-1}(\alpha I - H)(\alpha I + H)^{-1}(\alpha I - S) \tag{4}$$

is the iteration matrix and

$$\mathcal{G}(\alpha) = 2\alpha(\alpha I + S)^{-1}(\alpha I + H)^{-1}.$$

Note that the iteration (3) may be considered as a splitting iteration induced from the splitting

$$A = M(\alpha) - N(\alpha) \tag{5}$$

of the matrix A, where

$$\begin{cases} M(\alpha) = \dfrac{1}{2\alpha}(\alpha I + H)(\alpha I + S), \\ N(\alpha) = \dfrac{1}{2\alpha}(\alpha I - H)(\alpha I - S). \end{cases} \tag{6}$$

The following theorem established in [1] describes the convergence property of the HSS iteration methods.

Theorem 1. *Let $A \in \mathbb{C}^{n\times n}$ be a non-Hermitian positive definite matrix, $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ be its Hermitian and skew-Hermitian parts, respectively, and α be a positive constant. Then the spectral radius $\rho(\mathcal{M}(\alpha))$ of the iteration matrix $\mathcal{M}(\alpha)$ of the HSS iteration (see (4)) is bounded by*

$$\sigma(\alpha) = \max_{\lambda_j \in \lambda(H)} \frac{|\alpha - \lambda_j|}{|\alpha + \lambda_j|}, \tag{7}$$

where $\lambda(\cdot)$ represents the spectrum of the corresponding matrix. Consequently, we have

$$\rho(\mathcal{M}(\alpha)) \le \sigma(\alpha) < 1 \quad \text{for all } \alpha > 0,$$

i.e., the HSS iteration converges to the exact solution of the system of linear equations (1).

Further applications and generalizations of the HSS iteration methods can be found in [4–13]. From the above discussions, we noticed that the parameter α plays an important role in the HSS iteration methods. Since the convergence rate of the HSS iteration methods depends on the value of α, an accurate approximation to the optimal value of α can significantly speed up the convergence of the HSS iteration methods. The optimal parameter $\alpha_* = \operatorname{argmin}_\alpha \rho(\mathcal{M}(\alpha))$ is not easily obtainable in practice due to the complicated computation and the high computing cost. Therefore, a good strategy for approximating the optimal parameter α is an important task when the HSS iteration methods are applied to solve the linear system (1). Many works have been devoted to the selection of the optimal parameter α in the HSS iteration methods. For example, in [14], Bai computed the quasi-optimal parameters for solving the large sparse saddle-point problems. In [1,15], the authors suggested an estimation $\bar{\alpha} = \sqrt{\gamma_{\min}\gamma_{\max}}$ for α_* by minimizing the upper bound $\sigma(\alpha)$ in (7), where $\gamma_{\min}$ and $\gamma_{\max}$ are the lower and the upper bounds of the eigenvalues of the matrix H. They specially proposed the optimal parameter estimation methods for two-by-two block real coefficient matrices with specific structures. In the estimation methods proposed in [1,15], the extremal eigenvalues and determinants of some matrices are required in order to compute the optimal parameter and which may greatly decrease the computing efficiency of the HSS iteration methods. In addition, the computation of these eigenvalues and determinants is usually a difficult task to complete. In this paper, we concentrate on the determination of the optimal parameter problem and propose a new strategy for approximating the optimal parameter in the HSS iteration methods, which is simple, practical and less costly.

The outline of the paper is as follows. In Section 2, we describe the strategy for estimating the optimal parameter in detail. In Section 3, we use some numerical experiments to show the effectiveness of the strategy. Finally, in Section 4, we end the paper with some conclusions.

2. Estimation strategy for the optimal parameter

For simplicity, we denote by

$$\begin{cases} \tilde{M}(\alpha) = (\alpha I + H)(\alpha I + S), \\ \tilde{N}(\alpha) = (\alpha I - H)(\alpha I - S). \end{cases}$$

Then the splitting (5) can be rewritten as

$$2\alpha A = \tilde{M}(\alpha) - \tilde{N}(\alpha).$$

Hence, it holds that

$$2\alpha\tilde{M}(\alpha)^{-1}A = I - \tilde{M}(\alpha)^{-1}\tilde{N}(\alpha).$$

If α is such a value that $\tilde{M}(\alpha)$ is close to $2\alpha A$ as much as possible, then $\tilde{M}(\alpha)^{-1}\tilde{N}(\alpha)$ should be approaching to zero. This could be possible when $\tilde{N}(\alpha) \approx 0$. It then follows that the HSS iteration methods will converge fast and the HSS-preconditioned matrix will have clustered eigenvalue distribution when α minimizes the function:

$$\Phi(\alpha) := \|\tilde{N}(\alpha)\|_F^2.$$

By direct computations we have

$$\begin{aligned}\Phi(\alpha) &= \|\tilde{N}(\alpha)\|_F^2 \\ &= \|(\alpha I - H)(\alpha I - S)\|_F^2 \\ &= \mathrm{tr}\{[(\alpha I - H)(\alpha I - S)][(\alpha I - H)(\alpha I - S)]^*\} \\ &= n\alpha^4 - 2\mathrm{tr}(H)\alpha^3 + [\mathrm{tr}(H^2) - \mathrm{tr}(S^2)]\alpha^2 + 2\mathrm{tr}(HS^2)\alpha - \mathrm{tr}(H^2S^2) \\ &= n\alpha^4 + \eta\alpha^3 + \omega\alpha^2 + \gamma\alpha + \delta,\end{aligned}$$

where $\mathrm{tr}(\cdot)$ denotes the trace of a matrix, and

$$\eta = -2\mathrm{tr}(H), \qquad \omega = \mathrm{tr}(H^2) - \mathrm{tr}(S^2), \qquad \gamma = 2\mathrm{tr}(HS^2), \qquad \delta = -\mathrm{tr}(H^2S^2).$$

Moreover, we can obtain

$$\varphi(\alpha) := \frac{\mathrm{d}\Phi(\alpha)}{\mathrm{d}\alpha} = 4n\alpha^3 + 3\eta\alpha^2 + 2\omega\alpha + \gamma. \tag{8}$$

The stationary points of $\Phi(\alpha)$ are the roots of $\varphi(\alpha) = 0$. Note that the cubic polynomial equation $\varphi(\alpha) = 0$ has at least one real root. In addition, the matrix $A \in \mathbb{C}^{n\times n}$ is non-Hermitian positive definite, so the coefficients of $\varphi(\alpha)$ satisfy:

- $\eta = -2\mathrm{tr}(H) < 0$, and
- $\omega = \mathrm{tr}(H^2) - \mathrm{tr}(S^2) > 0$.

Lemma 1. *Suppose $U \in \mathbb{C}^{n\times n}$ is a Hermitian positive definite matrix, $V \in \mathbb{C}^{n\times n}$ is a skew-Hermitian matrix. Then $\mathrm{tr}(UV^2) \le 0$. Particularly, let $A \in \mathbb{C}^{n\times n}$ in (1) be a positive definite matrix, $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ be its Hermitian and skew-Hermitian parts. Then $\mathrm{tr}(HS^2) \le 0$.*

Proof. Since U is Hermitian positive definite, it can be rewritten as $U = U^{\frac{1}{2}}U^{\frac{1}{2}}$. Therefore,

$$\begin{aligned}\mathrm{tr}(UV^2) &= \sum_{j=1}^{n}\lambda_j(UV^2) = \sum_{j=1}^{n}\lambda_j(U^{\frac{1}{2}}V^2U^{\frac{1}{2}}) \\ &= -\sum_{j=1}^{n}\lambda_j(U^{\frac{1}{2}}V^*VU^{\frac{1}{2}}) = -\sum_{j=1}^{n}\lambda_j((VU^{\frac{1}{2}})^*(VU^{\frac{1}{2}})) \\ &\le 0,\end{aligned}$$

where $\lambda_j(\cdot)$ represents the jth eigenvalue of the corresponding matrix. Since H and S are Hermitian and skew-Hermitian parts of the non-Hermitian positive definite matrix A, H is Hermitian positive definite and S is skew-Hermitian, it easily follows that $\mathrm{tr}(HS^2) \le 0$. □

Based on Lemma 1, we have $\mathrm{tr}(HS^2) \le 0$. In the case of $\mathrm{tr}(HS^2) < 0$, we will have the conclusion that $\varphi(\alpha) = 0$ has at least one positive solution. Otherwise, suppose the real solutions of $\varphi(\alpha) = 0$ are negative or zero, we denote them by α'. Then according to the properties of the coefficients of the equation $\varphi(\alpha) = 0$ listed above, we easily get $\varphi(\alpha') < 0$, which contradicts the fact that α' is a root of the equation $\varphi(\alpha) = 0$. For the case $\mathrm{tr}(HS^2) = 0$, 0 is a root of the cubic equation $\varphi(\alpha) = 0$. And the other two roots are the solutions of the quadratic equation $4n\alpha^2 + 3\eta\alpha + 2\omega = 0$, whose discriminant is $\triangle = 9\mathrm{tr}^2(H) - 8n(\mathrm{tr}(H^2) - \mathrm{tr}(S^2))$. If $\triangle \ge 0$, there exists at least one positive solution for $\varphi(\alpha) = 0$.

The above analysis can be summarized as the following theorem.

Theorem 2. *Let $A \in \mathbb{C}^{n\times n}$ in (1) be a positive definite matrix, $H = \frac{1}{2}(A + A^*)$ and $S = \frac{1}{2}(A - A^*)$ be its Hermitian and skew-Hermitian parts. If $\mathrm{tr}(HS^2) < 0$ or $\triangle = 9\mathrm{tr}^2(H) - 8n(\mathrm{tr}(H^2) - \mathrm{tr}(S^2)) \ge 0$, then the equation $\varphi(\alpha) = 0$ has at least one positive solution, where $\varphi(\alpha)$ is defined in (8).*

For example, when $H = I$, we specially have $\eta = -2n$, $\omega = n - \mathrm{tr}(S^2)$, $\gamma = 2\mathrm{tr}(S^2)$, $\delta = -\mathrm{tr}(S^2)$. It follows that $\alpha = 1$ is a solution of $\varphi(\alpha) = 0$, and the other two solutions are those of the quadratic equation $2n\alpha^2 - n\alpha - \mathrm{tr}(S^2) = 0$. When $n < -8\mathrm{tr}(S^2)$, $\alpha = 1$ is the only positive solution, therefore, the optimal parameter for the case $H = I$ is 1, which is consistent with the conclusions in [15,16]. For general non-Hermitian positive definite matrices, since the HSS iteration parameter must be positive, those solutions of $\varphi(\alpha) = 0$ that are complex or negative will not be considered. If more than one solutions are positive numbers, then by simple calculation, the global minimizer of $\Phi(\alpha)$ is the one we want.

Table 1
Example 1, α versus $\rho(\mathcal{M}(\alpha))$ when $m = 32$.

β	100	500	1000
α_{opt}	5.1536	10.2948	15.0075
$\rho(\mathcal{M}(\alpha_{opt}))$	0.4771	0.6374	0.7179
α_{est}	3.2621	3.9358	3.9830
$\rho(\mathcal{M}(\alpha_{est}))$	0.4790	0.7345	0.8191

Table 2
Example 1, IT and CPU when $m = 32$.

	β	100	500	1000
opt	IT	45	55	72
	CPU	0.9126	1.162	1.356
est	IT	35	49	66
	CPU	0.7516	0.9172	1.0260

The determination of the parameter α_* is a tough task in the HSS iteration methods. The above proposed strategy is simple to approximate the optimal parameter when the HSS iteration methods are applied to solve a system of linear equations with non-Hermitian positive definite coefficient matrix, and in the proposed strategy, only several traces of some matrices are required. So when the proposed strategy is utilized to estimate the optimal parameter, less computation is required in the estimation.

3. Numerical experiments

In this section, we give some examples to illustrate the effectiveness of the proposed strategy to estimate the HSS iteration parameters. The computing results are compared with the estimated optimal HSS parameters obtained by using the method proposed in [15]. For simplicity, we call the estimated optimal HSS parameter method in [15] as "*opt*". We use "*est*" to represent our proposed strategy in the subsequent discussions.

Remark 1. In the "*opt*" method, in order to get the estimated optimal parameter, the minimum and the maximum eigenvalues of the symmetric matrix H and the norm of the skew-symmetric matrix S must be known, but in many applications, these numbers are difficult to be obtained. However, the proposed estimation of the HSS parameter is much simpler in its calculation, since the estimated parameter is just the positive solution of a cubic equation as we have discussed above.

In our experiments, all runs are initiated from the initial zero vector $x^{(0)}$ and terminated if the current iteration satisfies

$$r = \frac{\|b - Ax^{(k)}\|_2}{\|b - Ax^{(0)}\|_2} \leq 10^{-6}.$$

The numerical experiments are run in MATLAB(version 7.8) with a machine precision 10^{-16}. The machine used is a Intel(R)2.40 GHz personal computer with 2GB memory. In all our examples, the right-hand side vector b is chosen such that the exact solution of the system of linear equations is $(1, 1, \ldots, 1)^T \in \mathbb{R}^n$. In Tables 1, 3, 5 and 7, we list the estimated values of the optimal HSS parameter and the corresponding spectral radius of the HSS iteration matrix, where α_{opt} denotes the estimated optimal parameter α determined by Estimation 5.1 in [15], $\rho(\mathcal{M}(\alpha_{opt}))$ is the corresponding spectral radius of the HSS iteration matrix. α_{est} is the HSS parameter estimated by using the proposed strategy and $\rho(\mathcal{M}(\alpha_{est}))$ is the corresponding spectral radius of the HSS iteration matrix. In Tables 2, 4, 6 and 8, the number of iterations "IT" and the CPU times "CPU" of the HSS iterations are shown when the "*opt*" parameter and the proposed parameter are used in the HSS iteration methods.

Example 1. We consider the two-dimensional convection–diffusion equation

$$-(u_{xx} + u_{yy}) + \beta(u_x + u_y) = g(x, y)$$

defined on the unit square $[0, 1] \times [0, 1]$ with Dirichlet-type boundary conditions. Here the coefficient β is a constant. When the five-point centered finite difference discretization is applied, a system with the coefficient matrix

$$A = T \otimes I + I \otimes T$$

can be obtained, where the equidistant stepsize $h = \frac{1}{m+1}$ is used in the discretization on both directions and the natural lexicographic ordering is employed to the unknowns. In addition, $\otimes$ denotes the Kronecker product, T is a tridiagonal matrix given by

$$T = \text{tridiag}(-1 - R_e, 2, -1 + R_e),$$

Table 3
Example 2, α versus $\rho(\mathcal{M}(\alpha))$.

m	8	16	32
α_{opt}	118.9241	422.8556	1708.1598
$\rho(\mathcal{M}(\alpha_{opt}))$	0.9796	0.9947	0.9987
α_{est}	11.8624	20.5710	34.9678
$\rho(\mathcal{M}(\alpha_{est}cc))$	0.8604	0.9297	0.9709

Table 4
Example 2, IT and CPU.

	m	8	16	32
opt	IT	409	1217	3310
	CPU	0.0455	1.4677	30.8464
est	IT	90	164	408
	CPU	0.0141	0.2224	3.8330

Table 5
Example 3 for $K_{i,j}^{(1)}$, α versus $\rho(\mathcal{M}(\alpha))$, when $\mu = 0.1$ and $\mu = 0.01$.

μ	m	8	16	32
0.1	α_{opt}	0.7240	0.7060	0.7072
	$\rho(\mathcal{M}(\alpha_{opt}))$	0.7809	0.8361	0.8479
	α_{est}	0.2975	0.2933	0.2959
	$\rho(\mathcal{M}(\alpha_{est}))$	0.5860	0.6798	0.6816
0.01	α_{opt}	0.7159	0.7060	0.7072
	$\rho(\mathcal{M}(\alpha_{opt}))$	0.9573	0.9699	0.9717
	α_{est}	0.2513	0.2477	0.2506
	$\rho(\mathcal{M}(\alpha_{est}))$	0.9229	0.9223	0.9232

Table 6
Example 3 for $K_{i,j}^{(1)}$, IT and CPU, when $\mu = 0.1$ and $\mu = 0.01$.

μ		m	8	16	32
0.1	*opt*	IT	49	66	68
		CPU	0.0288	0.5414	22.5268
	est	IT	23	30	28
		CPU	0.02155	0.2593	10.6380
0.01	*opt*	IT	273	295	353
		CPU	0.0991	1.2231	30.9297
	est	IT	145	131	137
		CPU	0.0381	0.5653	13.6267

Table 7
Example 3 for $K_{i,j}^{(2)}$, α versus $\rho(\mathcal{M}(\alpha))$, when $\mu = 0.01$ and $\mu = 0.001$.

μ	m	8	16	32
0.01	α_{opt}	0.7326	0.7326	0.7326
	$\rho(\mathcal{M}(\alpha_{opt}))$	0.9708	0.9709	0.9711
	α_{est}	0.0091	0.0091	0.0090
	$\rho(\mathcal{M}(\alpha_{est}))$	0.9640	0.9638	0.9677
0.001	α_{opt}	0.7324	0.7325	0.7325
	$\rho(\mathcal{M}(\alpha_{opt}))$	0.9950	0.9950	0.9952
	α_{est}	0.0089	0.0090	0.0088
	$\rho(\mathcal{M}(\alpha_{est}))$	0.9782	0.9781	0.9794

and

$$R_e = \frac{\beta h}{2}$$

is the mesh Reynolds number.

We show the computing results with different β in Tables 1 and 2 when the discretization grid is 32.

Table 8
Example 3 for $K_{i,j}^{(2)}$, IT and CPU, when $\mu = 0.01$ and $\mu = 0.001$.

μ		m	8	16	32
0.01	*opt*	IT	418	413	421
		CPU	0.0834	1.3503	16.7188
	est	IT	363	351	373
		CPU	0.0532	1.1112	14.9680
0.001	*opt*	IT	2256	2241	2370
		CPU	0.4664	8.1641	145.0514
	est	IT	628	594	642
		CPU	0.1266	2.4332	68.4683

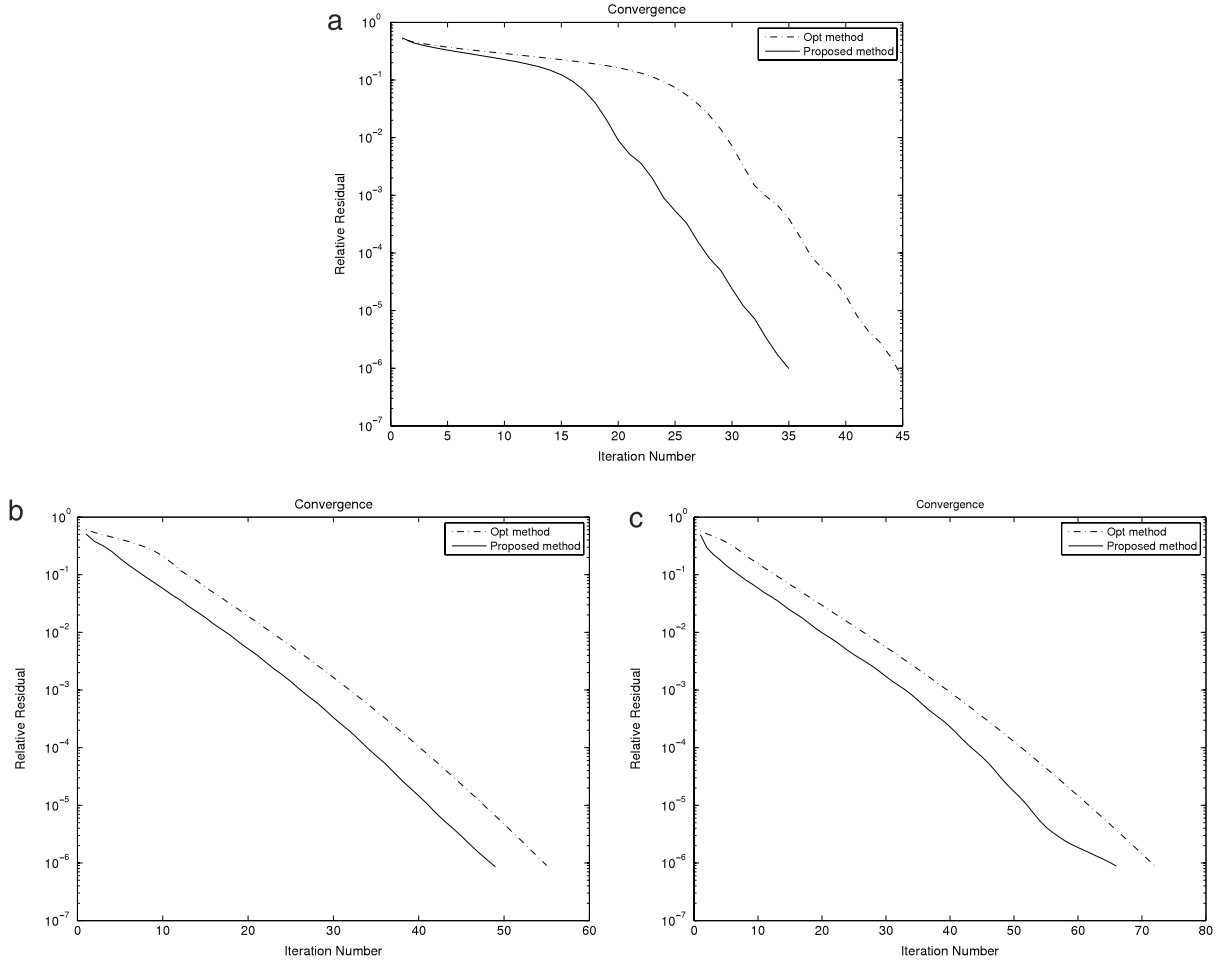

Fig. 1. The convergence history for Example 1 corresponding to Table 2, (a) $\beta = 100$, (b) $\beta = 500$, (c) $\beta = 1000$.

From Tables 1 and 2, we can see for different β, the proposed strategy requires less CPU times and smaller iteration numbers than the "*opt*" method does to achieve the same computing accuracy. However, the calculation of the HSS parameter is much simpler in the proposed strategy than the "*opt*" method. By using the computing results of the Example 1 for different β, Fig. 1 clearly depicts the superiority of the proposed strategy in the convergence of the HSS iterations.

Example 2. We solve the linear system $Ax = b$, where A is a 2-by-2 block matrix of the form

$$A = \begin{bmatrix} B & E \\ -E' & C \end{bmatrix}, \tag{9}$$

B and C are m^2-by-m^2 and m-by-m real symmetric positive definite matrices, respectively, and E is an m^2-by-m real rectangular matrix.

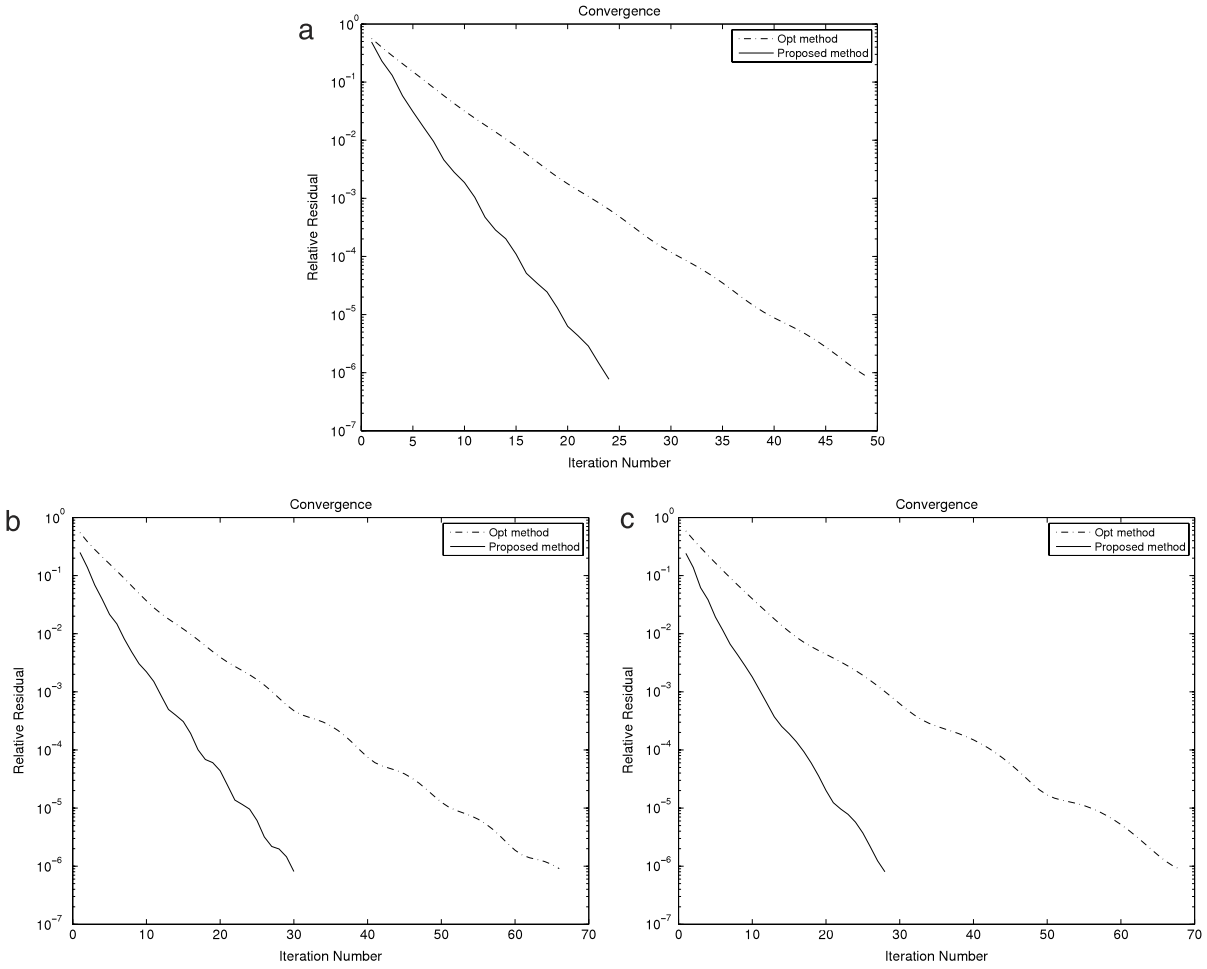

Fig. 2. The convergence history for Example 3 corresponding to Table 6 when $\mu = 0.1$, (a) $m = 8$, (b) $m = 16$, (c) $m = 32$.

In our experiments, the matrices B and C are randomly generated by using the MATLAB command "randn". These two matrices should not be thought of as a sequence of increasingly finer discretizations of a continuous problem. In our experiments, for any random matrices B, C and E with the same value m, the iteration counts were found to be very stable. So we only report the computing results in Tables 3 and 4 for one example with the fixed matrices B, C and E. These tables show that the proposed parameters are much more effective than those obtained by the "*opt*" method in terms of the iteration numbers and CPU times.

Example 3. We solve the example adopted from [17], the coefficient matrices of the linear system is of the form

$$A = \begin{bmatrix} W & K \\ -K' & \mu I \end{bmatrix}, \tag{10}$$

here $W \in \mathbb{R}^{n\times n}$ is a positive diagonal matrix, $K \in \mathbb{R}^{n\times n}$ is a Toeplitz matrix, and the entries of K are given by

$$K^{(1)}_{i,j} = \frac{1}{\sqrt{|i-j|+1}}$$

or

$$K^{(2)}_{i,j} = \frac{1}{\sqrt{2\pi}}\sigma e^{-|i-j|^2/(2\sigma^2)}.$$

In $K^{(2)}_{i,j}$, choosing $\sigma = 2$ results in a highly ill-conditioned Toeplitz matrix with rapidly decaying singular values.

Tables 5–6 show the values of α and $\rho(\mathcal{M}(\alpha))$, the iteration numbers and the CPU times for $K^{(1)}_{i,j}$ when $\mu = 0.1$ and $\mu = 0.01$, Tables 7–8 are the computing results for $K^{(2)}_{i,j}$ when $\mu = 0.01$ and $\mu = 0.001$. Fig. 2 is the convergence curve

for $K_{i,j}^{(1)}$ when $\mu = 0.1$. These tables and figure further show the superiority of the proposed strategy in estimating the HSS iteration parameter.

4. Conclusion

In this paper, a new and simple strategy is proposed to determine the optimal parameter arising in the HSS iteration methods only by solving a cubic polynomial equation. Numerical experiments show that the proposed strategy is superior to the estimate given in [15] in terms of both the computation cost and iteration numbers.

Acknowledgments

The author would like to thank the anonymous referees for their valuable comments and constructive suggestions for the final version of this paper. Research was supported in part by NSFC Grant No. 11101195 and China Postdoctoral Science Foundation funded project No. 2011M501488.

References

[1] Z.-Z. Bai, G.H. Golub, M.K. Ng, Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems, SIAM J. Matrix Anal. Appl. 24 (2003) 603–626.
[2] G.H. Golub, A.J. Wathen, An iteration for indefinite systems and its application to the Navier–Stokes equations, SIAM J. Sci. Comput. 19 (1998) 530–539.
[3] C.-L. Wang, Z.-Z. Bai, Sufficient conditions for the convergent splittings of non-Hermitian positive definite matrices, Linear Algebra Appl. 330 (2001) 215–218.
[4] Z.-Z. Bai, G.H. Golub, M.K. Ng, On successive-overrelaxation acceleration of the Hermitian and skew-Hermitian splitting iterations, Numer. Linear Algebra Appl. 14 (2007) 319–335.
[5] M.K. Ng, Circulant and skew-circulant splitting methods for Toeplitz systems, J. Comput. Appl. Math. 159 (2003) 101–108.
[6] M. Benzi, G.H. Golub, A preconditioner for generalized saddle point problems, SIAM J. Matrix Anal. Appl. 26 (2004) 20–41.
[7] V. Simoncini, M. Benzi, Spectral properties of the Hermitian and skew-Hermitian splitting preconditioner for saddle point problems, SIAM J. Matrix Anal. Appl. 26 (2004) 377–389.
[8] Z.-Z. Bai, G.H. Golub, L.-Z. Lu, J.-F. Yin, Block triangular and skew-Hermitian splitting methods for positive-definite linear systems, SIAM J. Sci. Comput. 26 (2005) 844–863.
[9] J.-Y. Pan, Z.-Z. Bai, M.K. Ng, Two-step waveform relaxation methods for implicit linear initial value problems, Numer. Linear Algebra Appl. 12 (2005) 293–304.
[10] D. Bertaccini, G.H. Golub, S. Serra-Capizzano, C.T. Possio, Preconditioned HSS methods for the solution of non-Hermitian positive definite linear systems and applications to the discrete convection–diffusion equation, Numer. Math. 99 (2005) 441–484.
[11] J.-Y. Pan, M.K. Ng, Z.-Z. Bai, New preconditioners for saddle point problems, Appl. Math. Comput. 172 (2006) 762–771.
[12] M.-K. Ho, M.K. Ng, Splitting iterations for circulant-plus-diagonal systems, Numer. Linear Algebra Appl. 12 (2005) 779–792.
[13] M. Benzi, M.K. Ng, Preconditioned iterative methods for weighted Toeplitz least squares problems, SIAM J. Matrix Anal. Appl. 27 (2006) 1106–1124.
[14] Z.-Z. Bai, Optimal parameters in the HSS-like methods for saddle-point problems, Numer. Linear Algebra Appl. 16 (2009) 447–479.
[15] Z.-Z. Bai, G.H. Golub, C.-K. Li, Optimal parameter in Hermitian and skew-Hermitian splitting method for certain two-by-two block matrices, SIAM J. Sci. Comput. 28 (2006) 583–603.
[16] Z.-Z. Bai, G.H. Golub, J.-Y. Pan, Preconditioned Hermitian and skew-Hermitian splitting methods for non-Hermitian positive semidefinite linear systems, Numer. Math. 98 (2004) 1–32.
[17] L.C. Chan, M.K. Ng, N.K. Tsing, Spectral analysis for HSS preconditioners, Numer. Math. Theory Methods Appl. 15 (2006) 1–18.