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Quasi-Chebyshev accelerated iteration methods based on optimization for linear systems[☆]

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ABSTRACT

In this paper, we present a quasi-Chebyshev accelerated iteration method for solving a system of linear equations. Compared with the Chebyshev semi-iterative method, the main difference is that the parameter ω is not obtained by a Chebyshev polynomial but by optimization models. We prove that the quasi-Chebyshev accelerated iteration method is unconditionally convergent if the original iteration method is convergent, and also discuss the convergence rate. Finally, three numerical examples indicate that our method is more efficient than the Chebyshev semi-iterative method.

1. Introduction and preliminaries

For many years, deriving an efficient iteration method for solving a system of linear equations has been an important and active topic. Given a large system of linear equations

$$Ax = b, \quad A = (a_{ij}) \in \mathbb{R}^{n\times n} \text{ nonsingular}, \ b \in \mathbb{R}^n, \tag{1.1}$$

to solve the system iteratively we always split the coefficient matrix $A \in \mathbb{R}^{n\times n}$ into

$$A = M - N, \tag{1.2}$$

where M is a nonsingular matrix, and then define the corresponding iteration formula as

$$x^{(k)} = M^{-1}Nx^{(k-1)} + M^{-1}b, \quad k = 1, 2, \ldots. \tag{1.3}$$

If the iteration method is convergent, the general interest is how to find a way to accelerate its convergence rate and/or to modify the iteration method to construct an extrapolated iteration so as to improve its efficiency (see [1–9]). To our knowledge, the Chebyshev semi-iterative method is one of the best accelerated methods; it is based on an optimal polynomial (see [9]):

$$y^{(m+1)} = \omega_{m+1}(Ty^{(m)} + M^{-1}b - y^{(m-1)}) + y^{(m-1)}, \tag{1.4}$$

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where

$$\omega_{m+1} = \frac{2C_m\left(\frac{1}{\rho}\right)}{\rho C_{m+1}\left(\frac{1}{\rho}\right)},$$
$$C_m(x) = 2xC_{m-1}(x) - C_{m-2}(x), \qquad C_0(x) = 1, \qquad C_1(x) = x,$$
$$y^{(0)} \in \mathbb{R}^n, \qquad y^{(1)} = Ty^{(0)} + M^{-1}b.$$
$$T = M^{-1}N, \qquad \rho = \rho(T) \text{ (the spectral radius)}.$$

Since the spectral radius ρ is usually unknown in advance, ρ is often replaced by some lower and upper bounds (see [5]):

$$y^{(m+1)} = \omega_{m+1}(\gamma z^{(m)} + y^{(m)} - y^{(m-1)}) + y^{(m-1)}, \tag{1.5}$$

where

$$\omega_{m+1} = 2 \cdot \frac{2-\beta-\delta}{\beta-\delta} \cdot \frac{C_m(\mu)}{C_{m+1}(\mu)},$$
$$C_m(x) = 2xC_{m-1}(x) - C_{m-2}(x), \qquad C_0(x) = 1, \qquad C_1(x) = x,$$
$$Mz^{(m)} = b - Ay^{(m)}, \qquad \gamma = \frac{2}{2-\beta-\delta}, \qquad \mu = 1 + 2\frac{1-\beta}{\beta-\delta},$$
$$y^{(0)} \in \mathbb{R}^n, \qquad y^{(1)} = Ty^{(0)} + M^{-1}b,$$

where β and δ are the upper and the lower bounds for ρ, respectively.

Better upper and lower bounds have been obtained only in a few structured problems. In particular, if $\rho \approx 1$, then a better upper bound is harder to obtain. This implies that the Chebyshev semi-iterative method is not always feasible in practice. For example, the acceleration ratio of the Chebyshev iterative method is

$$r = \frac{\ln \frac{\beta}{1-\sqrt{1-\beta^2}}}{\ln \rho}, \quad \text{if } \delta = 0.$$

Set $\rho = 0.9908$. Then $\beta = 0.999$ implies that $r = 4.76$, and $\beta = 0.9999$ implies that $r = 1.53$.

In this paper, we will establish optimization models to determine an optimal ω_{m+1}, instead of ω_{m+1} generated by the Chebyshev polynomial. The new acceleration methods based on the optimal ω_{m+1} are called quasi-Chebyshev accelerated (QCA) methods. We determine ω_{m+1} by finding the optimal solution in the following hyperplane:

$$P = \{y | y = \omega(Ty^{(m)} + M^{-1}b - y^{(m-1)}) + y^{(m-1)}, \omega \in \mathbb{R}\}. \tag{1.6}$$

We present some notations and preliminaries. As usual, $\mathbb{R}^{n\times n}$ denotes the $n \times n$ real matrix set and $\mathbb{R}^n$ the n-dimensional Euclidean vector space. X^T denotes the transpose of the matrix or vector X. X^H denotes the conjugate transpose of the matrix or vector X. $\langle x, y\rangle$ denotes the angle between the vectors x and y. The spectral radius of a square matrix A is denoted by $\rho(A)$. $\|\cdot\|$ denotes the Euclidean norm, with special explanation given later.

$A = M - N$ is called a splitting of the matrix A if $M \in \mathbb{R}^{n\times n}$ is nonsingular. The splitting is convergent if $\rho(M^{-1}N) < 1$. If A is symmetric positive definite and $M^T + N$ is positive definite, then we call the splitting $A = M - N$ a P-regular splitting (see [10,11]).

When A is a nonsymmetric positive definite matrix, the HSS (or PSS) iteration method, given by Bai et al. (see [12,13]), is an efficient method. Let

$$A = H + S, \tag{1.7}$$

where

$$H = \frac{1}{2}(A + A^H), \qquad S = \frac{1}{2}(A - A^H). \tag{1.8}$$

Given an initial guess $x^{(0)}$, for $k = 0, 1, 2, \ldots$, until $\{x^{(k)}\}$ converges, compute

$$\begin{cases} (\alpha I + H)x^{\left(k+\frac{1}{2}\right)} = (\alpha I - S)x^{(k)} + b, & \text{(a)} \\ (\alpha I + S)x^{(k+1)} = (\alpha I - H)x^{\left(k+\frac{1}{2}\right)} + b, & \text{(b)} \end{cases} \tag{1.9}$$

where α is a given positive constant.

Let

$$M = \frac{1}{2\alpha}(\alpha I + H)(\alpha I + S), \qquad N = \frac{1}{2\alpha}(\alpha I - H)(\alpha I - S). \tag{1.10}$$

Then the splitting $A = M - N$ induces the HSS iteration method.

Let x_* be the solution of (1.1), and let $\varepsilon^{(k,0)} = x^{(k,0)} - x_*$. If there exists a symmetric positive definite matrix P such that

$$\|P\varepsilon^{(k,0)}\| \le \|M^{-1}N\|^k \|P\varepsilon^{(0,0)}\|,$$

then the convergence rate is

$$r = -\ln(\|M^{-1}N\|). \tag{1.11}$$

The rest of the paper is organized as follows. We derive the quasi-Chebyshev accelerated iteration schemes of the splitting iteration methods in Section 2. The convergence and the convergence rate of the quasi-Chebyshev accelerated methods are discussed in Section 3. Finally, we compare the new methods with the Chebyshev semi-iterative method by using them on three numerical examples in Section 4.

2. The QCA method

In this section, we describe the quasi-Chebyshev accelerated method. Let

$$A = M - N, \tag{2.1}$$

and let the iteration matrix be

$$T = M^{-1}N. \tag{2.2}$$

Method 2.1 (*Quasi-Chebyshev Accelerated Method*). Let an initial point x_0 and a precision $\epsilon > 0$ be given. For $k = 1, 2, \ldots$ until convergence, do the following.

Step 1. Let $\bar{x}_{k+1} = M^{-1}Nx_k + M^{-1}b$. Solve the following system of linear equations:

$$x_{k+1} = \omega_{k+1}(\bar{x}_{k+1} - x_{k-1}) + x_{k-1}, \tag{2.3}$$

where

$$x_0 \in \mathbb{R}^n, \quad x_1 = M^{-1}Nx_0 + M^{-1}b.$$

ω_{k+1} is the solution of the following optimization problems:

(a) when A is a symmetric positive definite matrix, set $x = \omega(\bar{x}_{k+1} - x_{k-1}) + x_{k-1}$,

$$\min_{\omega} \frac{1}{2}x^TAx - x^Tb; \tag{2.4}$$

(b) when A is not symmetric positive definite, set $r = Ax - b$, $x = \omega(\bar{x}_{k+1} - x_{k-1}) + x_{k-1}$,

$$\min_{\omega} r^T(\alpha I + H)^{-2}r. \tag{2.5}$$

Step 2. If $\|r_{k+1}\| < \epsilon$, stop; otherwise, $k \Leftarrow k + 1$ and go to Step 1.

In fact, the solutions of the optimization problems (2.4) and (2.5) are given in the following, respectively:

$$\omega_{k+1} = ((\bar{x}_{k+1} - x_{k-1})^TA(\bar{x}_{k+1} - x_{k-1}))^{-1}(b - Ax_{k-1})^T(\bar{x}_{k+1} - x_{k-1}), \tag{2.6}$$

$$\omega_{k+1} = ((\bar{r}_{k+1} - r_{k-1})^T(\alpha I + H)^{-2}(\bar{r}_{k+1} - r_{k-1}))^{-1}(b - Ax_{k-1})^T(\alpha I + H)^{-2}(\bar{r}_{k+1} - r_{k-1}). \tag{2.7}$$

3. Convergence analysis

In this section, we study the convergence of Method 2.1 corresponding to (2.4) and (2.5), respectively.

Lemma 3.1 ([14]). *Let A be a symmetric positive definite matrix. If $A = M - N$ is a P-regular splitting, then there exists a positive number r such that*

$$\left\| A^{\frac{1}{2}}TA^{-\frac{1}{2}} \right\| \le r < 1.$$

Lemma 3.2. *Let A be a symmetric positive definite matrix. If $A = M - N$ is a symmetric P-regular splitting, then*

$$\left\| A^{\frac{1}{2}}TA^{-\frac{1}{2}} \right\| = \rho(T). \tag{3.1}$$

Proof. Since

$$A^{\frac{1}{2}}TA^{-\frac{1}{2}} = I - A^{\frac{1}{2}}M^{-1}A^{\frac{1}{2}},$$

$A^{\frac{1}{2}}TA^{-\frac{1}{2}}$ is symmetric, which implies (3.1).

Theorem 3.3. *Let A be a symmetric positive definite matrix, and let $A = M - N$ be a P-regular splitting. Suppose that $\{\omega_k\}$ is generated by the quadratic programming (2.4). Assume that x_* is the unique solution of linear system (1.1). If $\langle A^{\frac{1}{2}}(\bar{x}_{k+1} - x_*), A^{\frac{1}{2}}(\bar{x}_{k+1} - x_{k-1})\rangle \geq \theta$, then the sequence $\{x_k\}$ generated by Method 2.1 converges to x_*. Further, the convergence rate r has the following lower bound:*

$$r \geq -\left(\ln \left\| A^{\frac{1}{2}} T A^{-\frac{1}{2}} \right\| + \ln(\sin\theta)\right). \tag{3.2}$$

Furthermore, if M is symmetric, then

$$r \geq -(\ln \rho(T) + \ln(\sin\theta)). \tag{3.3}$$

Proof. Let x_* be the unique solution of linear system (1.1). Obviously, (2.4) is equivalent to the following problem:

$$\min_{\alpha} \frac{1}{2}(x - x_*)^T A (x - x_*). \tag{3.4}$$

According to (2.6), we have

$$\begin{aligned}(x_{k+1} - x_*)^T A(x_{k+1} - x_{k-1}) &= (x_{k+1} - x_{k-1} + x_{k-1} - x_*)^T A(x_{k+1} - x_{k-1}) \\ &= (x_{k+1} - x_{k-1})^T A(x_{k+1} - x_{k-1}) + (x_{k-1} - x_*)^T A(x_{k+1} - x_{k-1}) \\ &= \omega_{k+1}^2 (\bar{x}_{k+1} - x_{k-1})^T A(\bar{x}_{k+1} - x_{k-1}) + \omega_{k+1}(\bar{x}_{k+1} - x_{k-1})^T (Ax_{k-1} - b) \\ &= 0,\end{aligned}$$

which gives

$$(x_{k+1} - x_*)^T A(x_{k+1} - \bar{x}_{k+1}) = 0.$$

From the orthogonality between $A^{\frac{1}{2}}(x_{k+1} - x_*)$ and $A^{\frac{1}{2}}(x_{k+1} - \bar{x}_{k+1})$ as shown in Fig. 3.1, we have

$$(\bar{x}_{k+1} - x_*)^T A(\bar{x}_{k+1} - x_*) = (x_{k+1} - x_*)^T A(x_{k+1} - x_*) + (\bar{x}_{k+1} - x_{k+1})^T A(\bar{x}_{k+1} - x_{k+1}). \tag{3.5}$$

Let

$$\varepsilon_k = x_k - x_*.$$

Method 2.1 yields

$$\bar{x}_{k+1} - x_* = T\varepsilon_k. \tag{3.6}$$

Thus, (3.5) yields

$$\begin{aligned}\left\| A^{\frac{1}{2}} \varepsilon_{k+1} \right\| &= \left\| \sin(\theta_k) A^{\frac{1}{2}} T \varepsilon_k \right\| \\ &= \left\| \sin(\theta_k) A^{\frac{1}{2}} T A^{-\frac{1}{2}} A^{\frac{1}{2}} \varepsilon_k \right\| \\ &\leq \left\| \sin(\theta_k) A^{\frac{1}{2}} T A^{-\frac{1}{2}} \right\| \left\| A^{\frac{1}{2}} \varepsilon_k \right\| \\ &\leq \cdots \\ &\leq |\sin^k(\theta)| \left\| A^{\frac{1}{2}} T A^{-\frac{1}{2}} \right\|^k \left\| A^{\frac{1}{2}} \varepsilon_0 \right\|.\end{aligned}$$

Since $\|A^{\frac{1}{2}} T A^{-\frac{1}{2}}\| \leq r < 1$, this implies that $\lim_{k\to\infty} \varepsilon_k^T A \varepsilon_k = 0$, and

$$\lim_{k\to\infty} \varepsilon_k = 0.$$

On the other hand, from the assumption, we have

$$\left\| A^{\frac{1}{2}} \varepsilon_k \right\| \leq \sin^k\theta \left\| A^{\frac{1}{2}} T A^{-\frac{1}{2}} \right\|^k \left\| A^{\frac{1}{2}} \varepsilon_0 \right\|.$$

Based on the above inequality and (1.11), we see that the convergence rate r has the following lower bound:

$$r \geq -\left(\ln \left\| A^{\frac{1}{2}} T A^{-\frac{1}{2}} \right\| + \ln(\sin\theta)\right).$$

Since M is symmetric, (3.2) is obtained from Lemma 3.2. The theorem is proved.

Remark. As we know, the Chebyshev semi-iterative method is convergent under some conditions (the coefficient matrix leading to the splitting should be symmetrizable), but Method 2.1 is unconditionally convergent if the original iteration method is convergent.

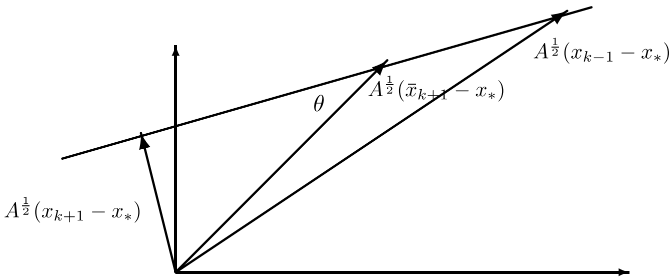

Fig. 3.1. The orthogonality between $A^{\frac{1}{2}}(x_{k+1}-x_*)$ and $A^{\frac{1}{2}}(x_{k+1}-\bar{x}_{k+1})$.

Lemma 3.4. *Let A be a nonsymmetric positive definite matrix. Assume that M, N are given by (1.10). Then*

$$\|(\alpha I+H)^{-1}NM^{-1}(\alpha I+H)\| < 1. \tag{3.7}$$

Furthermore, if A is a normal matrix, then

$$\|(\alpha I+H)^{-1}NM^{-1}(\alpha I+H)\| = \rho(M^{-1}N). \tag{3.8}$$

Proof. Since $(\alpha I+H)^{-1}NM^{-1}(\alpha I+H) = (\alpha I+H)^{-1}(\alpha I-H)(\alpha I-S)(\alpha I+S)$, $\|(\alpha I+H)^{-1}NM^{-1}(\alpha I+H)\| = \|(\alpha I+H)^{-1}(\alpha I-H)(\alpha I-S)(\alpha I+S)\| < 1$.

On the other hand, since A is a normal matrix,

$$HS = SH.$$

So $(\alpha I+H)^{-1}NM^{-1}(\alpha I+H)$ is a normal matrix, which implies that

$$\|(\alpha I+H)^{-1}NM^{-1}(\alpha I+H)\| = \rho(NM^{-1}).$$

(3.8) holds from $\rho(NM^{-1}) = \rho(M^{-1}N)$.

Theorem 3.5. *Let A be a nonsymmetric positive definite matrix. Assume that M, N are given by (1.10), and that ω_k is generated by the quadratic programming (2.5). Let $r = Ax - b$. If $\langle(\alpha I+H)^{-1}\bar{r}_{k+1}, (\alpha I+H)^{-1}(\bar{r}_{k+1}-r_{k+1})\rangle \geq \theta$, then $\{x_k\}$ generated by Method 2.1 converges to the unique solution of the system of linear equations (1.1). Further, the convergence rate satisfies*

$$r \geq -(\ln\|(\alpha I+H)^{-1}NM^{-1}(\alpha I+H)\| + \ln(\sin\theta)). \tag{3.9}$$

Furthermore, If A is a normal matrix, then

$$r \geq -(\ln\rho(M^{-1}N) + \ln(\sin\theta)). \tag{3.10}$$

Proof. From (2.3), we know that

$$r_{k+1} = \omega_{k+1}(\bar{r}_{k+1} - r_{k-1}) + r_{k-1},$$

and

$$\bar{r}_{k+1} = NM^{-1}r_k.$$

(2.7) yields

$$r_{k+1}^T(\alpha I+H)^{-2}(\bar{r}_{k+1} - r_{k-1}) = 0,$$

which generates

$$r_{k+1}^T(\alpha I+H)^{-2}(\bar{r}_{k+1} - r_{k+1}) = 0.$$

Hence, $(\alpha I+H)^{-1}r_{k+1}$ and $(\alpha I+H)^{-1}(\bar{r}_{k+1}-r_{k+1})$ are orthogonal, as shown in Fig. 3.2.

Thus, which leads to the same technique as the proof of Theorem 3.3, we have

$$\begin{aligned}
\|(\alpha I+H)^{-1}r_{k+1}\| &= \|\sin(\theta_k)(\alpha I+H)^{-1}\bar{r}_{k+1}\| \\
&= \|\sin(\theta_k)(\alpha I+H)^{-1}NM^{-1}(\alpha I+H)(\alpha I+H)^{-1}r_k\| \\
&\leq \|\sin(\theta_k)\|\,\|(\alpha I+H)^{-1}NM^{-1}(\alpha I+H)\|\,\|(\alpha I+H)^{-1}r_k\| \\
&\leq \cdots \\
&\leq \Pi_{i=0}^{k}|\sin(\theta_i)|\,\|(\alpha I+H)^{-1}NM^{-1}(\alpha I+H)\|^k\|(\alpha I+H)^{-1}r_0\|.
\end{aligned}$$

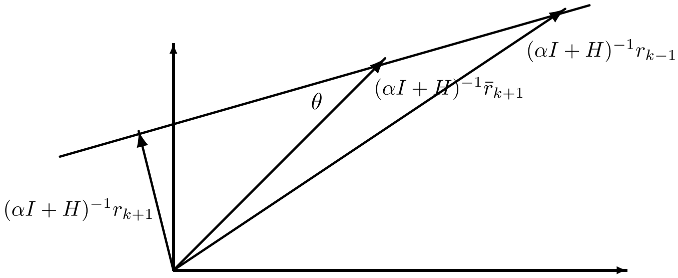

Fig. 3.2. The orthogonality between $(\alpha I + H)^{-1} r_{k+1}$ and $(\alpha I + H)^{-1}(\bar{r}_{k+1} - r_{k-1})$.

Lemma 3.4 yields

$$\lim_{k\to\infty} (\alpha I + H)^{-1} r_{k+1} = 0,$$

which implies that r_{k+1} converges to zero. On the other hand, from the assumption, we have

$$\|(\alpha I + H)^{-1} r_{k+1}\| \le \sin^k \theta \|(\alpha I + H)^{-1} N M^{-1} (\alpha I + H)\|^k \|(\alpha I + H)^{-1} r_0\|.$$

Thus, based on the above inequality and (1.11), we obtain (3.9). Obviously, if A is a normal matrix, (3.10) holds.

4. Numerical experiments

In this section, we report the results of numerical computations by implementing our Method 2.1 in Matlab to solve a system of linear equations.

In our computations, we start from the zero vector and terminate when the current iterate satisfies $\|r^{(k)}\| < 10^{-6}$, where $r^{(k)}$ is the residual of the kth iteration, or we allow that the number of iterations can be up to 30 000. For the latter, the iteration is failing. The running time and the number of iterations of the methods for cases of different size are given in the tables.

Example 4.1. Consider the system of linear equations in which $A = (a_{k,j}) \in \mathbb{R}^{n\times n}$ is defined as follows:

$$a_{k,j} = \begin{cases} 8, & \text{for } j = k, \\ -1, & \text{for } \max\{1, k-4\} \le j \le \min\{n, k+4\}, \quad k = j = 1, 2, \ldots, n. \\ 0, & \text{otherwise.} \end{cases}$$

The right-hand side $b = (1, 1, \ldots, 1)^T$. It is easy to verify that the matrix is very ill conditioned when n is large.

Let

$$A = D - L - L^T, \tag{4.1}$$

where $-L$ is the strictly lower triangular part of A, and $D = \mathrm{diag}(A)$.

Let $A = M - N$, where

$$M = (D - L) D^{-1} (D - L^T), \qquad N = L D^{-1} L^T. \tag{4.2}$$

Set $\delta = -\beta$. In Tables 4.1.1 and 4.1.2, we give the computation results.

Example 4.2. Consider the system of linear equations $Ax = b$ obtained from the nine-point finite difference discretization of the Poisson equations, where

$$A = \begin{bmatrix} D_m & B_m & & & \\ B_m & D_m & B_m & & \\ & \ddots & \ddots & \ddots & \\ & & B_m & D_m & B_m \\ & & & B_m & D_m \end{bmatrix}, \tag{4.3}$$

$$D_m = \begin{bmatrix} 20 & -4 & & & \\ -4 & 20 & -4 & & \\ & \ddots & \ddots & \ddots & \\ & & -4 & 20 & -4 \\ & & & -4 & 20 \end{bmatrix}, \qquad B_m = \begin{bmatrix} -4 & -1 & & & \\ -1 & -4 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & -4 & -1 \\ & & & -1 & -4 \end{bmatrix},$$

and the right-hand side $b = (1, 2, \ldots, n)^T$. We again use the same splitting as Example 4.1 with block form.

Table 4.1.1
The results obtained by the Chebyshev semi-iterative method with respect to (4.2).

n		Splitting (4.2)	$\beta = \rho(T)$	$\beta = 0.9$	$\beta = 0.99$	$\beta = 0.999$	$\beta = 0.9999$
200	IT	3911	177	–	497	319	956
	CPU(s)	0.311371	0.016222		0.044104	0.029548	0.084529
400	IT	15790	324	–	2176	457	973
	CPU(s)	2.210774	0.050387		0.336017	0.071265	0.150603
600	IT	–	517	–	–	1418	1052
	CPU(s)		0.114850			0.311664	0.231407
800	IT	–	729	–	–	2695	1048
	CPU(s)		0.208272			0.771236	0.298482
1000	IT	–	993	–	–	4341	1072
	CPU(s)		0.347769			1.540632	0.377746
1200	IT	–	1076	–	–	6361	1091
	CPU(s)		0.448230			2.653571	0.451586
1400	IT	–	1263	–	–	8761	2115
	CPU(s)		0.612357			4.241915	1.025501
1600	IT	–	1375	–	–	11541	3058
	CPU(s)		0.748913			6.335064	1.675106

Notes: the symbol – indicates that the iteration is failing.

Table 4.1.2
The results obtained by scheme (2.4) with respect to (4.2) for the QCA method.

n	200	400	600	800	1000	1200	1400	1600
IT	97	172	244	317	389	463	535	606
CPU(s)	0.010202	0.031204	0.062256	0.103508	0.155293	0.218713	0.292537	0.376958

Table 4.2.1
The results of the Chebyshev semi-iterative method with respect to (4.2).

n		Splitting (4.2)	$\beta = \rho(T)$	$\beta = 0.9$	$\beta = 0.99$	$\beta = 0.999$	$\beta = 0.9999$
100	IT	60	22	38	127	397	1259
	CPU(s)	0.074048	0.002784	0.004254	0.012068	0.036644	0.111741
400	IT	227	48	71	141	449	1410
	CPU(s)	1.223532	0.028875	0.042542	0.083272	0.263699	0.820607
900	IT	515	75	202	153	477	1474
	CPU(s)	7.478776	0.196082	0.530462	0.407387	1.245470	3.917370
1600	IT	932	103	384	157	486	1543
	CPU(s)	25.208226	0.655146	2.448177	1.024642	3.304377	10.242416
2500	IT	1482	134	624	162	512	1609
	CPU(s)	67.243682	1.614042	7.808740	1.907480	6.368588	19.532858
3600	IT	2168	162	922	165	527	1621
	CPU(s)	157.779993	3.573473	21.970256	3.520621	11.047421	34.754341
4900	IT	2993	193	1282	311	542	1684
	CPU(s)	303.721214	6.260179	45.398099	10.958538	20.767474	53.053443

Let $\delta = -\beta$. Tables 4.2.1 and 4.2.2 show the computational results ($m^2 = n$).

Example 4.3 (*See [13]*). Consider the generalized convection–diffusion equations in a two-dimensional case. The equation is

$$-\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial y^2} + q * \exp(x+y) * x * \frac{\partial u}{\partial x} + q * \exp(x+y) * y * \frac{du}{dy} = f, \tag{4.4}$$

with the homogeneous Dirichlet boundary condition. We use the standard Ritz–Galerkin finite-element method with a P_1-conforming triangular element to approximate the continuous solutions $u = x * y * (1-x) * (1-y)$ in the domain $\Omega = [0, 1] \times [0, 1]$; the step sizes along both x and y directions are the same, that is, $h = \frac{1}{2^m}$, $m = 5, 6, 7$. We apply the HSS iteration method to this problem. For this nonsymmetric system of linear equations, we use the optimization model (2.5).

Let $\delta = -\beta$ and $q = 1$. For the parameter α, we choose different values. Here, the optimal parameter is $\alpha^* = \sqrt{\lambda_{\max}(H)\lambda_{\min}(H)}$ (see [13]). Tables 4.3.1–4.3.3 show the computation results.

The Chebyshev acceleration method applied to the HSS iteration process fails if $\beta = 0.9, 0.99, 0.999, 0.9999$. But Method 2.1 is convergent. The computation results are given in Tables 4.3.1–4.3.3.

Table 4.2.2
The results of scheme (2.4) with respect to (4.2) for the QCA method.

n	100	400	900	1600	2500	3600	4900
IT	16	31	44	59	77	96	105
CPU(s)	0.002176	0.020331	0.125794	0.397888	0.965362	2.098523	3.835547

Table 4.3.1
The results of scheme (2.5) with respect to the HSS iteration method for the QCA method ($h = \frac{1}{128}$).

Iteration method		$\alpha = 2.5$	$\alpha = 1.2$	$\alpha^* = 0.1$
HSS	IT	19 942	9572	799
	CPU(s)	5767.0955	2867.3193	172.6504
QCA	IT	422	582	464
	CPU(s)	171.4245	226.7251	187.2008

Table 4.3.2
The results of scheme (2.5) with respect to the HSS iteration method for the QCA method ($h = \frac{1}{64}$).

Iteration method		$\alpha = 2$	$\alpha = 1$	$\alpha^* = 0.2$
HSS	IT	3880	1940	388
	CPU(s)	269.8594	130.7204	32.1606
QCA	IT	225	401	275
	CPU(s)	23.4207	42.2414	27.7141

Table 4.3.3
The results of scheme (2.5) with respect to the HSS iteration method for the QCA method ($h = \frac{1}{32}$).

Iteration method		$\alpha = 2$	$\alpha = 1$	$\alpha^* = 0.3$
HSS	IT	957	479	171
	CPU(s)	7.7579	4.0048	1.3985
QCA	IT	101	160	152
	CPU(s)	1.0654	1.8264	1.6555

From the three examples above, we see the quasi-Chebyshev accelerated method is always convergent and that the number of iterations and the CPU time (given in seconds) for it are less than those for the original methods and those for the Chebyshev semi-iterative method. For the HSS iteration method, the quasi-Chebyshev accelerated method is more effective when the HSS iteration method is not effective. On the other hand, as we know, when the parameter α is larger, the conjugate gradient method to solve the linear systems (1.9) becomes stable and speeds up its convergence; this is because the condition number of $\alpha I + H$ becomes smaller. Hence, the quasi-Chebyshev accelerated method is effective and valuable.

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References

[1] Z.-Z. Bai, X.-B. Chi, Asymptotically optimal successive overrelaxation methods for systems of linear equations, J. Comput. Math. 21 (5) (2003) 603–612.
[2] Z.-Z. Bai, G.H. Golub, Accelerated Hermitian and skew-Hermitian splitting iteration methods for saddle-point problems, IMA J. Numer. Anal. 27 (1) (2007) 1–23.
[3] Z.-Z. Bai, G.H. Golub, M.K. Ng, On successive overrelaxation acceleration of the Hermitian and skew-Hermitian splitting iterations, Numer. Linear Algebra Appl. 14 (4) (2007) 319–335.
[4] Z.-Z. Bai, B.N. Parlett, Z.-Q. Wang, On generalized successive overrelaxation methods for augmented linear systems, Numer. Math. 102 (1) (2005) 1–38.
[5] G.H. Golub, C.F. Van Loan, Matrix Computations, The Johns Hopkin University Press, 1996.
[6] G.H. Golub, R.S. Varga, Chebyshev semi-iterative methods, successive over-relaxation iterative methods, and second-order Richardson iterative methods, Parts I and II, Numer. Math. 3 (1) (1961) 147–168.
[7] Z. Li, C.-J. Li, D.J. Evans, Chebyshev acceleration for SOR-like method, Int. J. Comput. Math. 82 (5) (2005) 583–593.
[8] C.-L. Wang, G.-Y. Meng, A practical two-term acceleration algorithm for linear systems, Numer. Linear Algebra Appl. 19 (2012) 571–583.
[9] R.S. Varga, Matrix Iterative Analysis, second ed., Springer, Berlin, 2000.
[10] A. Berman, R.J. Plemmons, Nonnegative Matrices in the Mathematical Science, Academic Press, New York, 1979.
[11] R. Nabben, A note on comparison theorems for splitting and multisplitting of Hermitian positive definite matrices, Linear Algebra Appl. 233 (1) (1996) 67–80.

[12] Z.-Z. Bai, G.H. Golub, L.-Z. Lu, J.-F. Yin, Block triangular and skew-Hermitian splitting methods for positive-definite linear systems, SIAM J. Sci. Comput. 26 (3) (2005) 844–863.
[13] Z.-Z. Bai, G.H. Golub, M.K. Ng, Hermitian and skew-Hermitian splitting methods for non-Hermitian positive definite linear systems, SIAM J. Matrix Anal. Appl. 24 (3) (2003) 603–626.
[14] A. Frommer, D.B. Syzld, Weighted max norms, splittings, and overlapping additive Schwarz iterations, Numer. Math. 75 (1) (1997) 48–62.