and $t$ dependence, we are unable to complete the separation process successfully. That is, we are unable to get all of the $x$ dependence on one side of the equation and all of the $t$ dependence on the other side. Thus, we suggest seeking $u$ in the form

$$u(x,t) = u_s(x) + X(x)T(t) \tag{15.5}$$

instead, where $u_s(x)$ is the steady-state solution. In steady state, $u_t \to 0$ and $u(x,t) \to u_s(x)$, so (15.1) and (15.2) give

$$\alpha^2 u_s''(x) = -F; \qquad u_s(0) = 0, \quad u_s(L) = 50. \tag{15.6}$$

Putting (15.5) into (15.1), show that the $u_s$ term cancels the troublesome $F$ term, so that this time the separation can be successfully completed to yield the familiar result

$$\frac{X''}{X} = \frac{1}{\alpha^2}\frac{T'}{T} = \text{constant} = -\kappa^2. \tag{15.7}$$

Then solve (15.6) for $u_s$ and (15.7) for $X$ and $T$, and impose the conditions (15.2) and (15.3) on $u(x,t) = u_s(x) + X(x)T(t)$. Regarding the form of (15.5), observe that, in *physical* terms, $u_s$ is the steady-state solution and $XT$ is a transient quantity needed to match the initial condition $u(x,0) = f(x)$ with the steady-state solution $u_s(x)$. In *mathematical* terms, $u_s$ is a particular solution (i.e., a solution of the full equation $\alpha^2 u_{xx} - u_t = -F$) and $XT$ is a solution of the associated homogeneous equation $\alpha^2 u_{xx} - u_t = 0$.

**16.** Repeat Exercise 15 with (15.2) and (15.3) changed as follows.

(a) $u(0,t) = u(L,t) = u(x,0) = 0$
(b) $u_x(0,t) = u(L,t) = u(x,0) = 0$
(c) $u(0,t) = u_x(L,t) = u(x,0) = 0$
(d) $u(0,t) = 0, u_x(L,t) = -20, u(x,0) = 0$

**17.** (*Presence of nonconstant source term*) In Exercise 15 we include a source term $F$ that is a constant, although the solution method outlined therein would have worked even if $F$ were a nonconstant function of $x$. In this exercise we consider the problem

$$\begin{aligned} \alpha^2 u_{xx} &= u_t - F(x,t), && (0 < x < L,\ 0 < t < \infty) \\ u(0,t) &= u(L,t) = 0, && (0 < t < \infty) \\ u(x,0) &= 0, && (0 < x < L) \end{aligned} \tag{17.1}$$

where the source term $F$ is allowed to be a function of $x$ and $t$. To solve, we can use essentially the same **eigenvector expansion method** that we use in Section 11.3.2 to solve the nonhomogeneous matrix problem $\mathbf{Ax}=\Lambda\mathbf{x} + \mathbf{c}$. [In this case the matrix operator $\mathbf{A}$ is analogous to the partial differential operator $\alpha^2\partial^2/\partial x^2 - \partial/\partial t$, $\mathbf{c}$ is analogous to $-F(x,t)$, and $\Lambda = 0$.] There, we expand $\mathbf{x}$ and $\mathbf{c}$ in terms of the basis provided by the eigenvectors of $\mathbf{A}$. In the present case that step amounts to expanding $u(x,t)$ and $F(x,t)$ in terms of the $\sin\frac{n\pi x}{L}$ eigenfunctions provided by the relevant Sturm-Liouville problem. Thus, the problem that we pose is as follows.

(a) Solve for $u(x,t)$ by seeking

$$u(x,t) = \sum_{n=1}^{\infty} g_n(t) \sin\frac{n\pi x}{L} \tag{17.2}$$

and expanding

$$F(x,t) = \sum_{n=1}^{\infty} F_n(t) \sin\frac{n\pi x}{L}, \tag{17.3}$$

where the

$$F_n(t) = \frac{2}{L}\int_0^L F(x,t)\sin\frac{n\pi x}{L}\,dx \tag{17.4}$$

coefficients are considered as known (i.e., computable) functions of $t$. Thus, show that

$$u(x,t) = \sum_{n=1}^{\infty}\left[\int_0^t F_n(\tau)e^{(n\pi\alpha/L)^2(\tau-t)}\,d\tau\right]\sin\frac{n\pi x}{L}. \tag{17.5}$$

(b) Work out the solution (17.5) for the case where $F(x,t) = e^{-t}$.

(c) Modify the solution procedure described in part (a) if the left end condition is changed from $u(0,t) = 0$ to $u_x(0,t) = 0$.

**18.** (*Superposition*) Show that the solution to the problem

$$\begin{aligned} \alpha^2 u_{xx} &= u_t + g(x,t), \qquad (0 < x < L,\ 0 < t < \infty) \\ u(0,t) &= p(t), \quad u(L,t) = q(t), \quad u(x,0) = f(x) \end{aligned} \tag{18.1}$$

can be expressed as $u = u_1 + u_2 + u_3 + u_4$, where $u_1, \ldots, u_4$ are solutions of the four problems

$$\begin{aligned} &\alpha^2 u_{1xx} = u_{1t} + g(x,t), \\ &u_1(0,t) = u_1(L,t) = u_1(x,0) = 0, \\ &\alpha^2 u_{2xx} = u_{2t}, \\ &u_2(0,t) = p(t), \quad u_2(L,t) = u_2(x,0) = 0, \\ &\alpha^2 u_{3xx} = u_{3t}, \end{aligned}$$

$$u_3(0,t) = 0, \quad u_3(L,t) = q(t), \quad u_3(x,0) = 0,$$

$$\alpha^2 u_{4xx} = u_{4t},$$
$$u_4(0,t) = u_4(L,t) = 0, \quad u_4(x,0) = f(x),$$

each on the domain $0 < x < L, \ 0 < t < \infty$.

**19.** (*Nonhomogeneous Neumann conditions*) Solve the conduction problem

$$\begin{aligned} &\alpha^2 u_{xx} = u_t, \qquad (0 < x < L, \ 0 < t < \infty) \\ &u_x(0,t) = -1, \quad u_x(L,t) = 0, \quad u(x,0) = 0 \end{aligned} \tag{19.1}$$

for $u(x,t)$. HINT: You should find that the standard separation-of-variables procedure has difficulty coping with the boundary conditions if $u_x(0,t) \neq u_x(L,t)$, as is true in this case. To proceed successfully, we suggest that you change from $u(x,t)$ to $v(x,t)$ according to

$$u(x,t) = \frac{(x-L)^2}{2L} + v(x,t).$$

Then, show that $v$ can be split by superposition into $v = v_1 + v_2$, where

$$\alpha^2 v_{1xx} = v_{1t}, \qquad (0 < x < L, \ 0 < t < \infty)$$
$$v_{1x}(0,t) = v_{1x}(L,t) = 0, \quad v_1(x,0) = -\frac{(x-L)^2}{2L}$$

and

$$\alpha^2 v_{2xx} = v_{2t} - \frac{\alpha^2}{L}, \qquad (0 < x < L, \ 0 < t < \infty)$$
$$v_{2x}(0,t) = v_{2x}(L,t) = v_2(x,0) = 0.$$

The solution for $v_2$ can be found easily as a function of $t$ alone. Solve for $v_2$. (But you need not solve for $v_1$.)

**20.** (*Variable end conditions*) Thus far, our Dirichlet-type boundary conditions have been of constant type, e.g., $u(0,t) = 50$. Here, we consider nonconstant conditions. Consider the problem

$$\begin{aligned} &\alpha^2 u_{xx} = u_t, \qquad (0 < x < L, \ 0 < t < \infty) \\ &u(0,t) = p(t), \quad u(L,t) = q(t), \quad u(x,0) = f(x), \end{aligned} \tag{20.1}$$

where $p(t), q(t)$, and $f(x)$ are prescribed. Changing dependent variables from $u(x,t)$ to $v(x,t)$ according to

$$u(x,t) = v(x,t) + \left(1 - \frac{x}{L}\right) p(t) + \frac{x}{L} q(t), \tag{20.2}$$

show that the problem governing $v$ is precisely of the type treated in Exercise 17. NOTE: Observe how an "input" can be moved from the boundary conditions to the PDE. In the present case, the PDE on $u$ was homogeneous and the boundary conditions were nonhomogeneous; following the change of variables (20.2), you should find that the PDE on $v$ is nonhomogeneous and the boundary conditions are homogeneous.

**21.** (*Newton cooling*) Consider the conduction of heat in a rod, the lateral surface of which is not insulated. If heat is convected from the rod to the environment, the PDE governing the temperature $u(x,t)$ is

$$\alpha^2 u_{xx} = u_t + h(u - u_\infty), \tag{21.1}$$

where the constants $h$ and $u_\infty$ are the convective heat transfer coefficient and the ambient temperature, respectively. Our interest in (21.1) lies in the Newton cooling term $h(u - u_\infty)$. Although it is not essential, one normally begins by eliminating the $u_\infty$ term by setting $v(x,t) = u(x,t) - u_\infty$ and considering, instead, $\alpha^2 v_{xx} = v_t + hv$.

(a) Solve the Newton cooling problem

$$\begin{aligned} &\alpha^2 v_{xx} = v_t + hv, \qquad (0 < x < L, \ 0 < t < \infty) \\ &v(0,t) = v(L,t) = 50 \quad v(x,0) = f(x) \end{aligned} \tag{21.2}$$

by separation of variables, leaving expansion coefficients in integral form. HINT: Seeking $v(x,t) = v_s(x) + X(x)T(t)$, where $v_s(x)$ is the steady-state temperature distribution, show that

$$\frac{X''}{X} = \frac{1}{\alpha^2} \frac{T' + hT}{T} = \text{constant} = -\kappa^2. \tag{21.3}$$

(b) Solve (21.2) by omitting the $v_s(x)$ term and seeking $v(x,t) = X(x)T(t)$. That is, show that the inclusion of the $v_s(x)$ term in the solution form is not essential.

**22.** Show that the change of variables

$$w(x,t) = e^{ht}[u(x,t) - u_\infty]$$

reduces (21.1), above, to the simpler and more familiar form

$$\alpha^2 w_{xx} = w_t.$$

**23.** (*Flow of electricity in a cable*) The voltage $v(x,t)$ (volts) and the current $I(x,t)$ (amperes) in a long underground insulated cable are governed by the PDE's

$$\begin{aligned} v_{xx} &= LCv_{tt} + (rC + Lg)v_t + rgv, \\ I_{xx} &= LCI_{tt} + (rC + Lg)I_t + rgI, \end{aligned} \tag{23.1,2}$$

where $L, C, r, g$ are positive constants: $L$ is the inductance (henries/kilometer), $C$ is the capacitance to ground (farads/kilometer), $r$ is the resistance (ohms/kilometer), and $g$ is the leakance to ground (mhos/kilometer). These PDE's are called the **telephone equations** and are seen to be of wave (hyperbolic) type. Often, as in telegraph transmission, $L$ and $g$ can be neglected, in which case (23.1) and (23.2) reduce to the **telegraph equations**

$$v_{xx} = rCv_t,$$
$$I_{xx} = rCI_t,$$

which are of diffusion type. In the present example we suppose only that $L \approx 0$. Considering only the voltage $v$, we then have

$$v_{xx} = rCv_t + rgv. \tag{23.3}$$

[Comparing (23.3) with the PDE in Exercise 21(a), we see that the two phenomena are analogous, with the lateral heat loss to the environment corresponding to the voltage loss due to leakage to the ground.] Suppose that the line is of length $L$, the voltage at $x = 0$ is maintained (for a "long time") at 12 volts, the voltage at $x = L$ is maintained (for a "long time") at 6 volts, and then, beginning at $t = 0$, the left end is grounded. Thus,

$$\begin{aligned} v(0,t) &= 0, \quad (0 < t < \infty) \\ v(L,t) &= 6. \quad (0 < t < \infty) \end{aligned} \tag{23.4,5}$$

The initial condition $v(x,0)$ is not given but can be deduced from (23.3) together with the information that the ends have been maintained at 12 and 6 volts, respectively, for a long time.

(a) Determine $v(x,0)$.
(b) Determine the steady-state solution $v_s(x)$.
(c) Solve for $v(x,t)$ by separation of variables. Fourier expansion coefficients may be left in integral form. HINT: As in Exercise 21, it will be most convenient to seek $v(x,t) = v_s(x) + X(x)T(t)$.

**24.** (*Conduction in a sphere*) Consider the radial conduction of heat within a solid sphere. If the temperature $u$ is a function only of the spherical polar coordinate $\rho$ and the time $t$, then $\alpha^2\nabla^2 u = u_t$ becomes

$$\alpha^2\left(u_{\rho\rho} + \frac{2}{\rho}u_\rho\right) = u_t. \tag{24.1}$$

(a) Setting $u(\rho,t) = v(\rho,t)/\rho$, show that $v$ needs to satisfy the more familiar PDE

$$\alpha^2 v_{\rho\rho} = v_t. \tag{24.2}$$

(b) Use the idea contained in part (a) to solve the problem

$$\alpha^2\left(u_{\rho\rho} + \frac{2}{\rho}u_\rho\right) = u_t, \qquad (0 < \rho < a,\ 0 < t < \infty)$$
$$u(a,t) = 0, \quad (0 < t < \infty)$$
$$u(\rho,0) = f(\rho), \quad (0 < \rho < a)$$

where $u(0,t)$ is bounded. Expansion coefficients may be left in integral form. (Exercise 28 reexamines this problem, using the Sturm–Liouville theory.)

## EXERCISES FOR THE OPTIONAL SECTIONS 18.3.2, 18.3.3

**25.** (*Uniqueness*) (a) Consider the problem

$$\alpha^2 u_{xx} = u_t + f(x,t), \qquad (0 < x < L,\ 0 < t < T)$$
$$u(0,t) = p(t), \quad u(L,t) = q(t), \quad u(x,0) = r(x), \tag{25.1,2}$$

where $T$ is arbitrarily large. To establish the *uniqueness* of the solution, suppose that $u_1(x,t)$ and $u_2(x,t)$ are two solutions, and define

$$w(x,t) = u_1(x,t) - u_2(x,t). \tag{25.3}$$

Show that $w$ satisfies the "homogenized" problem

$$\alpha^2 w_{xx} = w_t, \qquad (0 < x < L,\ 0 < t < T)$$
$$w(0,t) = 0, \quad w(L,t) = 0, \quad w(x,0) = 0.$$

Proceeding formally, show that

$$\frac{d}{dt}\int_0^L w^2\,dx = 2\int_0^L ww_t\,dx = 2\alpha^2\int_0^L ww_{xx}\,dx. \tag{25.4}$$

Integrating the last integral by parts and then integrating both sides of the equation on $t$ from 0 to $t$, show that

$$\int_0^L w^2(x,t)\,dx = -\alpha^2\int_0^t\int_0^L w_x^2(x,\tau)\,dx\,d\tau. \tag{25.5}$$

Explain why it follows from (25.5) that $w(x,t) = 0$ throughout $0 < x < L$, $0 < t < T$. Thus, it must be true that $u_1(x,t) = u_2(x,t)$ for any solutions $u_1$ and $u_2$, so that the solution to (25.1) and (25.2) must be unique.
(b) Repeat part (a), but change $u(L,t) = q(t)$ in equation (25.2) to $u_x(L,t) = q(t)$.

(c) Establish the uniqueness of the solution to the problem

$$\alpha^2 u_{xx} = u_t + f(x,t), \qquad (0 < x < L,\ 0 < t < T)$$
$$u(0,t) = p(t), \quad u(L,t) + \beta u_x(L,t) = q(t),$$
$$u(x,0) = r(x),$$

where $T$ is arbitrarily large and $\beta > 0$, following essentially the same lines in part (a).

**26.** With $M_n$ defined by (50), use the ratio test to show that $\sum_{n=1}^{\infty} M_n$ converges, as we claimed.

**27.** (a)–(n) The problems in Exercise 6 are to be solved using half- or quarter-range expansions. Solve the corresponding problem in Exercise 6 again, this time using the Sturm–Liouville theory.

**28.** (*Conduction in a sphere*) In Exercise 24(b) we suggest using the change of variables $u(\rho,t) = v(\rho,t)/\rho$ to reduce the PDE to the form $\alpha^2 v_{\rho\rho} = v_t$ that is studied in this section. Here we ask you to solve the problem in Exercise 24(b) directly by seeking $u(\rho,t) = R(\rho)T(t)$ and using separation of variables. HINT: The ODE governing $R(\rho)$ can be solved in terms of Bessel functions using the formulas given in Section 4.6.6, and these results can be simplified using the formulas

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\sin x, \qquad J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\,\cos x.$$

You will need to distinguish the cases $\kappa \neq 0$ and $\kappa = 0$.

**29.** (*Conduction in a metal ring*) Here we reconsider the problem of Exercise 14(a). There we consider the $x$ domain to be $-\infty < x < \infty$, so $u(x,t)$ and $f(x)$ were $L$-periodic in $x$ and could be expanded in Fourier series. Alternatively, think of the $x$ domain as finite: $0 < x < L$. That step creates the boundaries $x = 0$ and $x = L$, so we need boundary conditions there. Although we don't know $u$ or $u_x$ there, we do know that, physically, the ends $x = 0$ and $x = L$ are abutting, so both the temperature $u$ and the heat flux (proportional to $u_x$) must be continuous there. Thus, we can pose the problem (on the *finite* interval $0 < x < L$) as

$$\alpha^2 u_{xx} = u_t, \qquad (0 < x < L,\ 0 < t < \infty)$$
$$u(0,t) = u(L,t), \quad u_x(0,t) = u_x(L,t), \quad u(x,0) = f(x).$$

Solve that problem by separation of variables, leaving expansion coefficients in integral form. HINT: The Sturm–Liouville problem that arises on $X(x)$ will have periodic boundary conditions.

**30.** (a) Solve the problem

$$u_{xx} - 2u_x = u_t, \qquad (0 < x < L,\ 0 < t < \infty)$$
$$u(0,t) = u(L,t) = 50, \quad u(x,0) = 0$$

(30.1)

by separation of variables. HINT: Seeking $u(x,t) = X(x)T(t)$, obtain

$$X'' - 2X' + \kappa^2 X = 0,$$
$$T' + \kappa^2 T = 0. \qquad (30.2,3)$$

With $X(x) = e^{\lambda x}$, obtain $\lambda = 1 \pm \sqrt{1-\kappa^2}$, so

$$X(x) = e^x\left(C_1 e^{\sqrt{1-\kappa^2}\,x} + C_2 e^{-\sqrt{1-\kappa^2}\,x}\right),$$
$$T(t) = C_3 e^{-\kappa^2 t}. \qquad (30.4,5)$$

However, expecting oscillatory functions (for the eventual expansion that will be needed), anticipate that the $\kappa$'s will be greater than unity and write $\lambda = 1 \pm i\sqrt{\kappa^2 - 1}$ instead, so

$$X(x) = e^x\left(C_3 \cos\sqrt{\kappa^2-1}\,x + C_2 \sin\sqrt{\kappa^2-1}\,x\right).$$

(30.6)

Distinguish the case $\kappa = 1$ because if $\kappa = 1$ then (30.6) reduces to $X(x) = C_3 e^x$, which falls short of being a general solution of (30.2). We don't *need* to also distinguish the case $\kappa = 0$ because if $\kappa = 0$ then (30.4) and (30.5) do give the general solutions of (30.2) and (30.3), respectively. However, the case $\kappa = 0$ is of special importance because it gives the steady-state part of the solution [since it reduces $T(t)$ to a constant]. Thus, use the "three-tier" solutions

$$X(x) = \begin{cases} e^x\left(A\cos\omega x + B\sin\omega x\right), & \kappa \neq 0,1 \\ e^x(C + Dx), & \kappa = 1 \\ E + Fe^{2x}, & \kappa = 0 \end{cases}$$

$$T(t) = \begin{cases} Ge^{-\kappa^2 t}, & \kappa \neq 0,1 \\ He^{-t}, & \kappa = 1 \\ I, & \kappa = 0 \end{cases}$$

where $\omega \equiv \sqrt{\kappa^2 - 1}$, for brevity, and form $u(x,t)$ as the sum of their respective products.

(b) Same as part (a), but with $u(0,t) = 0$ and $u(L,t) = 50$. Leave expansion coefficients in integral form.

**31.** (a) Show that

$$\int_0^{z_n} xJ_0(x)\,dx = z_n J_1(z_n) \qquad (31.1)$$

where $z_n$ is any root of $J_0(x) = 0$. HINT: Integrate the

Bessel equation $(xJ_0')' + xJ_0 = 0$ from 0 to $z_n$ and use the relation $J_0'(x) = -J_1(x)$ [from Exercise 4, Section 4.6].

(b) Then, use (31.1) to verify the last step in (83).

## 18.4 Fourier and Laplace Transforms (Optional)

In Section 18.3 the $x$ domain is always finite, namely, $0 < x < L$. Semi-infinite $(0 < x < \infty)$ and infinite $(-\infty < x < \infty)$ domains are also important, and it is these cases that we now address. We organize our discussion around two examples.

**EXAMPLE 1.** *Heat Conduction in an Infinite Rod.* Let us begin with the problem

$$\alpha^2 u_{xx} = u_t, \qquad (-\infty < x < \infty,\ 0 < t < \infty) \tag{1a}$$

$$u(x,0) = f(x), \qquad (-\infty < x < \infty) \tag{1b}$$

summarized in Fig. 1.

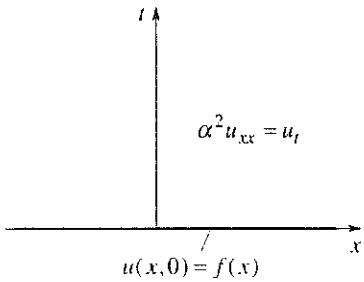



**Figure 1.** Infinite rod problem.

If we regard the infinite rod as the limiting case of a finite rod, on $-L < x < L$, as $L \to \infty$, and recall that boundary conditions are needed for the finite rod, we might well anticipate that some form of boundary conditions will be needed for the infinite rod at $x = \pm\infty$. But since a suitable form for those boundary conditions may not yet be apparent, let us defer that issue for the moment, in the hope that the solution process itself may provide a clue.

In selecting a solution technique, remember that if we use the method of separation of variables then we need, in the final step, to expand the $f(x)$ in (1b) in a Fourier-type series. If the $x$ domain is finite, then that series will be a half- or quarter-range cosine or sine series, or a generalized Fourier series containing the orthogonal eigenfunctions of a relevant Sturm–Liouville problem. If the $x$ domain is infinite, then we can still use separation of variables, provided that $f(x)$ is periodic with finite period, in which case we can expand $f(x)$ in a classical Fourier series of cosines and sines.

However, in this example we have an infinite $x$ domain and are interested in $f$'s that are *not* periodic, such as $e^{-|x|}$ (Fig. 2a) and the rectangular pulse shown in Fig. 2b. Such $f$'s can be represented not by Fourier series but by Fourier integrals, which fact suggests seeking a solution for $u$ in Fourier integral form or, equivalently and more conveniently, using a Fourier transform.

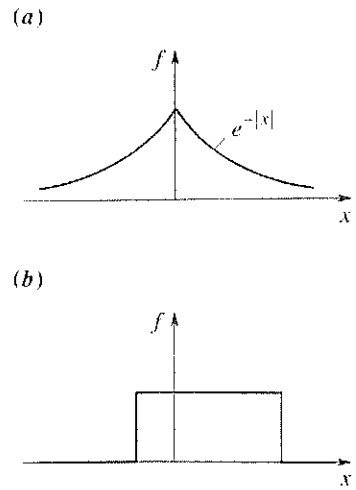



**Figure 2.** Nonperiodic $f$'s.

Thus, let us Fourier transform (1a) with respect to $x$:

$$F\{\alpha^2 u_{xx}\} = F\{u_t\}, \tag{2a}$$

$$\alpha^2 F\{u_{xx}\} = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t} e^{-i\omega x}\, dx, \tag{2b}$$

$$\alpha^2 (i\omega)^2 \hat{u} = \frac{d}{dt}\int_{-\infty}^{\infty} u(x,t) e^{-i\omega x}\, dx \tag{2c}$$

$$= \frac{d\hat{u}}{dt}, \tag{2d}$$

so

$$\frac{d\hat{u}}{dt} + \alpha^2\omega^2\hat{u} = 0, \tag{3}$$

with solution

$$\hat{u} = Ae^{-\alpha^2\omega^2 t}. \tag{4}$$

In passing from (2a) to (2b) we used the linearity of $F\{\ \}$ on the left, and the definition of the Fourier transform on the right. Next, we used

$$F\{u_{xx}\} = (i\omega)^2 F\{u\} = (i\omega)^2\hat{u} \tag{5}$$

on the left, and the Leibniz differentiation formula

$$\frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)e^{-i\omega x}\,dx = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}\,e^{-i\omega x}\,dx \tag{6}$$

on the right. For (5) to hold, we need

$$u \to 0 \text{ and } u_x \to 0 \text{ as } x \to \pm\infty, \tag{7}$$

so let us adopt (7) as our boundary conditions. For the output $u$ (and $u_x$) to tend to zero as $x \to \pm\infty$, we expect that we will need to restrict the input $f$ to die out sufficiently fast, as $x \to \pm\infty$, as well. However, as suggested in Section 18.3 let us proceed *formally* to a solution without getting bogged down with such technical points. If we like, we can then rigorously verify that that solution does satisfy the PDE and any boundary and initial conditions (if indeed it does).

Notice carefully that by Fourier transforming (1a) with respect to $x$ we convert the *partial* differential equation (1a) on $u(x,t)$ to the *ordinary* differential equation (3) on $\hat{u}(\omega,t)$, in which $\omega$ appears as a parameter – that is, as a constant.

To evaluate the integration constant $A$ in (4) we impose the initial condition (1b), but first we must take the Fourier transform of (1b): $\hat{u}(\omega,0) = \hat{f}(\omega)$. Thus,

$$\hat{u}\Big|_{t=0} = \hat{f}(\omega) = \left(Ae^{-\alpha^2\omega^2 t}\right)\Big|_{t=0} = A, \tag{8}$$

so $A = \hat{f}(\omega)$,* and (4) becomes

$$\hat{u}(\omega,t) = \hat{f}(\omega)e^{-\alpha^2\omega^2 t}. \tag{9}$$

Finally, using entry 6 of Appendix D and the Fourier convolution property (entry 21), it follows from (9) that

$$u(x,t) = f(x) * \frac{1}{2\alpha\sqrt{\pi t}}\,e^{-x^2/(4\alpha^2 t)} \tag{10}$$

or

$$u(x,t) = \frac{1}{2\alpha\sqrt{\pi t}}\int_{-\infty}^{\infty} f(\xi)e^{-(x-\xi)^2/(4\alpha^2 t)}\,d\xi. \tag{11}$$

*You may be concerned that $A$ was to be a constant, yet $\hat{f}(\omega)$ is a function of $\omega$. The point to keep in mind is that (3) is a differential equation in $t$, and $\omega$ is regarded there as a constant.

COMMENT 1. As a general principle it is important to check one's result for any special cases for which the solution is known. The simplest case that comes to mind is the case where

$$f(x) = \text{constant} \equiv F, \tag{12}$$

for in that case it is obvious that the solution to (1) is simply $u(x,t) = F$. In fact, if we set $f(\xi) = F$ in (11) and evaluate the integral we do obtain $u(x,t) = F$ (Exercise 1). It is striking that (11) gives the correct result for the case where $f(x) = F$ because in that case $\hat{f}(\omega)$ does not even exist, and the solution $u(x,t) = F$ violates the assumption that $u \to 0$ as $x \to \pm\infty$.*

COMMENT 2. As a more interesting special case, let

$$f(x) = \begin{cases} F, & x > 0 \\ 0, & x < 0 \end{cases} = FH(x), \tag{13}$$

where $F$ is a constant and $H(x)$ is the Heaviside function. In this case, putting (13) into (11) gives (Exercise 2)

$$u(x,t) = \frac{F}{2}\left[1 + \operatorname{erf}\left(\frac{x}{2\alpha\sqrt{t}}\right)\right], \tag{14}$$

where

$$\boxed{\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-\xi^2}\,d\xi} \tag{15}$$

is a tabulated function known as the **error function**. That is, the integral in (15) cannot be evaluated in terms of the so-called elementary functions, so it is given its own name, the "error function," and its properties, tabulated values, and computational formulas for it can be found in the literature.† The factor $2/\sqrt{\pi}$ is included to "normalize" $\operatorname{erf}(x)$ so that $\operatorname{erf}(\infty) = 1$ because

$$\boxed{\int_0^\infty e^{-\xi^2}\,d\xi = \frac{\sqrt{\pi}}{2},} \tag{16}$$

*The greater-than-expected validity of (11) can be traced to the inversion of the order of integration that is inherent in the Fourier convolution step.

†The trend in computing has been away from tabulations and toward approximate expressions, not only for $\operatorname{erf}(x)$ but for the various special functions: Bessel functions, the gamma function, and so on. For instance, the formula

$$\operatorname{erf}(x) \approx 1 - (a_1 p + a_2 p^2 + a_3 p^3)e^{-x^2},$$

where

$$p = \frac{1}{1 + 0.47047x}, \quad a_1 = 0.3480242, \quad a_2 = -0.0958798, \quad a_3 = 0.7478556,$$

developed by C. Hastings, Jr., is uniformly accurate, over $0 \le x < \infty$, to $\pm 0.000025$. The latter formula is an example of a common form of approximation known as **rational function approximation** because the function $\exp(x^2)[1 - \operatorname{erf}(x)]$ is being approximated by a rational function of $x$, namely, the ratio of two polynomials (as can be seen by combining the terms in $a_1 p + a_2 p^2 + a_3 p^3$ over a common denominator). For approximations such as this, see M. Abramowitz and I. Stegun (eds.), National Bureau of Standards Applied Math Series, 1964, or Y. L. Luke, *Mathematical Functions and Their Approximation* (New York: Academic Press, 1975).

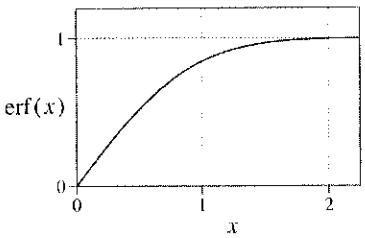



**Figure 3.** The error function.

derivation of which result is outlined in Exercise 9 of Section 4.5. The graph of $\operatorname{erf}(x)$ is shown in Fig. 3 for $x > 0$; for $x < 0$ we rely on the fact that the error function is an odd function (Exercise 4), so that $\operatorname{erf}(-x) = -\operatorname{erf}(x)$. Besides the integral from 0 to $x$, in (15), we also encounter the integral from $x$ to $\infty$ so frequently that we also define a **complementary error function**,

$$\begin{aligned}\operatorname{erfc}(x) &= \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-\xi^2}\,d\xi = \operatorname{erf}(\infty) - \operatorname{erf}(x)\\ &= 1 - \operatorname{erf}(x).\end{aligned} \tag{17}$$

The solution (14) is plotted in Fig. 4 at representative times. Observe carefully how the initially discontinuous temperature distribution smooths out as $t$ increases. This result illustrates the fact that, in physical terms, diffusion is a *smoothing process*.

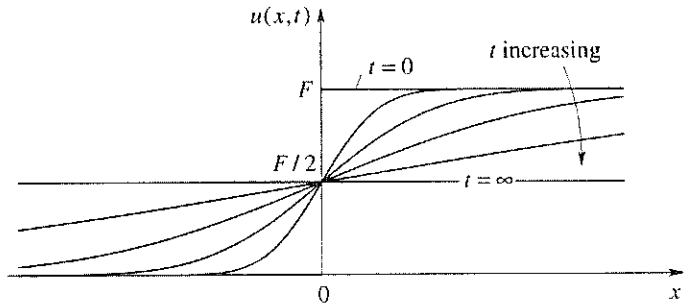



**Figure 4.** The solution (14).

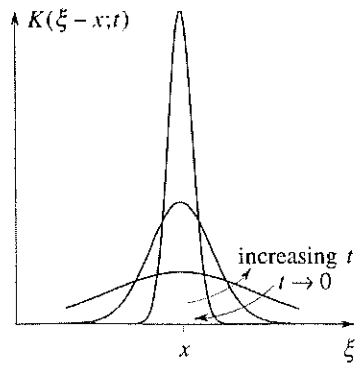



**Figure 5.** The delta-sequence behavior of $K$ as $t \to 0$, and as $t$ increases.

COMMENT 3. It is instructive to write (11) in the form

$$u(x,t) = \int_{-\infty}^{\infty} f(\xi)K(\xi - x;t)\,d\xi, \tag{18a}$$

where*

$$K(\xi - x;t) = \frac{e^{-(x-\xi)^2/(4\alpha^2 t)}}{2\alpha\sqrt{\pi t}} \tag{18b}$$

is called the **kernel**. That is, it comprises everything in the integrand other than the input $f(\xi)$. The kernel $K(\xi - x;t)$ happens to be a **normal** (or **Gaussian**) **probability distribution** centered at $\xi = x$, which result correctly suggests that diffusion is an essentially statistical phenomenon. The area between the graph of $K$ and the $\xi$ axis is unity for all $t$ (Exercise 3), and that graph becomes more and more focused as $t \to 0$ (Fig. 5). If you studied Section 5.6 you will see that $K$ appears to approach a delta function at $x$, $\delta(\xi - x)$, as $t \to 0$ because it has unit area for each $t$, and it becomes focused at $x$ as $t \to 0$. In fact,

*Here, $\xi$ is the active variable since it is the variable of integration; $x$ and $t$ are regarded as fixed in (18a). If we write $K(\xi; x, t)$ rather than $K(\xi, x, t)$, we are emphasizing that $\xi$ is the active variable and that $x$ and $t$ are, at least for the moment, fixed. Even so, we have written $K(\xi - x;t)$ instead because only the difference $\xi - x$ occurs in (18b). We say that $K(\xi - x;t)$ is a **difference kernel**. The present example is similar to Example 8 of Section 17.10, which we urge you to review when you have finished reading this example.

that observation is confirmed by the initial condition, which is

$$\lim_{t\to 0} u(x,t) = \lim_{t\to 0} \int_{-\infty}^{\infty} f(\xi)K(\xi - x; t)\, d\xi = f(x). \tag{19}$$

That is, the initial condition (19) is satisfied *by virtue* of the fact that $K(\xi - x; t)$ tends to a delta function $\delta(\xi - x)$ as $t \to 0$.

COMMENT 4. Since the kernel $K$ is evidently important, we would do well to try to understand its physical significance. To do so, let the input itself be a delta function at some point $x_0$,

$$f(x) = \delta(x - x_0). \tag{20}$$

In a crude way, this case can be conceptualized as corresponding to the application of a welding torch to the rod at $x_0$ for a brief instant. Then (18a) gives

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{\infty} \delta(\xi - x_0)K(\xi - x; t)\, d\xi \\ &= K(x_0 - x; t) = K(x - x_0; t). \end{aligned} \tag{21}$$

The second equality in (21) follows from the fundamental property

$$\int_{-\infty}^{\infty} \delta(\xi - a)g(\xi)\, d\xi = g(a) \tag{22}$$

of the delta function, and the third equality in (21) follows from the fact that $K$ is an even function of $x_0 - x$ [because it contains $(x_0 - x)^2$]. The upshot is that the kernel $K(x - x_0; t)$ is itself a solution of the heat conduction equation (1a), corresponding to an initial temperature distribution $\delta(x - x_0)$. Thus, $K(\xi - x; t)$, the graph of which is shown in Fig. 5, is the temperature distribution in the rod that results from an initial temperature distribution that is a delta function at $x$. Once again, we see the smoothing nature of the diffusion process, for beginning with the spike-like temperature profile $\delta(\xi - x)$ the temperature distribution $u(x,t)$ smooths out more and more as $t$ increases (Fig. 5). Finally, we can now understand the superposition nature of (18a), for if $K(x - \xi; t)$ is the temperature response to an initial temperature that is a delta function (hence having unit area) at $x$, then the response $du(x,t)$ to the rectangular-pulse initial temperature shown in Fig. 6, having area $f(\xi)\, d\xi$, is $K(x - \xi; t)$ scaled by $f(\xi)\, d\xi$,

$$du(x,t) = K(x - \xi; t)[f(\xi)\, d\xi] = K(\xi - x; t)[f(\xi)\, d\xi]. \tag{23}$$

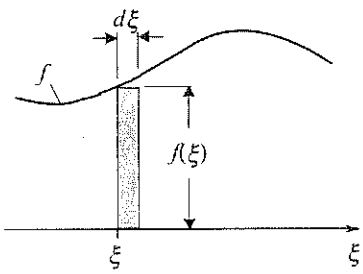



**Figure 6.** Breaking $f$ into rectangular pulses.

Adding such results for all of the rectangular pulses that comprise $f$ gives the integral (18a).

COMMENT 5. As a final observation about the physics, observe that the solution (18) indicates the spreading of information, by diffusion, with an *infinite velocity*. For instance, in Fig. 4 we see that some of the heat that was initially confined to the interval $0 < x < \infty$ diffuses to the interval $-\infty < x < 0$ over any arbitrarily small time $t$. This spreading can occur only if the speed of propagation is infinite. Inasmuch as it is agreed that energy cannot propagate faster than the speed of light, this result evidently reveals a flaw in our diffusion

equation $\alpha^2 u_{xx} = u_t$, one that is discussed in the literature,* and which is inconsequential in practical applications of the theory.

COMMENT 6. We were able to use a Fourier transform on $x$ to solve (1) because the latter was a boundary value problem on $-\infty < x < \infty$. Alternatively, we could have used a Laplace transform on $t$ because (1) was an initial value problem on $0 < t < \infty$. The latter approach proves to be less attractive because we obtain a *non*homogeneous ODE on $\overline{u}(x, s)$ rather than the homogeneous ODE (3) on $\hat{u}(\omega, t)$. (See Exercise 5.) ∎

In Example 1 we illustrate the use of the Fourier transform for a problem on the conduction of heat in an infinite rod ($-\infty < x < \infty$). In Example 2 we illustrate the use of the Laplace transform for a semi-infinite rod ($0 < x < \infty$) problem.

**EXAMPLE 2.** *Heat Conduction in a Semi-Infinite Rod.* This time, we consider the problem

$$\alpha^2 u_{xx} = u_t, \qquad (0 < x < \infty,\ 0 < t < \infty) \tag{24a}$$

$$u(x, 0) = 0, \qquad (0 < x < \infty) \tag{24b}$$

$$u(0, t) = g(t), \qquad (0 < t < \infty) \tag{24c}$$

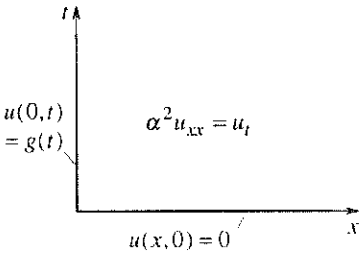



**Figure 7.** Semi-infinite rod problem.

as summarized in Fig. 7. The rod is initially at 0° throughout, we subject the left end ($x = 0$) to a prescribed temperature $g(t)$, and we seek the temperature distribution $u(x, t)$ that develops. We will also need a boundary condition at the right end ($x = \infty$), which we take to be $u(\infty, t) = 0$ for all $t$; that is,

$$\lim_{x \to \infty} u(x, t) = 0. \qquad (0 < t < \infty) \tag{24d}$$

In this case a Fourier transform is inappropriate because the domain is $0 < x < \infty$ rather than $-\infty < x < \infty$, so let us try a Laplace transform on $t$.* Thus, Laplace transform (24a) with respect to $t$:

$$L\{\alpha^2 u_{xx}\} = L\{u_t\}. \tag{25}$$

Now,

$$\begin{aligned} L\{\alpha^2 u_{xx}\} &= \alpha^2 L\{u_{xx}\} \qquad \text{(linearity of } L\text{)} \\ &= \alpha^2 \int_0^\infty \frac{\partial^2 u}{\partial x^2} e^{-st}\, dt \qquad \text{(definition of transform)} \\ &= \alpha^2 \frac{d^2}{dx^2} \int_0^\infty u(x, t) e^{-st}\, dt \qquad \text{(Leibniz rule)} \\ &= \alpha^2 \frac{d^2}{dx^2}\, \overline{u} \\ &= \alpha^2 \overline{u}_{xx}(x, s), \end{aligned} \tag{26}$$

*See, for example, P. M. Morse and H. Feshbach, *Methods of Theoretical Physics*, Part I (New York: McGraw-Hill, 1953), pp. 865–869.

*Alternatively, we could use a semi-infinite Fourier transform, namely, a Fourier cosine or sine transform. These are studied in the optional Section 17.11.

and

$$\begin{aligned} L\{u_t\} &= s\overline{u}(x,s) - u(x,0) \\ &= s\overline{u}(x,s), \end{aligned} \tag{27}$$

so (25) becomes $\alpha^2\overline{u}_{xx} = s\overline{u}$, or

$$\overline{u}_{xx} - \frac{s}{\alpha^2}\,\overline{u} = 0, \tag{28}$$

which is a second-order ODE with respect to $x$, $t$ having been eliminated by the Laplace transform process. Solving (27),

$$\overline{u}(x,s) = Ae^{\sqrt{s}\,x/\alpha} + Be^{-\sqrt{s}\,x/\alpha}. \tag{29}$$

We have already used (24a) and (24b). To solve for $A$ and $B$ we use the boundary conditions (24c) and (24d), but first we need to express those conditions in terms of $s$ rather than $t$. For the condition at $x = \infty$ observe that

$$\begin{aligned} \lim_{x\to\infty} \overline{u}(x,s) &= \lim_{x\to\infty} \int_0^\infty u(x,t)e^{-st}\,dt \\ &= \int_0^\infty \lim_{x\to\infty} u(x,t)e^{-st}\,dt = \int_0^\infty 0\,dt = 0; \end{aligned} \tag{30}$$

that is, $u(x,t) \to 0$ as $x \to \infty$ implies that

$$\overline{u}(x,s) \to 0 \ \text{ as } \ x \to \infty \tag{31}$$

as well. [Note that the second equality in (30) was carried out only formally inasmuch as we did not rigorously justify the interchange in the order of the two limit processes: the limit $x \to \infty$ and the limit process that lies behind the Riemann integral.] Applying (31) to (29) gives $A = 0$, so

$$\overline{u}(x,s) = Be^{-\sqrt{s}\,x/\alpha}. \tag{32}$$

To express (24c) in terms of $s$, take its Laplace transform:

$$\int_0^\infty u(0,t)e^{-st}\,dt = \int_0^\infty g(t)e^{-st}\,dt,$$

or

$$\overline{u}(0,s) = \overline{g}(s), \tag{33}$$

and imposing the latter upon (32) gives

$$\overline{u}(0,s) = \overline{g}(s) = Be^0, \tag{34}$$

so $B = \overline{g}(s)$ and

$$\overline{u}(x,s) = \overline{g}(s)e^{-\sqrt{s}\,x/\alpha}. \tag{35}$$

To invert, use entry 21 of Appendix C [with $a = x/(2\alpha)$] and the convolution property (entry 28):

$$\begin{aligned} u(x,t) &= g(t) * \frac{xe^{-x^2/(4\alpha^2 t)}}{2\alpha\sqrt{\pi}\,t^{3/2}} \\ &= \frac{x}{2\alpha\sqrt{\pi}} \int_0^t g(t-\tau)\,\frac{e^{-x^2/(4\alpha^2\tau)}}{\tau^{3/2}}\,d\tau, \end{aligned} \tag{36}$$

or equivalently,

$$u(x,t) = \frac{x}{2\alpha\sqrt{\pi}} \int_0^t g(\tau)\, \frac{e^{-x^2/[4\alpha^2(t-\tau)]}}{(t-\tau)^{3/2}}\, d\tau \tag{37}$$

if we prefer.

COMMENT 1. The application of (31) to the solution (29) was easy: the positive exponential grows with $x$ and the negative exponential dies out, so $A = 0$ and $B$ remained arbitrary. What if we had used the solution form

$$\overline{u}(x,s) = C\cosh\frac{\sqrt{s}\,x}{\alpha} + D\sinh\frac{\sqrt{s}\,x}{\alpha} \tag{38}$$

instead of (29), for *both* terms on the right-hand side tend to infinity as $x \to \infty$? Let us see:

$$\begin{aligned}\overline{u}(x,s) &= \frac{C}{2}\left(e^{\sqrt{s}\,x/\alpha} + e^{-\sqrt{s}\,x/\alpha}\right) + \frac{D}{2}\left(e^{\sqrt{s}\,x/\alpha} - e^{-\sqrt{s}\,x/\alpha}\right)\\ &= \frac{C+D}{2}e^{\sqrt{s}\,x/\alpha} + \frac{C-D}{2}e^{-\sqrt{s}\,x/\alpha}.\end{aligned} \tag{39}$$

To eliminate the positive exponential, set $D = -C$. Then (39) becomes $\overline{u}(x,s) = C\exp-(\sqrt{s}\,x/\alpha)$, which is equivalent to (32). The point is that either form will work, (29) or (38), but (29) is more convenient insofar as the application of the condition (31).

COMMENT 2. We did not specify the function $g(t)$. Observe that if $g(t)$ *were* specified, it would have been foolish to work out its transform $\overline{g}(s)$ because when we apply the convolution theorem to (35) we invert $\overline{g}(s)$ and get back the given function $g(t)$. Thus, it is best to merely call the transform $\overline{g}(s)$, as we did.

COMMENT 3. Observe from (37) that $u(x,t)$ depends upon the boundary data $g(t)$ only over $0 < \tau < t$, not over $0 < \tau < \infty$. This result is entirely reasonable since how could the temperature distribution $u(x,t)$ today depend upon the boundary values $g(t)$ to be imposed tomorrow?

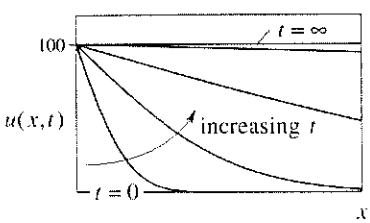



**Figure 8.** $u(x,t)$ given by (40).

COMMENT 4. Finally, let us use (37) to determine $u(x,t)$ for a specific case, say $g(t) = 100°$. With $g(t)$ a constant, it is easier to use (36) than (37), and the latter gives (Exercise 13)

$$\begin{aligned}u(x,t) &= \frac{100x}{2\alpha\sqrt{\pi}} \int_0^t \frac{e^{-x^2/(4\alpha^2\tau)}}{\tau^{3/2}}\, d\tau\\ &= 100\,\mathrm{erfc}\left(\frac{x}{2\alpha\sqrt{t}}\right),\end{aligned} \tag{40}$$

which is plotted in Fig. 8 at representative times. ∎

**Closure.** In Section 18.3 we learn how to solve the diffusion equation by separation of variables, but the $x$ domain is always finite and the boundary conditions do not vary with $t$ (although certain more complicated cases are outlined in the end-of-section exercises). The Fourier and Laplace transforms enable us to deal with

semi-infinite ($0 < x < \infty$) and infinite ($-\infty < x < \infty$) domains, as well as nonconstant boundary conditions.

Application of the Laplace and Fourier transform to the diffusion equation $\alpha^2 u_{xx} = u_t$ proceeds very much along the same lines as for ordinary differential equations, but instead of producing an algebraic equation it produces an ordinary differential equation. Specifically, Laplace transforming on $t$ produces the ODE $\alpha^2 \overline{u}_{xx} - s\overline{u} = -u(x,0)$ on $\overline{u}(x,s)$, and Fourier transforming on $x$ (if $-\infty < x < \infty$) produces the ODE $\hat{u}_t + \alpha^2\omega^2\hat{u} = 0$ on $\hat{u}(\omega,t)$.

## EXERCISES 18.4

NOTE: Exercises 1–10 relate to Example 1; the remainder to Example 2.

**1.** Show that for the case where $f(x) = \text{constant} \equiv F$, (11) gives $u(x,t) = F$. HINT: Make the change of variables $(x-\xi)^2/(4\alpha^2 t) = \mu^2$ in the integral, and use the known integral $\int_{-\infty}^{\infty} \exp(-\xi^2)\,d\xi = \sqrt{\pi}$.

**2.** (a) Show that for the case where $f(x) = FH(x)$, in (13), (11) gives (14) as the solution.
(b) Verify, directly, that (14) satisfies the PDE (1a) and the initial condition (1b).

**3.** Prove the claim, made in Comment 3 of Example 1, that $\int_{-\infty}^{\infty} K(\xi - x; t)\,d\xi = 1$ for all $t$, where $K$ is given by (18b). HINT: Use the change of variables suggested in Exercise 1.

**4.** Show that $\operatorname{erf}(x)$, defined by (15), is an odd function, as is claimed in Comment 2 of Example 1.

**5.** We used the Fourier transform to solve (1). Use the Laplace transform instead, and obtain the ODE

$$\overline{u}_{xx} - \frac{s}{\alpha^2}\overline{u} = -\frac{1}{\alpha^2}f(x).$$

**6.** Verify, by direct substitution, that the kernel $K$ given by (18b) satisfies the diffusion equation (1a), as was claimed in Example 1.

**7.** Use (18) to show that if the initial temperature $f(x)$ is
(a) a periodic function of $x$, with period $\tau$, then so is the solution $u(x,t)$.
(b) an odd function of $x$, then so is the solution $u(x,t)$.
(c) an even function of $x$, then so is the solution $u(x,t)$.

**8.** (*Inclusion of a source term*) In Example 3 of Section 16.8 we find that if there is a heat source distribution $F(x,t)$ within the medium, then in place of the homogeneous field equation $L[u] = \alpha^2 u_{xx} - u_t = 0$ we have the nonhomogeneous equation $L[u] = \alpha^2 u_{xx} - u_t = -F(x,t)$; $F$ acts as a source where $F > 0$ and as a sink where $F < 0$.
(a) Then the problem

$$\begin{aligned} \alpha^2 u_{xx} - u_t &= -F(x,t), && (-\infty < x < \infty,\ 0 < t < \infty) \\ u(x,0) &= f(x), && (-\infty < x < \infty) \end{aligned} \tag{8.1}$$

together with suitable boundary conditions at $x = \pm\infty$, has two inputs, the initial temperature $f(x)$ and the source distribution $F(x,t)$. By linearity, the response $u(x,t)$ should be the sum of the responses to these individual inputs. Specifically, show that if $v(x,t)$ and $w(x,t)$ are solutions to the problems

$$\begin{aligned} \alpha^2 v_{xx} - v_t &= 0, && (-\infty < x < \infty,\ 0 < t < \infty) \\ v(x,0) &= f(x) && (-\infty < x < \infty) \end{aligned} \tag{8.2}$$

and

$$\begin{aligned} \alpha^2 w_{xx} - w_t &= -F(x,t), && (-\infty < x < \infty,\ 0 < t < \infty) \\ w(x,0) &= 0, && (-\infty < x < \infty) \end{aligned} \tag{8.3}$$

respectively, then $u(x,t) = v(x,t) + w(x,t)$. Since the $v$ problem is already solved in Example 1, the remainder of this exercise is devoted to the $w$ problem.
(b) Consider the case where $F = F(t)$ is a function of $t$ alone. Choosing between the Fourier and Laplace transforms, solve for $w$. (The answer should be in the form of an integral.) Explain why you selected the transform that you did, and not the other.
(c) Now consider the case where $F = F(x)$ is a function of $x$ alone. Using a Fourier transform, show that

$$w(x,t) = F(x) * F^{-1}\left\{\frac{1-e^{-\alpha^2\omega^2 t}}{\alpha^2\omega^2}\right\}. \tag{8.4}$$

(d) The inverse needed in (8.4) is not found in our brief table (Appendix D). Nonetheless, show that

$$\begin{aligned} &F^{-1}\left\{\frac{1-e^{-\alpha^2\omega^2 t}}{\alpha^2\omega^2}\right\} \\ &= \frac{1}{\alpha}\sqrt{\frac{t}{\pi}}\,e^{-x^2/(4\alpha^2 t)} - \frac{x}{2\alpha^2}\operatorname{erfc}\left(\frac{x}{2\alpha\sqrt{t}}\right), \end{aligned} \tag{8.5}$$

so

$$\begin{aligned} w(x,t) = \frac{1}{\alpha}\int_{-\infty}^{\infty} F(x-\xi)\Bigg[&\sqrt{\frac{t}{\pi}}\,e^{-\xi^2/(4\alpha^2 t)} \\ &-\frac{\xi}{2\alpha}\operatorname{erfc}\left(\frac{\xi}{2\alpha\sqrt{t}}\right)\Bigg]\,d\xi. \end{aligned} \tag{8.6}$$

HINT: Letting $\hat{g} = (1-e^{-\alpha^2\omega^2 t})/(\alpha^2\omega^2)$, show that $\hat{g}_t = e^{-\alpha^2\omega^2 t} = \hat{g}_t$, so that $g_t = \{\exp[-x^2/(4\alpha^2 t)]\}/(2\alpha\sqrt{\pi t})$, with $g|_{t=0} = 0$. Thus,

$$g(x,t) = \int_0^t \frac{e^{-x^2/(4\alpha^2\tau)}}{2\alpha\sqrt{\pi\tau}}\,d\tau.$$

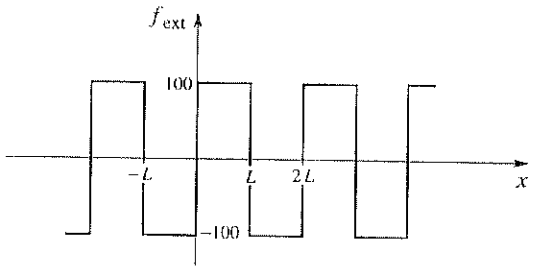
Then, use change of variables and integration by parts.

**9.** (*Small-t solution for finite rod*) In Section 18.3 we obtain, by separation of variables, the solution

$$u(x,t) = \frac{400}{\pi}\sum_{n=1,3,\ldots}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{L}\,e^{-(n\pi\alpha/L)^2 t} \tag{9.1}$$

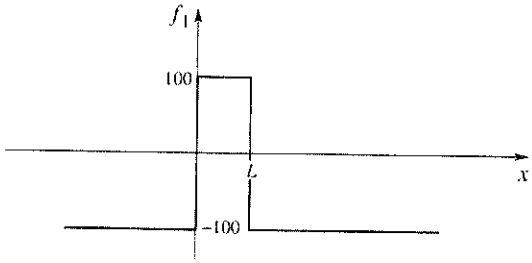
to the problem of heat conduction in a rod of length $L$ initially at a uniform temperature $u(x,0) = 100°$, with both ends held subsequently at 0°. It was pointed out that (9.1) converges rapidly if $t$ is large and slowly if $t$ is small. Our object in this exercise is to show how to use the results obtained in this section to obtain a complementary result: a series solution that *converges rapidly for small* $t$ (and slowly for large $t$). First, observe that our use of the half-range sine series to solve the stated finite-rod problem, on $0 < x < L$, is equivalent to solving the periodically extended *infinite*-rod problem

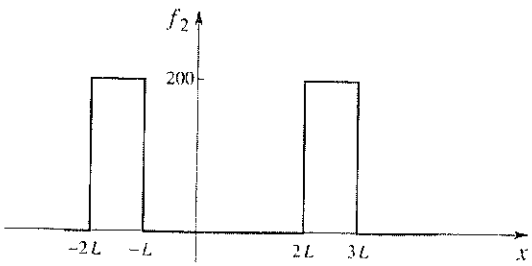
$$\begin{aligned} &\alpha^2 u_{xx} = u_t, \qquad (-\infty < x < \infty,\ 0 < t < \infty) \\ &u(x,0) = f_{\text{ext}}(x), \end{aligned}$$




where $f_{\text{ext}}$ is the square wave shown here. In effect, the separation-of-variables solution amounts to expanding $f_{\text{ext}}$ in a Fourier sine series, finding the response due to each sine term, and then adding them. Alternatively, suppose that we expand $f_{\text{ext}}(x)$ as $f_{\text{ext}}(x) = f_1(x) + f_2(x) + \cdots$, where the $f_j$'s are as shown. That is,

$$\begin{aligned} f_{\text{ext}}(x) = \{&-100 + 200[H(x) - H(x-L)]\} \\ &+200\{H(x+2L) - H(x+L) \\ &+H(x-2L) - H(x-3L)\} \\ &+200\{H(x+4L) - H(x+3L) \\ &+H(x-4L) - H(x-5L)\} + \cdots. \end{aligned} \tag{9.2}$$




(a) Recalling the solution (14), for the case where $f(x)$ is given by (13), show that the responses to $f_1, f_2, \ldots$ are

$$\begin{aligned} &u_1(x,t) \\ &= 100\left[\operatorname{erf}\left(\frac{x}{2\alpha\sqrt{t}}\right) - \operatorname{erf}\left(\frac{x-L}{2\alpha\sqrt{t}}\right) - 1\right] \end{aligned} \tag{9.3}$$

and

$$u_j(x,t) = 100\left[\operatorname{erf}\left(\frac{x+2(j-1)L}{2\alpha\sqrt{t}}\right) - \operatorname{erf}\left(\frac{x+(2j-3)L}{2\alpha\sqrt{t}}\right) + \operatorname{erf}\left(\frac{x-2(j-1)L}{2\alpha\sqrt{t}}\right) - \operatorname{erf}\left(\frac{x-(2j-1)L}{2\alpha\sqrt{t}}\right)\right], \tag{9.4}$$

for $j \geq 2$.

(b) Explain, in simple physical terms, why the series solution

$$u(x,t) = u_1(x,t) + u_2(x,t) + \cdots \tag{9.5}$$

should converge rapidly for small $t$.

(c) To illustrate, compute $u(x,t)$ at $x = 1$, $t = 0.1$, with $\alpha^2 = 1.14$ and $L = 10$, using equation (9.1), and again, using equation (9.5). Two-significant-figure accuracy will suffice. NOTICE: To calculate the error functions, use either the Hastings formula that we gave in a footnote, or use computer software such as the evalf *Maple* command.

(d) In (b) we stated that (9.5) should converge rapidly for small $t$. What do we really mean by small $t$? $t < 1$ sec? $t < [L/(4\alpha)]^2$? Propose some such inequality that can be used as a guarantee that (9.5) will indeed be rapidly convergent.

(e) Obtain a computer plot of the solution at $t = 0.1, 0.5$, and 1, using the approximation $u(x,t) \approx u_1(x,t)$, where $u_1$ is given in (9.3). Take $\alpha^2 = 1.14$ and $L = 10$, as in part (c).

**10.** (*Translating rod*) We saw in Section 18.2 that if the rod is translating rightward with constant speed $v$, then the PDE becomes $\alpha^2 u_{xx} = u_t + V u_x$, where $V = v/c$ and $c$ is the specific heat of the material. Use the Fourier transform to solve the problem.

$$\begin{aligned} \alpha^2 u_{xx} &= u_t + V u_x, \qquad (-\infty < x < \infty,\ 0 < t < \infty) \\ u(x,0) &= f(x), \qquad (-\infty < x < \infty) \end{aligned}$$

where $u \to 0$ and $u_x \to 0$ as $x \to \pm\infty$. Of course, your result should reduce to (18) for the case where $V = 0$.

**11.** Rework Example 2 with the initial condition changed to $u(x,0) = \text{constant} = u_0$, and show that

$$u(x,t) = u_0 + (u_1 - u_0)\operatorname{erfc}\left(\frac{x}{2\alpha\sqrt{t}}\right).$$

**12.** (*Oscillatory temperature at the left end*) If an oscillatory temperature is maintained at the left end of a semi-infinite rod, then we expect the solution $u(x,t)$ to approach a steady-state oscillation. Specifically, we have

$$\begin{aligned} \alpha^2 u_{xx} &= u_t, \qquad (0 < x < \infty,\ 0 < t < \infty) \\ u(0,t) &= u_0 \cos\omega t, \qquad (0 < t < \infty) \\ u &\to 0 \text{ as } x \to \infty. \qquad (0 < t < \infty) \end{aligned} \tag{12.1}$$

Derive the steady-state oscillatory solution

$$u(x,t) = u_0 e^{-rx}\cos(\omega t - rx), \tag{12.2}$$

where $r = \alpha\sqrt{\omega/2}$. HINT: The simplest approach is to consider, instead, the problem

$$\begin{aligned} \alpha^2 v_{xx} &= v_t \qquad (0 < x < \infty,\ 0 < t < \infty) \\ v(0,t) &= u_0 e^{i\omega t}, \qquad (0 < t < \infty) \\ v &\to 0 \text{ as } x \to \infty, \qquad (0 < t < \infty) \end{aligned} \tag{12.3}$$

which can then be solved by seeking $v$ in the form

$$v(x,t) = X(x)e^{i\omega t}. \tag{12.4}$$

Then $u$ is found as the real part of $v$:

$$u(x,t) = \operatorname{Re} v(x,t). \tag{12.5}$$

Such a **complex function method** of solution, for differential equations with oscillatory forcing terms, is the subject of Exercise 12 in Section 3.8. Alternatively, we could anticipate the phase shift caused by the $u_t$ term in the PDE and seek $u$, directly, in the form

$$u(x,t) = A(x)\cos\omega t + B(x)\sin\omega t,$$

but the complex function method is more attractive since it permits us to work with a single quantity $e^{i\omega t}$ rather than with a cosine and sine. Observe that there is no initial condition $u(x,0)$ included in (12.1) because we are concerned here only with the steady-state solution. NOTE: Besides the heat conduction application considered here, the problem (12.1) also arises (with the $t$'s changed to $y$'s) in fluid mechanics regarding the viscous flow, in the upper half plane $y > 0$, that is caused by harmonic oscillation of an infinite flat plate at $y = 0$. There, the problem is known as Stokes's second problem and was studied also by Lord Rayleigh.

**13.** (a) Show that the integral in (40) does give $u(x,t) = 100\operatorname{erfc}(x/(2\alpha\sqrt{t}))$.

(b) Use computer software to evaluate the latter at $x = 0, 2, 4, 6, \ldots, 20$, with $t = 10$ and $\alpha^2 = 1.14$.

**14.** (*Heat flux at left end*) The problem

$$\begin{aligned} \alpha^2 u_{xx} &= u_t, \qquad (0 < x < \infty,\ 0 < t < \infty) \\ u(x,0) &= 0, \qquad (0 < x < \infty) \\ u_x(0,t) &= -Q, \qquad (0 < t < \infty) \end{aligned} \tag{14.1}$$

with $u \to 0$ as $x \to \infty$, differs from (24) in that the boundary condition at $x = 0$ is of Neumann type rather than of Dirichlet type. Physically, $u_x(0,t) = -Q$ (where $Q$ is a prescribed constant) corresponds to the maintaining of a constant heat flux into the rod at $x = 0$, for all $t > 0$ (i.e., into the rod if $Q > 0$, out of the rod if $Q < 0$).

(a) Using the Laplace transform, derive the result

$$u(x,t) = \frac{\alpha Q}{\sqrt{\pi}} \int_0^t \frac{e^{-x^2/(4\alpha^2\tau)}}{\sqrt{\tau}}\, d\tau. \tag{14.2}$$

In particular, use (14.2) to solve for the temperature at the left end, $u(0,t)$, and sketch its graph.

(b) Show that the integral in (14.2) can be evaluated, and that we obtain

$$u(x,t) = Q\left[2\alpha\sqrt{\frac{t}{\pi}}e^{-x^2/(4\alpha^2 t)} - x\,\mathrm{erfc}\left(\frac{x}{2\alpha\sqrt{t}}\right)\right]. \tag{14.3}$$

**15.** We claimed, in a footnote, that Hastings's approximate formula for $\mathrm{erf}(x)$ is uniformly accurate, over $0 \le x < \infty$, to $\pm 0.000025$. Check that claim for $x = 0.5, 1$, and 2 by using that formula to compute $\mathrm{erf}(x)$, and comparing those results with results obtained either from computer software (such as the *Maple* evalf command) or from tables.

## 18.5 The Method of Images (Optional)

The method of images is a method of fictitiously extending the problem domain so as to satisfy homogeneous boundary conditions by means of symmetries or antisymmetries. In this section we not only illustrate the method for the diffusion equation, we also establish a class of PDE's for which the method works.

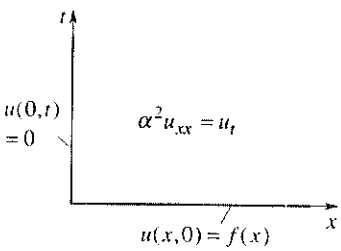



**Figure 1.** The problem (1).

**18.5.1. Illustration of the method.** To illustrate the method of images, consider the diffusion problem

$$\alpha^2 u_{xx} = u_t, \qquad (0 < x < \infty,\ 0 < t < \infty) \tag{1a}$$

$$u(x,0) = f(x), \qquad (0 < x < \infty) \tag{1b}$$

$$u(0,t) = 0, \qquad (0 < t < \infty) \tag{1c}$$

where $u \to 0$ and $u_x \to 0$ as $x \to \infty$ as depicted in Fig. 1.

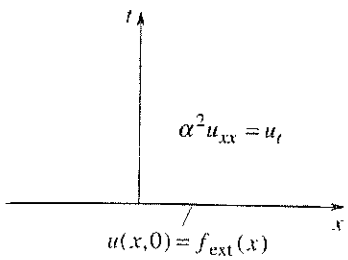



**Figure 2.** Extended problem.

The idea behind the image method is to extend the problem domain to $x = -\infty$ as shown in Fig. 2. In doing so, we also need to extend the initial condition to $x = -\infty$. Calling the extended function $f_{\text{ext}}$, we need $f_{\text{ext}}(x) = f(x)$ for $x > 0$, but for $x < 0$ we can define $f_{\text{ext}}(x)$ in any way we choose. Let us choose $f_{\text{ext}}(x)$ to be the *odd* extension of $f(x)$. For instance, if $f(x)$ is as shown in Fig. 3a, then $f_{\text{ext}}(x)$ is as shown in Fig. 3b. With $u(x,0) = f_{\text{ext}}(x)$ being an odd function of $x$ (i.e., antisymmetric about $x = 0$), we expect $u(x,t)$ to *remain* an odd function of $x$ for all $t > 0$, * and if $u(x,t)$ is an odd function of $x$ for all $t > 0$ then $u(0,t) = 0$ for all $t > 0$. That is, by building antisymmetry about $x = 0$ into the extended

*If $u(x,t)$ is an odd function of $x$, then $u(-x,t) = -u(x,t)$, so $u(0,t) = -u(0,t)$. Hence, $2u(0,t) = 0$, so $u(0,t) = 0$.

problem we are able to automatically satisfy the $u(0,t) = 0$ boundary condition in (1c).

Thus, the solution of the extended problem (Fig. 2) is also the solution of the original problem (Fig. 1). In fact, the extended problem was already solved in Example 1 of Section 18.4, so the desired solution of (1) is given by

$$u(x,t) = \int_{-\infty}^{\infty} f_{\text{ext}}(\xi) K(\xi - x; t)\, d\xi, \tag{2a}$$

where

$$K(\xi - x; t) = \frac{e^{-(\xi - x)^2/(4\alpha^2 t)}}{2\alpha\sqrt{\pi t}}. \tag{2b}$$

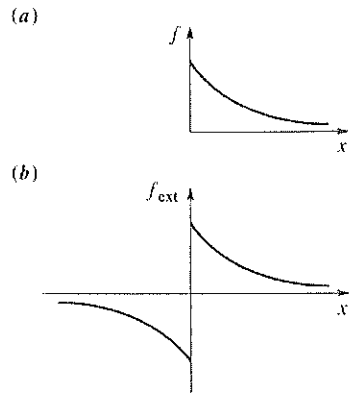



**Figure 3.** Odd extension.

Since we use (2) to compute $u$ only over the actual domain, $0 < x < \infty$, it is preferable (though not necessary) to re-express (2) so as to eliminate all reference to the fictitious extension, which extension is referred to as the **image system**. To do so, write

$$\begin{aligned} u(x,t) &= \int_{-\infty}^{0} f_{\text{ext}}(\xi) K(\xi - x; t)\, d\xi + \int_{0}^{\infty} f_{\text{ext}}(\xi) K(\xi - x; t)\, d\xi \\ &= \int_{\infty}^{0} f_{\text{ext}}(-\mu) K(-\mu - x; t)\,(-d\mu) + \int_{0}^{\infty} f_{\text{ext}}(\xi) K(\xi - x; t)\, d\xi \\ &= -\int_{\infty}^{0} [-f_{\text{ext}}(\mu)] K(\mu + x; t)\, d\mu + \int_{0}^{\infty} f_{\text{ext}}(\xi) K(\xi - x; t)\, d\xi, \end{aligned} \tag{3}$$

where we set $\xi = -\mu$ in the first integral and used the oddness of $f_{\text{ext}}$ [i.e., $f_{\text{ext}}(-\mu) = -f_{\text{ext}}(\mu)$] and the evenness of $K$ [i.e., $K(-\mu - x; t) = K(\mu + x; t)$] in the fifth integral. Finally, set $\mu = \xi$ and note that $f_{\text{ext}}(\xi) = f(\xi)$ over $0 < \xi < \infty$, and obtain

$$\begin{aligned} u(x,t) &= \int_{0}^{\infty} f(\xi)[K(\xi - x; t) - K(\xi + x; t)]\, d\xi \\ &= \int_{0}^{\infty} f(\xi) \frac{e^{-\frac{(\xi - x)^2}{4\alpha^2 t}} - e^{-\frac{(\xi + x)^2}{4\alpha^2 t}}}{2\alpha\sqrt{\pi t}}\, d\xi, \end{aligned} \tag{4}$$

which, as we desired, contains no reference to the image system.

Recall that we said that we "expect" $u(x,t)$ to remain an odd function of $x$ for all $t > 0$ and to thereby satisfy the condition (1c), that $u(0,t) = 0$ for all $t > 0$. With (4) in hand, we can now verify that claim since (4) gives

$$u(0,t) = \int_{0}^{\infty} f(\xi)(0)\, d\xi = 0. \tag{5}$$

**EXAMPLE 1.** For instance, let $f(x) = 100$ in (4). Then (4) gives (Exercise 1)

$$u(x,t) = 100 \operatorname{erf}\left(\frac{x}{2\alpha\sqrt{t}}\right). \tag{6}$$

We've plotted (6), at representative times, in Fig. 4, along with the fictitious extension (shown as dashed). Observe how the antisymmetry of $u$, with respect to $x$, ensures the satisfaction of the boundary condition $u(0,t) = 0$ for all $t > 0$. ∎

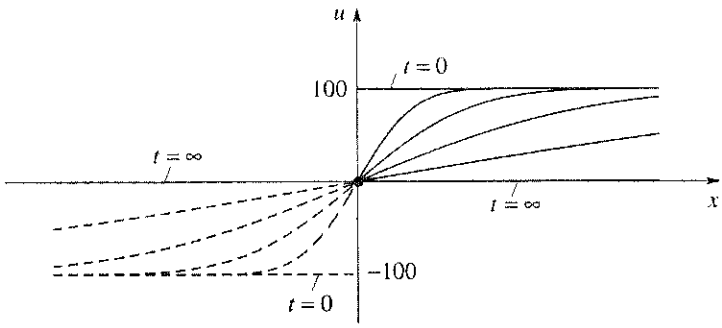



**Figure 4.** Solution for $f(x) = 100$.

Suppose that instead of the homogeneous Dirichlet condition (1c) we have the homogeneous Neumann condition

$$u_x(0,t) = 0. \qquad (0 < t < \infty) \tag{7}$$

In physical terms, instead of applying ice to the left end of the rod we insulate it so there is no heat flux across the face $x = 0$. In that case we extend $f(x)$ so as to be an *even* function, symmetric about $x = 0$. Then, we expect $u(x,t)$ to be a symmetric function of $x$ for all $t > 0$ so that, by virtue of that symmetry, we will have $u_x(0,t) = 0$ for all $t > 0$ (Exercise 2).

**18.5.2. Mathematical basis for the method.** The key point, in applying the method of images is that if

$$L[u] = \alpha^2 u_{xx} - u_t = 0 \qquad (-\infty < x < \infty,\ 0 < t < \infty) \tag{8a}$$

and

$$u(x,0) = f(x), \qquad (-\infty < x < \infty) \tag{8b}$$

together with suitable boundary conditions at $x = \pm\infty$, then $f(x)$ being odd implies that the solution $u(x,t)$ is an odd function of $x$ for all $t > 0$, and $f(x)$ being even implies that $u(x,t)$ is an even function of $x$ for all $t > 0$. Let us explain the mathematical basis for that claim, not just for the diffusion equation but for other PDE's as well.

We will draw upon the following elementary results, proof of which are left for the exercises.

1. Any function $F(x)$ can be split into even and odd parts, $F_e(x)$ and $F_o(x)$, respectively, as

$$F(x) = \underbrace{\frac{F(x) + F(-x)}{2}}_{F_e(x)} + \underbrace{\frac{F(x) - F(-x)}{2}}_{F_o(x)}. \tag{9}$$

2. The algebra of even and odd functions is as follows:

$$\begin{aligned} \text{even} \times \text{even} &= \text{even}, \\ \text{even} \times \text{odd} &= \text{odd}, \\ \text{odd} \times \text{odd} &= \text{even}. \end{aligned} \tag{10}$$

3. If $E_1(x)$ and $E_2(x)$ are even, and $O_1(x)$ and $O_2(x)$ are odd, then

$$E_1(x) + O_1(x) = E_2(x) + O_2(x) \tag{11}$$

implies that

$$E_1(x) = E_2(x) \qquad \text{and} \qquad O_1(x) = O_2(x). \tag{12}$$

4. If $F(x)$ is even, then $F'(0) = 0$.
5. If $F(x)$ is odd, then $F(0) = 0$.
6. If $F(x)$ is even, then $F(x), F''(x), F''''(x), \ldots$ are even and $F'(x)$, $F'''(x), \ldots$ are odd.
7. If $F(x)$ is odd, then $F(x), F''(x), F''''(x), \ldots$ are odd and $F'(x)$, $F'''(x), \ldots$ are even.

The foregoing results hold even if $F$ depends on other variables as well, such as $t$. For instance, if $F(x,t)$ is an even function of $x$, then $F(x,t), F_{xx}(x,t), F_{xxxx}(x,t)$, $\ldots$ are even functions of $x$, and $F_x(x,t), F_{xxx}(x,t), \ldots$ are odd functions of $x$.

To proceed, it will be useful to consider the problem

$$L[u] = \alpha^2 u_{xx} - V u_x - u_t = Q(x,t), \qquad (|x| < \infty, \ \ t > 0) \tag{13a}$$

$$u(x,0) = f(x), \qquad (|x| < \infty) \tag{13b}$$

which is more general than (1) by virtue of including the $Vu_x$ term (associated with translation of the rod at a constant speed) and the $Q$ term (associated with the presence of a distributed heat source or sink along the rod). Of course, (13) reduces to (1) if we set $V = 0$ and $Q(x,t) = 0$.

To track the development of the even and odd parts of $u$, let us break $u$ into the sum of its even and odd parts, say $E$ and $O$, respectively:

$$u(x,t) = E(x,t) + O(x,t). \tag{14}$$

Putting (14) into (13a), and also splitting $Q(x,t)$ and $f(x)$ into their even and odd parts, gives

$$\underbrace{\alpha^2 E_{xx}}_{e} + \underbrace{\alpha^2 O_{xx}}_{o} - \underbrace{V E_x}_{o} - \underbrace{V O_x}_{e} - \underbrace{E_t}_{e} - \underbrace{O_t}_{o} = \underbrace{Q_e}_{e} + \underbrace{Q_o}_{o}, \tag{15}$$

where $e$ or $o$ shown below each term indicates whether that term is even or odd, respectively. For instance, the $\alpha^2 O_{xx}$ term is odd according to item 7 above, and the $-VE_x$ term is odd according to item 6. Partial differentiation with respect to $t$ does not alter evenness or oddness (Exercise 7), so $E_t$ is even and $O_t$ is odd. Next, from item 3 it follows from (15) that $\alpha^2 E_{xx} - VO_x - E_t = Q_e$, and $\alpha^2 O_{xx} - VE_x - O_t = Q_o$. Similarly, it follows from (13b) that $E(x,0) = f_e(x)$ and $O(x,0) = f_o(x)$, so (13) can be split into the two problems

$$\alpha^2 E_{xx} - E_t = Q_e + VO_x, \qquad (-\infty < x < \infty,\ 0 < t < \infty) \tag{16a}$$
$$E(x,0) = f_e(x), \qquad (-\infty < x < \infty) \tag{16b}$$

and

$$\alpha^2 O_{xx} - O_t = Q_o + VE_x, \qquad (-\infty < x < \infty,\ 0 < t < \infty) \tag{17a}$$
$$O(x,0) = f_o(x). \qquad (-\infty < x < \infty) \tag{17b}$$

Suppose, first, that $V = 0$. Then (16) and (17) are uncoupled: (16) contains only $E(x,t)$ and (17) contains only $O(x,t)$. If both inputs $f$ and $Q$ are even, then $f_o = 0$ and $Q_o = 0$, the problem (17) on $O$ is homogeneous, and $O(x,t) \equiv 0$. *Thus, if the inputs are even, then the solution $u(x,t)$ is even.* If, on the other hand, $f$ and $Q$ are odd, then $f_e = 0$ and $Q_e = 0$, the problem on $E$ is homogeneous, and $E(x,t) \equiv 0$. *Thus, if the inputs are odd, then the solution $u(x,t)$ is odd.* The two italicized results enable us to use the method of images, as we did in Section 18.5.1.

However, if $V \neq 0$ then (16) and (17) are coupled by virtue of the $VO_x$ and $VE_x$ terms. For instance, even if $f$ and $Q$ are even, so $f_o = 0$ and $Q_o = 0$, (17) is still nonhomogeneous due to the $VE_x$, which acts as a source term and causes the development of a nonzero $O(x,t)$. Thus, we cannot use the method of images if $V \neq 0$, which result makes sense physically since translation of the rod (to the right if $V > 0$ and to the left if $V < 0$) will surely destroy any symmetry or antisymmetry in the solution about $x = 0$.

The upshot is that sometimes we can use the method of images and sometimes we cannot, depending on the operator $L$. What we need for the method of images to work is for $L$ to be both *linear* and *even*. We say that an operator $L$ is **even** if $L[u]$ is even whenever $u$ is even and odd whenever $u$ is odd; that is, an even operator preserves evenness and oddness.

Let us limit our attention to the linear operator

$$L = A\frac{\partial^2}{\partial x^2} + B\frac{\partial^2}{\partial t^2} + C\frac{\partial^2}{\partial x \partial t} + D\frac{\partial}{\partial x} + E\frac{\partial}{\partial t} + F, \tag{18}$$

where $A, \ldots, F$ are functions of $x$ and $t$. From the discussion above, we can see that $L$ will be an even operator if $A, B, E, F$ are even functions of $x$ and if $C, D$ are odd functions of $x$. In that case, let us re-express $L$ as

$$\boxed{L = E_1\frac{\partial^2}{\partial x^2} + E_2\frac{\partial^2}{\partial t^2} + O_1\frac{\partial^2}{\partial x \partial t} + O_2\frac{\partial}{\partial x} + E_3\frac{\partial}{\partial t} + E_4.} \tag{19}$$

This case is typical rather than exceptional.

**EXAMPLE 2.** For instance, consider the classical one-dimensional wave equation $c^2 u_{xx} = u_{tt}$ (studied in Chapter 19). In this case

$$L = c^2 \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial t^2}, \tag{20}$$

which is even because $A = c^2$ and $B = -1$ are even, $C = D = 0$ are odd, and $E = F = 0$ are even. (Remember that the zero function is both even and odd; it is even because its odd part is zero, and it is odd because its even part is zero.) ∎

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**THEOREM 18.5.1** *Applicability of the Method of Images*
Let $L$ be of the form (19) and therefore an even operator. Suppose that the problem

$$L[u] = Q(x,t), \qquad (-\infty < x < \infty,\ 0 < t < \infty) \tag{21a}$$
$$u(x,0) = f(x), \qquad (-\infty < x < \infty) \tag{21b}$$

where $u \to 0$ and $u_x \to 0$ as $x \to \pm\infty$, admits a unique solution $u(x,t)$. If $f$ and $Q$ are even functions of $x$ then so is $u$, and if $f$ and $Q$ are odd functions of $x$ then so is $u$.

---

*Proof*: Let $u(x,t) = E(x,t) + O(x,t)$, where $E$ is even and $O$ is odd. Then, $L[u] = L[E + O] = L[E] + L[O]$ because $L$ is linear. Further, $L[E]$ is even and $L[O]$ is odd because $L$ is even. Thus, (21) splits into the problems

$$L[E] = Q_e(x,t), \qquad (-\infty < x < \infty,\ 0 < t < \infty) \tag{22a}$$
$$E(x,0) = f_e(x), \qquad (-\infty < x < \infty) \tag{22b}$$

and

$$L[O] = Q_o(x,t), \qquad (-\infty < x < \infty,\ 0 < t < \infty) \tag{23a}$$
$$O(x,0) = f_o(x), \qquad (-\infty < x < \infty) \tag{23b}$$

where $E, E_x, O$, and $O_x$ tend to zero as $x \to \pm\infty$. If $f$ and $Q$ are even, then $f_o = 0$ and $Q_o = 0$, and (23) gives the unique solution $O(x,t) = 0$, so $u(x,t)$ is even. If $f$ and $Q$ are odd, then $f_e = 0$ and $O_e = 0$ and (22) gives the unique solution $E(x,t) = 0$, so $u(x,t)$ is odd. ∎

**Closure.** The method of images involves the fictitious extension of the problem domain such that homogeneous boundary conditions, which fall within the interior of the extended domain, are automatically satisfied by virtue of the symmetry or

antisymmetry of the various inputs. For the method to work, we need the symmetry (or antisymmetry) of the inputs to imply the symmetry (or antisymmetry) of the output $u$. Such symmetry (or antisymmetry) will, indeed, be passed on to $u$ if the operator $L$ is linear and even where, by $L$ being even, we mean that if $u$ is even then $L[u]$ is even and if $u$ is odd then $L[u]$ is odd. The linear operator $L$ in (18) is even if it is of the form (19), where $E_1(x,t), \ldots, E_4(x,t)$ are even functions of $x$ and $O_1(x,t), O_2(x,t)$ are odd functions of $x$.

## EXERCISES 18.5

**1.** Show that if $f(x) = 100$, then (4) gives the result stated in (6).

**2.** (a) Suppose that in place of the homogeneous Dirchlet condition (1c) we have the homogeneous Neumann condition (7). This time use an *even* extension of $f$ and, carrying out steps analogous to those in (3), derive the result

$$u(x,t) = \int_0^\infty f(\xi)\,\frac{e^{-\frac{(\xi-x)^2}{4\alpha^2 t}} + e^{-\frac{(\xi+x)^2}{4\alpha^2 t}}}{2\alpha\sqrt{\pi t}}\,d\xi. \quad (2.1)$$

(b) Verify that (2.1) does satisfy (7).
(c) Use (2.1) to determine $u(x,t)$ for the case where $f(x) = 100$.

**3.** Prove that (11) does imply (12).

**4.** Prove item 4, that if $F(x)$ is even then $F'(0) = 0$.

**5.** Prove item 6, that if $F(x)$ is even, then $F(x), F''(x), F''''(x), \ldots$ are even and $F'(x), F'''(x), \ldots$ are odd.

**6.** Prove item 7, that if $F(x)$ is odd, then $F(x), F''(x), F''''(x), \ldots$ are odd and $F'(x), F'''(x), \ldots$ are even.

**7.** (a) Prove that if $F(x,t)$ is an even function of $x$, then so is $F_t(x,t)$.
(b) Prove that if $F(x,t)$ is an odd function of $x$, then so is $F_t(x,t)$.

**8.** State whether or not $L$ is of the linear and even form (19), and briefly state your reasoning.

(a) $L[u] = u_{xx} - u_t - 3u$
(b) $L[u] = u_{xx} + e^{-x}u_t$
(c) $L[u] = u_{xx} + u_{tt} + u_{xt} + u_x + u_t + u$
(d) $L[u] = u_{xx} - u_t + u^2$
(e) $L[u] = u_{xx} + (\sin x)u_x - u_{tt}$
(f) $L[u] = u_{xx} + (\cos x)u_x - 6u_t$
(g) $L[u] = u_{xx} - u_{tt} + 2u$
(h) $L[u] = u_{xx} + u_{tt} - u_t + 5u$
(i) $L[u] = u_{xx} - u_{tt} - (3\cos 5x)u_t$
(j) $L[u] = u_{xx} - (\sin t)u_t$
(k) $L[u] = u_{xx} + xtu_x - u_t$
(l) $L[u] = (x^2u_x)_x - u_t$
(m) $L[u] = (x^3u_x)_x - u_{tt} + u$

## 18.6 Numerical Solution

**18.6.1. The finite-difference method.** We have seen that a wide variety of diffusion problems can be solved analytically using separation of variables or an integral transform. In more complicated cases one often gives up the hope of obtaining analytical solutions and turns, instead, to a numerical solution technique. For instance, the problem

$$\alpha^2 u_{xx} = u_t, \qquad (0 < x < L,\ 0 < t < \infty) \quad (1a)$$

$$u(0,t) = p(t), \qquad (0 < t < \infty) \quad (1b)$$

$$u(L,t) = q(t), \qquad (0 < t < \infty) \tag{1c}$$
$$u(x,0) = f(x) \qquad (0 < x < L) \tag{1d}$$

is readily solved by separation of variables if $p(t)$ and $q(t)$ are constants, but is so much more difficult if they are not constants (see Exercise 20, Section 18.3) that we might very well turn to numerical solution instead.

In this section we introduce one of the most important techniques for the numerical solution of PDE's, the **finite-difference method**. To explain that method, let us use the representative problem (1).

Our first step is to discretize the problem so that we seek $u(x,t)$ not over the entire $x,t$ domain but only at discrete *grid points* or *nodal points*, with coordinates $x_j, t_k$ in the $x,t$ plane. That is, we divide $L$ into $N$ equal parts, of length $\Delta x = L/N$, and define $x_j = j\Delta x$, for $j = 0,1,\ldots,N$. Further, we choose a time increment $\Delta t$ and define $t_k = k\Delta t$, for $k = 0,1,2,\ldots$. The resulting set of grid points, known as the *computational grid*, is shown in Fig. 1. At the open circle points $u$ is known, from the initial or boundary conditions, and we wish to solve for $u$ at the solid circle points.

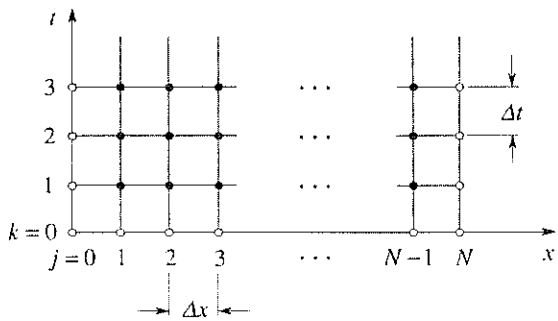


**Figure 1.** The computational grid.

Next, we seek a finite-difference approximation of the PDE (1a) that will relate $u$ at the various grid points. For the $u_t$ term in (1a) we use the difference quotient approximation

$$u_t(x,t) = \lim_{\Delta t \to 0} \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} \approx \frac{u(x,t+\Delta t) - u(x,t)}{\Delta t}. \tag{2}$$

That is, we do not take the limit as $\Delta t \to 0$; we choose a small $\Delta t$ and accept the error that results. We can treat the $u_{xx}$ term in the same manner if we deal with one derivative at a time. Accordingly, write

$$u_{xx} = (u_x)_x \approx \frac{u_x(x+\Delta x,t) - u_x(x,t)}{\Delta x} \tag{3}$$

and then approximate each of the derivatives in the numerator in the same way. Thus,

$$u_{xx}(x,t) \approx \frac{\dfrac{u(x+\Delta x,t) - u(x,t)}{\Delta x} - \dfrac{u(x,t) - u(x-\Delta x,t)}{\Delta x}}{\Delta x} \tag{4}$$

or*

$$u_{xx}(x,t) \approx \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{(\Delta x)^2}. \tag{5}$$

Putting (2) and (5) into (1a), with $x = x_j$, $x+\Delta x = x_{j+1}$, $x-\Delta x = x_{j-1}$, $t = t_k$, and $t + \Delta t = t_{k+1}$, gives

$$\alpha^2 \frac{u(x_{j+1},t_k) - 2u(x_j,t_k) + u(x_{j-1},t_k)}{(\Delta x)^2} \approx \frac{u(x_j,t_{k+1}) - u(x_j,t_k)}{\Delta t} \tag{6}$$

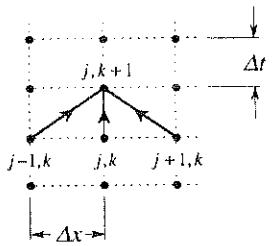


**Figure 2.** Schematic of (8).

or

$$\alpha^2 \frac{U_{j-1,k} - 2U_{j,k} + U_{j+1,k}}{(\Delta x)^2} = \frac{U_{j,k+1} - U_{j,k}}{\Delta t} \tag{7}$$

as our **finite-difference approximation** of the PDE (1a). Although not essential, we distinguish $u(x_j,t_k)$ and $U_{jk}$ as follows: $u(x_j,t_k)$ is the exact solution of the PDE (1a) at $x_j, t_k$, whereas $U_{jk}$ is the exact solution of the difference equation (7) at $x_j, t_k$. Since the two are not (in general) identical, because of the difference quotient approximations (2) and (5), it is useful to denote them by different letters, as we have.

If we solve (7) for $U_{j,k+1}$ we obtain†

$$\boxed{U_{j,k+1} = rU_{j-1,k} + (1-2r)U_{j,k} + rU_{j+1,k},} \tag{8}$$

where

$$r \equiv \alpha^2 \frac{\Delta t}{(\Delta x)^2}. \tag{9}$$

Equation (8) enables us to compute $U$ at a given grid point as a linear combination of $U$'s at the preceding time as indicated by the arrows in Fig. 2. Thus, it provides us with a "marching scheme" with which we can march out a solution, one line at a time, subject to the initial and boundary conditions

$$U_{j,0} = f(j\Delta x) \equiv f_j, \qquad (j = 1,2,\ldots,N-1) \tag{10a}$$
$$U_{0,k} = p(k\Delta t) \equiv p_k, \qquad (k = 1,2,\ldots) \tag{10b}$$
$$U_{N,k} = q(k\Delta t) \equiv q_k. \qquad (k = 1,2,\ldots) \tag{10c}$$

That is, beginning at the initial time $k = 0$ we can use (8), together with (10), to compute the $U$'s all along the line $k = 1$, then along the line $k = 2$, and so on.

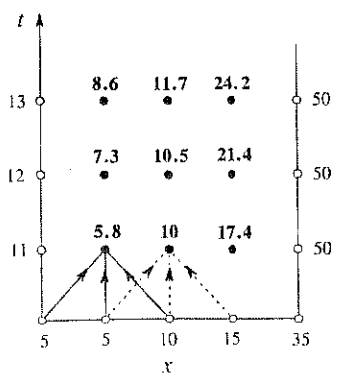


**Figure 3.** Example 1.

**EXAMPLE 1.** In (1), let $\alpha^2 = 1$, $L = 1$, $p(t) = 10 + 100t$, $q(t) = 50$, and $f(x) = 20x$. For purposes of illustration, let it suffice to take $N = 4$ (so $\Delta x = 0.25$) and $\Delta t = 0.01$. Then the grid and the initial and boundary values are as shown in Fig. 3. Where did

*We use a *forward* difference quotient (forward and backward difference quotients are defined in Section 16.4) in (3) and then *backward* difference quotients in the numerator of the right-hand side of (4) so that the result (5) is a *centered* formula; i.e., centered about $x$. These choices are discussed further in Exercise 1.

†The finite-difference scheme (8) is generally attributed to *E. Schmidt* (1924) and *L. Binder* (1911).

we get the value $u(0,0) = 5$ in the figure? The data is discontinuous at that point because $u(x,0) = f(x) = 20x = 0$ there, whereas $u(0,t) = 10+100t = 10$ there. Consistent with the fact that diffusion is a smoothing process, it seems reasonable to use the average value $(0+10)/2 = 5$ there. Similarly, we use the average value $u(1,0) = (20+50)/2 = 35$ at the other corner. From (9), $r = (1)(0.01)/(0.25)^2 = 0.16$, so (8) becomes

$$U_{j,k+1} = 0.16U_{j-1,k} + 0.68U_{j,k} + 0.16U_{j+1,k}. \tag{11}$$

Sweeping across the first time line, (11) gives

$$\begin{aligned} U_{1,1} &= 0.16U_{0,0} + 0.68U_{1,0} + 0.16U_{2,0} \\ &= 0.16(5) + 0.68(5) + 0.16(10) = 5.8, \\ U_{2,1} &= 0.16U_{1,0} + 0.68U_{2,0} + 0.16U_{3,0} \\ &= 0.16(5) + 0.68(10) + 0.16(15) = 10, \\ U_{3,1} &= 0.16U_{2,0} + 0.68U_{3,0} + 0.16U_{4,0} \\ &= 0.16(10) + 0.68(15) + 0.16(35) = 17.4. \end{aligned} \tag{12}$$

Moving up to the second time line, (11) gives

$$\begin{aligned} U_{1,2} &= 0.16(11) + 0.68(5.8) + 0.16(10) = 7.3, \\ U_{2,2} &= 0.16(5.8) + 0.68(10) + 0.16(17.4) = 10.5, \\ U_{3,2} &= 0.16(10) + 0.68(17.4) + 0.16(50) = 21.4, \end{aligned} \tag{13}$$

and so on.

COMMENT. Since the difference approximations (2) and (5) become exact only as $\Delta t \to 0$ and $\Delta x \to 0$, respectively, it is clear that we need $\Delta t$ and $\Delta x$ to be sufficiently small if we are to expect accurate results. Are $\Delta t = 0.01$ and $\Delta x = 0.25$ sufficiently small? Remember that smallness (or largeness) is always *relative* to some reference. Although there is no ready-made reference time with which to compare $\Delta t$, observe from the PDE (1a) that $\alpha^2/L^2$ has the dimensions of $1/\text{time}$. Thus, we can use $T = L^2/\alpha^2$ as a reference time. And as a reference length we can simply use $L$. Consequently, for our results to be accurate we need both

$$\frac{\Delta t}{T} = \frac{\Delta t}{L^2/\alpha^2} \ll 1 \qquad \text{and} \qquad \frac{\Delta x}{L} \ll 1 \tag{14}$$

or, since $\alpha = L = 1$ in this example,

$$\Delta t \ll 1 \qquad \text{and} \qquad \Delta x \ll 1. \tag{15}$$

Although $\Delta t = 0.01$ is much smaller than unity, $\Delta x = 0.25$ is not, so we can expect our results to provide only a rough approximation of the exact solution. However, realize that (14) is only a rule of thumb. Typically, one runs the calculation, then reduces $\Delta t$ and $\Delta x$ and runs it again, until a sufficient degree of convergence is achieved. ∎

Example 1 illustrates the numerical implementation of (8). In Example 2 let us test the method by choosing $\Delta x$ and $\Delta t$ that easily satisfy (15) and comparing

the numerical results with a known exact solution.

**EXAMPLE 2.** *Test Case.* Let us apply (8) to the case where $\alpha^2 = 1$, $L = 1$, $p(t) = q(t) = 0$, and $f(x) = 100$ because this case can be solved exactly by separation of variables [since $p(t)$ and $q(t)$ are constants]. Specifically, we have the exact solution

$$u(x,t) = \frac{400}{\pi} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n} \sin n\pi x \; e^{-(n\pi)^2 t} \tag{16}$$

with which to compare our numerical solution. With (15) in mind, let us choose $\Delta x = 0.02$ (i.e., $N = 50$) and $\Delta t = 0.00018$, in which case $r = 0.00018/(0.02)^2 = 0.45$.

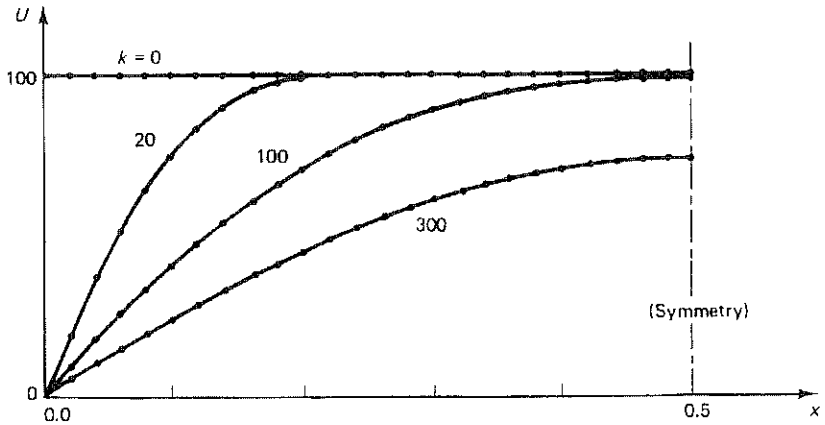


**Figure 4.** Application of (8), for the case $p(t) = q(t) = 0$, $f(x) = 100$, $L = 1$, $\alpha^2 = 1$, $\Delta x = 0.02$, $\Delta t = 0.00018$ ($r = 0.45$).

In this case our computational grid is quite fine so there are too many calculations to do by hand. However, it is easy to program the calculation (11) with a "do loop" on $k$ and with the initial and boundary conditions

$$U_{j,0} = 100, \qquad (j = 1, 2, \ldots, 49) \tag{17a}$$

$$U_{0,k} = 0, \qquad (k = 1, 2, \ldots) \tag{17b}$$

$$U_{50,k} = 0. \qquad (k = 1, 2, \ldots) \tag{17c}$$

For the corner points (where the data are discontinuous) use average values as in Example 1: $U_{0,0} = 50$, $U_{50,0} = 50$. The numerical results are shown as dots in Fig. 4, and the solid curves correspond to the exact solution given by (16). We have plotted the results only for $k = 20, 100$, and 300 and only over $0 < x < 0.5$ because the solution is symmetric about the midpoint $x = 0.5$. Although the exact and numerical results are not identical, they can hardly be distinguished in Fig. 4. ∎

The results of the test case in Example 2 are encouraging. In fact, if we change the boundary temperatures $p(t)$ and $q(t)$ from constants to time-varying functions, then the analytical separation of variables solution is made much more difficult,

yet the numerical finite-difference solution based upon equation (8) is unchanged, except for the data on the right-hand sides of (17b) and (17c).

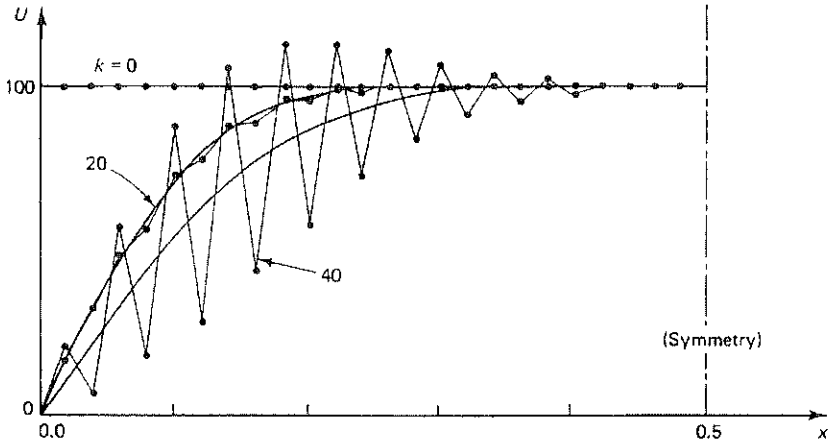


**Figure 5.** Example 2, with $\Delta t$ increased to 0.00022.

However, we must qualify our endorsement of the finite difference scheme given by equation (8). For suppose we change $\Delta t$ even slightly, from 0.00018 to 0.00022. We see from Fig. 5 that the results quickly degenerate: after only 20 time steps a deviation from the exact solution is apparent, and by the time $k$ = 40 the results are worthless. The error is oscillatory, both spatially ( with period $2\Delta x$) and temporally (with period $2\Delta t$, although that fact is not observable in the figure because the plots are not for consecutive $k$ values). Even if we do not rely on the exact solution for comparison (indeed, in real applications we do not know the exact solution) it seems clear that the oscillations are some sort of numerical instability rather than a faithful representation of a physical reality because it would surely be unlikely that the spatial and temporal periodicity of such a physical event would exactly equal $2\Delta x$ and $2\Delta t$, respectively. To test this assertion we can halve $\Delta x$, say, and rerun the calculation. Sure enough, an oscillation will result once again, this time with spatial period $2\Delta x'$, where $\Delta x' = \Delta x/2$ is the new spacing. Furthermore, we know that diffusion is a smoothing process, whereas the numerical results in Fig. 5 reveal quite the opposite tendency. Thus, even if we do not have the exact solution for comparison, it is clear that the oscillations imply some sort of numerical instability.

Short of a detailed analysis, let us briefly explain the breakdown observed in Fig. 5. Recall that we have used different letters to distinguish the exact solution $u(x,t)$ from the approximate solution $U_{j,k}$ generated by the finite-difference equation (8). We call the difference $u(x_j, t_k) - U_{j,k}$ the **accumulated truncation error** at the $j, k$ grid point, namely, the error incurred by replacing $u_t$ and $u_{xx}$ in (1a) by the finite-difference approximations (2) and (5), respectively. Besides the accumulated truncation error there is an additional error called the **accumulated roundoff error**, incurred because the computer rounds off numbers after a finite number of significant figures. Thus, if we further distinguish $U_{j,k}$ as the values computed by a "perfect computer" (one that keeps an infinite number of significant figures), and

$U^*_{j,k}$ as the actual printout of the real computer, then we can express the total error as

$$\begin{aligned}\text{total error} &= u(x_j, t_k) - U^*_{j,k} \\ &= [u(x_j, t_k) - U_{j,k}] + [U_{j,k} - U^*_{j,k}] \\ &= [\text{accumulated truncation error}] \\ &\quad + [\text{accumulated roundoff error}]. \end{aligned} \tag{18}$$

Regarding the accumulated truncation error, two closely related questions come to mind.

1. With $\Delta x$ and $\Delta t$ fixed, what is the behavior of the accumulated truncation error as $k \to \infty$?

2. At the fixed points in the $x, t$ domain, does the accumulated truncation error tend to zero as the mesh is continually refined?

Let us rephrase the second question, which is crucial: At any chosen fixed point in the $x, t$ domain, is it possible to reduce the accumulated truncation error to be smaller in magnitude than any prescribed number by sufficiently refining the grid, that is, by sufficiently reducing $\Delta x$ and $\Delta t$? If so, we say that the finite-difference scheme is **convergent**.

Since the roundoff error enters randomly, we simply ask that the accumulated roundoff error remain small – for instance, that it remain bounded as $k \to \infty$. If so, we say that the scheme is **stable**. (Be aware that these definitions are not entirely standard from one text to another. At least, our terminology here is consistent with the terminology used in our analogous discussion for ODE's in Section 6.5.2.)

Analysis reveals that the finite-difference method (8) is both convergent and stable if $\Delta x$ and $\Delta t$ satisfy the criterion

$$\boxed{r = \alpha^2 \frac{\Delta t}{(\Delta x)^2} \le \frac{1}{2},} \tag{19}$$

and is both divergent and unstable if $r > \frac{1}{2}$. [Sure enough, the excellent results displayed in Fig. 4 correspond to $r = (1)(0.00018)/(0.02)^2 = 0.45$, which satisfies (19), whereas those displayed in Fig. 5 correspond to $r = (1)(0.00022)/(0.02)^2 = 0.55$, which does not.] The analysis behind (19) is beyond our present scope,† but we do outline the stability part of the analysis in the exercises and urge you to study the latter because it is a typical and powerful application of the matrix eigenvalue problem to the analysis of finite-difference methods.

The restriction (19) may be quite severe, for if $\Delta x$ is chosen small for the sake of accuracy, then the maximum $\Delta t$ allowed by (19) may be so small that it necessitates a great many time steps and, consequently, considerable computer time. In

† See, for instance, G. D. Smith, *Numerical Solution of Partial Differential Equations* (New York: Oxford University Press, 1965) or R. D. Richtmyer and K. W. Morton, *Difference Methods for Initial-Value Problems*, 2nd ed. (New York: Interscience, 1967).

the optional remainder of this section we show how to modify the method so as to relieve us of the restriction (19).

**18.6.2. Implicit methods: Crank–Nicolson, with iterative solution. (Optional)** Equation (8) is by no means the *only* possible finite-difference method for the diffusion equation (1a). For example, the $x$-wise differencing in (4) is at the initial time $t$, whereas some weighted average over the time interval ($t$ to $t + \Delta t$) should be more accurate, namely,

$$u_{xx}(x,t) \approx (1-\theta)\,\frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{(\Delta x)^2} + \theta\,\frac{u(x+\Delta x,t+\Delta t) - 2u(x,t+\Delta t) + u(x-\Delta x,t+\Delta t)}{(\Delta x)^2}, \tag{20}$$

where the number $\theta$ is specified so that $0 \le \theta \le 1$. Then, in place of (7) we have

$$\alpha^2\left[(1-\theta)\,\frac{U_{j-1,k} - 2U_{j,k} + U_{j+1,k}}{(\Delta x)^2} + \theta\,\frac{U_{j-1,k+1} - 2U_{j,k+1} + U_{j+1,k+1}}{(\Delta x)^2}\right] = \frac{U_{j,k+1} - U_{j,k}}{\Delta t}, \tag{21}$$

which reduces to (7) if $\theta = 0$. It turns out that the parameter $\theta$ gives us the desired control. Specifically, it can be shown that if $\theta \ge \frac{1}{2}$, then (21) is convergent and stable for *all* $r > 0$; that is, if we use any $\theta$ greater than or equal to 1/2, then the condition (19) can be discarded. The borderline case $\theta = \frac{1}{2}$ gives the well-known **Crank–Nicolson** scheme

$$\boxed{\begin{aligned} -rU_{j-1,k+1} + 2(1+r)U_{j,k+1} - rU_{j+1,k+1} \\ = rU_{j-1,k} + 2(1-r)U_{j,k} + rU_{j+1,k} \end{aligned}} \tag{22}$$

for $j = 1, 2, \ldots, N-1$ and $k = 0, 1, 2, \ldots$, where $r = \alpha^2\Delta t/(\Delta x)^2$, as before.

Given the initial and boundary values, we can use (22) to compute the first line of unknowns $U_{1,1}, U_{2,1}, \ldots, U_{N-1,1}$, then the second line of unknowns $U_{1,2}, U_{2,2}, \ldots, U_{N-1,2}$, and so on, just as we did using (8). However, (22) presents a difficulty which we illustrate by writing it out for the first line of unknowns, with $r = 1$ and $N = 5$, say, for definiteness.

$$\begin{aligned} j = 1: &\quad -\underline{U_{0,1}} + 4U_{1,1} - U_{2,1} = U_{0,0} + U_{2,0}, \\ j = 2: &\quad -U_{1,1} + 4U_{2,1} - U_{3,1} = U_{1,0} + U_{3,0}, \\ j = 3: &\quad -U_{2,1} + 4U_{3,1} - U_{4,1} = U_{2,0} + U_{4,0}, \\ j = 4: &\quad -U_{3,1} + 4U_{4,1} - \underline{U_{5,1}} = U_{3,0} + U_{5,0}. \end{aligned} \tag{23}$$

The two underlined terms are known boundary values, so let us move them to the right with the other known terms. Then (23) becomes

$$\begin{aligned} 4U_{1,1} - U_{2,1} &= U_{0,1} + U_{0,0} + U_{2,0}, \\ -U_{1,1} + 4U_{2,1} - U_{3,1} &= U_{1,0} + U_{3,0}, \\ - U_{2,1} + 4U_{3,1} - U_{4,1} &= U_{2,0} + U_{4,0}, \\ - U_{3,1} + 4U_{4,1} &= U_{3,0} + U_{5,0} + U_{3,1}. \end{aligned} \tag{24}$$

The difficulty, which we can see clearly in (24), is that the equations (24) are *coupled*. Thus, we need to solve the matrix equation (24) for the first line of unknowns $U_{1,1}, U_{2,1}, U_{3,1}, U_{4,1}$, then increment $k$ by 1 in (22) and solve for a similar matrix equation for the second line of unknowns $U_{1,2}, U_{2,2}, U_{3,2}, U_{4,2}$, and so on. In contrast, the scheme (8) gives *uncoupled* equations for $U_{1,k+1}, U_{2,k+1}, \ldots, U_{N-1,k+1}$. Thus, we call (22) an **implicit** scheme, whereas (8) is an **explicit** scheme. Graphically, the Crank–Nicolson scheme (22) corresponds to the computational pattern shown in Fig. 6, which reveals the *nearest-neighbor coupling* and is in contrast with the pattern shown in Fig. 2 for the scheme (8).

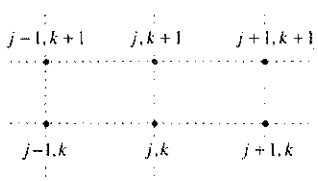


**Figure 6.** Implicit scheme.

Expressing (22) in matrix form and recalling the initial and boundary conditions given by (10) gives

$$\begin{bmatrix} 2(1+r) & -r & & \cdots & 0 \\ -r & 2(1+r) & -r & & \vdots \\ & & \ddots & & \\ \vdots & & -r & 2(1+r) & -r \\ 0 & \cdots & & -r & 2(1+r) \end{bmatrix} \begin{bmatrix} U_{1,k+1} \\ U_{2,k+1} \\ \vdots \\ U_{N-2,k+1} \\ U_{N-1,k+1} \end{bmatrix}$$

$$= \begin{bmatrix} rp_{k+1} + rp_k + 2(1-r)U_{1,k} + rU_{2,k} \\ rU_{1,k} + 2(1-r)U_{2,k} + rU_{3,k} \\ \vdots \\ rU_{N-3,k} + 2(1-r)U_{N-2,k} + rU_{N-1,k} \\ rU_{N-2,k} + 2(1-r)U_{N-1,k} + rq_{k+1} + rq_k \end{bmatrix}. \tag{25}$$

The system (25) is of the matrix form

$$\mathbf{A}\mathbf{U}_{k+1} = \mathbf{c}, \tag{26}$$

where $\mathbf{A}$ is *tridiagonal* (due to the nearest-neighbor coupling) and *symmetric*. $\mathbf{A}$ and $\mathbf{c}$ are known and $\mathbf{U}_{k+1}$ is the unknown. The idea is to set $k = 0$ and solve (25) for the entire line of unknowns $U_{1,1}, U_{2,1}, \ldots, U_{N-1,1}$, then set $k = 1$ and solve for the line $U_{1,2}, U_{2,2}, \ldots, U_{N-1,2}$, and so on.

Notice carefully that the situation is not as bad as it may appear to be because $\mathbf{A}$ is strongly diagonal. That is, the off-diagonal terms ($-r$'s and 0's) are small compared to the diagonal terms [$2(1+r)$'s], so (25) is *almost* uncoupled. Thus, we say that (25) is only *weakly coupled*. To take advantage of this circumstance, let us split $\mathbf{A}$ into its diagonal part plus the deviation from that,

$$\mathbf{A} = \begin{bmatrix} 2(1+r) & 0 & & \cdots & 0 \\ 0 & 2(1+r) & & & \vdots \\ & & \ddots & & \\ \vdots & & & 2(1+r) & \\ 0 & \cdots & & 0 & 2(1+r) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -r & 0 & & \cdots & 0 \\ -r & 0 & -r & & & \vdots \\ & & & \ddots & & \\ \vdots & & & -r & 0 & -r \\ 0 & \cdots & & 0 & -r & 0 \end{bmatrix}$$

$$\equiv 2(1+r)\mathbf{I} + \mathbf{A}', \tag{27}$$

where $\mathbf{I}$ is an $(N-1) \times (N-1)$ identity matrix and $\mathbf{A}'$ is the "deviation" matrix. Then (26) becomes

$$[2(1+r)\mathbf{I} + \mathbf{A}']\mathbf{U}_{k+1} = \mathbf{c} \tag{28}$$

or

$$\mathbf{U}_{k+1} = \frac{1}{2(1+r)}\,\mathbf{c} - \frac{1}{2(1+r)}\,\mathbf{A}'\mathbf{U}_{k+1}. \tag{29}$$

Since $\mathbf{A}'$ is small compared to $2(1+r)\mathbf{I}$, in (28), we could neglect it altogether and obtain, from (29),

$$\mathbf{U}_{k+1} \approx \frac{1}{2(1+r)}\,\mathbf{c}. \tag{30}$$

Better yet, we could accept (30) as an initial approximation

$$\mathbf{U}_{k+1}^{(0)} = \frac{1}{2(1+r)}\,\mathbf{c} \tag{31}$$

and obtain an improved result as

$$\mathbf{U}_{k+1}^{(1)} = \frac{1}{2(1+r)}\,\mathbf{c} - \frac{1}{2(1+r)}\,\mathbf{A}'\mathbf{U}_{k+1}^{(0)}. \tag{32}$$

In fact, repeating this procedure, we have the **iterative** algorithm

$$\boxed{\mathbf{U}_{k+1}^{(n+1)} = \frac{1}{2(1+r)}\left[\mathbf{c} - \mathbf{A}'\mathbf{U}_{k+1}^{(n)}\right],} \tag{33}$$

where $\mathbf{U}_{k+1}^{(n+1)}$ is the $(n+1)$st iterate of $\mathbf{U}_{k+1}$. With $k$ fixed, we carry out (33) for $n = 0, 1, 2, \ldots$ [where $\mathbf{U}_{k+1}^{(0)}$ is given by (31)], until suitable convergence is attained. For example, we might continue the iteration until each element of $\mathbf{U}_{k+1}^{(n+1)}$ and the corresponding element of $\mathbf{U}_{k+1}^{(n)}$ differ in magnitude by less than $10^{-5}$.

The scheme given by (33) and (31) is called **Jacobi iteration**, and it is shown in the exercises that Jacobi iteration converges to the exact solution of $\mathbf{A}\mathbf{U}_{k+1} = \mathbf{c}$ for all finite values of $r$. In fact, we can improve upon (33) slightly, by using the $(n+1)$st components as soon as they become available. To elaborate, let us write out the Jacobi scheme (33):

$$\begin{aligned}
\mathbf{U}_{1,k+1}^{(n+1)} &= \frac{1}{2(1+r)}\left[c_1 - a'_{1,1}\mathbf{U}_{1,k+1}^{(n)} - \cdots - a'_{1,N-1}\mathbf{U}_{N-1,k+1}^{(n)}\right] \\
\mathbf{U}_{2,k+1}^{(n+1)} &= \frac{1}{2(1+r)}\left[c_2 - a'_{2,1}\mathbf{U}_{1,k+1}^{(n)} - a'_{2,2}\mathbf{U}_{2,k+1}^{(n)}\right. \\
&\qquad\qquad \left. - \cdots - a'_{2,N-1}\mathbf{U}_{N-1,k+1}^{(n)}\right] \\
&\qquad\qquad \vdots \\
\mathbf{U}_{N-1,k+1}^{(n+1)} &= \frac{1}{2(1+r)}\left[c_{N-1} - a'_{N-1,1}\mathbf{U}_{1,k+1}^{(n)}\right. \\
&\qquad\qquad \left. - \cdots - a'_{N-1,N-2}\mathbf{U}_{N-2,k+1}^{(n)} - a'_{N-1,N-1}\mathbf{U}_{N-1,k+1}^{(n)}\right].
\end{aligned} \tag{34}$$

Having computed $\mathbf{U}_{1,k+1}^{(n+1)}$ from the first equation in (34), let us use that value in place of the less-up-to-date value $\mathbf{U}_{1,k+1}^{(n)}$ that appears in the right-hand side of the second equation. Similarly, let us use the already computed values $\mathbf{U}_{1,k+1}^{(n+1)}, \mathbf{U}_{2,k+1}^{(n+1)}$ in place of $\mathbf{U}_{1,k+1}^{(n)}, \mathbf{U}_{2,k+1}^{(n)}$ in the right-hand side of the third equation, and so on. This idea produces the **Gauss–Seidel** scheme

$$\boxed{\mathbf{U}_{k+1}^{(n+1)} = \frac{1}{2(1+r)}\left[\mathbf{c} - \mathbf{L}\mathbf{U}_{k+1}^{(n+1)} - \mathbf{M}\mathbf{U}_{k+1}^{(n)}\right],} \tag{35}$$

where $\mathbf{L}$ and $\mathbf{M}$ are the lower and upper parts, respectively, of $\mathbf{A}'$. That is, $\mathbf{A}' = \mathbf{L} + \mathbf{M}$, where

$$\mathbf{L} = \begin{bmatrix} 0 & & & \cdots & 0 \\ -r & 0 & & & \vdots \\ 0 & -r & 0 & & \\ \vdots & & & \ddots & \\ 0 & \cdots & 0 & -r & 0 \end{bmatrix} \quad \text{and} \quad \mathbf{M} = \begin{bmatrix} 0 & -r & 0 & \cdots & 0 \\ & 0 & -r & & \vdots \\ & & 0 & & \\ \vdots & & & \ddots & -r \\ 0 & \cdots & & & 0 \end{bmatrix}. \tag{36}$$

Like the Jacobi method, the Gauss–Seidel method converges for all finite $r$, and approximately twice as fast.

Finally, there is another simple improvement that will further increase the speed of convergence. Re-expressing (35) as

$$\begin{aligned}\mathbf{U}_{k+1}^{(n+1)} &= \mathbf{U}_{k+1}^{(n)} + \frac{1}{2(1+r)}\left\{\mathbf{c} - \mathbf{L}\mathbf{U}_{k+1}^{(n+1)} - [\mathbf{M} + 2(1+r)\mathbf{I}]\mathbf{U}_{k+1}^{(n)}\right\} \\ &\equiv \mathbf{U}_{k+1}^{(n)} + \boldsymbol{\Delta}\mathbf{U}_{k+1}^{(n)},\end{aligned} \tag{37}$$

we insert a numerical "control parameter" $\omega$:

$$\mathbf{U}_{k+1}^{(n+1)} = \mathbf{U}_{k+1}^{(n)} + \omega\boldsymbol{\Delta}\mathbf{U}_{k+1}^{(n)}. \tag{38}$$

The idea is general and can be applied to any iterative scheme: $\boldsymbol{\Delta}\mathbf{U}_{k+1}^{(n)}$ is a "correction term," and adjusting the size of the correction, by means of $\omega$, might speed the convergence. Often, one chooses $\omega$ based on numerical experimentation, but in the present case it can be shown analytically that the optimum $\omega$ is given by

$$\omega_{\text{opt}} = \frac{2}{1+\sqrt{1-\mu^2}}, \qquad \text{where} \qquad \mu = \frac{r}{1+r}\cos\frac{\pi}{N}. \tag{39}$$

We see from (39) that $\omega_{\text{opt}}$ lies somewhere between 1 and 2. Since $\omega_{\text{opt}} > 1$, the modified Gauss–Seidel scheme (38) is known as **successive overrelaxation**, or **SOR** for brevity.

**EXAMPLE 3.** To illustrate the Jacobi, Gauss–Seidel, and SOR methods, consider the problem where $u(0,t) = u(L,t) = 0$ and $u(x,0) = 100$, and choose $N = 4$ and $r = 1$. Then the Crank–Nicolson scheme (22) gives these simultaneous equations for the unknown values $U_{1,1}$, $U_{2,1}$, $U_{3,1}$ in the first "time-line":

$$\begin{aligned} 4U_{1,1} - U_{2,1} &= 150, \\ -U_{1,1} + 4U_{2,1} - U_{3,1} &= 200, \\ -U_{2,1} + 4U_{3,1} &= 60. \end{aligned} \tag{40}$$

The latter is readily solved, and we obtain

$$U_{1,1} = 55.54, \quad U_{2,1} = 72.14, \quad U_{3,1} = 33.04, \tag{41}$$

but let us use (40) to illustrate the three methods of iterative solution.

*Jacobi*:

$$\begin{aligned} U_{1,1}^{(n+1)} &= \frac{1}{4}(U_{2,1}^{(n)} + 150), \\ U_{2,1}^{(n+1)} &= \frac{1}{4}(U_{1,1}^{(n)} + U_{3,1}^{(n)} + 200), \\ U_{3,1}^{(n+1)} &= \frac{1}{4}(U_{2,1}^{(n)} + 60), \end{aligned} \tag{42}$$

so

$$
\begin{aligned}
U_{1,1}^{(0)} &= \frac{1}{4}(0+150) = \underline{37.5}, \\
U_{2,1}^{(0)} &= \frac{1}{4}(0+0+200) = \underline{50}, \\
U_{3,1}^{(0)} &= \frac{1}{4}(0+60) = \underline{15},
\end{aligned}
\tag{43}
$$

$$
\begin{aligned}
U_{1,1}^{(1)} &= \frac{1}{4}(50+150) = \underline{50}, \\
U_{2,1}^{(1)} &= \frac{1}{4}(37.5+15+200) = \underline{63.13}, \\
U_{3,1}^{(1)} &= \frac{1}{4}(50+60) = \underline{27.5},
\end{aligned}
\tag{44}
$$

and so on.
*Gauss–Seidel*:

$$
\begin{aligned}
U_{1,1}^{(n+1)} &= \frac{1}{4}(U_{2,1}^{(n)} + 150), \\
U_{2,1}^{(n+1)} &= \frac{1}{4}(U_{1,1}^{(n+1)} + U_{3,1}^{(n)} + 200), \\
U_{3,1}^{(n+1)} &= \frac{1}{4}(U_{2,1}^{(n+1)} + 60),
\end{aligned}
\tag{45}
$$

so

$$
\begin{aligned}
U_{1,1}^{(0)} &= \frac{1}{4}(0+150) = \underline{37.5}, \\
U_{2,1}^{(0)} &= \frac{1}{4}(0+0+200) = \underline{50}, \\
U_{3,1}^{(0)} &= \frac{1}{4}(0+60) = \underline{15},
\end{aligned}
\tag{46}
$$

$$
\begin{aligned}
U_{1,1}^{(1)} &= \frac{1}{4}(50+150) = \underline{50}, \\
U_{2,1}^{(1)} &= \frac{1}{4}(50+15+200) = \underline{66.25}, \\
U_{3,1}^{(1)} &= \frac{1}{4}(66.25+60) = \underline{31.56},
\end{aligned}
\tag{47}
$$

and so on.
*SOR*: First, we need to compute $\omega_{opt}$ : $\mu = \frac{1}{2}\cos\frac{\pi}{4} = \sqrt{2}/4$, $\omega_{opt} = 2/(1+\sqrt{1-\mu^2})$ $= 1.03$. As in (43) and (46),

$$
U_{1,1}^{(0)} = \underline{37.5}, \quad U_{2,1}^{(0)} = \underline{50}, \quad U_{3,1}^{(0)} = \underline{15}. \tag{48}
$$

Next, make a "tentative" Gauss–Seidel step using (45) (with $n = 0$) and the values given by (48):

$$
U_{1,1}^{(1)} = 50 \quad \text{so} \quad \Delta U_{1,1}^{(0)} = 50 - 37.5 = 12.5,
$$

$$U_{2,1}^{(1)} = 66.25 \quad \text{so} \quad \Delta U_{2,1}^{(0)} = 66.25 - 50 = 16.25, \tag{49}$$
$$U_{3,1}^{(1)} = 31.56 \quad \text{so} \quad \Delta U_{3,1}^{(0)} = 31.56 - 15 = 16.56.$$

Now make the SOR step:

$$\begin{aligned} U_{1,1}^{(1)} &= U_{1,1}^{(0)} + \omega \Delta U_{1,1}^{(0)} = 37.5 + 1.03(12.5) = \underline{50.38}, \\ U_{2,1}^{(1)} &= U_{2,1}^{(0)} + \omega \Delta U_{2,1}^{(0)} = 50 + 1.03(16.25) = \underline{66.74}, \\ U_{3,1}^{(1)} &= U_{3,1}^{(0)} + \omega \Delta U_{3,1}^{(0)} = 15 + 1.03(16.56) = \underline{32.06}. \end{aligned} \tag{50}$$

Carrying out one more iteration (Exercise 16) and comparing the successive $U_{2,1}^{(n)}$ values, which should be representative, gives the results presented in Table 1. ∎

**Table 1.** Successive $U_{2,1}^{(n)}$ values.

| | Jacobi | Gauss–Seidel | SOR |
|---|---|---|---|
| $n = 0$ | 50 | 50 | 50 |
| $n = 1$ | 63.13 | 66.25 | 66.74 |
| $n = 2$ | 69.38 | 71.41 | 71.70 |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| Exact | 72.14 | 72.14 | 72.14 |

**Closure.** The finite-difference method developed quickly beginning around 1950 when digital computers became widely available, and is one of the most important methods for solving partial differential equations. In this text we discuss it in the present section for the diffusion equation and in Section 20.5 for the Laplace equation. The basic idea is to discretize the problem so that instead of seeking $u(x, t)$ over the given $x, t$ domain we seek $u$ only at a finite set of grid points. The PDE is discretized by replacing the various partial derivations by approximate difference quotients so, in place of the PDE, we end up with linear algebraic equations on the unknown $U_{j,k}$ values at the grid points. There are many possible finite-difference schemes for a given PDE, depending upon the forms chosen for the difference quotients. The simplest one for our diffusion equation is given by (8) and is called an explicit scheme because it gives $U_{j,k+1}$ explicitly in terms of known values. However, this scheme is invalid if $r = \alpha^2 \Delta t/(\Delta x)^2$ exceeds 1/2, so that if we choose $\Delta x$ small, for accuracy, then we may need $\Delta t$ to be so small that a prohibitively large number of time steps may be required. In the optional second part of this section we turn to implicit schemes to eliminate the $r \leq 1/2$ restriction. The price that we pay for this improvement is that at each time step we need to solve an $(N-1) \times (N-1)$ matrix equation for the $N-1$ values of $U$ along that line. Fortunately, the matrix is strongly diagonal, so that efficient iterative methods can be used for the solution.

## EXERCISES 18.6

**1.** We use a forward difference quotient in (3) and then backward difference quotients in the numerator of the right-hand side of (4).

(a) Show that if we use forward difference quotients in (3) and (4), then we obtain the scheme

$$U_{j,k+1} = (1+r)U_{j,k} - 2rU_{j+1,k} + rU_{j+2,k} \qquad (1.1)$$

in place of (8).
(b) Show that if we use backward difference quotients in (3) and (4), then we obtain the scheme

$$U_{j,k+1} = (1+r)U_{j,k} - 2rU_{j-1,k} + rU_{j-2,k}. \qquad (1.2)$$

(c) Discuss any advantages or disadvantages that occur to you for the schemes (1.1) and (1.2) in comparison with (8).

**2.** Continue the hand calculation begun in Example 1. Specifically, determine $U_{1,3}, U_{2,3}, U_{3,3}$, and $U_{1,4}, U_{2,4}, U_{3,4}$.

**3.** Consider the problem $\alpha^2 u_{xx} = u_t$ $(0 < x < 10,\ 0 < t < \infty)$ with the boundary and initial conditions $u(0,t) = 100$, $u(10,t) = 0$, and $u(x,0) = 0$. With $\Delta x = 2.5$, $\Delta t = 2$, and $\alpha^2 = 1$, use (8) to compute the first three "lines" of $U_{j,k}$'s: the nine values $U_{1,1}$ through $U_{3,3}$.

**4.** Show that if the PDE (1a) is modified to include a Newton cooling term $Hu$ and a heat source distribution term $F(x,t)$, as

$$\alpha^2 u_{xx} = u_t + Hu - F(x,t), \quad (0 < x < L,\ 0 < t < \infty) \qquad (4.1)$$

then in place of (8) we obtain

$$U_{j,k+1} = rU_{j-1,k} + (1 - 2r - H\Delta t)U_{j,k} + rU_{j+1,k} + F_{j,k}\Delta t, \qquad (4.2)$$

where $F_{j,k}$ denotes $F(x_j, t_k)$ and $H$ is a constant.

**5.** In Exercise 4, let $\alpha^2 = 1$, $L = 1$, $u(0,t) = 0, u(x,0) = 0, u(1,t) = 0$, $H = 0$, and $F(x,t) = 10$. With $\Delta t = 0.02$ and $\Delta x = 0.25$, use (4.2) to compute the first three "lines" of $U_{j,k}$'s, i.e., the nine values $U_{1,1}$ through $U_{3,3}$.

**6.** Repeat Exercise 5, with these changes: $u(0,t) = 100$, $F(x,t) = 10 \sin \pi x$.

**7.** Repeat Exercise 5, with these changes: $u(x,0) = 100$, $H = 4$, $F(x,t) = 0$.

**8.** To see the smoothing nature of the diffusion process in a simple numerical example, consider the problem

$$\alpha^2 u_{xx} = u_t, \qquad (-\infty < x < \infty,\ 0 < t < \infty)$$
$$u(x,0) = 100H(x),$$

where $H(x)$ is the Heaviside function, and the diffusivity is $\alpha^2 = 0.2$. Use (8), with $\Delta t = 0.5$ and $\Delta x = 1$, to compute the $U_{j,k}$ values for the first three lines; i.e., through $k = 3$. Plot $U_{j,k}$ versus $j$, for $k = 0, 1, 2, 3$.

**9.** (a) To see the effect of the condition (19) in a simple calculation, use (8) for the problem

$$u_{xx} = u_t, \qquad (-\infty < x < \infty,\ 0 < t < \infty)$$
$$u(x,0) = 100H(x), \qquad (9.1)$$

where $H(x)$ is the Heaviside function. With $\Delta t = 0.2$ and $\Delta x = 1$, compute the $U_{j,k}$ values for the first five lines; i.e., through $k = 5$. Keeping $\Delta x = 1$, repeat the calculation with $\Delta t = 0.4, 0.6$, and $0.8$.
(b) Plot your results and discuss them.
(c) For the four different cases ($\Delta t = 0.2, 0.4, 0.6, 0.8$), plot your final temperature distribution (i.e., at $k = 5$) and also the exact solution (which can be found in Section 18.4).

**10.** In (1) let $L = 12$, $p(t) = 100$, $q(t) = 0$, and $f(x) = 0$, and suppose the rod is comprised of two different materials, with $\alpha^2 = 1.8$ over $0 < x < 6$ and $\alpha^2 = 0.2$ over $6 < x < 12$. With $\Delta t = 0.5$ and $\Delta x = 2$, use (8) to compute the $U_{j,k}$ values for the first four lines; i.e., through $k = 4$. Note that (8) holds over $x < 6$ and over $x > 6$ but not at the junction $x = 6$. There, use the fact that the heat flux crossing $x = 6$ from the left must equal the heat flux crossing $x = 6$ toward the right or, from the Fourier law of heat conduction,

$$K_L \left.\frac{\partial u}{\partial x}\right|_{x=6-} = K_R \left.\frac{\partial u}{\partial x}\right|_{x=6+}, \qquad (10.1)$$

where $K_L = 25$ is the thermal conductivity of the material to the left of $x = 6$ and $K_R = 3$ is the thermal conductivity of the material to the right of $x = 6$. You will need to express (10.1) in finite-difference form.

**11.** The problem

$$u_{xx} = u_t, \qquad (0 < x < 1,\ 0 < t < \infty)$$
$$u(0,t) = u(1,t) = 0, \quad u(x,0) = 100 \sin \pi x \qquad (11.1)$$

admits the exact one-term solution $u(x,t) = 100(\sin \pi x)\exp(-\pi^2 t)$.

(a) Use (8), with $\Delta t = 0.02$ and $\Delta x = 0.25$ to compute the first three lines of $U_{j,k}$'s (i.e., through $k = 3$) and compare your results with the exact values.
(b) Use (8) to evaluate $u(0.5, 0.06)$, with $\Delta t$ and $\Delta x$ sufficiently small so that your value of $u(0.5, 0.06)$ is correct to four significant figures. (Use a computer.)
(c) Use (8) to evaluate (by computer) the $U_{j,k}$'s through $k = 45$, with $\Delta t = 0.0010$ and $\Delta x = 0.05$. Plot both your computed solution and the exact solution at $t = 0.045$ (i.e., at $k = 45$).
(d) Use (8) to evaluate (by computer) the $U_{j,k}$'s through $k = 30$, and $\Delta t = 0.0015$ and $\Delta x = 0.05$. Plot both your computed solution and the exact solution at $t = 0.045$ (i.e., at $k = 30$).
(e) Use (8) to evaluate (by computer) the $U_{j,k}$'s through $k = 15$, with $\Delta t = 0.0030$ and $\Delta x = 0.05$. Plot both your computed solution and the exact solution at $t = 0.045$ (i.e., at $k = 15$). Interpret your results in the light of your results to parts (c) and (d) if, indeed, you worked those parts.

**12.** (*Use of Taylor series*) To derive the finite-difference approximation (5) we used the classical difference quotient definition of the derivative. Derive (5) using Taylor series instead. HINT: Expand $u(x + \Delta x, t)$ about $x$:

$$u(x+\Delta x, t) = u(x,t) + u_x(x,t)\Delta x + \frac{1}{2}u_{xx}(x,t)(\Delta x)^2 + \cdots.$$

Similarly, expand $u(x - \Delta x, t)$ about $x$. Add those two formulas, cut off the series on the right-hand side after the first couple of terms (as an approximation), and solve for $u_{xx}(x,t)$.

**13.** (*Deriving the stability criterion (19)*) We stated that the finite-difference scheme (8) is both convergent and stable if $r \le \frac{1}{2}$, and is both divergent and unstable if $r > \frac{1}{2}$. Here, we outline a proof of the stability part of that claim and ask you to write out the steps and to supply any missing steps or reasoning. To begin, show that (8) can be expressed in matrix form as

$$[U_{1,k+1}, \ldots, U_{N-1,k+1}]^{\mathrm{T}} = \begin{bmatrix} 1-2r & r & 0 & \cdots & 0 \\ r & 1-2r & r & & \vdots \\ & & \ddots & & \\ \vdots & & r & 1-2r & r \\ 0 & \cdots & 0 & r & 1-2r \end{bmatrix} \begin{bmatrix} U_{1,k} \\ U_{2,k} \\ \vdots \\ U_{N-2,k} \\ U_{N-1,k} \end{bmatrix}$$

$$+ [rU_{0,k}, 0, \ldots, 0, rU_{N,k}]^{\mathrm{T}}, \tag{13.1}$$

where we use the transpose notation to save vertical space. More compactly, express (13.1) as

$$\mathbf{U}_{k+1} = \mathbf{A}\mathbf{U}_k + \mathbf{c}_k. \tag{13.2}$$

In stability analyses it is commonly assumed that roundoff errors occur only along the first line ($k = 0$), say, and then to see whether the errors remain bounded as $k$ increases. Thus, in place of the exact values $\mathbf{U}_0$, $\mathbf{c}_0$, the initial roundoff errors result in the actual values $\mathbf{U}_0^*$, $\mathbf{c}_0^*$. The machine proceeds to compute values $\mathbf{U}_1^*, \mathbf{U}_2^*, \ldots$ according to

$$\mathbf{U}_{k+1}^* = \mathbf{A}\mathbf{U}_k^* + \mathbf{c}_k^*, \tag{13.3}$$

rather than exact values $\mathbf{U}_1, \mathbf{U}_2, \ldots$ according to (13.2). Show that the roundoff error $\mathbf{e}_k \equiv \mathbf{U}_k - \mathbf{U}_k^*$ propagates according to

$$\mathbf{e}_{k+1} = \mathbf{A}\mathbf{e}_k + \mathbf{b}_k, \tag{13.4}$$

where $\mathbf{b}_k = \mathbf{c}_k - \mathbf{c}_k^* = [re_{0,k}, 0, \ldots, 0, re_{N,k}]^{\mathrm{T}}$. From (13.4) show that

$$\mathbf{e}_k = \mathbf{A}^k\mathbf{e}_0 + \mathbf{A}^{k-1}\mathbf{b}_0. \tag{13.5}$$

Show that the $(N-1) \times (N-1)$ matrix $\mathbf{A}$ must have $N-1$ orthogonal eigenvectors, say $\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_{N-1}$, so that $\mathbf{e}_0$ and $\mathbf{b}_0$ can be expressed in the form

$$\begin{aligned} \mathbf{e}_0 &= \alpha_1\mathbf{\Phi}_1 + \cdots + \alpha_{N-1}\mathbf{\Phi}_{N-1}, \\ \mathbf{b}_0 &= \beta_1\mathbf{\Phi}_1 + \cdots + \beta_{N-1}\mathbf{\Phi}_{N-1}. \end{aligned} \tag{13.6,7}$$

Putting these into (13.5), show that

$$\begin{aligned} \mathbf{e}_k = {} & (\alpha_1\lambda_1^k + \beta_1\lambda_1^{k-1})\mathbf{\Phi}_1 + \cdots \\ & + (\alpha_{N-1}\lambda_{N-1}^k + \beta_{N-1}\lambda_{N-1}^{k-1})\mathbf{\Phi}_{N-1}, \end{aligned} \tag{13.8}$$

where $\lambda_1, \ldots, \lambda_{N-1}$ are the eigenvalues of $\mathbf{A}$. Explain why it follows from (13.8) that for stability it is necessary and sufficient that

$$|\lambda_1| \le 1, \ldots, |\lambda_{N-1}| \le 1. \tag{13.9}$$

Since $\mathbf{A}$ is a function of $r$ its $\lambda$'s are too. Hence, the most restrictive of the conditions (13.9) should give (19). Show that (13.9) does, indeed, give $r \le \frac{1}{2}$ or, more precisely,

$$r \le \frac{1}{1 - \cos\left(\frac{N-1}{N}\pi\right)}. \tag{13.10}$$

For large $N$ the right-hand side is approximately $\frac{1}{2}$; e.g., for $N = 50$ (13.10) gives $r \le 0.50049$. HINT: The $\mathbf{A}$ matrix is tridiagonal, of the type shown in Exercise 7 of Section 11.2 with "$a$" $= r$, "$b$" $= 1-2r$, and "$c$" $= r$. Thus, its eigenvalues are, according to equation (7.1) of that exercise,

$$\lambda_n = 1 - 2r + 2r\cos\frac{n\pi}{N} \tag{13.11}$$

for $n = 1, 2, \ldots, N-1$. Further, note that each inequality $|\lambda_j| \le 1$ in (13.9) amounts to the statement $-1 \le \lambda_j \le 1$, hence the *two* inequalities $-1 \le \lambda_j$ and $\lambda_j \le 1$. Of the $2N-2$ inequalities in (13.9), you should find that the most restrictive condition on $r$ is given by (13.10).

**14.** (*Stability of implicit scheme*) The stability of the implicit scheme (21) for all $r$ (if $\theta \ge \frac{1}{2}$) is a striking result. Proof of that result would follow the same lines as the proof that is outlined in Exercise 13 for the explicit scheme. However, for pedagogical purposes it might be better to consider the simpler case of an *ordinary* differential equation. Specifically, consider the simple *test equation*

$$u_t = -Au, \tag{14.1}$$

where $A$ is a prescribed positive constant, with the implicit finite-difference approximation

$$\frac{U_{k+1} - U_k}{\Delta t} = -(1-\theta)AU_k - \theta AU_{k+1}. \tag{14.2}$$

Following the same lines as in Exercise 13, show that the roundoff error $e_k \equiv U_k - U_k^*$ propagates according to

$$e_{k+1} = Ke_k, \tag{14.3}$$

where

$$K = \frac{1 - A(1-\theta)\Delta t}{1 + A\theta\Delta t}. \tag{14.4}$$

Thus, show that the scheme is stable if and only if

$$A(1-2\theta)\Delta t \le 2. \tag{14.5}$$

NOTE: Suppose that $\theta = 0$ so the scheme (14.2) is explicit. Then (14.5) tells us that for stability we must choose $\Delta t \le 2/A$. In some problems, such as occur in the study of chemical kinetics, $A$ can be extremely large, so that $\Delta t \le 2/A$ forces us to use extremely small time steps. However, observe that if we use the implicit scheme (14.2) with $\theta \ge \frac{1}{2}$, then (14.5) is satisfied with no restriction on $\Delta t$.

**15.** Use the Crank–Nicolson scheme (22) to solve the problem (11.1) in Exercise 11 through the first three lines of $U_{j,k}$'s (i.e., through $k = 3$), with $\Delta t = 0.1$ and $\Delta x = 0.25$, using computer software (such as the *Maple* linsolve command) to solve the matrix equation obtained at each time step. Compare your results at $t = 0.3$ with the exact solution.

**16.** In Example 3 we used the Jacobi, Gauss–Seidel, and SOR methods to work out the iterates $U_{1,1}^{(0)}$, $U_{2,1}^{(0)}$, $U_{3,1}^{(0)}$ and $U_{1,1}^{(1)}$, $U_{2,1}^{(1)}$, $U_{3,1}^{(1)}$. Continuing the calculation, work out $U_{1,1}^{(2)}$, $U_{2,1}^{(2)}$, $U_{3,1}^{(2)}$. NOTE: For $U_{2,1}^{(2)}$ your three values should agree with the values 69.38, 71.41, 71.70 given in Table 1.

**17.** In (28) let $r = 2$ and $\mathbf{c} = [1, 1, 1, 1]^{\mathrm{T}}$. (Thus, $N = 5$.)

(a) Solve for $\mathbf{U}_{k+1}$ using Jacobi iteration, terminating the iterations when $\left|\mathbf{U}_{j,k+1}^{(n+1)} - \mathbf{U}_{j,k+1}^{(n)}\right| < 0.0001$, say, for each $j$ ($j = 1, 2, 3, 4$). Record the number of iterations needed to achieve that accuracy.

(b) Same as part (a) but using Gauss–Seidel iteration instead.

(c) Same as part (a) but using the SOR method. Use $\omega = 0.9, 1.0, 1.1$, and $1.2$, and compare the optimum $\omega$ with that predicted by (39).

**18.** (*Convergence of Jacobi iteration*) It is stated below (33) that the Jacobi iteration (33) converges to the solution of (26) for all finite values of $r$. Prove that claim. HINT: Denote the eigenvalues and eigenvectors of $\mathbf{A}'$ as $\lambda_1, \ldots, \lambda_{N-1}$ and $\mathbf{\Phi}_1, \ldots, \mathbf{\Phi}_{N-1}$. Expanding $\mathbf{c}$ as

$$\mathbf{c} = \sum_1^{N-1} c_j \mathbf{\Phi}_j, \tag{18.1}$$

(31) and (33) give

$$\mathbf{U}_{k+1}^{(0)} = \beta \sum_1^{N-1} c_j \mathbf{\Phi}_j, \qquad \left(\beta \equiv \frac{1}{2(1+r)}\right)$$

$$\mathbf{U}_{k+1}^{(1)} = \beta \sum_1^{N-1} c_j \mathbf{\Phi}_j - \beta^2 \sum_1^{N-1} c_j \lambda_j \mathbf{\Phi}_j$$

$$= \beta \sum_1^{N-1} (1 - \beta\lambda_j) c_j \mathbf{\Phi}_j,$$

$$\mathbf{U}_{k+1}^{(2)} = \beta \sum_1^{N-1} (1 - \beta\lambda_j + \beta^2\lambda_j^2) c_j \mathbf{\Phi}_j,$$

and so on. Use (13.11) in Exercise 13 to show that $|\beta\lambda_j| < 1$ for each $j$, for all $r < \infty$. Thus, show that

$$\mathbf{U}_{k+1}^{(n)} \to \beta \sum_{1}^{N-1} \frac{1}{1+\beta\lambda_j} \mathbf{\Phi}_j \tag{18.2}$$

as $n \to \infty$, and verify that (18.2) satisfies (26).

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# Chapter 18 Review

The central problem of this chapter is the one-dimensional diffusion equation

$$\alpha^2 u_{xx} = u_t \tag{1}$$

on a finite $x$ interval, with constant Dirichlet ($u$ given) or Neumann ($u_x$ given) boundary conditions. The problem can be solved by the method of separation of variables, the key point of which is the reduction of the PDE to two ODE's governing the factors $X(x)$ and $T(t)$. We emphasize that the boundary conditions are to be applied before the initial condition and we show how to use Fourier series to satisfy the initial condition. Our approach is only formal, but the issue of rigorous justification is addressed in the brief optional Section 18.3.2.

In other cases Fourier series (i.e., the half- and quarter-range formulas and the full Fourier series of a periodic function) may not suffice – for instance, if we have Robin boundary conditions (a linear combination of $u$ and $u_x$ given) or if we have axisymmetric heat conduction in a circular disk or cylinder, governed by the PDE

$$\alpha^2 \left( u_{rr} + \frac{1}{r} u_r \right) = u_t. \tag{2}$$

These cases can be handled using the more powerful Sturm–Liouville theory, as discussed in the optional Section 18.3.3, and the series expansion required for satisfaction of the initial condition is in terms of the eigenfunctions of the relevant Sturm–Liouville problem.

Additional generalizations of (1) are contained in the exercises for Section 18.3, such as the inclusion of one or more of the terms $Vu_x$ (due to axial convection of the medium), $Hu$ (a Newton cooling term due to lateral heat loss), and $F(x,t)$ (due to a distributed source within the medium) in the equation

$$\alpha^2 u_{xx} = u_t + Vu_x + Hu - F(x,t). \tag{3}$$

The Fourier and Laplace transforms enable us to handle problems for which the basic separation of variable method fails or is awkward – for instance, problems on a semi-infinite ($0 < x < \infty$) or infinite ($-\infty < x < \infty$) $x$ domain, or with nonconstant boundary conditions or a distributed source term. These cases are discussed in Section 18.4.

The optional Section 18.5 explains a useful method known as the method of images, which is based on the idea of satisfying homogeneous boundary conditions

by fictitiously extending the problem so as to build in symmetry or antisymmetry about the boundary in question. The method is used again in Chapters 19 and 20, but the idea is simple enough so that Section 18.5 is *not* a prerequisite for those chapters.

The final section, 18.6, explains the numerical solution of the diffusion equation by the method of finite differences and gives a glimpse of the power of numerical simulation. We find that the simple explicit method is limited by the condition that

$$r = \alpha^2 \frac{\Delta t}{(\Delta x)^2} \leq \frac{1}{2} \tag{4}$$

for convergence and stability. To remove the condition (4) we can use an implicit method instead, such as the Crank–Nicolson method, but the price that we pay is that to generate each time line of $U_{j,k}$ values we need to solve an $(N-1) \times (N-1)$ matrix equation, where $N$ is the number of divisions of $L$ (i.e., $\Delta x = L/N$). Fortunately, the matrix is strongly diagonal, so the equation can be solved efficiently by iteration. The mathematics of the finite-difference method is linear algebra since one is replacing a linear PDE by a large system of linear algebraic equations. Questions of convergence, stability, and efficiency are best dealt with using linear algebra methods such as are covered here in Chapters 8–12.

Important general features of the diffusion equation are as follows:

1. The diffusion equation is an *initial-value problem* in $t$, as is especially clear in Section 18.6 where we see from the "marching" scheme

$$U_{j,k+1} = rU_{j-1,k} + (1-2r)U_{j,k} + rU_{j+1,k} \tag{5}$$

that the solution along the line $t = (k+1)\Delta t$ is implied by the solution along the preceding line $t = k\Delta t$.

2. Diffusion is a *smoothing process*, as seen from our various solution plots and also from the finite-difference formula (5), since $U_{j,k+1}$ is thereby given as a weighted average of the preceding values $U_{j-1,k}$, $U_{j,k}$ and $U_{j+1,k}$, "average" because the coefficients $r, 1-2r, r$ in (5) sum to unity.

Recall from Section 18.2 that the linear PDE

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = f \tag{6}$$

is **parabolic** if $B^2 - AC = 0$, and that the diffusion equation (1) is a simple or canonical form of the general second-order parabolic PDE (with the $y$'s changed to $t$'s). Thus, understand (1) to be *representative* of that entire class of PDE's.

# Chapter 19

# Wave Equation

## 19.1 Introduction

The **wave equation**

$$\boxed{c^2\nabla^2 u = u_{tt}} \tag{1}$$

governs a wide variety a wave phenomena such as electromagnetic waves, water waves, supersonic flow, pulsatile blood flow, acoustics, elastic waves in solids, and vibrating strings and membranes. In this introductory section we derive the wave equations governing the vibrating string and vibrating membrane and outline, in the exercises, several other such cases leading to wave equations.

By a *vibrating string* we mean a taut string, such as a guitar string, undergoing a planar vibratory motion. The problem of the vibrating string is of historical importance since it was studied extensively by the great mathematicians *Leonhard Euler* (1707–1783), *Jean Le Rond d'Alembert* (1717–1783), *Daniel Bernoulli* (1700–1782), and *Joseph-Louis Lagrange* (1736–1813). That work gave birth to the subject of partial differential equations. For example, it was in his study of the vibrating string that d'Alembert developed the method of separation of variables used in Chapter 18 and which is now a standard tool in the solution of PDE's. Likewise, he found it necessary to represent the initial shape of the string by what is now known as a Fourier series. Whether such representation is possible, for any given initial shape of the string, was the subject of heated debate, a debate that continued well into the nineteenth century. The theory of Fourier series that emerged is the subject of Chapter 17.

Beginning our derivation, consider a flexible string stretched under tension $\tau$ newtons between fixed endpoints at $x = 0$ and $x = L$ on a horizontal $x$ axis (Fig. 1). Let the mass per unit length of the string be $\sigma$, a constant, and suppose that the string supports a distributed load $f(x,t)$ newtons per unit $x$ length, counted as positive if it acts downward. Considering the plane vertical motion of the string (in the $x, y$ plane), the desired unknown is the displacement $y(x,t)$.

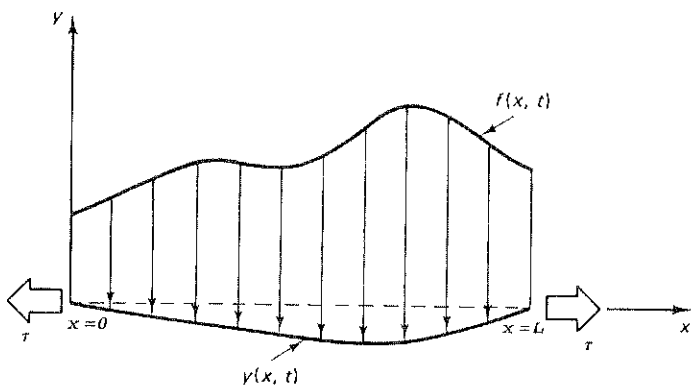



**Figure 1.** Loaded vibrating string.

We assume that

1. the slope $\partial y/\partial x$ is uniformly small over the length of the string (i.e., $|\partial y/\partial x| \ll 1$);

2. acting on each cross section of the string is the tangentially oriented tension force $\tau$, but no shear force or bending moment as is explained in Example 2 of Section 1.3.

The relevant physical principle is Newton's second law of motion, which we apply to the vertical motion of an element of the string between $x$ and $x + \Delta x$ (Fig. 2):

$$\begin{aligned} &\tau \sin\theta(x + \Delta x, t) - \tau \sin\theta(x, t) - f(x + \alpha\Delta x, t)\Delta x \\ &\quad = \sigma\Delta s \frac{\partial^2 y}{\partial t^2}(x + \beta\Delta x, t), \end{aligned} \tag{2}$$

where $\Delta s$ is the arc length. That is, the sum of the vertical forces [on the left-hand side of (2)] equals the mass $\sigma\Delta s$ times the vertical acceleration of the mass center at $x + \beta\Delta x$ (for some $\beta$ such that $0 \le \beta \le 1$). We have assumed that $f$ is a continuous function of $x$ (although that condition could be relaxed) so that there is a point $x + \alpha\Delta x$ (for some $\alpha$ between 0 and 1) at which $f$ takes on its average value over the $(x, x + \Delta x)$ interval.

According to Assumption 1, $\theta$ is small so we have these approximations from the Taylor series of $\sin\theta$ and $\tan\theta$:

$$\sin\theta = \theta - \frac{1}{3!}\theta^3 + \frac{1}{5!}\theta^5 - \cdots \approx \theta \tag{3}$$

and

$$\tan\theta = \theta + \frac{1}{3!}\theta^3 + \frac{2}{15}\theta^5 + \cdots \approx \theta. \tag{4}$$

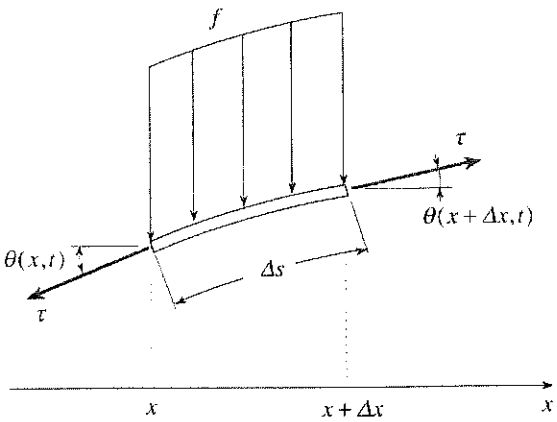



**Figure 2.** String element.

Hence, for small $\theta$ we have $\sin\theta \approx \tan\theta$, the truth of which can also be seen graphically in Fig. 3. But $\tan\theta$ is the slope $\partial y/\partial x$. It also follows from assumption 1 that $\cos\theta = 1 - \frac{1}{2!}\theta^2 + \cdots \approx 1$, so $\Delta s = \Delta x/\cos\theta \approx \Delta x$. Thus, we can eliminate the temporary variables $\theta$ and $s$ in favor of $y$ and $x$ and re-express (2) as

$$\tau\,\frac{\dfrac{\partial y}{\partial x}(x+\Delta x,t) - \dfrac{\partial y}{\partial x}(x,t)}{\Delta x} - f(x+\alpha\Delta x,t) = \sigma\,\frac{\partial^2 y}{\partial t^2}(x+\beta\Delta x,t). \tag{5}$$

Finally, letting $\Delta x \to 0$ in (5) gives the desired PDE

$$\tau\,\frac{\partial^2 y}{\partial x^2}(x,t) - f(x,t) = \sigma\,\frac{\partial^2 y}{\partial t^2}(x,t) \tag{6}$$

governing $y(x,t)$. If $f$ is simply the gravitational force on the string (i.e., the weight force per unit length of the string), then $f(x,t) = \sigma g =$ constant, where $g$ is the acceleration due to gravity, and (6) becomes

$$\tau\,\frac{\partial^2 y}{\partial x^2} = \sigma\,\frac{\partial^2 y}{\partial t^2} + \sigma g. \tag{7}$$

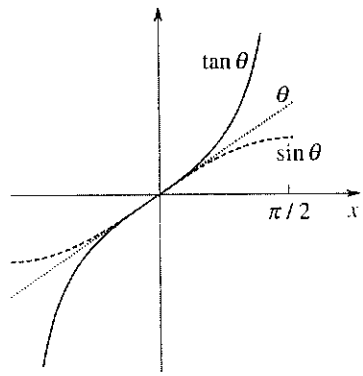



**Figure 3.** $\sin\theta \approx \theta \approx \tan\theta$ for small $\theta$.

If the gravitational term is negligible, as is surely the case for a guitar string,* then (7) reduces to

$$\tau\,\frac{\partial^2 y}{\partial x^2} = \sigma\,\frac{\partial^2 y}{\partial t^2} \tag{8}$$

or, setting $c \equiv \sqrt{\tau/\sigma}$ and using the more compact subscript notation for partial derivatives,

$$\boxed{c^2 y_{xx} = y_{tt},} \tag{9}$$

*A guitar sounds the same if it is played horizontally [so the $\sigma g$ term is present in (7)] or vertically [ so the $\sigma g$ term is *not* present in (7)]. Applying dimensional reasoning to (7), we find that the $\sigma g$ term can be neglected if $\sigma g << \tau/L$.

which is the classical *one-dimensional wave equation* governing $y(x,t)$.

It is always important to tie together the mathematics and the physics. Accordingly, how are we to understand the terms in (8)? Essentially, (8) is of the form force = mass × acceleration.* The $\tau y_{xx}$ term is the net vertical force on the element due to the tension $\tau$. To understand its form, recall from the calculus that the local radius of curvature $R$ and curvature $\kappa$ for a plane curve $y(x)$ (Fig. 4) are given by

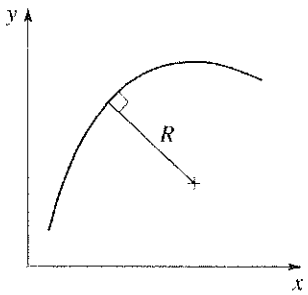



**Figure 4.** Radius of curvature $R$.

$$\kappa = \frac{1}{R} = \frac{y''}{(1+y'^2)^{3/2}}. \tag{10}$$

In our case $|y'| \ll 1$, so (10) becomes $\kappa \approx y''$. Thus, within our assumption of small deflection and small slope the $\tau y_{xx}$ term in (8) is the product of the tension and the curvature. That result makes sense physically because it is through the curvature that the two tension forces in Fig. 2 are misaligned and therefore have a nonzero vertical resultant.

We close this section with an introduction to the *two*-dimensional wave equation. Specifically, consider the two-dimensional version of a vibrating string, a vibrating *membrane* such as a drumhead. We assume that the membrane is stretched uniformly under a tension $\tau$ per unit length. That is, at each point of the membrane the tension per unit length along any straight line through that point, independent of the orientation of the line, is $\tau$. If, for example, we stretch a rectangular membrane horizontally (Fig. 5) and then clamp the four edges, then the membrane would *not* be stretched uniformly, for the tension per unit length along any vertical line would be $\tau$ whereas along any horizontal line it would be zero.

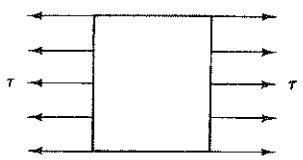



**Figure 5.** *Non*uniformly stretched membrane.

Denoting the displacement of the membrane out of the $x, y$ plane as $w(x,y,t)$, we proceed essentially as before. Thus, we assume that the slopes $\partial w/\partial x$ and $\partial w/\partial y$ are uniformly small over the domain (i.e., $|\partial w/\partial x| \ll 1$ and $|\partial w/\partial y| \ll 1$), and that the membrane is perfectly flexible, so that only the tangential tensile force $\tau$ acts. Then, applying Newton's second law to a membrane element lying between $x$ and $x+\Delta x$ and between $y$ and $y+\Delta y$ (Fig. 6), gives

$$\begin{aligned}\tau\Delta y \sin\theta\Big|_{x+\Delta x} - \tau\Delta y\sin\theta\Big|_x \\ +\tau\Delta x\sin\phi\Big|_{y+\Delta y} - \tau\Delta x\sin\phi\Big|_y - f\Delta x\Delta y = \sigma\Delta A\,\frac{\partial^2 w}{\partial t^2},\end{aligned} \tag{11}$$

where $\sigma$ is the mass per unit area of the membrane, $\Delta A$ is the surface area of the element under consideration, and $f(x,y,t)$ is a distributed load counted as positive if it acts downward. By virtue of the stated assumptions, $\sin\theta \approx \theta \approx \tan\theta = \partial w/\partial x$, $\sin\phi \approx \phi \approx \tan\phi = \partial w/\partial y$, and $\Delta A \approx \Delta x\Delta y$, so (11) becomes

$$\tau\,\frac{w_x|_{x+\Delta x} - w_x|_x}{\Delta x} + \tau\,\frac{w_y|_{y+\Delta y} - w_y|_y}{\Delta y} - f = \sigma w_{tt}, \tag{12}$$

*More precisely, (8) is on a per unit $x$-length basis: the vertical force per unit $x$ length is equal to the mass per unit $x$ length times the vertical acceleration.

and letting $\Delta x \to 0$ and $\Delta y \to 0$ gives the PDE

$$\tau(w_{xx} + w_{yy}) - f(x, y, t) = \sigma w_{tt}. \tag{13}$$

If $f = 0$ and if we define $c \equiv \sqrt{\tau/\sigma}$, then (13) becomes the classical *two-dimensional wave equation*

$$\boxed{c^2(w_{xx} + w_{yy}) = w_{tt}} \tag{14}$$

governing $w(x, y, t)$.

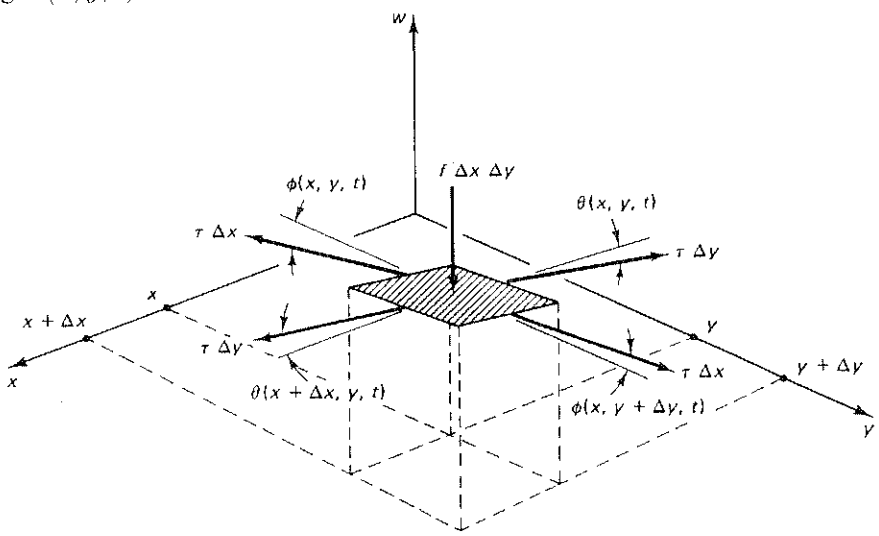



**Figure 6.** Membrane element.

In the next two sections we solve the vibrating string and vibrating membrane equations by separation of variables.

**Closure.** In this section we derive the one- and two-dimensional wave equations (9) and (14) governing vibrating strings and vibrating membranes, respectively, subject to the assumptions that the string or membrane is flexible and that the slopes are small. Several additional problems governed by wave equations are given in the exercises.

## EXERCISES 19.1

1. (*Hanging chain*) Let a flexible chain hang from the ceiling. Measure $x$ downward from the ceiling and let $y(x, t)$ be its lateral displacement. Modifying our derivation of (9) as appropriate, show that the PDE governing $y(x, t)$ is

$$g[(L - x)y_x]_x = y_{tt}, \tag{1.1}$$

where $g$ is the acceleration due to gravity and $L$ is the length

of the chain.

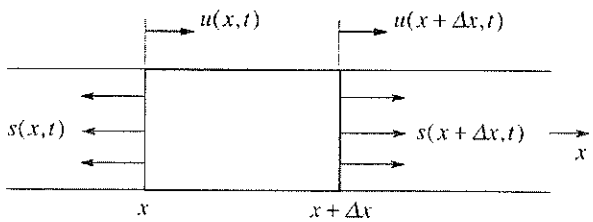
**2.** (*Longitudinal waves in a rod*) Consider a uniform metal bar, of cross-sectional area $A$ and mass per unit length $\sigma$, with a stress distribution $s(x,t)$ and a resulting longitudinal displacement $u(x,t)$. Applying Newton's second law to an element of the rod between $x$ and $x+\Delta x$, shown in the figure, show that

$$A\frac{\partial s}{\partial x} = \sigma\frac{\partial^2 u}{\partial t^2}. \tag{2.1}$$

Suppose that the material admits a linear stress-strain relationship $s = E\epsilon$, where the constant of proportionality $E$ is Young's modulus, and the strain $\epsilon$ is defined as the "stretch per unit length," so that $\epsilon = \dfrac{u(x+\Delta x,t)-x(x,t)}{\Delta x}$. From these relations, show that $s$ and $u$ both satisfy one-dimensional wave equations

$$c^2 u_{xx} = u_{tt} \qquad \text{and} \qquad c^2 s_{xx} = s_{tt}, \tag{2.2}$$

where $c \equiv \sqrt{EA/\sigma}$.




**3.** (*Electromagnetic waves*) (a) Electromagnetic fields in free space (i.e., in a vacuum) are governed by the famous **Maxwell's equations**:

$$\begin{aligned} \nabla\times\mathbf{H} &= \epsilon_0\frac{\partial\mathbf{E}}{\partial t},\\ \nabla\times\mathbf{E} &= -\mu_0\frac{\partial\mathbf{H}}{\partial t},\\ \nabla\cdot\mathbf{H} &= 0,\\ \nabla\cdot\mathbf{E} &= 0, \end{aligned} \tag{3.1,2,3,4}$$

where $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic field intensities, respectively, and where $\epsilon_0$ and $\mu_0$ are the permittivity and permeability of free space, respectively. Show that it follows from (3.1)–(3.4) that $\mathbf{E}$ and $\mathbf{H}$ (which are functions of $x, y, z$ and the time $t$) satisfy the wave equations

$$c^2\nabla^2\mathbf{H} = \mathbf{H}_{tt} \qquad \text{and} \qquad c^2\nabla^2\mathbf{E} = \mathbf{E}_{tt}, \tag{3.5}$$

where $c \equiv 1/\sqrt{\epsilon_0\mu_0}$. HINT: Recall the identity $\nabla\times(\nabla\times\mathbf{v}) = \nabla(\nabla\cdot\mathbf{v}) - \nabla^2\mathbf{v}$ from Section 16.6.

(b) Write the $x, y, z$ components of the vector wave equations (3.5).

**4.** (*Current and magnetic field*) Suppose that the "string" is actually a flexible wire of mass per unit length $\sigma$, under tension $\tau$, carrying a current $I$ in the presence of a uniform magnetic field of magnetic flux density $\mathbf{B} = B_1\hat{\mathbf{i}} + B_2\hat{\mathbf{j}} + B_3\hat{\mathbf{k}}$. Since a charge $Q$ moving with velocity $\mathbf{U}$ in such a field experiences a force $\mathbf{F} = Q\mathbf{U}\times\mathbf{B}$, we may need to revise the vibrating string equation to include lateral magnetic forcing terms.

(a) Show that within the usual vibrating string approximations, the magnetic force exerted on an element of the wire is

$$\Delta\mathbf{F} \approx qA\Delta x\left[U\hat{\mathbf{i}} + (Uy_x + y_t)\hat{\mathbf{j}} + (Uz_x + z_t)\hat{\mathbf{k}}\right]\times\mathbf{B}, \tag{4.1}$$

where the constants $q, A, U$ are the charge per unit volume within the wire, cross-sectional area of the wire, and velocity of the charges within the wire, respectively. HINT: The $x, y, z$ velocity components of an element of charge within the wire are

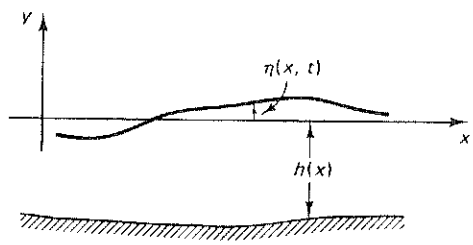
$$\begin{aligned} \frac{dx}{dt} &= U,\\ \frac{dy}{dt}(x(t),t) &= y_x\frac{dx}{dt} + y_t = Uy_x + y_t,\\ \frac{dz}{dt}(x(t),t) &= z_x\frac{dx}{dt} + z_t = Uz_x + z_t. \end{aligned}$$

(b) Working out the cross product in (4.1) and noting that $qAU$ is the current $I$, show that the equations of (the not necessarily planar) motion of the wire are

$$\begin{aligned} \tau y_{xx} - IB_3 + (Iz_x + qAz_t)B_1 &= \sigma y_{tt},\\ \tau z_{xx} + IB_2 - (Iy_x + qAy_t)B_1 &= \sigma z_{tt}, \end{aligned} \tag{4.2}$$

where $x, y, z$ are right-handed Cartesian coordinates. Notice that these equations are *coupled* due to the $z_x, z_t, y_x$, and $y_t$ terms. Naturally, in a particular application one or more of the additional magnetic-field terms might be negligible.

**5.** (*Water waves*) Consider plane water waves in water of depth $h(x)$ as shown in the figure. If the wavelength is much greater



than $h$ (as is true for ocean tides and certain waves in shallow water), the governing equations are found to be

$$\begin{aligned} u_t + uu_x &= -g\eta_x, \\ [u(\eta + h)]_x &= -\eta_t, \end{aligned} \tag{5.1,2}$$

where $u$ is the $x$ velocity [which is approximately constant with respect to $y$ so $u = u(x,t)$], $\eta(x,t)$ is the free-surface elevation relative to the undisturbed water level, and $g$ is the acceleration due to gravity. If we restrict our attention to small amplitude motions – so that the "second-order" term $uu_x$ can be neglected relative to the "first-order" terms $u_t$ and $g\eta_x$, and $\eta$ can be neglected relative to $h$ – then show that $\eta$ satisfies the equation $g(h\eta_x)_x = \eta_{tt}$ or, if $h(x)$ is a constant,

$$c^2\eta_{xx} = \eta_{tt}, \tag{5.3}$$

where $c \equiv \sqrt{gh}$.

## 19.2 Separation of Variables; Vibrating String

**19.2.1. Solution by separation of variables.** In Section 19.1 we derive the wave equation governing the motion of a vibrating string. In the present section we complete the formulation by appending boundary and initial conditions and then show how to solve for $y(x,t)$ by the method of separation of variables.

Specifically, let us consider a finite string extending over $0 < x < L$, tied at its ends, and having initial displacement $y(x,0) = f(x)$ and initial velocity $y_t(x,0) = g(x)$, where $f$ and $g$ are prescribed. Thus, the complete problem statement is

$$c^2 y_{xx} = y_{tt}, \qquad (0 < x < L,\ 0 < t < \infty) \tag{1a}$$

$$y(0,t) = 0,\ \ y(L,t) = 0, \qquad (0 < t < \infty) \tag{1b}$$

$$y(x,0) = f(x),\ \ y_t(x,0) = g(x) \qquad (0 < x < L) \tag{1c}$$

and is summarized in Fig. 1.

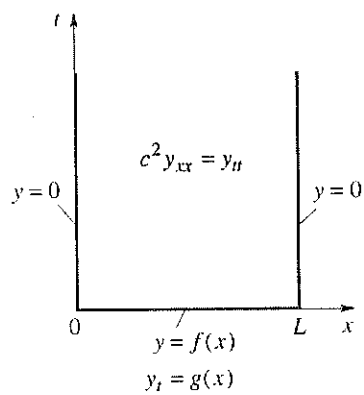



**Figure 1.** Problem (1) in the $x, t$ plane.

Notice carefully that for the diffusion equation $\alpha^2 u_{xx} = u_t$ (Chapter 18) we prescribe only $u$ initially, whereas for the wave equation we prescribe both $y$ *and* $y_t$ initially. Intuitively, it certainly seems reasonable that to predict $y(x,t)$ we will need to know how the string is set in motion, namely, both the initial displacement $y(x,0)$ and the initial velocity $y_t(x,0)$. From a mathematical point of view we are guided by the fact that the wave equation (1a) is a second-order equation with respect to $t$ (whereas the diffusion equation is of first order), so we expect to need two initial conditions. Verification that the problem statement (1) *uniquely* determines $y(x,t)$ is left for the exercises. Here, we limit our attention to *finding* the solution.

Using separation of variables, seek

$$y(x,t) = X(x)T(t). \tag{2}$$

Putting (2) into (1a) and separating the variables gives

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T}. \tag{3}$$

Since the left-hand side of (3) is a function of $x$ alone and the right-hand side is a function of $t$ alone, it follows from (3) (as discussed in Section 18.3) that both sides are constants, say $-\kappa^2$, so

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} = \text{constant} = -\kappa^2. \tag{4}$$

Thus,

$$X'' + \kappa^2 X = 0, \tag{5a}$$
$$T'' + \kappa^2 c^2 T = 0, \tag{5b}$$

and

$$X = \begin{cases} A + Bx, & \kappa = 0 \\ D\cos\kappa x + E\sin\kappa x, & \kappa \neq 0 \end{cases} \tag{6}$$

$$T = \begin{cases} H + It, & \kappa = 0 \\ J\cos\kappa ct + K\sin\kappa ct, & \kappa \neq 0. \end{cases} \tag{7}$$

Our motivation for writing the minus sign in (4) is that the resulting ODE (5b) admits $\cos\kappa ct$ and $\sin\kappa ct$ solutions, which look correct since we anticipate a vibratory motion; if we write $\kappa^2$ in (4) instead of $-\kappa^2$, then we obtain $T'' - \kappa^2 c^2 T = 0$, instead of (5b), with nonvibratory exponential solutions.*

Thus, we have found product solutions of the form $(A + Bx)(H + It)$ and $(D\cos\kappa x + E\sin\kappa x)(J\cos\kappa ct + K\sin\kappa ct)$ and, relying on the linearity of $L[y] = c^2 y_{xx} - y_{tt} = 0$, we can use superposition and write

$$\begin{aligned} y(x,t) = {} & (A + Bx)(H + It) \\ & + (D\cos\kappa x + E\sin\kappa x)(J\cos\kappa ct + K\sin\kappa ct). \end{aligned} \tag{8}$$

As in Chapter 18, we apply the boundary conditions before the initial conditions. Accordingly,

$$y(0,t) = 0 = A(H + It) + D(J\cos\kappa ct + K\sin\kappa ct). \tag{9}$$

Since the right-hand side of (9) is a linear combination of the linearly independent functions $1, t, \cos\kappa ct, \sin\kappa ct$, it follows from (9) that we must have either $A = 0$ or $H = I = 0$, and either $D = 0$ or $J = K = 0$. We choose $A = 0$ and $D = 0$ so as to be left with as robust a solution as possible, namely,

$$y(x,t) = Bx(H + It) + E\sin\kappa x(J\cos\kappa ct + K\sin\kappa ct) \tag{10}$$

or, combining $BH$ as $P$, $BI$ as $Q$, $EJ$ as $R$, and $EK$ as $S$, for brevity,

$$y(x,t) = x(P + Qt) + \sin\kappa x(R\cos\kappa ct + S\sin\kappa ct). \tag{11}$$

*As discussed in Exercise 2 of Section 18.3, we can survive writing $\kappa^2$ instead of $-\kappa^2$, but the choice is less convenient because $\kappa$ will then end up being purely imaginary rather than real.

In case it is not clear how the choice $A = D = 0$ gives "as robust a solution as possible," let us return to (9) and focus on the $D(J\cos\kappa ct + K\sin\kappa ct)$ term. If we choose $J = K = 0$, then we lose the entire $(D\cos\kappa x + E\sin\kappa x)(J\cos\kappa ct + K\sin\kappa ct)$ term in (8), whereas if we choose $D = 0$ as, indeed, we did, then we are left with $E\sin\kappa x(J\cos\kappa ct + K\sin\kappa ct)$. Similarly, if we infer from (9) that $H = I = 0$, then we lose the entire $(A + Bx)(H + It)$ term in (8), whereas if we infer that $A = 0$, then we are left with $Bx(H + It)$.

Next,

$$y(L,t) = 0 = L(P + Qt) + \sin\kappa L(R\cos\kappa ct + S\sin\kappa ct) \tag{12}$$

so we need

$$LP = 0 \quad \text{and} \quad LQ = 0 \tag{13}$$

as well as

$$R\sin\kappa L = 0 \quad \text{and} \quad S\sin\kappa L = 0. \tag{14}$$

Since $L \neq 0$, (13) gives $P = Q = 0$. And if we are to avoid having $R = S = 0$, we must choose

$$\sin\kappa L = 0. \tag{15}$$

Thus, $\kappa = n\pi/L$ for $n = 1, 2, \ldots$. Putting these results into (11) and using superposition gives

$$y(x,t) = \sum_{n=1}^{\infty} \sin\frac{n\pi x}{L}\left(R_n\cos\frac{n\pi ct}{L} + S_n\sin\frac{n\pi ct}{L}\right). \tag{16}$$

[If any steps in deriving (16) are unclear, we urge you to review Example 1 in Section 18.3]

Our expectation is that the $R$'s and $S$'s can now be determined from the initial conditions (1c). Imposing those conditions gives

$$y(x,0) = f(x) = \sum_{n=1}^{\infty} R_n\sin\frac{n\pi x}{L} \qquad (0 < x < L) \tag{17a}$$

and

$$y_t(x,0) = g(x) = \sum_{n=1}^{\infty} \frac{n\pi c}{L} S_n\sin\frac{n\pi x}{L}. \qquad (0 < x < L) \tag{17b}$$

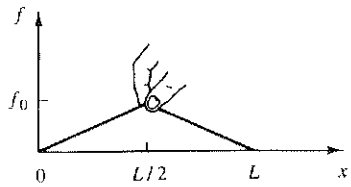



**Figure 2.** Plucked string.

We can identify each of (17a,b) as a half-range sine series so (→ pág 870)

$$R_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \tag{18a}$$

and

$$\frac{n\pi c}{L}S_n = \frac{2}{L}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx$$

or

$$S_n = \frac{2}{n\pi c}\int_0^L g(x)\sin\frac{n\pi x}{L}\,dx. \tag{18b}$$

Hence, the solution of (1) is given by (16) with the $R_n$'s and $S_n$'s computed according to (18a) and (18b).

To illustrate, let $f(x)$ be as shown in Fig. 2 and let $g(x) = 0$. That is, we pull the string up at its midpoint and then release it from rest. Then (18a,b) give

$$R_n = \frac{8f_0}{n^2\pi^2}\sin\frac{n\pi}{2} \quad \text{and} \quad S_n = 0, \tag{19}$$

so

$$y(x,t) = \frac{8f_0}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}. \tag{20}$$

The right-hand side of (20) is a superposition of distinct **modes** of vibration $\sin(n\pi x/L)\cos(n\pi ct/L)$, each of which is a **standing wave** of spatial frequency $n\pi/L$ and temporal frequency $n\pi c/L$. The first three mode shapes are depicted in Fig. 3. The points $x = 0$ and $x = L$ (heavy dots in Fig. 3) are called **nodal points** of the first mode because $y = 0$ there for all $t$; $x = 0$, $L/2$, and $L$ are nodal points of the second mode, and so on. Further, we say that the modes are **orthogonal** inasmuch as their shapes satisfy the **orthogonality relation**

$$\int_0^L \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = 0 \tag{21}$$

for any pair of integers $m$ and $n$ with $m \neq n$ [equation (24b), Section 17.3].

It is interesting to see what (20) can tell us about the musical quality of a violin string plucked in the manner shown in Fig. 2. Suppose, for definiteness, that we tune the string so that its fundamental frequency $\pi c/L$ corresponds to the lowest $A$ on a piano, say $A_0$. Since $A_0$'s frequency is 27.5 cycles/sec, we accomplish that tuning by adjusting the tension $\tau$ so that

$$\frac{\pi c}{L} = \frac{\pi}{L}\sqrt{\frac{\tau}{\sigma}}\,\frac{\text{rad}}{\text{sec}} = \left(27.5\,\frac{\text{cycles}}{\text{sec}}\right)\left(\frac{2\pi\,\text{rad}}{\text{cycle}}\right), \tag{22}$$

or

$$\tau = (55L)^2\sigma. \tag{23}$$

Then the first several terms in (20) correspond to the combination of notes shown in Table 1, where $A_1$ is an $A$ one octave higher than $A_0$, and so on, and $C_3^{\#}$ is $C$ sharp in the third octave above the lowest. Observe that the overtones (with nonzero amplitude), $E_2, C_3^{\#}, \ldots$, do not occur at octaves above $A_0$. Thus, the sound is not a pristine $A_0$ with octave overtones but it is, nonetheless, fairly "clean" because of the relatively small amplitudes of the $E_2, C_3^{\#}, \ldots$ contributions. The mix of frequencies and amplitudes is different for different instruments and an $A_0$ played on

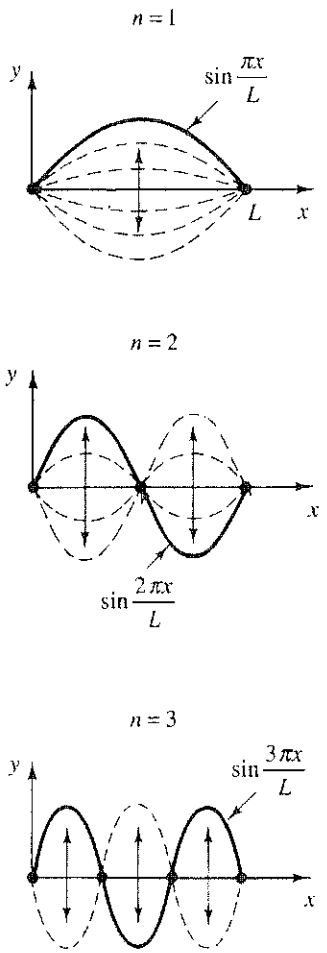



**Figure 3.** The first three modes.

**Table 1.** The first five notes in (20).

| $n$ | Frequency $n\dfrac{\pi c}{L}$ (cycles/sec) | Relative Amplitude $\dfrac{1}{n^2}\left\lvert\sin\dfrac{n\pi}{2}\right\rvert$ | Musical Note |
|---|---|---|---|
| 1 | 27.5 | 1 | $A_0$ |
| 2 | 55.0 | 0 | $A_1$ |
| 3 | 82.5 | $\frac{1}{9}$ | $\approx E_2$ |
| 4 | 110.0 | 0 | $A_2$ |
| 5 | 137.5 | $\frac{1}{25}$ | $\approx C_3^{\#}$ |

a violin sounds different from an $A_0$ played on a tuba.*

**19.2.2. Traveling wave interpretation.** We have seen that if $y(x,0) = f(x)$ is prescribed and $y_t(x,0) = 0$, then $y(x,t)$ is given by

$$y(x,t) = \sum_{n=1}^{\infty} R_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}, \qquad (0 < x < L,\ \ 0 < t < \infty) \tag{24}$$

where the $R_n$'s are computed from the initial condition

$$y(x,0) = f(x) = \sum_{n=1}^{\infty} R_n \sin\frac{n\pi x}{L} \qquad (0 < x < L) \tag{25}$$

as

$$R_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx. \tag{26}$$

If we wish to plot $y(x,t)$ at various times, we can sum the series (24) at a number of different $x$'s and $t$'s. However, we can do much better as we will now show. First, use the trigonometric identity

$$\sin A\cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right] \tag{27}$$

to re-express (24) as

$$y(x,t) = \frac{1}{2}\sum_{n=1}^{\infty} R_n\left[\sin\frac{n\pi}{L}(x-ct) + \sin\frac{n\pi}{L}(x+ct)\right]$$

*Actually, the sound of a violin is due very little to the air being set in motion by the vibrating string. Rather,the string drives the sounding board, through its connection at the bridge, and it is the vibrating sounding board that sets the adjacent air in motion and creates an audible sound. Thus, a serious investigation of violin mechanics would lead immediately to a much more difficult analysis of the vibrating sounding board.

$$= \frac{1}{2}\left[\sum_{n=1}^{\infty} R_n \sin\frac{n\pi}{L}(x-ct) + \sum_{n=1}^{\infty} R_n \sin\frac{n\pi}{L}(x+ct)\right]. \tag{28}$$

Finally, comparing (28) with (25) we see that the two series in (28) *can be summed into closed form* in terms of $f$ as

$$y(x,t) = \frac{1}{2}[f(x-ct) + f(x+ct)]. \tag{29}$$

There is a difficulty associated with (29): for a given $x$ in the interval $(0, L)$ and a given $t$ in the $t$ interval $(0, \infty)$, one or both of the arguments $x - ct$ and $x + ct$ in (29) may lie outside the domain of definition of $f$, $0 \le x \le L$, in which event (29) is meaningless. However, recall from our study of half-range sine series (Section 17.4) that the half-range sine expansion of $f(x)$ on $0 < x < L$, given by (25), is also the Fourier series expansion of the extended function $f_{\text{ext}}(x)$ that is $2L$-periodic and antisymmetric about $x = 0$ and $x = L$. For example, if $f(x)$ is the function shown in Fig. 4a, then $f_{\text{ext}}(x)$ is the function shown in Fig. 4b. Thus, the right-hand side of (28) actually sums to

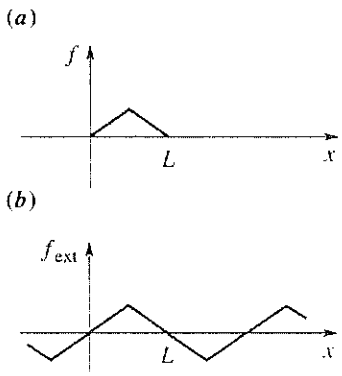



**Figure 4.** $f$ and $f_{\text{ext}}(x)$.

$$\boxed{y(x,t) = \frac{1}{2}[f_{\text{ext}}(x-ct) + f_{\text{ext}}(x+ct)]} \tag{30}$$

for *any* values of the arguments $x - ct$ and $x + ct$.

To illustrate the use of (30), let $L = 10$ and $c = 12$, let $f$ be as shown in Fig. 2 with $f_0 = 1$, and let us compute $y$ at $x = 2$ and $t = 3$. Using (30) and the fact that $f_{\text{ext}}$ is periodic with period $2L = 20$ and odd, we have

$$\begin{aligned}
y(2,3) &= \tfrac{1}{2}[f_{\text{ext}}(2-36) + f_{\text{ext}}(2+36)] \\
&= \tfrac{1}{2}[f_{\text{ext}}(-34) + f_{\text{ext}}(38)] \\
&= \tfrac{1}{2}[f_{\text{ext}}(-14) + f_{\text{ext}}(18)] && \text{(periodicity of } f_{\text{ext}}) \\
&= \tfrac{1}{2}[f_{\text{ext}}(6) + f_{\text{ext}}(-2)] && \text{(periodicity of } f_{\text{ext}}) \\
&= \tfrac{1}{2}[f(6) - f(2)] && [f_{\text{ext}} \text{ is odd, and } f_{\text{ext}}(x) = f(x) \\
& && \quad \text{on } 0 < x < 10] \\
&= \tfrac{1}{2}(\tfrac{4}{5} - \tfrac{2}{5}) && \text{(definition of } F \text{ in Fig. 2)} \\
&= \tfrac{1}{5}.
\end{aligned}$$

Besides using (30) to compute $y$ at any specific $x$ and $t$, we can use it to obtain the graph of $y(x,t)$ over the entire interval $0 < x < L$, at any given $t$, by observing that the graph of $f_{\text{ext}}(x - ct)$ plotted as a function of $x$ is simply the graph of $f_{\text{ext}}(x)$ translated to the right through a distance $ct$. Similarly, the graph of $f_{\text{ext}}(x + ct)$ is the graph of $f_{\text{ext}}(x)$ translated to the left through a distance $ct$. Thus, if $f$ is as shown in Fig. 2, then $f_{\text{ext}}(x)$ is as shown in Fig. 5a, $f_{\text{ext}}(x-ct)$ and $f_{\text{ext}}(x + ct)$ are as shown in Fig. 5b, and $y(x,t) = \frac{1}{2}[f_{\text{ext}}(x - ct) + f_{\text{ext}}(x + ct)]$ is (by addition of the two graphs in Fig. 5b and scaling by $1/2$) as shown in Fig. 5c.

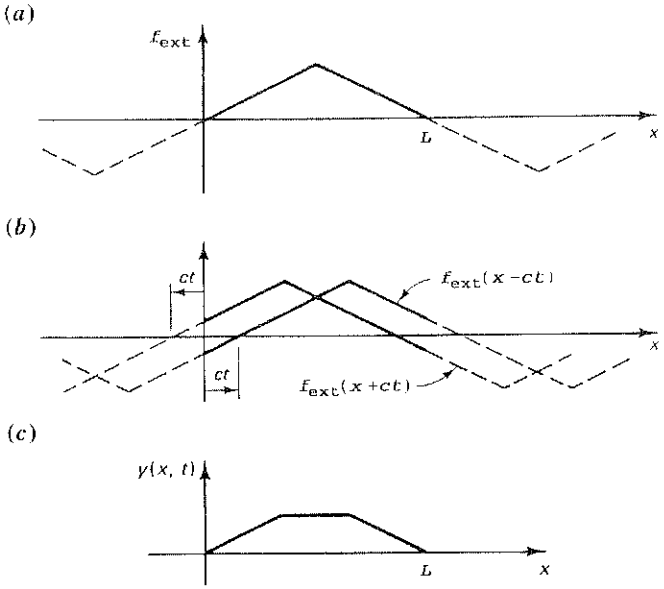



**Figure 5.** Graphical use of (30).

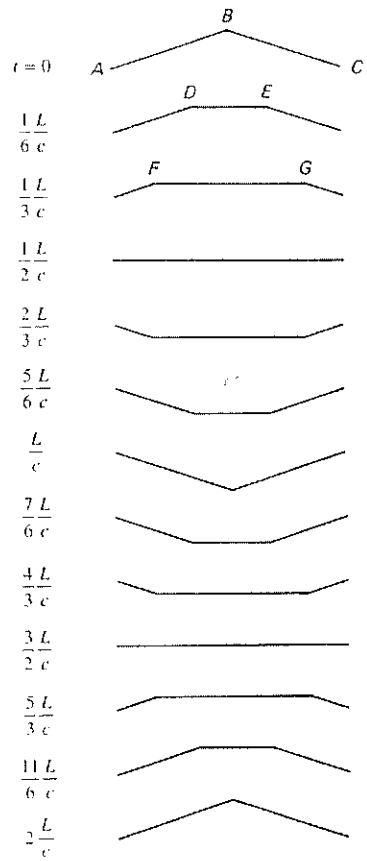



**Figure 6.** Solution sequence over one cycle.

Carrying out this graphical procedure at a number of different times over one complete cycle yields the solution sequence shown in Fig. 6. Consider the results shown in Fig. 6 in terms of the physics. At $t = 0$ the string segments $AB$ and $BC$ are straight, so each string element is in static equilibrium – except at $B$, which point is driven downward (Fig. 7). The plateau $DE$ (Fig. 6) moves downward at constant velocity (i.e., with no acceleration) since there is no net vertical force on the elements between $D$ and $E$. (Remember that we have neglected the effects of gravity.)

We can now explain why $c^2 y_{xx} = y_{tt}$ is called the "wave" equation for we see from (24) that $y(x,t)$ can be expressed as a superposition of **standing waves** of shape $\sin(n\pi x/L)$ and temporal frequency $n\pi c/L$ rad/sec. The sum of these waves is also a standing wave, of temporal frequency $\pi c/L$. Alternatively, (28) expresses $y(x,t)$ as a superposition of **traveling waves** traveling to the right and left with speed $c$. Thus, the parameter $c$ in the wave equation $c^2 y_{xx} = y_{tt}$ is now seen to be the **wave speed** – that is, the speed of propagation of the traveling waves. Recalling that $c = \sqrt{\tau/\sigma}$, it does seem reasonable physically that the wave speed should increase with the tension $\tau$ and decrease with the lineal mass density $\sigma$. To reiterate, (28) expresses $y$ as a superposition of sinusoidal traveling waves criss-crossing leftward and rightward with speed $c$. Over the physical interval $(0, L)$ these waves sum to a standing wave with nodes at $x = 0$ and $x = L$.

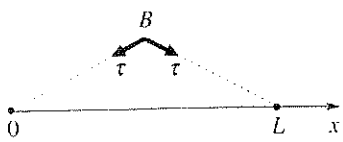



**Figure 7.** Downward force at $B$.

**19.2.3. Using Sturm–Liouville theory. (Optional)** If we rely on the Sturm–Liouville theory, then we understand (17a) and (17b) as eigenfunction expansions of $f$ and $g$ in terms of the orthogonal eigenfunctions $\sin(n\pi x/L)$ of the relevant

Sturm–Liouville problem, namely,

$$X'' + \kappa^2 X = 0, \qquad (0 < x < L) \tag{31a}$$

$$X(0) = 0, \quad X(L) = 0. \tag{31b}$$

Then the weight function in the inner product is 1, and

$$R_n = \frac{\langle F(x), \sin \frac{n\pi x}{L} \rangle}{\langle \sin \frac{n\pi x}{L}, \sin \frac{n\pi x}{L} \rangle} = \frac{\int_0^L F(x) \sin \frac{n\pi x}{L}\, dx}{\int_0^L \sin^2 \frac{n\pi x}{L}\, dx}$$

$$= \frac{2}{L} \int_0^L F(x) \sin \frac{n\pi x}{L}\, dx \tag{32}$$

as in (18a), and similarly for $S_n$.

**Closure.** Like the problem of heat conduction in a finite rod (Section 18.3), the vibrating finite string problem (1) is defined on a semi-infinite strip in the $x, t$ plane, with boundary conditions at $x = 0$ and $x = L$. However, the PDE is of second order with respect to $t$, so two initial conditions are appropriate rather than one. Solution by separation of variables proceeds in essentially the same manner as in Section 18.3 and, as for the heat equation, it is necessary to apply the boundary conditions before the initial conditions. The solution (24) is in the form of a superposition of orthogonal modes, each one being a standing wave. Alternatively, the trigonometric identity (27) enables us to re-express the solution in the form (28), namely, as a superposition of left- and right-running traveling waves with wave speed $c$. More simply, (30) gives the solution as the sum of *two* traveling waves, one left-running and one right-running.

From Fig. 6 we see that the initially "kinky" deflection $y(x, t)$ does not smooth out as $t$ increases, the way an initially kinky or discontinuous temperature distribution $u(x, t)$ does, and this is a major difference between the wave and diffusion processes. Rather, kinks and discontinuities in the initial conditions propagate into the $x, t$ domain.

## EXERCISES 19.2

**1.** Let $L = 10$, $c = 12$, and $f_0 = 1$ in (20). Use (20) to compute $y$ at the specified values of $x$ and $t$, to two significant figures. Then use (30) to compute $y$ and show that your two results agree.

(a) $y(5, 1)$ (b) $y(5, 2)$ (c) $y(5, 3)$
(d) $y(5, 4)$ (e) $y(5, 6)$ (f) $y(5, 10)$
(g) $y(5, 20)$ (h) $y(3, 1)$ (i) $y(3, 10)$
(j) $y(3, 40)$ (k) $y(1, 2)$ (l) $y(1, 10)$

**2.** Solve (1) for $y(x, t)$ for the case where $f(x) = 0$ and

(a) $g(x) = 50 \sin(\pi x/L)$
(b) $g(x) = 3 \sin(\pi x/L) - 5 \sin(4\pi x/L)$
(c) $g(x) = \sin(2\pi x/L) + \sin(3\pi x/L) + 4 \sin(8\pi x/L)$

**3.** Using the solution technique illustrated in Fig. 5, obtain the graphs shown in Fig. 6, for the given values of $t$, and label any key values.

(a) $t = \frac{1}{6}\frac{L}{c}$ (b) $t = \frac{1}{3}\frac{L}{c}$ (c) $t = \frac{1}{2}\frac{L}{c}$

(d) $t = \frac{2}{3}\frac{L}{c}$ (e) $t = \frac{5}{6}\frac{L}{c}$ (f) $t = \frac{L}{c}$

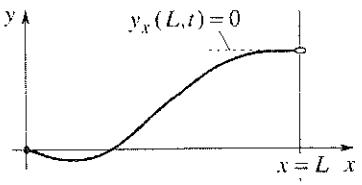
(g) $t = \frac{7}{6}\frac{L}{c}$ (h) $t = \frac{4}{3}\frac{L}{c}$ (i) $t = \frac{3}{2}\frac{L}{c}$

(j) $t = \frac{5}{3}\frac{L}{c}$ (k) $t = \frac{11}{2}\frac{L}{c}$ (l) $t = 2\frac{L}{c}$

**4.** Construct neat labeled sketches of the graph of $y(x,t)$ versus t for the oscillation depicted in Fig. 6, for $x = L/2$ and also for $x = L/4$.

**5.** If, instead of the string being tied at $x = L$ such that $y(L,t) = 0$, the string is looped around a vertical frictionless wire (as shown in the figure), then in place of $y(L,t) = 0$ the boundary condition becomes

$$y_x(L,t) = 0. \tag{5.1}$$




(a) Explain why (5.1) is true.
(b) Solve (1a) for $y(x,t)$ by separation of variables, with these boundary and initial conditions:

$$y(0,t) = 0, \quad y_x(L,t) = 0,$$
$$y(x,0) = f(x), \quad y_t(x,0) = 0,$$

leaving expansion coefficients in integral form.
(c) Solve (1a) for $y(x,t)$ by separation of variables, with these boundary and initial conditions:

$$y_x(0,t) = 0, \quad y_x(L,t) = 0,$$
$$y(x,0) = 0, \quad y_t(x,0) = V,$$

where $V$ is a constant.
(d) Solve (1a) for $y(x,t)$ by separation of variables, with these boundary and initial conditions.

$$y_x(0,t) = 0, \quad y(L,t) = 0,$$
$$y(x,0) = 0, \quad y_t(x,0) = g(x),$$

leaving expansion coefficients in integral form.

**6.** (*Inclusion of damping*) Unless the string vibrates in a vacuum, there will be some damping due to the movement of the string through the fluid (be it air, water, or whatever). If the damping force is proportional to the velocity $y_t$, the modified equation of motion becomes

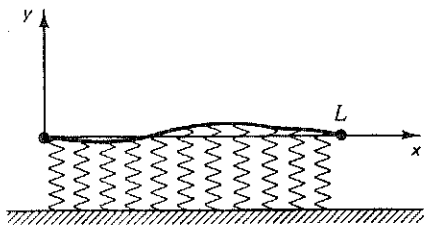
$$c^2 y_{xx} = y_{tt} + a y_t, \tag{6.1}$$

where $a$ is a known constant. For definiteness, suppose that $0 < a < 2\pi c/L$. Solve (6.1) by separation of variables, subject to the conditions.

$$y(0,t) = 0, \quad y(L,t) = 0,$$
$$y(x,0) = f(x), \quad y_t(x,0) = 0,$$

leaving expansion coefficients in integral form. Summarize, in words, the effect(s) of the damping term $ay_t$.

**7.** (*Inclusion of lateral spring*) If, as shown in the figure,




a lateral distributed spring is included, then the modified equation of motion for the vibrating string is $\tau y_{xx} - ky = \sigma y_{tt}$, where $k$ is the spring stiffness per unit length (newtons per meter per meter) or

$$c^2 y_{xx} - by = y_{tt}. \qquad \left(c^2 = \frac{\tau}{\sigma}, \ b = \frac{k}{\sigma}\right) \tag{7.1}$$

Solve (7.1) by separation of variables, subject to the conditions

$$y(0,t) = 0, \quad y(L,t) = 0,$$
$$y(x,0) = f(x), \quad y_t(x,0) = 0,$$

leaving expansion coefficients in integral form. Summarize, in words, the effect(s) of the spring term $by$ in (7.1).

**8.** (*Constant forcing function*) We saw in Section 19.1 that if the effects of gravity are included, then the governing PDE is

$$c^2 y_{xx} = y_{tt} + g. \tag{8.1}$$

That is, the PDE $L[y] = c^2 y_{xx} - y_{tt} - g$ is *non*homogeneous. Solve (8.1) subject to the conditions

$$y(0,t) = 0, \quad y(L,t) = 0,$$
$$y(x,0) = f(x), \quad y_t(x,0) = 0,$$

leaving expansion coefficients in integral form. HINT: The form $y(x,t) = X(x)T(t)$ gives

$$\frac{X''}{X} = \frac{1}{c^2}\frac{T''}{T} + \frac{g}{c^2 XT}. \tag{8.2}$$

Because of the last term in (8.2), which contains both $x$ and $t$ dependence, we are unable to successfully complete the separation process (i.e., we are unable to get all of the $x$ dependence on one side of the equation and all of the $t$ dependence on the other). Thus, we suggest seeking $y$ in the form

$$y(x,t) = y_p(x) + X(x)T(t) \tag{8.3}$$

instead. Putting (8.3) into (8.1), obtain

$$c^2 y_p'' + c^2 X''T = XT'' + g. \tag{8.4}$$

Thus, we can remove the unwelcome $g$ term by setting $c^2 y_p'' = g$. Then we can complete the separation of variable in (8.4) as usual. Mathematically, $y_p(x)$ is a **particular solution** of the nonhomogeneous equation (8.1) since it satisfies the full equation (7.1), and $XT$ is a solution of the associated homogeneous equation $c^2 y_{xx} = y_{tt}$. But in physical terms you will find that it is simply the "static sag" of the string due to gravity, satisfying the problem

$$c^2 y_p''(x) = g, \quad y_p(0) = 0, \quad y_p(L) = 0. \tag{8.5}$$

**9.** (*Nonconstant forcing function*) In Exercise 8 we included a forcing term that was a constant. The suggested solution technique would have worked even if the forcing term were a nonconstant function of $x$. But in this exercise we allow for $t$ dependence as well. Thus, consider the problem

$$\begin{aligned} c^2 y_{xx} &= y_{tt} + F(x,t), \\ y(0,t) = 0, &\quad y(L,t) = 0, \\ y(x,0) = f(x), &\quad y_t(x,0) = 0. \end{aligned} \tag{9.1}$$

To solve, we can use essentially the same **eigenvector expansion method** that is used in Section 11.3.2 to solve the nonhomogeneous matrix problem $\mathbf{Ax} = \Lambda\mathbf{x} + \mathbf{c}$, and again in Exercise 17 of Section 18.3 to solve the nonhomogeneous diffusion equation $\alpha^2 u_{xx} = u_t - F(x,t)$.

(a) Accordingly, solve (9.1) by seeking

$$y(x,t) = \sum_{n=1}^{\infty} h_n(t) \sin\frac{n\pi x}{L} \tag{9.2}$$

and expanding

$$F(x,t) = \sum_{n=1}^{\infty} F_n(t) \sin\frac{n\pi x}{L} \tag{9.3}$$

and

$$f(x) = \sum_{n=1}^{\infty} f_n \sin\frac{n\pi x}{L},$$

where the coefficients

$$F_n(t) = \frac{2}{L}\int_0^L F(x,t)\sin\frac{n\pi x}{L}\,dx \tag{9.4}$$

and

$$f_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx \tag{9.5}$$

are considered as known [i.e., computable from $F(x,t)$ and $f(x)$]. With $\omega_n = n\pi c/L$, show that

$$y(x,t) = \sum_{n=1}^{\infty}\left[f_n\cos\omega_n t + \frac{1}{\omega_n}\int_0^t F_n(\tau)\sin\omega_n(\tau - t)\,d\tau\right]\sin\frac{n\pi x}{L}, \tag{9.6}$$

(b) With the help of the Leibniz rule formally verify that (9.6) satisfies (9.1).
(c) Work out the solution (9.6) for the case where $F(x,t) = F_0 \sin\Omega t$ and $f(x) = 0$, assuming that the driving frequency does not equal any of the natural frequencies $\omega_n$.
(d) Same as (c), but where $\Omega$ equals one of the natural frequencies, say $\omega_k$.

**10.** (*Variable end conditions*) Thus far our boundary conditions have been constant in time. Here, we consider nonconstant conditions. Consider the problem

$$\begin{aligned} c^2 y_{xx} &= y_{tt}, \\ y(0,t) = p(t), &\quad y(L,t) = q(t), \\ y(x,0) = f(x), &\quad y_t(x,0) = 0. \end{aligned} \tag{10.1}$$

Changing dependent variables from $y(x,t)$ to $z(x,t)$ according to

$$y(x,t) = z(x,t) + \left(1 - \frac{x}{L}\right)p(t) + \frac{x}{L}q(t), \tag{10.2}$$

show that the problem governing $z(x,t)$ is of the type treated in Exercise 9. NOTE: Notice how an "input" can be moved – from the boundary conditions to a forcing term in the PDE. In the present case the PDE on $y$ was homogeneous and the boundary conditions were nonhomogeneous. Following the

change of variables you should find that the PDE on $z$ is nonhomogeneous and the boundary conditions are homogeneous.

**11.** The voltage $v(x,t)$ in an underground cable is governed by a PDE of the form

$$v_{xx} = Av_{tt} + Bv_t + Cv, \tag{11.1}$$

where $A, B, C$ are constants. It is proposed that if $v_1$ is a solution of $v_{xx} = Av_{tt} + Bv_t$ and $v_2$ is a solution of $v_{xx} = Av_{tt} + Cv$, then $v = v_1 + v_2$ is, by superposition, a solution of (10.1). Give a critical evaluation of that proposal.

**12.** (*Longitudinal waves in a rod*) First, read Exercise 2 of Section 19.1. Consider a rod of length $L$, cross-sectional area $A$, Young's modulus $E$, and mass per unit length $\sigma$. At $x = 0$ the rod is attached to a rigid wall and at $x = L$ the rod is free. Prior to time $t = 0$ we pull on the free end with a force $F_0$, so the rod is in static equilibrium, with a uniform stress $s_0 = F_0/A$. At $t = 0$ we remove the force.

(a) Show that the problem governing the displacement is

$$\begin{aligned} c^2 u_{xx} &= u_{tt}, \\ u(0,t) = 0, &\quad u_x(L,t) = 0, \\ u(x,0) = \frac{s_0}{E}x, &\quad u_t(x,0) = 0. \end{aligned} \tag{12.1}$$

(b) Solve (12.1) for $u(x,t)$.
(c) Determine the stress at the wall, $s(0,t)$.

**13.** (*Uniqueness*) In this section and in the preceding exercises we have developed solution techniques for wave problems, most of which are of the form

$$\begin{aligned} c^2 y_{xx} &= y_{tt} + F(x,t), \\ y(0,t) = p(t), &\quad y(L,t) = q(t), \\ y(x,0) = f(x), &\quad y_t(x,0) = g(x). \end{aligned} \tag{13.1}$$

Show that the solution to (13.1) is unique. HINT: As usual in uniqueness proofs, let and $y_1(x,t)$ and $y_2(x,t)$ be two solutions and consider the difference $w(x,t) = y_1(x,t) - y_2(x,t)$. Show that w satisfies the homogeneous version of (13.1),

$$\begin{aligned} c^2 w_{xx} &= w_{tt}, \\ w(0,t) = 0, &\quad w(L,t) = 0, \\ w(x,0) = 0, &\quad w_t(x,0) = 0. \end{aligned} \tag{13.2}$$

Considering the integral

$$I(t) = \int_0^L (w_t^2 + c^2 w_x^2)\, dx, \tag{13.3}$$

show, with the help of Leibniz differentiation and (13.2), that $dI/dt = 0$. Thus, show that $I(t) = 0$ for all $t \geq 0$ and hence that $w(x,t) = 0$ for all $0 < x < L$ and $0 < t < \infty$. Since $w(x,t) = y_1(x,t) - y_2(x,t) = 0$, $y_1$ and $y_2$ are necessarily identical, so there exists at most one solution to (13.1).

**14.** (*Vibrating beam*) It is known from mechanics that the free vibration of a uniform beam is governed by the fourth-order PDE*

$$y_{xxxx} + \frac{\sigma}{EI} y_{tt} = 0, \tag{14.1}$$

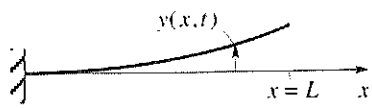
where $y(x,t)$ is the deflection, $\sigma$ is the mass per unit length, $E$ is Young's modulus of the material, and $I$ is the moment of inertia of the cross section about its neutral axis.

(a) If the beam is "cantilevered" (see figure), then the boundary conditions are

$$\begin{aligned} &y(0,t) = 0, \quad y_x(0,t) = 0, \\ &y_{xx}(L,t) = 0, \quad \text{(no bending moment at free end)} \\ &y_{xxx}(L,t) = 0, \quad \text{(no shear force at free end)} \end{aligned} \tag{14.2}$$

and if the beam is initially deflected and at rest, then

$$y(x,0) = f(x), \quad y_t(x,0) = 0. \tag{14.3}$$




Seeking $y(x,t) = X(x)T(t)$, derive the solution form

$$y(x,t) = \sum_{n=1}^{\infty} A_n X_n(x) \cos \omega_n t, \tag{14.4}$$

where the mode shapes are given by

$$\begin{aligned} X_n(x) = \sin\frac{z_n x}{L} - \sinh\frac{z_n x}{L} \\ + \left(\frac{\cos z_n + \cosh z_n}{\sin z_n - \sinh z_n}\right)\left(\cos\frac{z_n x}{L} - \cosh\frac{z_n x}{L}\right), \end{aligned} \tag{14.5}$$

where the frequencies are

*See for example, William T. Thomson, *Theory of Vibration with Applications*, 2nd ed. (Englewood Cliffs, NJ: Prentice Hall, 1981)

$$\omega_n = \sqrt{\frac{EI}{\sigma}}\left(\frac{z_n}{L}\right)^2, \tag{14.6}$$

and where the $z_n$'s are the solutions of the transcendental equation $\cos z \cosh z + 1 = 0$. NOTE: The initial condition

$$y(x,0) = f(x) = \sum_{n=1}^{\infty} A_n X_n(x) \tag{14.7}$$

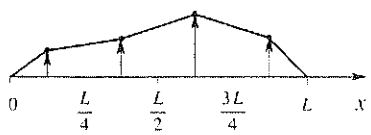
can then be used to evaluate the $A_n$'s, but we do not ask you to go that far. Indeed, the eigenvalue problem on $X(x)$, which yields the eigenfunctions $X_n(x)$, is not of Sturm–Liouville type, so we do not find (in this text) the theoretical foundation needed to guide us through the evaluation of the $A_n$'s in (14.7). Let it suffice to observe that (14.4) gives $y(x,t)$ as the superposition of infinitely many modes, having shape $X_n(x)$ and frequency $\omega_n$.

(b) If, instead of being cantilevered the beam is pinned at both ends, then the boundary conditions are

$$\begin{aligned} y(0,t) = 0, \quad y(L,t) = 0, \\ y_{xx}(0,t) = 0, \quad y_{xx}(L,t) = 0, \end{aligned} \tag{14.8}$$

in place of (14.2). Find the solution form analogous to the one cited in part (a), corresponding to the boundary conditions (14.8).

**15.** (*Lumped-parameter model*) It is sometimes useful to model a *continuous* system, such as the string in our vibrating string problem, approximately, by a *discrete* system. To illustrate, let us divide the string into four equal parts and focus the mass $\sigma L/4$ of each segment at the center of that segment. (Such a system is called a **lumped-parameter** system.) Thus, we have four "beads" of mass $\sigma L/4$ connected by massless string under tension $\tau$.




(a) Applying Newton's second law of motion to the first bead, show that

$$\frac{\sigma L}{4}\ddot{y}_1 \approx \tau\left(\frac{y_2 - y_1}{L/4} - \frac{y_1}{L/8}\right), \tag{15.1}$$

where dots denote time derivatives. Doing the same for the remaining three masses, show that the resulting ODE's can be expressed in matrix form as

$$\ddot{\mathbf{y}} + \mathbf{A}\mathbf{y} = 0, \tag{15.2}$$

where $\mathbf{y} = [y_1(t), \ldots, y_4(t)]^{\mathrm{T}}$ and

$$\mathbf{A} = \frac{16\tau}{\sigma L^2}\begin{bmatrix} 3 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 3 \end{bmatrix}. \tag{15.3}$$

(b) More generally, with $N$ beads one obtains (15.2), where $\mathbf{y} = [y_1(t), \ldots, y_N(t)]^{\mathrm{T}}$ and

$$\mathbf{A} = \left(\frac{N}{L}\right)^2 \frac{\tau}{\sigma}\begin{bmatrix} 3 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ 0 & -1 & 2 & -1 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & & -1 & 2 & -1 \\ 0 & \cdots & & 0 & -1 & 3 \end{bmatrix}. \tag{15.4}$$

(You need not derive this result.) Seeking a solution in the form

$$\begin{aligned} y_1(t) &= \eta_1 \sin(\omega t + \phi) \\ &\vdots \\ y_N(t) &= \eta_N \sin(\omega t + \phi) \end{aligned} \tag{15.5}$$

or $\mathbf{y}(t) = \boldsymbol{\eta}\sin(\omega t + \phi)$, show that

$$\mathbf{B}\boldsymbol{\eta} = \lambda\boldsymbol{\eta}, \tag{15.6}$$

where $\mathbf{B}$ is the matrix in (15.4), without the $(N/L)^2(\tau/\sigma)$ factor, and $\lambda = (\sigma/\tau)(L/N)^2\omega^2$.

(c) If, for a chosen value of $N$, one solves the eigenvalue problem (15.6), then one obtains approximations of the first $N$ orthogonal mode shapes and eigenfrequencies of the continuous string: the eigenvectors $\boldsymbol{\eta}_1, \ldots, \boldsymbol{\eta}_N$ give the approximate mode shapes and the eigenvalues $\lambda_1, \ldots, \lambda_N$ give the approximate eigenfrequencies $\omega_1 = (N/L)\sqrt{\tau/\sigma}\sqrt{\lambda_1}, \ldots, \omega_N = (N/L)\sqrt{\tau/\sigma}\sqrt{\lambda_N}$. Here is the problem: Use computer software to solve (15.6) for the case where $N = 4$. Compare the computed eigenfrequencies with the exact values $\omega_1 = \pi c/L = (\pi/L)\sqrt{\tau/\sigma}, \ldots, \omega_N = N\pi c/L = (N\pi/L)\sqrt{\tau/\sigma}$, and compare the $m$ computed mode shapes with the exact shapes $\sin(\pi x/L), \ldots, \sin(4\pi x/L)$.

(d) Same as (c), with $N = 10$.

## 19.3 Separation of Variables; Vibrating Membrane

In Section 19.1 we derive not only the equation $c^2 y_{xx} = y_{tt}$ governing the vibrating string but also the equation

$$\boxed{c^2(w_{xx} + w_{yy}) = w_{tt}} \tag{1a}$$

governing the vibrating membrane; $w(x, y, t)$ is the membrane deflection normal to the $x, y$ plane, and $c^2 = \tau/\sigma$ where $\tau$ is the (uniform) tension per unit length and $\sigma$ is the (uniform) mass per unit area. Let us consider the domain to be the rectangle $0 < x < a$ and $0 < y < b$ (Fig. 1), and let us solve (1) subject to the boundary conditions

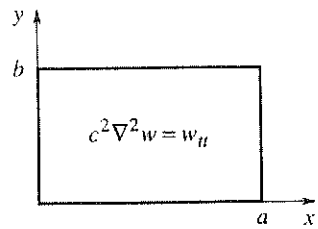



**Figure 1.** Rectangular membrane.

$$w(0, y, t) = w(a, y, t) = w(x, 0, t) = w(x, b, t) = 0 \tag{1b}$$

and the initial conditions

$$w(x, y, 0) = f(x, y), \quad w_t(x, y, 0) = 0, \tag{1c}$$

where $w(x, y, 0) = f(x, y)$ is the initial deflection and $w_t(x, y, 0)$ is the initial velocity. That is, the membrane is initially at rest.

The reason that we isolate this problem in a separate section is that it is our first problem with more than two independent variables – namely $x, y$, and $t$. Our approach here is intended to serve as a model for such cases.

To solve by separation of variables, seek

$$\boxed{w(x, y, t) = X(x)Y(y)T(t).} \tag{2}$$

Putting (2) into (1a) gives

$$c^2(X''YT + XY''T) = XYT''$$

or, dividing by $c^2XYT$,

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{c^2}\frac{T''}{T}. \tag{3}$$

The left-hand side is a function of $x$ and $y$, and the right-hand side is a function of $t$. Since $x, y, t$ are independent variables, it follows in the usual way that each side must be a constant. Thus,

$$\frac{X''}{X} + \frac{Y''}{Y} = \frac{1}{c^2}\frac{T''}{T} = \text{constant} \equiv -\kappa^2, \tag{4}$$

with the minus sign included so that the $T$ equation

$$T'' + \kappa^2 c^2 T = 0 \tag{5}$$

gives oscillatory solutions ($\cos \kappa ct$ and $\sin \kappa ct$), since we anticipate a vibratory motion. Next, we separate the $x$ and $y$ dependence in (4) by writing

$$\frac{X''}{X} = -\frac{Y''}{Y} - \kappa^2.$$

The left-hand side is a function of $x$ alone and the right-hand side is a function of $y$ alone so it follows, as usual, that

$$\frac{X''}{X} = -\frac{Y''}{Y} - \kappa^2 = \text{constant} \equiv -\alpha^2. \tag{6}$$

Hence,

$$X'' + \alpha^2 X = 0, \tag{7}$$

$$Y'' + (\kappa^2 - \alpha^2)Y = 0. \tag{8}$$

From (5), (7), and (8), we obtain

$$X = A\cos \alpha x + B \sin \alpha x, \tag{9a}$$

$$Y = D\cos \sqrt{\kappa^2 - \alpha^2}\, y + E \sin \sqrt{\kappa^2 - \alpha^2}\, y, \tag{9b}$$

$$T = F\cos \kappa ct + G \sin \kappa ct. \tag{9c}$$

The right-hand side of (9c) is the general solution of (5) only if $\kappa \neq 0$, for if $\kappa = 0$ then the sine term drops out. Consistent with the strategy that we have used until now, we should write, in place of (9c),

$$T = \begin{cases} F\cos \kappa ct + G \sin \kappa ct, & \kappa \neq 0 \\ H + It, & \kappa = 0. \end{cases} \tag{10}$$

Similarly for (9a) for the case $\alpha = 0$ and for (9b) for the case $\kappa = \alpha$. However, in this problem we anticipate that the $I$ in (10) will be found to be zero when we apply the boundary and initial conditions because it gives a linear variation in $t$, whereas we expect an oscillatory motion. Similarly, we do not expect to need the linear terms in $x$ and $y$ (corresponding to the special cases $\alpha = 0$ and $\kappa = \alpha$) because $w = 0$ on the boundary. Thus, let us proceed with the solution forms (9a,b,c) rather than carry extra terms, terms that we know will drop out later.

Next, we put (9a,b,c) into (2) and apply the boundary and initial conditions. However, it is more efficient to observe from the boundary condition

$$w(0, y, t) = 0 = X(0)Y(y)T(t) \tag{11}$$

that $X(0) = 0$, and to observe from the other boundary conditions in (1b) that $X(a) = 0$, $Y(0) = 0$, and $Y(b) = 0$.* Applying the boundary condition $X(0) = 0$

*Alternatively, (11) is satisfied by $Y(y) = 0$ or $T(t) = 0$, but we cannot tolerate these choices because they give $w(x, y, t) = 0$, which will not satisfy the initial condition $w(x, y, 0) = f(x, y)$. That is, as usual, we make such choices so as to maintain as robust a solution as possible.

to (9a) gives $A = 0$, so $X(x) = B \sin \alpha x$. Then, $X(a) = 0 = B \sin \alpha a$. We cannot afford the choice $B = 0$ for then $X(x) = 0$ and $w(x, y, z) = 0$, so we require that $\sin \alpha a = 0$. Hence,

$$\alpha = \frac{m\pi}{a} \tag{12}$$

for any integer $m = 1, 2, \ldots$. Similarly, $Y(0) = 0$ gives $D = 0$ and $Y(b) = 0$ gives $\sqrt{\kappa^2 - \alpha^2} = n\pi/b$ for any integer $n = 1, 2, \ldots$, or because of (12),

$$\kappa = \pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}. \tag{13}$$

Further, the initial condition

$$w_t(x, y, 0) = 0 = X(x)Y(y)T'(0) \tag{14}$$

gives $T'(0) = 0$, and application of this condition to (9c) gives $G = 0$.

Putting these results into (9a,b,c) and then putting the latter into (2) gives the product solution

$$w(x, y, t) = BEF \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}\, t \tag{15}$$

for any $m = 1, 2, \ldots$, $n = 1, 2, \ldots$, where $BEF \equiv H$, say, is arbitrary. Or, with the help of superposition,

$$\boxed{w(x, y, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} H_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \cos \omega_{mn} t,} \tag{16a}$$

where

$$\omega_{mn} = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}. \tag{16b}$$

Finally, the initial condition $w(x, y, 0) = f(x, y)$ requires that

$$\boxed{f(x, y) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} H_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}} \tag{17}$$

on the rectangular domain. The latter series is an example of a **double series**; more specifically, it is a **double Fourier series**.* Suppose that $f(x, y)$ can, indeed, be

*Recall that for "single series" we say that $\sum_{n=1}^{\infty} a_n$ converges to $s$ if to each $\epsilon > 0$ (no matter how small) there corresponds an integer $N_0(\epsilon)$ such that

$$\left| s - \sum_{n=1}^{\infty} a_n \right| < \epsilon$$

represented by such a series. Then we can compute the $H_{mn}$ coefficients formally as follows.

Re-express (17) as

$$f(x,y) = \sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} H_{mn}\sin\frac{m\pi x}{a}\right)\sin\frac{n\pi y}{b}$$
$$\equiv \sum_{n=1}^{\infty} R_n(x)\sin\frac{n\pi y}{b}. \tag{18}$$

For fixed $x$ ($a < x < b$) the latter is a half-range sine expansion of $f(x,y)$ on $0 < y < b$, so

$$R_n(x) = \frac{2}{b}\int_0^b f(x,y)\sin\frac{n\pi y}{b}\,dy. \tag{19}$$

In turn,

$$R_n(x) = \sum_{m=1}^{\infty} H_{mn}\sin\frac{m\pi x}{a} \tag{20}$$

is a half-range sine expansion of $R_n(x)$ on $0 < x < a$, so

$$H_{mn} = \frac{2}{a}\int_0^a R_n(x)\sin\frac{m\pi x}{a}\,dx. \tag{21}$$

Putting (19) into (21) gives the formula

$$\boxed{H_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy} \tag{22}$$

for the evaluation of the Fourier coefficients $H_{mn}$ in (17). Thus, the solution of (1) is given by (16) with the coefficients calculated according to (22). With these results in hand let us comment on their derivation and physical interpretation.

COMMENT 1. Our choice of the minus sign in (4) was dictated by our understanding that the motion is, indeed, a vibration; the minus sign in (4) led to a plus sign

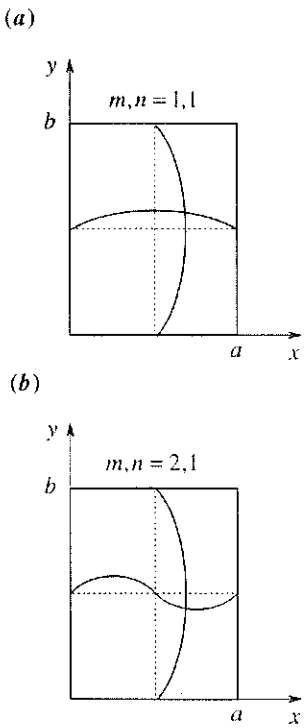



**Figure 2.** 1, 1 and 2, 1 mode shapes.

whenever $N > N_0$. Other definitions are used occasionally; this one is called **ordinary convergence** and we say that $\sum_{n=1}^{\infty} a_n$ converges to $s$ in the sense of ordinary convergence. If to each $\epsilon > 0$ there does not exist such an then the series is said to diverge. Analogously, the **double series** $\sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{mn}$ is said to converge to $s$, in the sense of *Pringsheim convergence*, if to each $\epsilon > 0$ (no matter how small) there correspond integers $M_0(\epsilon)$ and $N_0(\epsilon)$ such that

$$\left| s - \sum_{n=1}^{N}\sum_{m=1}^{M} a_{mn}\right| < \epsilon$$

whenever $M > M_0$ and $N > N_0$; otherwise the series is said to diverge. For further discussion, we refer the interested reader to P. W. Berg and J. L. McGregor, *Elementary Partial Differential Equations*, prelim. ed. (San Francisco: Holden–Day, 1964), Sec. 10.3.

in (5) and the oscillatory solutions $\cos \kappa ct$ and $\sin \kappa ct$. But what motivated us to choose the minus in front of the $\alpha^2$ in (6)? Once again, we need to look ahead. Specifically, the eventual Fourier series expansion of $f(x, y)$ dictates the need for cosine and sine solutions of (7) and (8). The $+\alpha^2$ in (7), which results from the $-\alpha^2$ in (6), does give $\cos \alpha x$ and $\sin \alpha x$ solutions. Further, observe that we wrote the $Y$ equation (8) in the form $Y'' + (\kappa^2 - \alpha^2)Y = 0$ rather than in the equally correct form $Y'' - (\alpha^2 - \kappa^2)Y = 0$ in order to force the cosine and sine solutions given in (9b) (see Exercise 1).

COMMENT 2. What do the individual modes look like? The shape of the $m, n$ mode is given by $\sin(m\pi x/a)\sin(n\pi y/b)$, which is modulated periodically in time by the $\cos \omega_{mn} t$ factor. For $m, n = 1, 1$ and $2, 1$, for instance, the mode shapes are as indicated schematically in Fig. 2. In the case of the 2, 1 mode $w = 0$ all along the line $x = a/2$ because the $\sin(2\pi x/a)$ factor is zero on that line. We call that line, or any curve within the domain and along which $w = 0$ for all time, a **nodal line**, and show it as solid in Fig. 2a. If we simply show the nodal lines and indicate the positive and negative deflections by $+$'s and $-$'s, then the first several mode patterns are as shown in Fig. 3. Of course, the $+$'s and $-$'s alternate in time due to the $\cos \omega_{mn} t$ factor.

COMMENT 3. Concerning the temporal frequencies, observe that whereas for the vibrating string the frequencies $\omega_n = n\pi c/L$ are integer multiples of a fundamental frequency $\pi c/L$, the frequencies

$$\omega_{mn} = \pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}} \ \frac{\text{rad}}{\text{sec}} \tag{23}$$

of the vibrating membrane are not. To illustrate, let $a = b$ and let us examine the musical notes corresponding to the various modes. For comparison with the analogous results for a violin string (Table 1, Section 19.2)), let us tune our "square drum" so that its fundamental frequency is 27.5 cycles/sec, corresponding to $A_0$, the lowest $A$ on a piano. That is, adjust the tension $\tau$ so that

$$\omega_{1,1} = \left(\pi c \sqrt{\frac{1^2}{a^2} + \frac{1^2}{a^2}} \ \frac{\text{rad}}{\text{sec}}\right)\left(\frac{1 \text{ cycle}}{2\pi \text{ rad}}\right) = \frac{c}{\sqrt{2}\, a} \ \frac{\text{cycles}}{\text{sec}} = 27.5,$$

where $c = \sqrt{\tau/\sigma}$. Then the frequencies are

$$\begin{aligned}\omega_{mn} &= \left(\pi c \sqrt{\frac{m^2}{a^2} + \frac{n^2}{a^2}} \ \frac{\text{rad}}{\text{sec}}\right)\left(\frac{1 \text{ cycle}}{2\pi \text{ rad}}\right) \\ &= \frac{c}{2\,a} \sqrt{m^2 + n^2} \ \frac{\text{cycles}}{\text{sec}} = 27.5 \sqrt{\frac{m^2 + n^2}{2}} \ \frac{\text{cycles}}{\text{sec}}\end{aligned}$$

and these are listed in ascending order in Fig. 4, along with the corresponding musical note. From the tabulation in Fig. 4 we can see why square drums are not prized as musical instruments for, beginning with $A_1$, virtually every note is

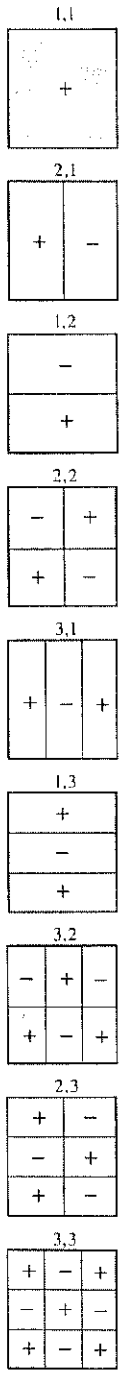



**Figure 3.** The first nine modes.

| $mn$ | $\omega_{mn}$ | Note |
|---|---|---|
| 11 | <u>27.5</u> | $A_0$ |
| 12,21 | 43.5 | $\approx F_1$ |
| 22 | <u>55.0</u> | $A_1$ |
| 13,31 | 61.5 | $\approx B_1$ |
| 23,32 | 70.1 | $\approx C_2^\#$ |
| 14,41 | 80.2 | $\approx E_2$ |
| 33 | 82.5 | $\approx E_2$ |
| 24,42 | 87.0 | $\approx F_2$ |
| 34,43 | 97.2 | $\approx G_2$ |
| 15,51 | 99.2 | $\approx G_2$ |
| 25,52 | 104.7 | $\approx G_2^\#$ |
| 44 | <u>110.0</u> | $A_2$ |
| 35,53 | 113.4 | $\approx A_2^\#$ |
| 16,61 | 118.3 | $\approx A_2^\#$ |
| 26,62 | 123.0 | $\approx B_2$ |
| 45,54 | 124.5 | $\approx B_2$ |
| 36,63 | 130.4 | $\approx C_3$ |
| 17,71,55 | 137.5 | $\approx C_3^\#$ |
| 46,64 | 140.2 | $\approx C_3^\#$ |

**Figure 4.** Square drum, octave overtones underlined.

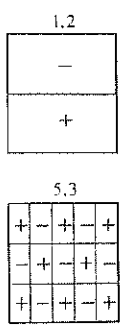



**Figure 5.** Modes in (30).

present. This profusion of notes is due to the dense values of $\sqrt{m^2+n^2}$. The result would be somewhat like playing the piano with our forearms rather than with our fingers. Circular drums are better and are discussed in the exercises.

COMMENT 4. Alternative to the product form (6) we can seek $w$ in the product form

$$\boxed{w(x,y,t) = W(x,y)T(t).} \tag{24}$$

Putting (4) into (1a) and separating gives

$$c^2\left(W_{xx}+W_{yy}\right)T = WT'', \tag{25a}$$

$$\frac{W_{xx}+W_{yy}}{W} = \frac{1}{c^2}\frac{T''}{T} = -\kappa^2, \tag{25b}$$

and the separated equations

$$\boxed{W_{xx}+W_{yy}+\kappa^2 W = 0} \tag{26}$$

on $W$, and $T''+\kappa^2c^2T=0$ on $T$, as before. Then we can seek

$$W(x,y) = X(x)Y(y) \tag{27}$$

in (26). The resulting ODE's on $X, Y, T$ are the same as before, but we mention this option for two reasons. First, some people prefer this method, and it is often used in textbooks. Second, (26) is a well known PDE, the **Helmholtz equation**, named after *Hermann von Helmholtz* (1821–1894) and studied by him in connection with acoustics.

**EXAMPLE 1.** Let the initial deflection be

$$w(x,y,0) = f(x,y) = 0.2\sin\frac{\pi x}{a}\sin\frac{2\pi y}{b} - 0.1\sin\frac{5\pi x}{a}\sin\frac{3\pi y}{b}. \tag{28}$$

We could put this $f$ into (22) and integrate, to determine the $H_{mn}$'s, but it is much simpler to "match" terms in (17):

$$\begin{aligned} &0.2\sin\frac{\pi x}{a}\sin\frac{2\pi y}{b} - 0.1\sin\frac{5\pi x}{a}\sin\frac{3\pi y}{b} \\ &\quad = H_{11}\sin\frac{\pi x}{a}\sin\frac{\pi y}{b} + H_{12}\sin\frac{\pi x}{a}\sin\frac{2\pi y}{b} + H_{21}\sin\frac{2\pi x}{a}\sin\frac{\pi y}{b} + \cdots. \end{aligned} \tag{29}$$

Thus, $H_{12} = 0.2$, $H_{53} = -0.1$, and all other $H_{mn}$'s are zero. Then, $\omega_{12}$ and $\omega_{53}$ are obtained from (16b), so the solution is

$$\begin{aligned} w(x,y,t) = {} & 0.2\sin\frac{\pi x}{a}\sin\frac{2\pi y}{b}\cos\left(\pi c\sqrt{\frac{1}{a^2}+\frac{4}{b^2}}\,t\right) \\ & -0.1\sin\frac{5\pi x}{a}\sin\frac{3\pi y}{b}\cos\left(\pi c\sqrt{\frac{25}{a^2}+\frac{9}{b^2}}\,t\right). \end{aligned} \tag{30}$$

In this example only two modes are present, and these have nodal lines as indicated in Fig. 5. What, if any, are the nodal lines of the solution $w(x, y, t)$, lines along which $w = 0$ for all $t$? Since the two cosine functions in (30) are linearly independent, $w = 0$ for all $t$ requires that both $\sin \frac{\pi x}{a} \sin \frac{2\pi y}{b}$ *and* $\sin \frac{5\pi x}{a} \sin \frac{3\pi y}{b}$ be zero. But the curves on which these products are zero are disjoint (Fig. 5), so the solution $w(x, y, t)$ has no nodal lines. In other cases $w(x, y, t)$ can, indeed, have nodal lines and these lines are not necessarily straight, as discussed in the exercises. ∎

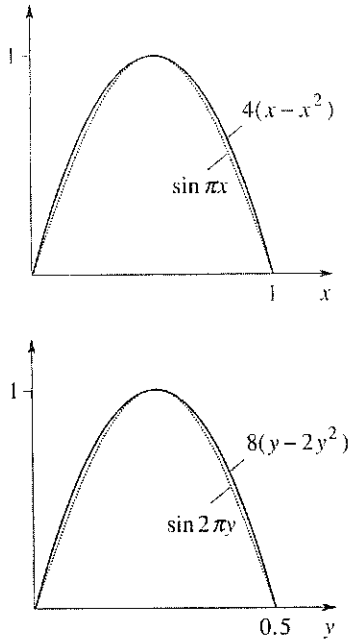



**Figure 6.** Similarity between the factors in (31) and the factors in the first mode.

**EXAMPLE 2.** Using the mks system of units (meters, kilograms, seconds, and newtons), let $a = 1$ m, $b = 0.5$ m, $\tau = 8$ nt/m, $\sigma = 0.02$ kg/m$^2$, and let the initial displacement be

$$w(x, y, 0) = f(x, y) = 0.02(x - x^2)(y - 2y^2) \text{ meters.} \tag{31}$$

Then $c = \sqrt{\tau/\sigma} = 20$ and, from (22),

$$\begin{aligned} H_{mn} &= 8 \int_0^{0.5} \int_0^1 0.02(x - x^2)(y - 2y^2) \sin m\pi x \sin 2n\pi y \, dx \, dy \\ &= 0.16 \int_0^1 (x - x^2) \sin m\pi x \, dx \int_0^{0.5} (y - 2y^2) \sin 2n\pi y \, dy \\ &= \frac{0.64}{m^3 n^3 \pi^6} \end{aligned} \tag{32}$$

if $m$ and $n$ are odd, and zero if $m$ and/or $n$ is even. Hence, the solution is

$$w(x, y, t) = \frac{0.64}{\pi^6} \sum_{n=1,3,\ldots}^{\infty} \sum_{m=1,3,\ldots}^{\infty} \frac{1}{m^3 n^3} \sin m\pi x \sin 2n\pi y \cos\left(20\pi\sqrt{m^2 + 4n^2}\right) t. \tag{33}$$

COMMENT. Observe that the terms diminish rapidly with $m$ and $n$ due to the $1/(m^3 n^3)$ factor, so that even the first term alone gives a reasonable approximation of the solution. That result is a consequence of the fact that the $(x - x^2)(y - 2y^2)$ product in (31) is approximated well by a scalar multiple of the first mode $\sin \pi x \sin 2\pi y$ (Fig. 6). ∎

**Closure.** To solve the wave equation in three independent variables $(x, y, t)$ by separation of variables we seek $w(x, y, t)$ in the form $X(x)Y(y)T(t)$. The separation process is successful, and we obtain ODE's on $X, Y$, and $T$, and two separation constants rather than one. The chief complication is that the solution is a double series rather a single series, with the coefficients given by a double integral rather than a single integral.

**Computer software.** Nodal lines are not always identified as readily as those in Fig. 3 but can be obtained using computer software. For instance, if $a = \pi$, $b = 2\pi$, $c = 1$, and

$$w(x, y, t) = (2 \sin 3x \sin y - \sin x \sin 3y) \cos(\sqrt{10}\, t), \tag{34}$$

then the nodal lines, if any, are given implicitly by the relation

$$2 \sin 3x \sin y - \sin x \sin 3y = 0. \tag{35}$$

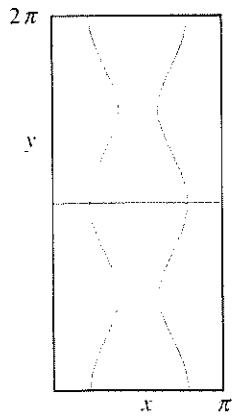



**Figure 7.** Nodal lines for (34).

Using *Maple*, enter

with (plots):

to access the plotting commands. Then use the **implicitplot** command as follows:

$$\text{implicitplot}\,(2 * \sin(3 * x) * \sin(y) - \sin(x) * \sin(3 * y) = 0,$$
$$x = 0..Pi, y = 2 * Pi, \text{grid} = [100, 200]);$$

The result is shown in Fig. 7. The "grid = [100, 200]" part is an option that we have used to set the plotting grid at 100 divisions of the $x$ interval and 200 divisions of the $y$ interval since $w$ the default grid, [25, 25], is too coarse in this case.

## EXERCISES 19.3

**1.** Obtain the solution to the initial-value problem (1) for the given $f(x, y)$, $a$, and $b$, and give the relations that define the nodal lines of $w$, if any. Let $c = 1$ in each case.

(a) $f(x, y) = 8 \sin 2x \sin 2y, \quad a = \pi, b = 2\pi$
(b) $f(x, y) = -5 \sin x \sin 4y, \quad a = 2\pi, b = \pi$
(c) $f(x, y) = \sin 3x \sin y - \sin x \sin 3y, \quad a = b = \pi$
(d) $f(x, y) = \sin 3x \sin y - \sin x \sin 3y, \quad a = \pi, b = 2\pi$
(e) $f(x, y) = 1.05 \sin 3x \sin y - \sin x \sin 3y, \quad a = \pi, b = 2\pi$
(f) $f(x, y) = 10 \sin 3x \sin y - \sin x \sin 3y, \quad a = \pi, b = 2\pi$
(g) $f(x, y) = 8 \sin x \sin 7y + \sin 5x \sin 5y, \quad a = b = \pi$
(h) $f(x, y) = 8 \sin 2x \sin 7y + \sin 5x \sin 5y, \quad a = b = \pi$
(i) $f(x, y) = 6 \sin \pi x \sin 7\pi y + \sin 5\pi x \sin 5\pi y, \quad a = 1, b = 3$
(j) $f(x, y) = \sin \pi x \sin 4\pi y - \sin 3\pi x \sin 5\pi y, \quad a = 1, b = 2$

**2.** Obtain the solution to the initial-value problem (1) for the case where $f(x, y) = 20 \sin 3\pi x \sin 4\pi y - 8 \sin 5\pi x \sin 12\pi y$ and $a = b = c = 1$, and determine the period of the motion.

**3.** Evaluate the following claim and reasoning and indicate whether it is correct or incorrect. Claim: Whereas the solution to the vibrating string problem [given by (16) in Section 19.2] is periodic in time (for any choice of the $R_n$'s and $S_n$'s), the solution to the vibrating membrane problem (1) [given by (16a) in this section] is *not*, in general, a periodic function of $t$. Reasoning: It will suffice to consider two terms in (16a) so that $w(x, y)$ is of the form $A \cos \omega_{kl} t + B \cos \omega_{mn} t$ for some choice of the integers $k, l, m, n$ and the constants $A$ and $B$. (Of course, $A$ and $B$ contain $x$ and $y$ dependence, but we can consider $x$ and $y$ as fixed here.) If the period is $T$, then there must be integers $M$ and $N$ such that $\omega_{kl} T = (M)(2\pi)$ and $\omega_{mn} T = (N)(2\pi)$. Division gives $\omega_{kl}/\omega_{mn} = M/N$. That is, $\omega_{kl}/\omega_{mn}$ must be a rational number. However, we see from (16b) that $\omega_{kl}/\omega_{mn}$ is, except for certain choices of $k, l, m$, and $n$, irrational.

**4.** (*Nonzero initial velocity*) We used $w_t(x, y, 0) = 0$ in (1c) merely for brevity.

(a) With

$$w(x, y, 0) = 0, \quad w_t(x, y, 0) = g(x, y)$$

in place of (1c), rederive the solution and obtain results analogous to (16) and (22).
(b) With

$$w(x, y, 0) = f(x, y), \quad w_t(x, y, 0) = g(x, y)$$

in place of (1c), re-derive the solution and obtain results analogous to (16) and (22).

**5.** (*Circular drum*) If the membrane is stretched over the circular disk $r < a$, then it is best to use the polar coordinates $r, \theta$ rather than the Cartesian coordinates $x, y$. Then the wave equation $c^2 \nabla^2 w = w_{tt}$ on $w(r, \theta, t)$ becomes

$$c^2 \left( w_{rr} + \frac{1}{r} w_r + \frac{1}{r^2} w_{\theta\theta} \right) = w_{tt}. \tag{5.1}$$

Let us consider only axisymmetric motions (i.e., we consider $w$ to be independent of $\theta$). Then the $w_{\theta\theta}$ term in (5.1) is zero, and (5.1) reduces to

$$c^2 \left( w_{rr} + \frac{1}{r} w_r \right) = w_{tt} \tag{5.2}$$

on $w(r, t)$.

(a) Seeking $w(r, t) = R(r)T(t)$, obtain

$$rR'' + R' + \kappa^2 rR = 0,$$
$$T'' + \kappa^2 c^2 T = 0$$

and hence the solution form

$$w(r,t) = (A + B\ln r)(D + Et) + [FJ_0(\kappa r) + GY_0(\kappa r)](H\cos\kappa ct + I\sin\kappa ct). \tag{5.3}$$

(b) If $w(a,t) = 0$ and $w(r,t)$ is to be bounded on $r < a$ (in particular at $r = 0$), show that we obtain

$$w(r,t) = \sum_{n=1}^{\infty} J_0\left(z_n \frac{r}{a}\right)(H_n \cos\omega_n t + I_n \sin\omega_n t), \tag{5.4}$$

where the $z_n$'s ($n = 1, 2, \ldots$) are the (known) zeros of $J_0$ [i.e., $J_0(z_n) = 0$] and $\omega_n = z_n c/a$.
(c) Let the initial conditions be

$$w(r,0) = f(r), \quad w_t(r,0) = g(r). \tag{5.5}$$

Imposing these conditions on (5.4), show that

$$H_n = \frac{2}{a^2[J_1(z_n)]^2}\int_0^a f(r)J_0\left(z_n\frac{r}{a}\right) r\,dr,$$
$$I_n = \frac{2}{\omega_n a^2[J_1(z_n)]^2}\int_0^a g(r)J_0\left(z_n\frac{r}{a}\right) r\,dr. \tag{5.6a,b}$$

HINT: This problem is similar to Example 5 in Section 18.3.
(d) Show that if we seek $w(r,\theta,t) = R(r)\Theta(\theta)T(t)$ in the full equation (5.1), then we obtain the ODE's

$$r^2R'' + rR' + (\kappa^2r^2 - \alpha^2)R = 0,$$
$$\Theta'' + \alpha^2\Theta = 0,$$
$$T'' + \kappa^2c^2T = 0,$$

where $\kappa$ and $\alpha$ are separation constants.

**6.** (*Two-dimensional diffusion*) In Chapter 18 we study one-dimensional diffusion phenomena such as the unsteady conduction of heat in a rod. Using the methods discussed in this section we can now return to Chapter 18 and solve two-dimensional problems as well. Specifically, consider the temperature field $u(x,y,t)$ in a rectangular plate ($0 < x < a$, $0 < y < b$), governed by the problem

$$\alpha^2(u_{xx} + u_{yy}) = u_t,$$
$$u(0,y) = u(a,y) = u(x,0) = u(x,b) = 0,$$
$$u(x,y,0) = f(x,y). \tag{6.1a,b,c}$$

(a) Derive the solution

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_{mn}\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,e^{-\kappa_{mn}^2\alpha^2 t}, \tag{6.2}$$

where

$$\kappa_{mn}^2 = \pi^2\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right) \tag{6.3}$$

and

$$A_{mn} = \frac{4}{ab}\int_0^b\int_0^a f(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy. \tag{6.4}$$

(b) Verify, formally, that (6.2)–(6.4) does satisfy (6.1).
(c) Evaluate the $A_{mn}$'s for the case where $f(x,y) = 100$.

## 19.4 Vibrating String; d'Alembert's Solution

**19.4.1. d'Alembert's solution.** For the wave equation

$$c^2y_{xx} = y_{tt} \tag{1}$$

there exists a striking solution form that is due to *Jean Le Rond d'Alembert* (1717–1783). D'Alembert's method is based on a change of independent variables, from $x$ and $t$ to $\xi$ and $\eta$, say, according to the simple relations

$$\boxed{\begin{aligned} \xi &= x - ct, \\ \eta &= x + ct. \end{aligned}} \tag{2a,b}$$

To re-express (1) in terms of $\xi$ and $\eta$, the needed "building blocks" are the $\partial/\partial x$ and $\partial/\partial t$ operators. According to the chain rule (Sections 13.4, 13.6.4),

$$\frac{\partial}{\partial x} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta} \tag{3}$$

and

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial}{\partial \eta}\frac{\partial \eta}{\partial t} = -c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}, \tag{4}$$

so (1) becomes

$$c^2\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)y = \left(-c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right)\left(-c\frac{\partial}{\partial \xi} + c\frac{\partial}{\partial \eta}\right)y$$

or

$$\left(\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)(y_\xi + y_\eta) = \left(-\frac{\partial}{\partial \xi} + \frac{\partial}{\partial \eta}\right)(-y_\xi + y_\eta)$$

or

$$y_{\xi\xi} + y_{\xi\eta} + y_{\eta\xi} + y_{\eta\eta} = y_{\xi\xi} - y_{\xi\eta} - y_{\eta\xi} + y_{\eta\eta}. \tag{5}$$

Assuming that $y$ is well enough behaved so that $y_{\xi\eta} = y_{\eta\xi}$,* (5) simplifies to $4y_{\xi\eta} = 0$, or

$$y_{\xi\eta} = 0. \tag{6}$$

The point, then, is that when expressed in terms of $\xi$ and $\eta$ the wave equation (1) simplifies dramatically to the form (6), and we say that (6) is the **canonical**, or simplest, form of (1). It is simple because it can be solved directly by integration. First, integrating (6) with respect to $\eta$ gives

$$y_\xi = \int 0\,\partial\eta = 0 + A(\xi) = A(\xi), \tag{7}$$

where the "constant of integration" $A$ is allowed to be an arbitrary function of $\xi$ since $\xi$ was held fixed during the integration on $\eta$. Next, integration of (7) with respect to $\xi$ gives $y = \int A(\xi)\,\partial\xi$ plus an arbitrary function of $\eta$, say $G(\eta)$. Since $A$ is arbitrary we might as well simply write $F(\xi)$ in place of $\int A(\xi)\,\partial\xi$. Thus,

$$y = F(\xi) + G(\eta) \tag{8}$$

or, returning to the original variables,

$$\boxed{y(x,t) = F(x - ct) + G(x + ct),} \tag{9}$$

*See Theorem 13.3.1.

where $F$ and $G$ are arbitrary functions [although they do need to be twice differentiable if (9) is to satisfy (1)]. For example, each of the three functions

$$6e^{x-ct}, \qquad \sin(x-ct) + 5\sin[3(x+ct)], \qquad 100 - 5(x-ct)^3 - 8e^{4(x+ct)^2} \tag{10}$$

is of the form (9) and it is readily verified that each one satisfies (1). To verify (9), in general, we can use the chain rule:

$$\begin{aligned} y_t &= F'(x-ct)\frac{\partial(x-ct)}{\partial t} + G'(x+ct)\frac{\partial(x+ct)}{\partial t} \\ &= -cF'(x-ct) + cG'(x+ct) \\ y_{tt} &= -cF''(x-ct)\frac{\partial(x-ct)}{\partial t} + cG''(x+ct)\frac{\partial(x+ct)}{\partial t} \\ &= c^2F''(x-ct) + c^2G''(x+ct). \end{aligned} \tag{11a}$$

Similarly, we obtain

$$y_{xx} = F''(x-ct) + G''(x+ct), \tag{11b}$$

so we see from (11) that (1) is satisfied. Remember that primes are standard notation for the derivative of a function of a single variable with respect to that variable; for functions of more than one variable we use the partial derivative notation. Thus, by $F'(x-ct)$, for example, we mean the ordinary derivative $\frac{dF(x-ct)}{d(x-ct)}$ of $F$ with respect to its single argument $x-ct$. To illustrate, if

$$F(x-ct) = 3\sin(x-ct)^2,$$

then

$$F'(x-ct) = 6(x-ct)\cos(x-ct)^2.$$

We say that (9) is the **general solution** of the wave equation (1). There is no analog of (9) in Chapter 18; nowhere in Chapter 18 did we find the general solution of the diffusion equation. Even the infinite series solutions that we found by separation of variables, which contained an infinite number of arbitrary constants, are not general solutions; they are simply comprehensive enough so as to be capable of satisfying the initial condition.

It is interesting to compare the form of (9) with the form $y(x) = C_1y_1(x) + C_2y_2(x)$ of the general solution to a linear homogeneous second-order *ordinary* differential equation; in place of an arbitrary constant $C_1$ *times* $y_1(x)$ we have an arbitrary *function* of $x-ct$, and in place of an arbitrary constant $C_2$ *times* $y_2(x)$ we have an arbitrary *function* of $x+ct$.

To illustrate the use of (9), consider the infinite string problem

$$c^2y_{xx} = y_{tt}, \qquad (-\infty < x < \infty,\ 0 < t < \infty) \tag{12a}$$

$$y(x,0) = f(x), \quad y_t(x,0) = g(x). \qquad (-\infty < x < \infty) \tag{12b}$$

Since we already have in (9) the general solution of (12a), we can bypass (12a) and immediately impose the initial conditions (12b) in order to determine the functions $F$ and $G$ in (9). Doing so gives

$$y(x,0) = f(x) = F(x) + G(x), \tag{13a}$$
$$y_t(x,0) = g(x) = -cF'(x) + cG'(x) \tag{13b}$$

as two equations on $F$ and $G$. Integrating (13b) from any fixed point, say 0, to $x$ gives

$$\int_0^x g(\xi)\,d\xi = -cF(x) + cF(0) + cG(x) - cG(0), \tag{14}$$

and solving (13a) and (14) for $F(x)$ and $G(x)$ gives

$$F(x) = \frac{f(x)}{2} - \frac{1}{2c}\int_0^x g(\xi)\,d\xi + \frac{F(0)-G(0)}{2}, \tag{15a}$$
$$G(x) = \frac{f(x)}{2} + \frac{1}{2c}\int_0^x g(\xi)\,d\xi - \frac{F(0)-G(0)}{2}. \tag{15b}$$

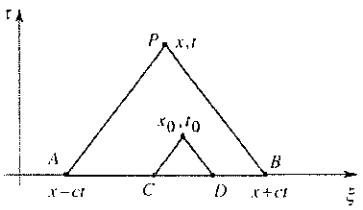



**Figure 1.** Interpretation of (17).

Since $F(x)$ is given by the right-hand side of (15a) $F(x-ct)$ is likewise given by the right-hand side of (15a), but with each of the two $x$'s changed to $x-ct$. Similarly, $G(x+ct)$ is obtained by changing each of the $x$'s in (15b) to $x+ct$. Doing so, we obtain

$$\begin{aligned} y(x,t) &= F(x-ct) + G(x+ct) \\ &= \frac{f(x-ct)}{2} - \frac{1}{2c}\int_0^{x-ct} g(\xi)\,d\xi + \frac{F(0)-G(0)}{2} \\ &\quad + \frac{f(x+ct)}{2} + \frac{1}{2c}\int_0^{x+ct} g(\xi)\,d\xi - \frac{F(0)-G(0)}{2}, \end{aligned} \tag{16}$$

or

$$\boxed{y(x,t) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\xi)\,d\xi.} \tag{17}$$

Let us interpret (17) in the $x,t$ plane. Since $x,t$ is the specific point at which $y$ is being evaluated, let us use dummy variables $\xi,\eta$ for the axes (Fig. 1). In words, (17) tells us that $y$ at $P$ is the average of the $f$ values at $A$ and $B$ plus $1/2c$ times the integral of $g$ from $A$ to $B$. Thus, the value of $y$ at $P$ depends on initial data only on the interval $AB$. Similarly, the value of $y$ at $x_0, t_0$ depends on initial data only on the interval $CD$, and so on. Thus, we call the triangular region $ABP$ the **domain of influence** of the interval $AB$; initial data on $AB$ determine the solution within the triangle $ABP$.

This result for the wave equation is in contrast with the result that the solution $u(x,t)$ of the diffusion equation depends on the initial data $u(x,0) = f(x)$ all

along the axis, as can be seen from (11) in Section 18.4 where the integration is from $\xi = -\infty$ to $\xi = +\infty$.

**EXAMPLE 1.** *The Case Where* $y_t(x, 0) = 0$. To illustrate (17), suppose we release the string from rest in the configuration $y(x, 0) = f(x)$, where $f$ is the triangular pulse shown shown in Fig. 2. Then $g(x) = 0$ and (17) gives

$$y(x, t) = \frac{f(x - ct) + f(x + ct)}{2}. \tag{18}$$

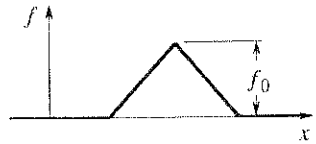



**Figure 2.** Initial shape of string: $y(x, 0) f(x)$.

At $t = 0$, (18) becomes $y(x, 0) = f(x)/2 + f(x)/2$, so we may regard the initial pulse $f(x)$ as the sum of two "half-pulses," $f(x)/2$ and $f(x)/2$. For $t > 0$ the $f(x - ct)/2$ term in (18) amounts to one of the half-pulses translated to the right through a distance $ct$, and the $f(x+ct)/2$ term amounts to the other half-pulse translated to the left through a distance $ct$, as depicted in Fig. 3. Evidently, the speed of these right- and left-running waves is $c$. Thus, the $c = \sqrt{\tau/\sigma}$ that appears in (1) is the **wave speed**.

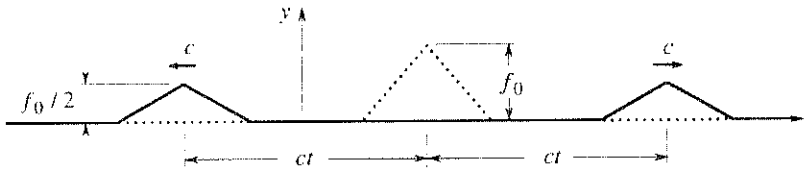



**Figure 3.** Right- and left-running waves.

Besides plotting $y$ versus $x$ as we have in Fig. 3, it is illuminating to display these results in the $x, t$ plane as we have in Fig. 4. Naturally, plotting $y(x, t)$ above the $x, t$ plane calls for three-dimensional graphics, but it is simpler to merely "lay the solution curves down" in the plane of the paper as we have in the figure. Since the $f(x - ct)/2$ term is constant along $x - ct =$ constant lines, the values of $f(x - ct)/2$ propagate along these lines without change. Similarly, the values of $f(x + ct)/2$ propagate along $x + ct =$ constant lines.

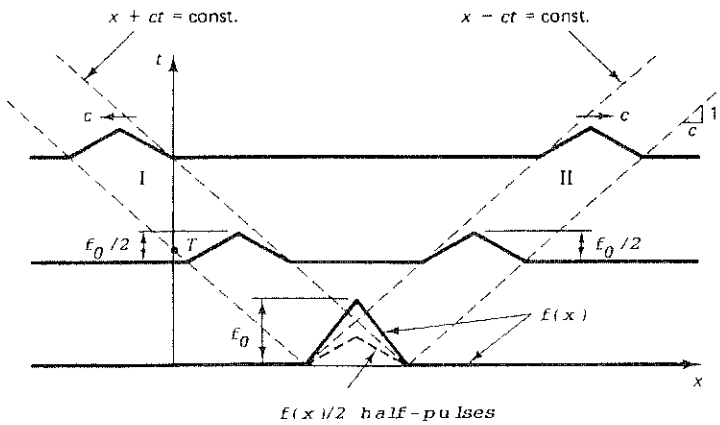



**Figure 4.** $x, t$ plane display of (18).

These special families of curves, the lines $x - ct =$ constant and $x + ct =$ constant, are known as **characteristics**, and it is along the characteristics that information (i.e., val-

ues) propagates. In Fig. 4 we have drawn only four characteristics, those that define the "channels" I and II in which the half-pulses propagate, but there are actually an infinite number of characteristics, all of the curves $\xi = x - ct =$ constant and all of the curves $\xi = x + ct =$ constant as suggested more fully in Fig. 5.

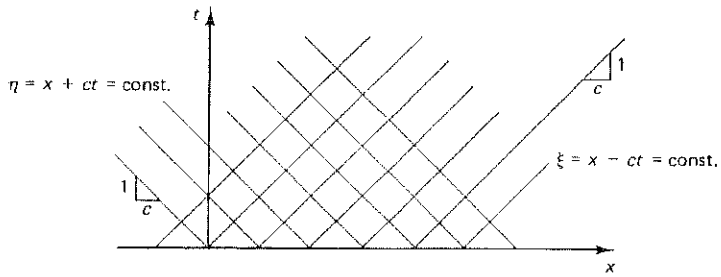



**Figure 5.** The two families of characteristics.

The upshot is that the initial $f$ pulse breaks into two half-pulses that travel outward, without change in shape, in the channels I and II that are bounded by characteristics. Outside of these channels $y(x,t) = 0$. ∎

**EXAMPLE 2.** *The Case Where* $y(x,0) = 0$. (This example can be omitted if you have not read the optional Section 5.6 on the Dirac delta function.) In Example 1 we took $g(x) = 0$ and $f(x)$ to be a simple triangular pulse in order to gain understanding of the $[f(x-ct) + f(x+ct)]/2$ term in (17). Now, to study the integral term let the initial displacement be $y(x,0) = f(x) = 0$ and let the initial velocity be

$$y_t(x,0) = g(x) = \delta(x - x_0), \tag{19}$$

a delta function at some point $x_0$. These initial conditions are similar to those to which a piano wire is subjected, for the initial displacement is zero but a localized velocity is imparted at time $t = 0$ when the string is struck by a narrow hammer.

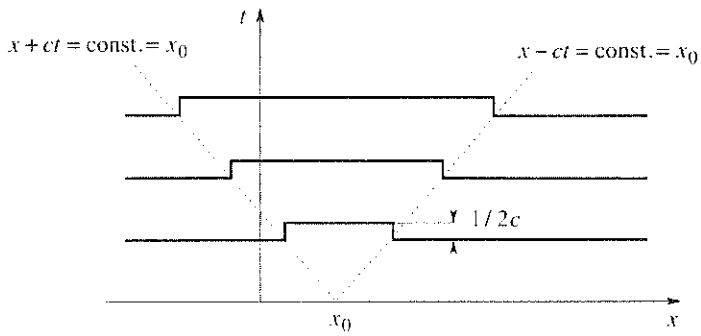



**Figure 7.** $x,t$ plane display of (20).

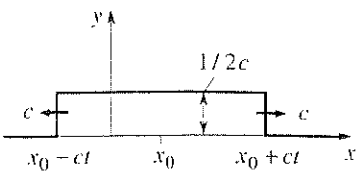



**Figure 6.** Response to hammer blow.

With $f(x) = 0$ and $g(x) = \delta(x - x_0)$, (17) gives

$$y(x,t) = 0 + \frac{1}{2c}\int_{x-ct}^{x+ct} \delta(\xi - x_0)\, d\xi. \tag{20}$$

From the definition of the delta function [see (13) in Section 5.6], the integral is 1 if $x - ct < x_0 < x + ct$ or, equivalently, if $x_0 - ct < x < x_0 + ct$. Hence, the solution (20) is the rectangular pulse shown in Fig. 6; in the $x, t$ plane the solution is as shown in Fig. 7. ∎

**19.4.2. Use of images.** (NOTE: The optional Section 18.5 on the method of images is not a prerequisite for this section.) Remember that the solution form (17) is for infinite strings, on $-\infty < x < \infty$. If we have a semi-infinite or finite string, then we have boundary conditions to deal with besides the two initial conditions. To illustrate, consider the following semi-infinite string problem:

$$c^2 y_{xx} = y_{tt}, \qquad (0 < x < \infty,\ 0 < t < \infty) \tag{21a}$$

$$y(0,t) = 0, \qquad (0 < t < \infty) \tag{21b}$$

$$y(x,0) = f(x),\ \ y_t(x,0) = 0, \qquad (0 < x < \infty) \tag{21c}$$

where $f$ is the triangular pulse shown in Fig. 4. Observe that the solution shown in Fig. 4 does indeed satisfy (21) but only up until time $T$, at which time the left-running wave in channel I reaches the end point $x = 0$ and upsets the boundary condition (21b).

We can overcome this difficulty by an artifice known as the **method of images**. Namely, consider an *infinite* string $(-\infty < x < \infty)$ with the initial conditions

$$y(x,0) = f_{\text{ext}}(x),\ \ y_t(x,0) = 0, \qquad (-\infty < x < \infty) \tag{22}$$

where $f_{\text{ext}}(x)$ is identical to $f(x)$ on $0 < x < \infty$ and is an antisymmetric extension of $f(x)$ for $-\infty < x < 0$ as shown in Fig. 8. The solution of the extended problem is

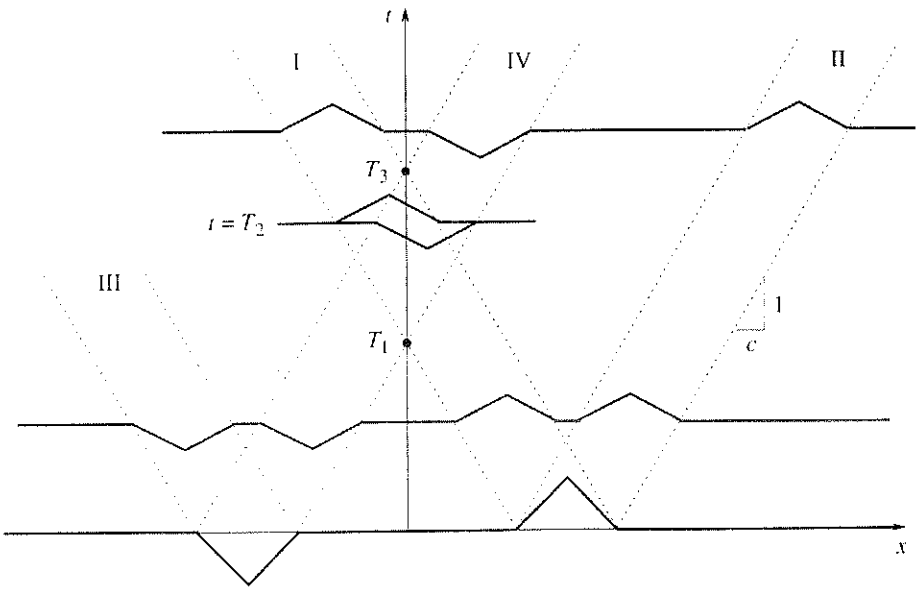



**Figure 8.** Image system.

$$y(x,0) = \frac{f_{\text{ext}}(x-ct) + f_{\text{ext}}(x+ct)}{2} \tag{23}$$

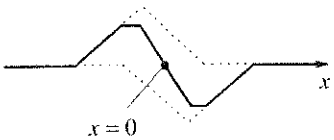



**Figure 9.** Cancellation at $x = 0$.

and the latter is comprised of two positive half-pulses in channels I and II plus the two negative half-pulses in channels III and IV. The point is that over the time interval $T_1 < t < T_3$ the waves in channels I and IV pass over each other, so that the negative wave in channel IV cancels the positive wave in channel I along the $t$ axis. At $t = T_2$, for instance, the situation is as shown in Fig. 9, where the sum of the two waves is indicated by the solid line. That part of the diagram that is in the second quadrant ($-\infty < x < 0, 0 < t < \infty$) is called the **image system** and is fictitious, serving only to automatically satisfy the boundary condition (21b) by virtue of the antisymmetry about $x = 0$. It can now be ignored, or even discarded, since the wave system in the first quadrant fully satisfies conditions (21a,b,c) and is the desired solution.

Since the image system is, after all, fictitious, to apprehend the physical event we look only at the first quadrant. We see that the initial pulse breaks into two half-pulses. One travels rightward indefinitely, while the left-running wave is both *reflected* (into channel $IV$) and *inverted* when it encounters the left end of the string, which is tied at $x = 0$.

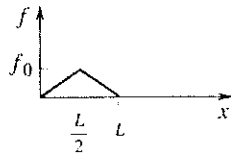



**Figure 10.** $f(x)$ in (24).

Having studied the infinite string and semi-infinite string, in that order, we can finally return to the finite string problem

$$c^2 y_{xx} = y_{tt}, \qquad (0 < x < L,\ 0 < t < \infty) \tag{24a}$$

$$y(0,t) = 0,\ \ y(L,t) = 0, \qquad (0 < t < \infty) \tag{24b}$$

$$y(x,0) = f(x),\ \ y_t(x,0) = 0 \qquad (0 < x < L) \tag{24c}$$

that is solved by separation of variables in Section 19.2, and solve it by d'Alembert's method – with help from the method of images. Let $f(x)$ be our usual generic triangular pulse (Fig. 10). Using the method of images, we consider instead the *infinite* string problem with initial conditions

$$y(x,0) = f_{\text{ext}}(x),\ \ y_t(x,0) = 0, \qquad (-\infty < x < \infty) \tag{25}$$

where $f_{\text{ext}}(x)$ is identical to $f(x)$ on $0 < x < L$ and is defined on $x < 0$ and on $x > L$ so as to be antisymmetric about *both* $x = 0$ and $x = L$. Thus, $f_{\text{ext}}(x)$ is the $2L$-periodic function shown in Fig. 11. Since the extended initial conditions (25) are antisymmetric about $x = 0$ and $x = L$, the solution $y(x,t)$ will be too. Hence, $y$ will be zero all along the $x = 0$ and $x = L$ lines in the $x, t$ plane. That is, the image system is designed so as to satisfy the boundary conditions (24b).

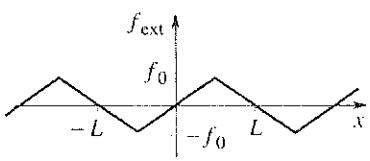



**Figure 11.** $f_{\text{ext}}(x)$, antisymmetric about $x = 0$ and $x = L$.

With $f_{\text{ext}}(x)$ as defined in Fig. 11, the d'Alembert solution (17) gives the closed form solution

$$y(x,t) = \frac{f_{\text{ext}}(x-ct) + f_{\text{ext}}(x+ct)}{2}. \tag{26}$$

To relate (26) to the separation of variables solution obtained in Section 19.2, expand the periodic function $f_{\text{ext}}(x)$ in a Fourier series as

$$f_{\text{ext}}(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \tag{27}$$

where

$$b_n = \frac{2}{L}\int_0^L f_{\text{ext}}(x)\sin\frac{n\pi x}{L}\,dx = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

since $f_{\text{ext}}(x) = f(x)$ on $0 < x < L$. Then (26) becomes

$$\begin{aligned} y(x,t) &= \frac{1}{2}\sum_{n=1}^{\infty} b_n\left[\sin\frac{n\pi}{L}(x-ct) + \sin\frac{n\pi}{L}(x+ct)\right] \\ &= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}, \end{aligned} \tag{28}$$

which is identical to the result that we obtained by separation of variables.

**19.4.3. Solution by integral transforms. (Optional)** Following d'Alembert, we derived the solution (17) of the infinite string problem (12) by using the general solution $y(x,t) = F(x-ct) + G(x+ct)$. Alternatively, we could solve (12) by Laplace transforming with respect to the $t$ variable or by Fourier transforming with respect to the $x$ variable.

Let us try the Laplace transform. Transforming (12a) gives

$$\begin{aligned} c^2\overline{y}_{xx} &= s^2\overline{y} - sy(x,0) - y_t(x,0) \\ &= s^2\overline{y} - sf(x) - g(x) \end{aligned}$$

or

$$\overline{y}_{xx} - \frac{s^2}{c^2}\overline{y} = -\frac{1}{c^2}[sf(x) + g(x)]. \tag{29}$$

The homogeneous solution of (29) is simple but obtaining a particular solution (e.g., by the method of variation of parameters) is messy, so let us see whether the Fourier transform is more convenient. Fourier transforming (12a) gives

$$c^2(i\omega)^2\hat{y} = \hat{y}_{tt}$$

or

$$\hat{y}_{tt} + \omega^2c^2\hat{y} = 0, \tag{30}$$

which is simpler than (29) because it is homogeneous, so let us continue. From (30),

$$\hat{y} = A\cos\omega ct + B\sin\omega ct. \tag{31}$$

To solve for $A$ and $B$ we impose Fourier transformed versions of (12b):

$$\hat{y}\Big|_{t=0} = \hat{f}(\omega) = A, \tag{32a}$$

$$\hat{y}_t\Big|_{t=0} = \hat{g}(\omega) = \omega cB, \tag{32b}$$

so $A = \hat{f}(\omega)$ and $B = \hat{g}(\omega)/\omega c$ and

$$\hat{y} = \hat{f}(\omega)\cos\omega ct + \hat{g}(\omega)\frac{\sin\omega ct}{\omega c}. \tag{33}$$

To invert the two terms on the right, use entries 15, 9, and 21 of Appendix D and obtain

$$F^{-1}\left\{\hat{f}(\omega)\cos\omega ct\right\} = \frac{f(x-ct)+f(x+ct)}{2}, \tag{34a}$$

$$\begin{aligned} F^{-1}\left\{\hat{g}(\omega)\frac{\sin\omega ct}{\omega c}\right\} &= g(x) * \frac{1}{2c}[H(x+ct)-H(x-ct)] \\ &= \frac{1}{2c}\int_{-\infty}^{\infty} g(x-\xi)[H(\xi+ct)-H(\xi-ct)]\,d\xi \\ &= \frac{1}{2c}\int_{-ct}^{ct} g(x-\xi)\,d\xi \qquad (x-\xi=\mu) \\ &= \frac{1}{2c}\int_{x-ct}^{x+ct} g(\mu)\,d\mu, \end{aligned} \tag{34b}$$

where the third equality in (34b) follows from the fact that $H(x+ct)-H(x-ct)$ is zero for $x < -ct$, unity for $-ct < x < ct$, and zero for $x > ct$. Thus,

$$y(x,t) = \frac{f(x-ct)+f(x+ct)}{2} + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\mu)\,d\mu,$$

which result is the same as (17).

**Closure.** Following d'Alembert, we change the independent variables from $x, t$ to $\xi, \eta$ according to the relations $\xi = x - ct$ and $\eta = x + ct$. That change reduces the wave equation $c^2 y_{xx} = y_{tt}$ to its canonical (i.e., simplest) form $y_{\xi\eta} = 0$, which can be solved by integration to give the general solution of the wave equation

$$y(x,t) = F(x-ct) + G(x+ct),$$

where $F$ and $G$ are arbitrary twice differentiable functions (Exercise 15). To illustrate the use of this general solution we solve an infinite string problem with prescribed initial displacement $f(x)$ and velocity $g(x)$ and obtain the solution (17).

The $\xi, \eta$ variables are the most "natural" independent variables and the $\xi =$ constant and $\eta =$ constant lines constitute two families of lines (Fig. 5) called characteristics. Rather than kinks and discontinuities in the initial conditions smoothing out, as occurs in the process of diffusion, they propagate into the solution domain along characteristics. And whereas information spreads to the left and right at an infinite speed for the diffusion equation, it spreads with a finite wave speed $c$ for the wave equation.

Remember that, while extremely important in their own right, the diffusion ($\alpha^2 u_{xx} = u_t$) and wave ($c^2 y_{xx} = y_{tt}$) equations also serve to represent the parabolic

and hyperbolic types of PDE's, respectively, and the main features of these two equations are shared by other PDE's of their type. Thus, every hyperbolic PDE has its own families of characteristics, which may be curved rather than straight and along which information propagates.

In Section 19.4.2 we use the infinite string solution (17) together with the method of images to solve semi-infinite and finite string problems. If you studied Section 18.5 you will understand that the method of images works, here, because the $L = c^2\partial^2/\partial x^2 - \partial^2/\partial t^2$ operator is linear and even, but it is not necessary to understand *why* the method works in these cases inasmuch as we can *see* that it does lead to satisfaction of the homogeneous boundary conditions.

Finally, in Section 19.4.3 we rederive the solution (17), this time using a Fourier transform on $x$.

## EXERCISES 19.4

**1.** Verify that (17) does satisfy (12a) and (12b).

**2.** Show that, like the wave equation, the given PDE is hyperbolic and find its general solution by introducing the suggested change of variables.

(a) $u_{xx} + 4u_{xy} + 3u_{yy} = 0$; $\xi = x - y$, $\eta = 3x - y$
(b) $u_{xx} - 4u_{xy} - 5u_{yy} = 0$; $\xi = x - y$, $\eta = 5x + y$
(c) $u_{xx} + 6u_{xy} + 8u_{yy} = 0$; $\xi = 4x - y$, $\eta = 2x - y$
(d) $u_{xx} + 4u_{xy} - 5u_{yy} = 0$; $\xi = x + y$, $\eta = 5x - y$
(e) $u_{xx} + 2u_{xy} - 3u_{yy} = 0$; $\xi = 3x - y$, $\eta = x + y$

**3.** Show that, like the wave equation, the given PDE is hyperbolic and find its general solution by introducing a suitable change of variables of the form $\xi = ax + by$, $\eta = cx + dy$.

(a) $u_{xx} + 8u_{xy} + 12u_{yy} = 0$
(b) $u_{xx} - 2u_{xy} - 3u_{yy} = 0$
(c) $u_{xx} - 10u_{xy} + 9u_{yy} = 0$
(d) $u_{xx} + 2u_{xy} - 8u_{yy} = 0$

**4.** Find the general solution of the first-order PDE $u_t + cu_x = 0$, where $c$ is a constant, by introducing the change of variables $\xi = x - ct$, $\eta = t$, and then use that general solution to solve the problem.

$$u_t + cu_x = 0, \qquad (-\infty < x < \infty,\ 0 < t < \infty)$$
$$u(x,0) = f(x).$$

**5.** In Figs. 3 and 4 we show $y(x,t)$ at times $t$ which were large enough so that the two half-pulses had completely separated (i.e., they do not overlap). In this exercise we examine the case in which they do overlap. Letting

$$f(t) = \begin{cases} 0, & x < 1 \\ 2x - 2, & 1 < x < 1.5 \\ 4 - 2x, & 1.5 < x < 2 \\ 0, & x > 2, \end{cases}$$

$g(x) = 0$, and $c = 20$, give labeled graphs of $y(x,t)$ at $t = 0.005$ and at $t = 0.02$.

**6.** With $c = 100$, sketch the solution to (12) at $t = 0.02$ and $t = 0.04$ using an $x, t$ plane display similar to those in Figs. 4 and 7, and labeling all key values. As usual, $H$ is the Heaviside function.

(a) $f(x) = H(x+1) - H(x-1)$, $g(x) = 0$
(b) $f(x) = H(x)$, $g(x) = 0$
(c) $f(x) = 0$, $g(x) = H(x+1) - H(x-1)$
(d) $f(x) = 0$, $g(x) = H(x)$

**7.** (a) Obtain the general solution for the nonhomogeneous wave equation $c^2 y_{xx} = y_{tt} - K$, where $K$ is a constant.
(b) Use the general solution obtained in part (a) to solve the problem

$$c^2 y_{xx} = y_{tt} - K, \qquad (-\infty < x < \infty,\ 0 < t < \infty)$$
$$y(x,0) = p(x),\ \ y_t(x,0) = q(x).$$

**8.** Use the general solution (9) to solve the problem

$$c^2 y_{xx} = y_{tt}, \qquad (0 < x < \infty,\ 0 < t < \infty)$$
$$y(x,0) = y_t(x,0) = 0, \qquad (0 < x < \infty)$$
$$y(0,t) = h(t), \qquad (0 < t < \infty)$$

where $h(t)$ is prescribed. [We can imagine taking the left end of the string between two fingers and, beginning at $t = 0$, "jiggling" it according to $y(0,t) = h(t)$.]

**9.** Same as Exercise 8, but solve by using the Laplace transform rather than (9). HINT: You may assume the truth of the statement

$$\lim_{x\to\infty} \overline{y}(x,s) = \lim_{x\to\infty} \int_0^\infty y(x,t)e^{-st}\,dt$$
$$= \int_0^\infty \lim_{x\to\infty} y(x,t)e^{-st}\,dt = \int_0^\infty 0\,dt = 0.$$

**10.** If spherical symmetry is present so that $u$ depends only on $\rho$ and $t$ (where $\rho, \phi, \theta$ are the spherical polar coordinates), then the wave equation $c^2\nabla^2 u = u_{tt}$ becomes

$$\frac{c^2}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right) = \frac{\partial^2 u}{\partial t^2}. \tag{10.1}$$

Show that (10.1) may be re-expressed as

$$c^2\frac{\partial^2}{\partial\rho^2}(\rho u) = \frac{\partial^2}{\partial t^2}(\rho u) \tag{10.2}$$

and thus derive the general solution

$$u(\rho,t) = \frac{1}{\rho}[F(\rho - ct) + G(\rho + ct)] \tag{10.3}$$

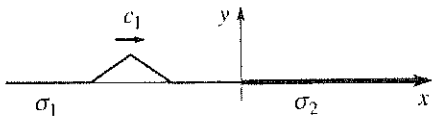
of (10.1), where $F$ and $G$ are arbitrary twice-differentiable functions. NOTE: Observe from the $1/\rho$ factor in (10.3) that in the case of spherical waves the wave amplitude tends to zero as $\rho \to \infty$.

**11.** Show whether traveling wave solutions of the form $F(x - at)$ are possible for the diffusion problem

$$\alpha^2 u_{xx} = u_t, \qquad (-\infty < x < \infty,\ 0 < t < \infty)$$
$$u(x,0) = f(x), \qquad (-\infty < x < \infty)$$

where $u(x,t)$ is bounded, for all $t$, as $x \to \pm\infty$.

**12.** (*Infinite string with density discontinuity*) Consider a string stretched over $-\infty < x < \infty$ under a tension $\tau$, where the density $\sigma(x)$ has a step discontinuity at $x = 0$; i.e., $\sigma = \sigma_1$ for $x < 0$ and $\sigma = \sigma_2$ for $x > 0$. Suppose a rightward-running wave $y(x,t) = F(x - c_1 t)$, such as a triangular pulse, is initiated in the $x < 0$ part of the string, where $c_1 = \sqrt{\tau/\sigma_1}$ is the wave speed for $x < 0$ and $c_2 = \sqrt{\tau/\sigma_2}$ is the wave speed for $x > 0$. What happens when this wave encounters the density discontinuity (see the accompanying figure)? That is, determine the solution $y(x,t)$. HINT: Use the general solution $y(x,t) = G(x - c_1 t) + H(x + c_1 t)$ for $x < 0$, and $y(x,t) = I(x - c_2 t) + J(x + c_2 t)$ for $x > 0$. The solution will be nonzero only within certain channels, in which you need to determine the functions $G, H, I$, and $J$. You will need to match the two solutions (i.e., for $x < 0$ and for $x > 0$) along the line $x = 0$ in the $x, t$ plane.




**13.** Use (17) to show that if $y(x,0) = f(x)$ and $y_t(x,0) = g(x)$ are even functions of $x$, then $y(x,t)$ remains an even function for all $t > 0$, and that if $f(x)$ and $g(x)$ are odd functions of $x$, then $y(x,t)$ remains an odd function for all $t > 0$.

**14.** We solve (21) by the method of images, where $f(x)$ is the triangular pulse shown in Fig. 4. Repeat that solution, with (21b) changed to $y_x(0,t) = 0$, and obtain an $x, t$ plane diagram analogous to that in Fig. 8. Is there a reflection and an inversion as in Fig. 8? Explain.

**15.** (*Breakdowns at kinks*) We have emphasized that kinks and discontinuities in the initial conditions $y(x,0) = f(x)$ and $y_t(x,0) = g(x)$ propagate into the solution domain as in Fig. 4, for instance, where the kinks in $f$ propagate along both right- and left-running characteristics. However, it must be confessed that at each point along those characteristics the PDE $c^2 y_{xx} = y_{tt}$ is not satisfied because both $y_{xx}$ and $y_{tt}$ fail to exist. (That $y_{xx}$ does not exist there should be evident from Fig. 4. Do you see why $y_{tt}$ fails to exist there as well?) Explain why results such as those displayed in Fig. 4 are acceptable nonetheless.

**16.** (*Finite-difference method*) In Section 18.6 we discretize the problem by means of the computational grid shown there in Fig. 1, and use difference quotient approximations of $u_{xx}$ and $u_t$ to obtain the computational formula $U_{j,k+1} = rU_{j-1,k} + (1-2r)U_{j,k} + rU_{j+1,k}$, where $U_{j,k}$ denotes the exact solution of the difference equation.

(a) Proceeding in the same manner, derive the scheme

$$Y_{i,j+1} = r^2 Y_{i-1,j} + 2(1-r^2)Y_{i,j} + r^2 Y_{i+1,j} - Y_{i,j-1} \tag{16.1}$$

for the wave equation problem

$$\begin{gathered} c^2 y_{xx} = y_{tt}, \\ y(0,t) = p(t),\ \ y(L,t) = q(t), \\ y(x,0) = f(x),\ \ y_t(x,0) = g(x) \end{gathered} \tag{16.2,3,4}$$

on $0 < x < L$, $0 < t < \infty$, where $r = c\Delta t/\Delta x$.

(b) Letting $c = 10$, $L = 1$, $\Delta x = 0.25$, $\Delta t = 0.02$, $p(t) = q(t) = f(x) = 0$, and $g(x) = \sin \pi x$, use (16.1) to evaluate the $Y_{i,j}$'s along the first two time lines, i.e., at $(i, j) = (1,1), (2,1), (3,1), (1,2), (2,2), (3,2)$. Compare your results with the exact solution. HINT: For points on the first time (16.1) gives

$$Y_{1,1} = r^2 Y_{0,0} + 2(1 - r^2)Y_{1,0} + r^2 Y_{2,0} - Y_{1,-1},$$
$$Y_{2,1} = r^2 Y_{1,0} + 2(1 - r^2)Y_{2,0} + r^2 Y_{3,0} - Y_{2,-1},$$
$$Y_{3,1} = r^2 Y_{2,0} + 2(1 - r^2)Y_{3,0} + r^2 Y_{4,0} - Y_{3,-1},$$

but how are we to deal with the $Y_{1,-1}, Y_{2,-1}$ and $Y_{3,-1}$ terms when the points $(i, j) = (1, -1), (2, -1)$, and $(3, -1)$ do not lie on the computational grid? Use a finite-difference version of the initial condition $y_t(x, 0) = g(x)$.

(c) Same as (b) but with $p(t) = 20t$.

(d) Same as (b) but with $q(t) = 1000(1 - \cos t)$.

(e) Same as (b) but with $f(x) = 10 \sin \pi x$ and $g(x) = 0$.

(f) It can be shown that for the stability and convergence of the scheme (16.1) we need $r \leq 1$. You need not derive this result; we merely ask you to interpret it graphically in terms of the concept of the domain of influence.

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# Chapter 19 Review

The wave and diffusion equations are similar in several ways:

1. For both the one-dimensional wave and diffusion equations the solution domain is, typically, a semi-infinite strip ($0 < x < L, 0 < t < \infty$), the quarter plane $0 < x < \infty, 0 < t < \infty$, or the half plane $-\infty < x < \infty, 0 < t < \infty$.

2. For both equations we can use the method of separation of variables, a Laplace transform on the $t$ variable, or a Fourier transform on the $x$ variable if the $x$ domain is $-\infty < x < \infty$.

3. When solving by separation of variables, the boundary conditions are to be applied before the initial conditions.

But the wave and diffusion equations also differ in some ways:

1. A single initial condition $u(x, 0) = f(x)$ is appropriate for the diffusion equation $\alpha^2 u_{xx} = u_t$, whereas the two initial conditions $y(x, 0) = f(x)$ and $y_t(x, 0) = g(x)$ are appropriate for the wave equation $c^2 y_{xx} = y_{tt}$.

2. Only for the wave equation do we find a *general solution*, namely,

$$y(x, t) = F(x - ct) + G(x + ct), \tag{1}$$

where $F$ and $G$ are arbitrary (twice differentiable) functions. The graphs of $F(x - ct)$ and $G(x + ct)$, plotted versus $x$, translate rightward and leftward, respectively, with speed $c$. Of course, $F$ and $G$ need not be single terms. For example, in the solution

$$y(x, t) = \frac{1}{2}\sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L}(x - ct) + \frac{1}{2}\sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{L}(x + ct) \tag{2}$$

of the problem

$$c^2 y_{xx} = y_{tt}, \qquad (-\infty < x < \infty,\ 0 < t < \infty) \tag{3a}$$
$$y(x,0) = f(x),\ \ y_t(x) = 0, \qquad (-\infty < x < \infty) \tag{3b}$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx,$$

each of the functions

$$F(x-ct) = \frac{1}{2}\sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{L}(x-ct), \tag{4a}$$

$$G(x+ct) = \frac{1}{2}\sum_{n=1}^{\infty} a_n \sin\frac{n\pi}{L}(x+ct) \tag{4b}$$

is a superposition of individual *traveling waves*. In this case the trigonometric identities $\sin(A \pm B) = \sin A\cos B \pm \sin B\cos A$ reduce (2) to the form

$$y(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}, \tag{5}$$

which is a superposition of *standing waves*. Thus, the wave equation inevitably gives traveling waves, by virtue of (1), and in some cases these traveling waves sum to standing waves. The variables $\xi = x - ct$ and $\eta = x + ct$ give two families of *characteristics* in the $x, t$ plane, the $\xi$ = constant and $\eta$ = constant lines, along which information propagates as can be seen from (1) because $F(x-ct)$ is constant along $x - ct$ = constant lines and $G(x+ct)$ is constant along $x + ct$ = constant lines.

3. Whereas diffusion is a smoothing process, we see in Chapter 19 that kinks and discontinuities in the initial conditions propagate into the $x, t$ solution domain indefinitely, along characteristics.

The *two*-dimensional wave equation

$$c^2(w_{xx} + w_{yy}) = w_{tt}, \tag{6}$$

such as governs the displacement $w(x,y,t)$ of a vibrating drumhead, can be solved by separation of variables by seeking

$$w(x,y,t) = W(x,y)T(t) \tag{7}$$

and obtaining

$$W_{xx} + W_{yy} + \kappa^2 W = 0, \tag{8a}$$
$$T'' + \kappa^2 c^2 T = 0, \tag{8b}$$

where (8a) is the two-dimensional *Helmholtz equation*. In turn, (8a) can be separated by seeking $W(x, y) = X(x)Y(y)$, or we could seek $w(x, y, t) = X(x)Y(y)T(t)$ right from the start. The result is a *double series*, and each initial condition $[w(x, y, 0) = f(x, y),\ w_t(x, y, 0) = g(x, y)]$ leads to a double Fourier series expansion. The same method can be applied to the two-dimensional diffusion equation $\alpha^2(u_{xx} + u_{yy}) = u_t$, the chief difference being that for the wave equation the T equation (8b) gives the oscillatory solutions $\cos \kappa ct$ and $\sin \kappa ct$, whereas for the diffusion equation the $T$ equation $T' + \kappa^2\alpha^2 T = 0$ gives the exponential decay $\exp(-\kappa^2\alpha^2 t)$.

Finally, note that in this chapter on the wave equation there is no section on numerical solution analogous to Section 18.6 on the numerical solution of the diffusion equation by the method of finite differences. The finite-difference method can indeed be applied to the wave equation, but if the initial conditions $y(x, 0) = f(x)$ and $y_t(x, 0) = g(x)$ are not smooth functions, then inaccuracies result from the discontinuities that propagate into the solution domain. In that case it is better to use the **method of characteristics**, which uses the more natural characteristic variables $\xi$ and $\eta$ in place of $x$ and $t$. Essentially, the calculation is carried out along the characteristics.*

*For an introduction to the method of characteristics see G. D. Smith, *Numerical Solution of Partial Differential Equations* (New York: Oxford University Press, 1965) or, for a more complete discussion, see E. Zauderer, *Partial Differential Equations of Applied Mathematics* (New York: Wiley, 1983).

# Chapter 20

# Laplace Equation

## 20.1 Introduction

We have already encountered the **Laplace equation**

$$\boxed{\nabla^2 u = 0} \tag{1}$$

in Chapter 16, as well as its nonhomogeneous version, the **Poisson equation**

$$\boxed{\nabla^2 u = f,} \tag{2}$$

where $f$ is a "source" function that is prescribed over the region in question. Recall, for example, the unsteady diffusion equation

$$\alpha^2 \nabla^2 u = u_t - F(x, y, z, t) \tag{3}$$

governing the temperature field $u(x, y, z, t)$, in which $F(x, y, z, t)$ is a heat source distribution. If $F$ does not vary with $t$ and if there exists a steady-state solution $u(x, y, z)$, then the latter satisfies the Poisson equation $\nabla^2 u = -F(x, y, z)/\alpha^2$. If there is no heat source distribution [i.e., if $F(x, y, z) = 0$], then the steady-state temperature distribution $u(x, y, z)$ satisfies the Laplace equation (1).

As a second example observe that the electric potential (i.e., the voltage) field $\Phi(x, y, z)$ is governed by the Poisson equation

$$\nabla^2 \Phi = -\frac{1}{\epsilon} q(x, y, z), \tag{4}$$

where $q(x, y, z)$ is the charge density distribution (which serves as a "source" for the electric potential) and $\epsilon$ is a physical constant known as the permittivity of the medium. If $q(x, y, z) = 0$ in the region, then $\Phi$ satisfies the Laplace equation $\nabla^2 \Phi = 0$. Since the presence of an electric field is solely attributable to the presence of charges, how can there be anything other than the trivial solution $\Phi(x, y, z) = 0$ in the event that $q(x, y, z) = 0$? The answer is that there may be charges *outside of* the region under consideration or on its boundary.

Finally, recall from Example 3 of Section 16.10 that the velocity potential $\Phi(x, y, z)$ for any irrotational incompressible fluid flow is governed by the Laplace equation

$$\nabla^2\Phi = 0,$$

where $\Phi$ is related to the velocity field $\mathbf{v}(x, y, z)$ by the formula $\mathbf{v} = \nabla\Phi$.

Emphasis in this chapter is on the Laplace equation, with the Poisson equation considered only within the exercises. Like Chapters 18 and 19, Chapter 20 is organized according to the various methods of solution. In Sections 20.2 and 20.3 we study the solution of the Laplace equation by separation of variables – using Cartesian coordinates in Section 20.2 and non-Cartesian coordinates in Section 20.3. Solution of certain problems by the Fourier transform is covered in Section 20.4, and the numerical finite difference method is the subject of Section 20.5.

## 20.2 Separation of Variables; Cartesian Coordinates

We limit our attention in this chapter to *two*-dimensional problems, so the domain $\mathcal{D}$ is some part of the $x, y$ plane. For the method of separation of variables to work, $\mathcal{D}$ must be bounded by constant-coordinate curves, so if we use the Cartesian coordinates $x, y$, then the generic domain is a rectangle, bounded by constant-$x$ and constant-$y$ lines.

**EXAMPLE 1.** *Dirichlet Problem for Rectangle.* Consider the boundary-value problem

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \quad \text{in } \mathcal{D}, \tag{1a}$$

$$u(0, y) = 0, \quad (0 < y < b) \tag{1b}$$

$$u(a, y) = f(y), \quad (0 < y < b) \tag{1c}$$

$$u(x, 0) = u(x, b) = 0, \quad (0 < x < a) \tag{1d}$$

where $\mathcal{D}$ is the rectangle shown in Fig. 1. Since all of the boundary conditions are of Dirichlet type (i.e., where $u$ is given), we call (1) a **Dirichlet problem**.

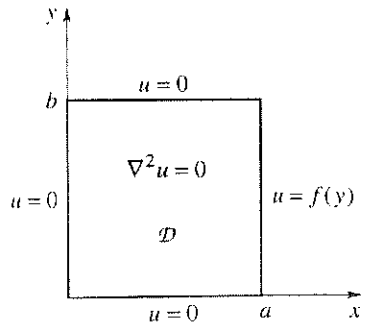



**Figure 1.** The Dirichlet problem (1).

To solve by separation of variables, seek

$$u(x, y) = X(x)Y(y). \tag{2}$$

Putting (2) into (1a) and separating the variables gives

$$\frac{X''}{X} = -\frac{Y''}{Y} = \text{constant} = \kappa^2, \tag{3}$$

so

$$X'' - \kappa^2 X = 0, \tag{4a}$$

$$Y'' + \kappa^2 Y = 0, \tag{4b}$$

and

$$X(x) = \begin{cases} A + Bx, & \kappa = 0 \\ C\cosh\kappa x + D\sinh\kappa x, & \kappa \neq 0 \end{cases} \tag{5a}$$

$$Y(y) = \begin{cases} E + Fy, & \kappa = 0 \\ G\cos\kappa y + H\sin\kappa y, & \kappa \neq 0 \end{cases} \tag{5b}$$

Why did we choose the constant in (3) as $+\kappa^2$ rather than as $-\kappa^2$? Because, looking ahead to the application of the boundary conditions we anticipate the eventual Fourier series expansion of $f(y)$. The choice $+\kappa^2$ does indeed lead to the $\cos\kappa y$ and $\sin\kappa y$ solutions in (5b) that will be needed for that Fourier series expansion; $-\kappa^2$ would have led to $\cosh\kappa y$ and $\sinh\kappa y$ instead.

Because the Laplace equation $\nabla^2 u = 0$ is linear, we can use superposition and combine the $\kappa = 0$ and $\kappa \neq 0$ product solutions as

$$\begin{aligned} u(x,y) = {} & (A + Bx)(E + Fy) \\ & +(C\cosh\kappa x + D\sinh\kappa x)(G\cos\kappa y + H\sin\kappa y). \end{aligned} \tag{6}$$

Saving the boundary condition at $x = a$ for last, we first apply the boundary conditions on the edges $y = 0$, $y = b$, and $x = 0$:

$$u(x,0) = 0 = (A + Bx)E + (C\cosh\kappa x + D\sinh\kappa x)G,$$

so (to retain as robust a solution as possible) set $E = 0$ (rather than $A = 0$ and $B = 0$) and $G = 0$ (rather than $C = 0$ and $D = 0$). Then (6) becomes

$$u(x,y) = (I + Jx)y + (P\cosh\kappa x + Q\sinh\kappa x)\sin\kappa y, \tag{7}$$

where we have combined $AF$ as $I$, $BF$ as $J$, $CH$ as $P$, and $DH$ as $Q$ for brevity. Next,

$$u(x,b) = 0 = (I + Jx)b + (P\cosh\kappa x + Q\sinh\kappa x)\sin\kappa b \tag{8}$$

gives $I = 0$, $J = 0$, and $\sin\kappa b = 0$. Hence,

$$\kappa = n\pi/b \quad (n = 1, 2, \ldots)$$

so, with the help of superposition, we can update (7) as

$$u(x,y) = \sum_{n=1}^{\infty}\left(P_n\cosh\frac{n\pi x}{b} + Q_n\sinh\frac{n\pi x}{b}\right)\sin\frac{n\pi y}{b}. \tag{9}$$

Next, the "western" boundary condition (i.e., at $x = 0$) gives

$$u(0,y) = 0 = \sum_{n=1}^{\infty} P_n\sin\frac{n\pi y}{b}, \qquad (0 < y < b) \tag{10}$$

which is satisfied by setting $P_n = 0$ for $n = 1, 2, \ldots$. Thus, (9) becomes

$$u(x,y) = \sum_{n=1}^{\infty} Q_n\sinh\frac{n\pi x}{b}\sin\frac{n\pi y}{b}. \tag{11}$$

Finally, the eastern boundary condition

$$u(a, y) = f(y) = \sum_{n=1}^{\infty} Q_n \sinh \frac{n\pi a}{b} \sin \frac{n\pi y}{b} \qquad (0 < y < b) \tag{12}$$

is seen to be a half-range sine series so

$$Q_n \sinh \frac{n\pi a}{b} = \frac{2}{b} \int_0^b f(y) \sin \frac{n\pi y}{b}\, dy,$$

or

$$Q_n = \frac{2}{b \sinh \dfrac{n\pi a}{b}} \int_0^b f(y) \sin \frac{n\pi y}{b}\, dy. \tag{13}$$

The solution to (1) is given by (11) and (13).

COMMENT 1. We stated that (10) is satisfied by setting $P_n = 0$ for $n = 1, 2, \ldots$. Obviously that is true, but there is a subtle question here: *must* the $P_n$'s be zero? That is, might the $\sin(n\pi y/b)$ terms cancel to zero on $0 < y < b$ without all the $P_n$'s being zero? Surely the $\sin(n\pi y/b)$ terms are linearly independent on $0 < y < b$, and if a *finite* linear combination of linearly independent functions is zero, then each coefficient must be zero. However, the sum in (10) is an infinite series, not a linear combination of a finite number of terms. The cleanest way to handle (10) is to see that it is actually a half-range sine expansion of the function $u(0, y) = 0$. Thus,

$$P_n = \frac{2}{b} \int_0^b (0) \sin \frac{n\pi y}{b}\, dy = 0,$$

as stated.

COMMENT 2. To make our results more concrete, consider a specific $f(y)$. For simplicity, let $f(y)$ be a constant, say

$$f(y) = 100.$$

Then (13) gives

$$Q_n = \frac{400}{n\pi \sinh \dfrac{n\pi a}{b}}$$

for $n = 1, 3, \ldots$ and 0 for $n = 2, 4, \ldots$, so (11) becomes

$$u(x, y) = \frac{400}{\pi} \sum_{n=1,3,\ldots}^{\infty} \frac{1}{n} \frac{\sinh(n\pi x/b)}{\sinh(n\pi a/b)} \sin \frac{n\pi y}{b}. \tag{14}$$

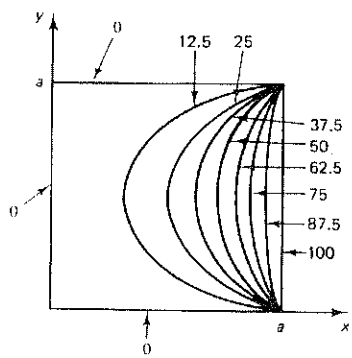



**Figure 2.** Selected isotherms for $f(y) = 100$ and $b = a$.

One useful way of presenting such two-dimensional results is to plot a number of $u =$ constant curves, isothermal curves if we consider (1) in the context of steady-state heat conduction. We have done so in Fig. 2 for the case where $b = a$. The $u = 0$ isotherm is the northern, western, and southern boundary; the $u = 100$ isotherm is the eastern edge; and all other isotherms spring from the corners $(a, 0)$ and $(a, a)$. As implied by Fig. 2, all $u$ values within the rectangle are between the minimum and maximum values of $u$ on the boundary, namely, 0 and 100, respectively. This result illustrates the important and

fundamental **maximum principle** of potential theory (i.e., the theory associated with the Laplace equation), which states that if the Laplace equation $\nabla^2 u = 0$ holds on a domain $\mathcal{D}$, then the maximum (and minimum) values of $u$ occur on the boundary of $\mathcal{D}$, not in its interior. This result is proved in the next section.

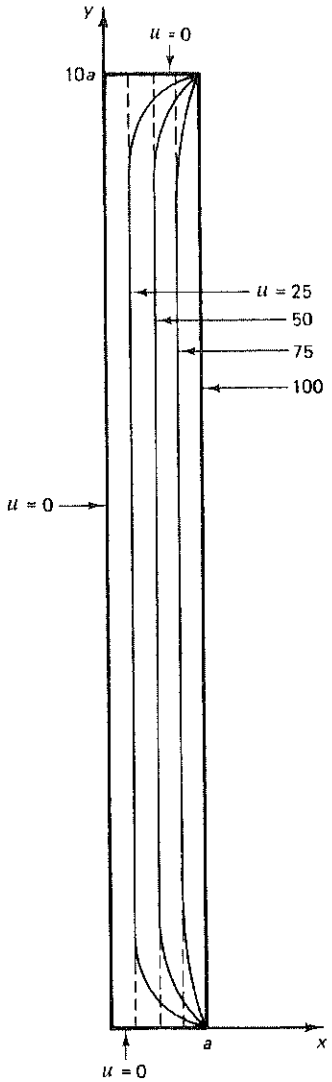



**Figure 3.** High-aspect-ratio case.

COMMENT 3. We have also plotted the isotherms corresponding to (14) for the high-aspect-ratio case where $b = 10a$, again with $f(y) = 100$ (Fig. 3). Observe that over $2a < y < 8a$, say, the problem is essentially one-dimensional, with $u$ varying with $x$ but hardly at all with $y$. If, accordingly, we neglect the $\partial^2 u/\partial y^2$ term in the Laplace equation, then we have the problem

$$\frac{\partial^2 u}{\partial x^2} \approx 0; \qquad u\Big|_{x=0} = 0, \;\; u\Big|_{x=a} = 100, \tag{15}$$

with solution

$$u(x,y) \approx 100\frac{x}{a}, \tag{16}$$

the isotherms of which are the lines $x$ = constant in Fig. 3. We see that (16) is an excellent approximation to $u(x,y)$ except near the ends, that is, except within one or two widths (the width being $a$) of the ends $y = 0$ and $y = 10a$. Indeed, if we are interested in the solution only within $2a < y < 8a$, say, then the simpler one-dimensional model and its solution (16) may well suffice in place of the two-dimensional model and the more cumbersome solution (14). Of course, (16) does not satisfy the boundary conditions $u(x,0) = 0$ and $u(x,10a) = 0$, so there are regions of adjustment near those two ends, where the approximate solution (16) needs to be blended with the boundary conditions $u(x,0) = 0$ and $u(x,10a) = 0$ in a way that satisfies the Laplace equation.

The results found in this example hold in general; namely, we can expect end effects to be significant only within one or two widths of the end. ∎

Remember that in Chapters 18 and 19 we always chose the separation constant to be $-\kappa^2$, and that we always applied the boundary conditions before the initial condition. For the Laplace equation, however, the sign of the separation constant and the sequencing of the boundary conditions needs to be decided on a case-by-case basis. We offer this rule of thumb as guidance:

*Anticipating the edge along which the eventual Fourier series will take place, choose the $+\kappa^2$ or $-\kappa^2$ so as to obtain oscillatory functions along that edge. Then, apply the boundary conditions adjacent to that edge first.*

For instance, in Example 1 we can anticipate that it is the boundary condition $u(a,y) = f(y)$ that will require the Fourier series, so we choose $+\kappa^2$ in (3) so as to obtain the oscillatory solutions $Y(y) = \cos\kappa y$ and $\sin\kappa y$ in the $y$ variable. Then, we apply the boundary conditions on the northern and southern edges, which are adjacent to that edge. Similarly in Chapters 18 and 19. There, the Fourier series is always along the southern edge in the $x,t$ plane (i.e., along the edge $t = 0$) so we chose $-\kappa^2$ in order to obtain $\cos\kappa x$ and $\sin\kappa x$; we applied the boundary conditions on the adjacent edges $x = 0$ and $x = L$ first, and on the edge $t = 0$ last.

What if the boundary conditions are, in place of those in Fig. 1, $u(0,y) = p(y)$, $u(a,y) = f(y)$, $u(x,0) = q(x)$, and $u(x,b) = g(x)$? Evidently *each* edge

will require a Fourier series expansion, so how can we apply the adjacent boundary conditions first if they require Fourier series expansions themselves? This difficulty can be resolved by using the concepts of linearity and superposition to break the problem into four simpler ones as indicated schematically in Fig. 4. The idea is to solve each of the four problems on the right along the lines indicated in Example 1 (using $+\kappa^2$ for the first and third problems and $-\kappa^2$ for the second and fourth) and then obtain $u$ as the sum

$$u(x, y) = u_1(x, y) + u_2(x, y) + u_3(x, y) + u_4(x, y). \tag{17}$$

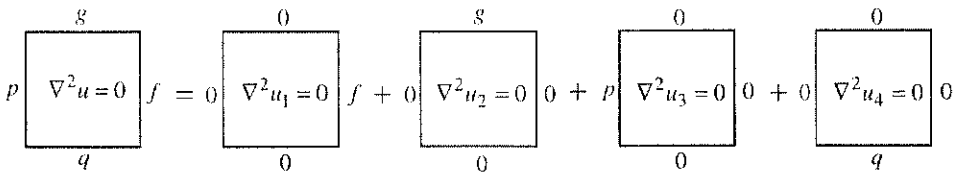



**Figure 4.** Use of superposition.

To see that (17) is true, add the equations $\nabla^2 u_1 = 0$, $\nabla^2 u_2 = 0$, $\nabla^2 u_3 = 0$, and $\nabla^2 u_4 = 0$, and obtain

$$\nabla^2 u_1 + \nabla^2 u_2 + \nabla^2 u_3 + \nabla^2 u_4 = 0. \tag{18}$$

But since the operator $\nabla^2$ is linear, it follows from (18) that

$$\nabla^2(u_1 + u_2 + u_3 + u_4) = 0,$$

so $u = u_1 + u_2 + u_3 + u_4$ does satisfy the Laplace equation, as required. Turning to the boundary conditions, consider the eastern condition. From (17) and the conditions imposed on $u_1, u_2, u_3$, and $u_4$,

$$\begin{aligned} u(a, y) &= u_1(a, y) + u_2(a, y) + u_3(a, y) + u_4(a, y) \\ &= f(y) + 0 + 0 + 0 \\ &= f(y), \end{aligned}$$

as required. Similarly for the other boundary conditions.

**EXAMPLE 2.** *Semi-Infinite Strip.* Consider the Dirichlet problem

$$\nabla^2 u = u_{xx} + u_{yy} = 0 \quad \text{in } \mathcal{D}, \tag{19a}$$

$$u(0, y) = 20, \quad u(5, y) = 50, \quad (0 < y < \infty) \tag{19b}$$

$$u(x, 0) = f(x), \quad (0 < x < 5) \tag{19c}$$

$$u(x, y) \quad \text{bounded as} \quad y \to \infty, \tag{19d}$$

where $\mathcal{D}$ is the semi-infinite strip $0 < x < 5$, $0 < y < \infty$ (Fig. 5).

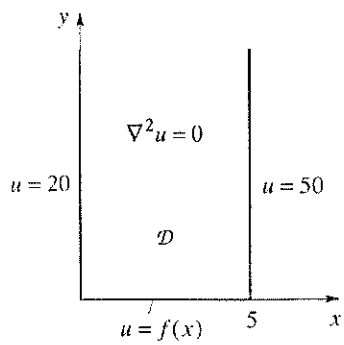



**Figure 5.** Semi-infinite strip.

Since physical objects are of finite size, why might we be interested in a semi-infinite strip that extends to $y = \infty$? The actual physical object might, for instance, be a slender

"fin" attached to a thick base as shown in Fig. 6. Then the problem is actually defined on a complicated three-dimensional domain. However, suppose that our interest is not in finding the temperature field everywhere in the object but only near the end of the fin, within the rectangle $ABCE$. And suppose that we know the boundary conditions $u = 20$ along $AE$, $u = 50$ along $BC$, and $u = f(x)$ along $EC$. We could take our domain $\mathcal{D}$ to be the rectangle $ABCE$, but we do not know a boundary condition along $AB$. To within a good approximation, we render the problem tractable by letting $\mathcal{D}$ be the entire semi-infinite strip shown in Fig. 5, with the boundedness condition (19d) as our missing boundary condition at $y = \infty$. Based on Comment 3 of Example 1, we expect the difference between the actual temperature field and the one defined by (19) to be very small within the region of interest, $ABCE$.

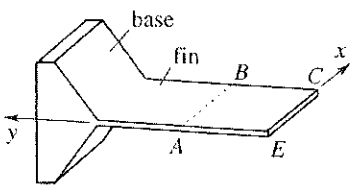



**Figure 6.** The actual object.

We anticipate that the eventual Fourier series expansion will be a half- or quarter-range expansion on the edge $y = 0$. Thus, to obtain oscillatory functions of $x$ (namely $\cos \kappa x$ and $\sin \kappa x$) rather than of $y$, write

$$\frac{X''}{X} = -\frac{Y''}{Y} = -\kappa^2 \tag{20}$$

in place of (3). Solving the resulting ODE's on $X$ and $Y$ and superimposing the $\kappa = 0$ and $\kappa \neq 0$ solutions gives

$$\begin{aligned} u(x,y) &= (A + Bx)(E + Fy) \\ &\quad + (C \cos \kappa x + D \sin \kappa x)(Ge^{\kappa y} + He^{-\kappa y}). \end{aligned} \tag{21}$$

Apply the boundedness condition (19d) first. Since the $y$ and $e^{\kappa y}$ terms in (21) grow unboundedly as $y \to \infty$, we must set $F = 0$ and $G = 0$ to eliminate those terms. Then (21) becomes

$$u(x,y) = I + Jx + (P \cos \kappa x + Q \sin \kappa x)e^{-\kappa y}, \tag{22}$$

where we have combined $AE$ as $I$, $BE$ as $J$, $CH$ as $P$, and $DH$ as $Q$. Since we anticipate the Fourier expansion to be on the $y = 0$ edge, we save that boundary condition for last. Next,

$$u(0,y) = 20 = I + Pe^{-\kappa y}, \tag{23}$$

and matching the coefficients of the (linearly independent) constant and $e^{-\kappa y}$ terms on both sides of (23) gives $20 = I$ and $0 = P$. Using these results to update (22) gives

$$u(x,y) = 20 + Jx + Q \sin \kappa x \, e^{-\kappa y}. \tag{24}$$

Next,

$$u(5,y) = 50 = 20 + 5J + Q \sin 5\kappa \, e^{-\kappa y}, \tag{25}$$

so $50 = 20 + 5J$ and $\sin 5\kappa = 0$. Thus, $J = 6$ and $\kappa = n\pi/5$ ($n = 1, 2, \ldots$). Putting these results into (24) we have, with the help of superposition,

$$u(x,y) = 20 + 6x + \sum_{n=1}^{\infty} Q_n \sin \frac{n\pi x}{5} e^{-n\pi y/5}. \tag{26}$$

Finally,

$$u(x,0) = f(x) = 20 + 6x + \sum_{n=1}^{\infty} Q_n \sin \frac{n\pi x}{5} \tag{27}$$

or, moving the (known) $20 + 6x$ terms to the left-hand side,

$$f(x) - 20 - 6x = \sum_{n=1}^{\infty} Q_n \sin \frac{n\pi x}{5}. \qquad (0 < x < 5) \tag{28}$$

The latter is a half-range sine expansion of $f(x) - 20 - 6x$, so we can compute the $Q_n$'s from

$$Q_n = \frac{2}{5} \int_0^5 [f(x) - 20 - 6x] \sin \frac{n\pi x}{5}\, dx. \tag{29}$$

Thus, the solution to (19) is given by (26), with the $Q_n$'s computed according to (29).

COMMENT 1. The $\kappa = 0$ term $(A + Bx)(E + Fy)$ in (21) contributes the $20 + 6x$ part of the final solution (26). The graph of that part of the solution is like a ramp, from $u = 20$ along the $x = 0$ edge to $u = 50$ along the $x = 5$ edge. In the language of Chapter 18 we can think of $20 + 6x$ as the "steady-state" solution, the solution that is approached as $y \to \infty$ (analogous to the limit $t \to \infty$ in Chapter 18), and we can think of the series in (26) as the "transient" that blends the "steady-state" $20 + 6x$ with the "initial condition" $u(x, 0) = f(x)$. We see from (26) that the transient part, or *end effect*, dies out exponentially with $y$; with $n = 1$, the $\exp(-n\pi y/5)$ factor is merely 0.043 at $y = 5$ and 0.0019 at $y = 10$, and with $n = 2, 3, \ldots$ it is even smaller. Only if $f(x)$ happens to equal $20 + 6x$ does the end effect vanish entirely, for then all the $Q_n$'s are zero.

COMMENT 2. In (21) we wrote $Ge^{\kappa y} + He^{-\kappa y}$ but could have written $R \cosh \kappa y + S \sinh \kappa y$, say, instead. The choice is immaterial since the two forms are equivalent, but the former is more convenient regarding the application of the boundedness condition, for it is clear that $e^{\kappa y}$ is a "bad" term (unbounded) and that $e^{-\kappa y}$ is a "good" term (bounded) so we set $G = 0$. Working with $\cosh \kappa y$ and $\sinh \kappa y$ instead would be awkward because *both* are unbounded (Fig. 7). However,

$$\begin{aligned} R \cosh \kappa y + S \sinh \kappa y &= \frac{R}{2}\left(e^{\kappa y} + e^{-\kappa y}\right) + \frac{S}{2}\left(e^{\kappa y} - e^{-\kappa y}\right) \\ &= \frac{R+S}{2} e^{\kappa y} + \frac{R-S}{2} e^{-\kappa y}, \end{aligned} \tag{30}$$

so we can arrange for the unbounded parts of the $\cosh \kappa y$ and $\sinh \kappa y$ to cancel by choosing $S = -R$, in which case (30) reduces to $Re^{-\kappa y}$. Thus, we arrive at the same place but the trip is more arduous.

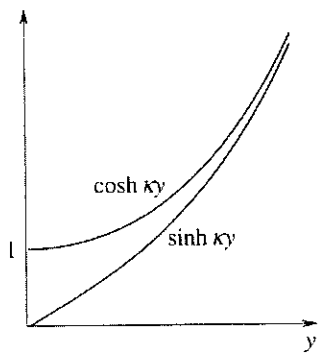



**Figure 7.** $\cosh \kappa y$ and $\sinh \kappa y$.

COMMENT 3. Since we had three nonzero boundary conditions (Fig. 5), why did we not break the problem into three sub-problems along the lines indicated in Fig. 4? We could have but did not need to because the boundary conditions along the edges $x = 0$ and $x = 5$ are merely constants and can therefore be handled by the $(A + Bx)(E + Fx)$ *ramp term* in the solution.

COMMENT 4. In these first two examples the Fourier expansions happened to be half-range sine series, but that will not always be the case and will depend on the boundary conditions. For instance, if we change the Dirichlet boundary condition $u(0, y) = 20$ to a Neumann boundary condition such as $u_x(0, y) = 3$, then in place of the half-range sine expansion (28) we would have a quarter-range cosine expansion. ∎

**Closure.** In this section we study the separation-of-variable solution of the two-dimensional Laplace equation $u_{xx} + u_{yy} = 0$ on domains bounded by constant-$x$ and constant-$y$ lines. Unlike Chapters 18 and 19, where we always take the separation constant to be $-\kappa^2$ and apply the boundary conditions at $x = 0$ and $x = L$ before the initial condition(s) at $t = 0$, here we need to choose the sign of $\kappa^2$ and the sequence of application of the boundary conditions on a case-by-case basis. The rule of thumb is to choose the separation constant as $+\kappa^2$ or $-\kappa^2$ so as to give oscillatory solutions (cosines and sines) along the edge where the eventual Fourier expansion will take place, and then to be sure to apply the boundary conditions adjacent to that edge first. (For example, if the Fourier expansion is on the eastern edge, then by the "adjacent" edges we mean the northern and southern ones.)

Consider the general Dirichlet problem on $u$ shown in Fig. 4. If all of the functions $p(y), g(x), f(y)$, and $q(x)$ are nonconstant, then we can break the problem into four sub-problems, as shown in the figure, and solve separately for $u_1(x, y)$, $u_2(x, y), u_3(x, y)$, and $u_4(x, y)$. For the $u_1$ and $u_3$ problems, set $X''/X = -Y''/Y = +\kappa^2$, so as to obtain cosine and sine solutions in the $y$ variable, and apply the southern and northern boundary conditions before attempting the nonhomogeneous boundary condition (eastern in the $u_1$ problem, western in the $u_3$ problem). In fact, you will find that if the boundary conditions on any two opposite boundaries are constants, then it is not necessary to break the problem down as we do in Fig. 4 (although we can if we wish). For instance, suppose that both $p(y)$ and $f(y)$ are constants. Then set $X''/X = -Y''/Y = -\kappa^2$ to obtain cosine and sine solutions in the $x$ variable, and apply the western and eastern boundary conditions first.

Similar statements apply if boundary conditions of Neumann or Robin type are present on one or more edges, but be careful because if all four edges have Neumann boundary conditions, then there may be no solution or a nonunique solution (Exercises 17 and 18).

In Example 2 we consider the domain to be an idealized semi-infinite strip and adopt a boundedness condition in lieu of the missing boundary condition at $y = \infty$. The Fourier expansion is necessarily on the finite edge (the edge $y = 0$ in Example 2). Given the choice between expressing $Y$ in terms of $\cosh \kappa y$ and $\sinh \kappa y$ or in terms of $e^{\kappa y}$ and $e^{-\kappa y}$, we choose the latter because of convenience in regard to the application of the boundedness condition.

Finally, and very important, is the fact that the Laplace and Poisson equations arise in the context of **boundary-value** problems; that is, a boundary condition is supplied on each of the four edges. In contrast, the diffusion and wave problems studied in Chapters 18 and 19 are of initial-value type (with respect to the $t$ variable) since conditions are given at $t = 0$ but not at a final time or at $t = \infty$. This boundary-value nature will be felt more acutely when we use the finite-difference method in Section 20.5.

## EXERCISES 20.2

**1.** Solve $u_{xx}+u_{yy}=0$ in the rectangle $0<x<3, 0<y<2$ by separation of variables, subject to the given boundary conditions. $H$ denotes the Heaviside function.

(a) $u(0,y) = u(x,2) = u(3,y) = 0,\ u(x,0) = 50\sin(\pi x/3)$
(b) $u(0,y) = u(x,0) = u(3,y) = 0,\ u(x,2) = 10\sin(\pi x/3) - 4\sin \pi x$
(c) $u(x,0) = u(3,y) = u(x,2) = 0,\ u(0,y) = 5\sin \pi y + 4\sin 2\pi y - \sin 3\pi y$
(d) $u(0,y) = u(x,2) = u(3,y) = 0,\ u(x,0) = 50H(x-2)$
(e) $u_x(0,y) = u(x,2) = u(3,y) = 0,\ u(x,0) = 50H(x-2)$
(f) $u(0,y) = u(x,2) = u_x(3,y) = 0,\ u(x,0) = 50H(x-2)$
(g) $u_x(0,y) = u(x,2) = u_x(3,y) = 0,\ u(x,0) = 50H(x-2)$
(h) $u_y(x,2) = u(3,y) = u(x,0) = 0,\ u(0,y) = H(y-1)$
(i) $u(0,y) = u(x,0) = u(3,y) = 0,\ u(x,2) = 5x$
(j) $u(0,y) = u(x,2) = u(3,y) = 0,\ u(x,0) = 5[H(x-1) - H(x-2)]$
(k) $u(x,2) = u(3,y) = u(x,0) = 0,\ u_x(0,y) = 5\sin 3\pi y$
(l) $u(x,2) = u(3,y) = u(x,0) = 0,\ u_x(0,y) = 20$
(m) $u(x,2) = u(3,y) = u_y(x,0) = 0,\ u_x(0,y) = 20$
(n) $u_y(x,2) = u(3,y) = u_y(x,0) = 0,\ u_x(0,y) = 20$

**2.** (a) The solution to Exercise 1(d) is given in the Answers to Selected Exercises. Using that solution and computer software, evaluate $u(2.5,1)$, $u(2.5,0.5)$, $u(2.5,0.2)$, and $u(2.5,0.1)$, correct to two decimal places. In each case, tell how many terms must be summed to achieve that accuracy. Explain why more terms are needed as the point approaches the $x$ axis.
(b) The same as part (a), but using the solution to Exercise 1(h).
(c) In Exercise 1(e) evaluate $u(1,1)$ to three significant figures.
(d) In Exercise 1(e) evaluate $u(0,y)$ at $y = 0.25, 0.5, 0.75, \ldots, 1.75$, to three significant figures, and plot $u(0,y)$ versus $y$, by hand or by computer.

**3.** Solve $u_{xx}+u_{yy}=0$ in the square $0<x<2$, $0<y<2$ by separation of variables, subject to the given boundary conditions. (You should be able to obtain the solution in closed form.) Then, obtain a computer plot of the $u = 10, 20, 30, \ldots, 90$ isotherms using software such as the *Maple* implicitplot command. NOTE: We urge you to try sketching the isotherms even before you solve the problem.

(a) $u(0,y) = u(x,2) = u(2,y) = 0,\ u(x,0) = 100\sin(\pi x/2)$
(b) $u(0,y) = u(2,y) = 0,\ u(x,2) = u(x,0) = 100\sin(\pi x/2)$
(c) $u(0,y) = u(x,2) = 0,\ u(2,y) = 100\sin(\pi y/2),\ u(x,0) = 100\sin(\pi x/2)$
(d) $u(0,y) = u(x,2) = 0,\ u_x(2,y) = 0,\ u(x,0) = 100\sin(\pi x/4)$
(e) $u(0,y) = u(x,2) = 0,\ u(x,0) = 100\sin(\pi x/2),\ u(2,y) = 20\sin(\pi y/2)$
(f) $u(0,y) = u(2,y) = 0,\ u(x,0) = 100\sin(\pi x/2),\ u(x,2) = 20\sin(\pi x/2)$
(g) $u(0,y) = u(2,y) = 100\sin(\pi y/2),\ u(x,0) = u(x,2) = 100\sin(\pi x/2)$

**4.** Solve $u_{xx}+u_{yy}=0$ in the rectangle $0<x<a$, $0<y<b$ subject to the boundary conditions $u(0,y)=p(y)$, $u(x,b)=u_2$, $u(a,y)=f(y)$, $u(x,0)=u_1$ *without* breaking the problem into sub-problems; $p(y)$ and $f(y)$ are prescribed functions and $u_1$ and $u_2$ are prescribed constants. HINT: Read the second paragraph of the closure.

**5.** Same as Exercise 4, but with these boundary conditions:

(a) $u(0,y) = u_1,\ u(x,b) = p(x),\ u(a,y) = u_2,\ u(x,0) = f(x)$
(b) $u_x(0,y) = p(y),\ u(x,b) = u_1,\ u(a,y) = f(y),\ u(x,0) = u_2$
(c) $u_x(0,y) = p(y),\ u(x,b) = u_1,\ u_x(a,y) = f(y),\ u(x,0) = u_2$

**6.** Solve $u_{xx}+u_{yy}=0$ in the rectangle $0<x<a, 0<y<b$ subject to the boundary conditions $u(x,0)=u_1$, $u(x,b)=u_2$, $u(0,y)=u_3$, $u(a,y)=u_4$, where $u_1,\ldots,u_4$ are constants. Do not break the problem into subproblems; you don't need to. Rather, choose $X''/X = -Y''/Y = +\kappa^2$ and apply the southern and northern boundary conditions first. (You may leave expansion coefficients in integral form.) Next, solve the problem again, this time choosing $X''/X = -Y''/Y = -\kappa^2$ and applying the western and eastern boundary conditions first. Your two solutions will look different but will merely be two different representations of the same function. If $b = 10a$, which of the two solution forms would you prefer – for purposes of calculation? Explain your reasoning.

**7.** Let $f(y) = 100$ in (1c), and let $b = a$. Without solving the problem (or using the solution in the text), show that $u(a/2, a/2) = 25$. HINT: Let $p(y) = g(x) = f(y) = q(x) = 100$ in Fig. 4.

**8.** To promote physical "intuition," we ask you to draw a neat, labeled sketch of representative isotherms for the problem con-

sisting of the Laplace equation on the square $0 < x < a$, $0 < y < a$ with the given boundary conditions.

(a) $u(0,y) = u(x,a) = u(a,y) = 0$, $u(x,0) = 100\sin(\pi x/a)$
(b) $u(0,y) = u(x,a) = u_x(a,y) = 0$, $u(x,0) = 100\sin(\pi x/2a)$
(c) $u(0,y) = u(x,0) = 0$, $u(a,y) = u(x,a) = 20$
(d) $u(0,y) = u(x,0) = u(a,y) = 0$, $u(x,a) = 100$ for $0 < x < a/2$ and 0 for $a/2 < x < a$
(e) $u(0,y) = u(x,0) = 0$, $u(x,a) = 0$, $u(a,y) = 20$

**9.** Show that the solution (11) of the problem (1) can be expressed in the form of an integration over the boundary data, namely,

$$u(x,y) = \int_0^b K(\eta; x, y) f(\eta)\, d\eta, \tag{9.1}$$

and give an expression for the kernel $K(\eta; x, y)$; it will be in the form of an infinite series.

**10.** Solve $u_{xx} + u_{yy} = 0$ in the semi-infinite strip $0 < x < \infty$, $0 < y < 1$ subject to the given boundary conditions plus the condition that $u$ is bounded as $x \to \infty$.

(a) $u(0,y) = 0$, $u(x,0) = 10$, $u_y(x,1) = 0$
(b) $u(0,y) = 100$, $u_y(x,0) = u_y(x,1) = 0$
(c) $u_x(0,y) = 5$, $u(x,0) = u(x,1) = 0$
(d) $u(0,y) = 0$, $u(x,0) = 50$, $u(x,1) = 10$
(e) $u(0,y) = 10y$, $u(x,0) = 20$, $u(x,1) = 50$

**11.** (a)–(e) Give a labeled sketch of representative isotherms for the corresponding problem in Exercise 10.

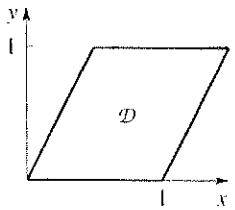
**12.** The problem

$$u_{xx} + u_{yy} = 0,$$
$$u(x,0) = 0, \quad u(x,b) = 50e^{-(x/10b)^2}$$

on $-\infty < x < \infty$, $0 < y < b$ admits a simple and accurate approximate solution, which we ask you to find. HINT: $\exp[-(x/10b)^2]$ is a slowly varying function of $x$.

**13.** In Example 2 we apply a boundedness condition on $u$ at $y = \infty$. Dropping that condition, put forward two or three solutions that are unbounded on the semi-infinite strip.

**14.** Consider the Laplace equation $u_{xx} + u_{yy} = 0$ on the parallelogram $\mathcal{D}$ shown, bounded by the lines $y = 0$, $y = 1$,




$y = 2x$, and $y = 2x - 2$, with boundary conditions given on the four edges. There is no future in seeking $u(x,y) = X(x)Y(y)$ and using separation of variables because the boundary is not comprised of constant coordinate curves. Specifically, the left and right edges are neither constant-$x$ nor constant-$y$ lines. One possibility seems to be a change of variables from $x, y$ to $\xi, \eta$ according to

$$\xi = y - 2x, \quad \eta = y \tag{14.1}$$

so that the new domain, in the $\xi, \eta$ plane, will be a rectangle bounded by constant-$\xi$ and constant-$\eta$ lines.

(a) Show that new domain in a labeled sketch.
(b) Show that in terms of $\xi$ and $\eta$ the Laplace equation becomes

$$5u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} = 0 \tag{14.2}$$

and that our plan fails because (14.2) is not separable. NOTE: Nonetheless, the idea is a good one; we simply need to figure out how to design a change of variables $\xi = F(x,y)$ so as to simplify the domain without at the same time complicating the PDE. How to do this, for the two-dimensional Laplace equation, is the subject of Chapter 22 on conformal mapping.

**15.** (*Poisson equation*) Consider the Poisson problem

$$\begin{gathered} u_{xx} + u_{yy} = f(x,y), \\ u(0,y) = u(x,b) = u(a,y) = u(x,0) = 0 \end{gathered} \tag{15.1}$$

on the rectangle $0 < x < a$, $0 < y < b$, where the source function $f(x,y)$ is prescribed.

(a) Solve (15.1) by separation of variables for the case where $f(x,y) = \text{constant} \equiv f$, leaving expansion coefficients in integral form. HINT: Noticing that $fx^2/2$ is a simple particular solution of (15.1a), seek $u$ in the form

$$u(x,y) = \frac{f}{2}x^2 + U(x,y). \tag{15.2}$$

Show that the "homogeneous solution" $U$ satisfies the problem

$$\begin{gathered} U_{xx} + U_{yy} = 0, \\ U(0,y) = 0, \quad U(a,y) = -fa^2/2, \\ U(x,0) = U(x,b) = -fx^2/2. \end{gathered} \tag{15.3a,b,c}$$

and solve for $U$ by separation of variables. NOTE: $fx^2/2$ is not the only particular solution that could be used; for instance, we could use $fy^2/2$, $fx^2/2 + 37xy - 5y + 6$, and so on, but $fx^2/2$ (or $fy^2/2$) seems a simple and natural choice.
(b) Observe that the method proposed in (a) will work not only when $f(x, y)$ is a constant, but also when it is a function only of $x$ or only of $y$. Here, we ask you to solve (15.1) for the case where $f(x, y)$ is not necessarily of that form. HINT: Use the **eigenvector expansion method** in very much the same manner as we did in Exercise 17 of Section 18.3. Essentially, we can use either the eigenfunctions $\sin n\pi x/a$ in the $x$ variable or the eigenfunctions $\sin n\pi y/a$ in the $y$ variable to expand "everything in sight." Specifically, expanding

$$f(x, y) = \sum_{n=1}^{\infty} f_n(x) \sin \frac{n\pi y}{b}, \tag{15.4}$$

where

$$f_n(x) = \frac{2}{b} \int_0^b f(x, y) \sin \frac{n\pi y}{b}\, dy, \tag{15.5}$$

and seeking

$$u(x, y) = \sum_{n=1}^{\infty} g_n(x) \sin \frac{n\pi y}{b}, \tag{15.6}$$

show that the $g_n$'s are found by solving the problems

$$g_n'' - \frac{n^2\pi^2}{b^2} g_n = f_n(x); \qquad g_n(0) = g_n(a) = 0 \tag{15.7}$$

for $n = 1, 2, \ldots$.
(c) Implement the method of part (b) for the case where $f(x, y) = xy$, and solve for $u(x, y)$.

**16.** (*Three-dimensional case*) Consider the *three*-dimensional problem

$$\nabla^2 u = u_{xx} + u_{yy} + u_{zz} = 0 \tag{16.1}$$

in the rectangular prism $0 < x < a$, $0 < y < b$, $0 < z < c$, where $u = 0$ on each of the six faces except for the face $z = c$, on which $u$ is a prescribed function $f(x, y)$,

$$u(x, y, c) = f(x, y), \qquad (0 < x < a,\ 0 < y < b) \tag{16.2}$$

(a) Use separation of variables to derive the solution

$$u(x, y, z) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} D_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \sinh \omega_{mn} z, \tag{16.3}$$

where

$$\begin{aligned} \omega_{mn} &= \pi \sqrt{\frac{m^2}{a^2} + \frac{n^2}{b^2}}, \\ D_{mn} &= \frac{4}{ab \sinh(\omega_{mn} c)} \int_0^b \int_0^a f(x, y) \\ &\qquad \times \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}\, dx\, dy. \end{aligned} \tag{16.4}$$

(b) For the case where $c = b = a$ and $f(x, y) = 100$, use (16.3) and (16.4) to evaluate $u(a/2, a/2, a/2)$.

**17.** (*A necessary condition for existence*) Consider the Poisson problem

$$\nabla^2 u = f(x, y, z) \tag{17.1}$$

in some three-dimensional domain $\mathcal{D}$ with surface $\mathcal{S}$. Integrating (17.1) over $\mathcal{D}$, show that

$$\int_{\mathcal{S}} \frac{\partial u}{\partial n}\, dA = \int_{\mathcal{D}} f\, dV. \tag{17.2}$$

NOTE: Thus, the boundary values of $\partial u/\partial n$ (whether they are specified or not) need to be consistent with the source $f$ in the sense of (17.2) if a solution to (17.1) is to exist. For instance, suppose a homogeneous Neumann condition is appended to (17.1), that $\partial u/\partial n = 0$ everywhere on $\mathcal{S}$. Then, (17.2) tells us that for a solution to exist the net source must be zero: $\int_{\mathcal{D}} f\, dV = 0$. That result makes sense physically because if the integral were positive, say, then the average temperature within $\mathcal{D}$ would be an increasing function of time, whereas (17.1) is based on *steady-state* conduction.

**18.** (*Uniqueness*) Suppose that $u(x, y, z)$ is $C^2$ and satisfies the Poisson equation (17.1) throughout a domain $\mathcal{D}$, together with a Dirichlet boundary condition $u = g(x, y, z)$ on the (piecewise smooth orientable) surface $\mathcal{S}$ of $\mathcal{D}$.

(a) Show that that solution is unique. HINT: Suppose that there are two such solutions, say $u_1$ and $u_2$. With $w \equiv u_1 - u_2$, show that $\nabla^2 w = 0$ in $\mathcal{D}$ and $w = 0$ on $\mathcal{S}$. With "$u$" = "$v$" = $w$ in Green's first identity, show that

$$\int_{\mathcal{D}} (w_x^2 + w_y^2 + w_z^2)\, dV = 0, \tag{18.1}$$

and conclude that $w_x = w_y = w_z$ so $w$ is at most a constant. Show that the constant must be zero, so $u_1 = u_2$ in $\mathcal{D}$.
(b) Repeat (a) with the Dirichlet condition replaced by the Neumann condition $\partial u/\partial n = g$. This time, show that the solution is unique only to within an arbitrary additive constant.

(c) Repeat (a) with the Dirichlet condition replaced by a mixed boundary condition whereby $u = g$ over part of $S$ and $\partial u/\partial n = h$ over the rest of $S$.

**19.** (*Application of Sturm–Liouville theory*) (a) Solve the problem

$$\begin{gathered} u_{xx} + u_{yy} = 0, \\ u(0,y) = u(x,0) = u(x,3) + 5u_y(x,3) = 0, \\ u(4,y) = 100 \end{gathered} \tag{19.1a,b,c}$$

on the rectangle $0 < x < 4$, $0 < y < 3$ by separation of variables, with the help of the Sturm–Liouville theory. Show that the values of the separation constant $\kappa$ are the (nonzero) roots of the equation

$$\tan 3\kappa = -5\kappa, \tag{19.2}$$

and denote those roots as $\kappa_n$ $(n = 1, 2, \ldots)$. Use computer software to evaluate $\kappa_1$ through $\kappa_5$ and use the first five terms of your series solution to estimate $u(2,1)$.

(b) Same as (a), with (19.1c) changed to $u_x(4,y) = 100$.

## 20.3 Separation of Variables; Non-Cartesian Coordinates

**20.3.1. Plane polar coordinates.** Let $r, \theta$ be the usual plane polar coordinates with $x = r\cos\theta$ and $y = r\sin\theta$. If the problem domain is bounded by constant $r$ and constant $\theta$ curves, then we must use $r$ and $\theta$ as our independent variables because, for the separation of variable method to work, we need the boundary conditions to be given on the constant coordinate curves. Since we will need to express the Laplace equation in terms of $r$ and $\theta$, we recall from (24) in Section 16.7 that for plane polar coordinates the Laplacian is

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{1}$$

**EXAMPLE 1.** Consider the Dirichlet problem

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad (a < r < b,\ 0 < \theta < \alpha) \tag{2a}$$

$$u(r,0) = u_1, \quad (a < r < b) \tag{2b}$$

$$u(r,\alpha) = u_2, \quad (a < r < b) \tag{2c}$$

$$u(a,\theta) = 0, \quad (0 < \theta < \alpha) \tag{2d}$$

$$u(b,\theta) = f(\theta) \quad (0 < \theta < \alpha) \tag{2e}$$

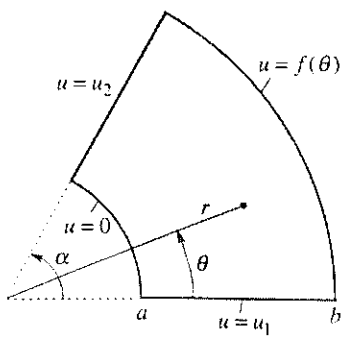



**Figure 1.** The Dirichlet problem (2).

shown in Fig. 1. Observe that this problem is similar to the basic Cartesian coordinate version (Fig. 1 of Section 20.2), with some "distortion," so we expect our solution steps to be similar as well.

According to the method of separation of variables, we seek $u$ in the product form

$$u(r,\theta) = R(r)\Theta(\theta). \tag{3}$$

Putting (3) into (2a) gives

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0,$$

and if we multiply by $r^2$ and divide by $R\Theta$ to separate the variables we obtain

$$\frac{r^2R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \text{constant} = \kappa^2. \tag{4}$$

Here, we choose $+\kappa^2$ so as to obtain $\cos\kappa\theta$ and $\sin\kappa\theta$ solutions for $\Theta$ (rather than $\cosh\kappa\theta$ and $\sinh\kappa\theta$) since we anticipate that to satisfy the $u(b,\theta) = f(\theta)$ boundary condition we will need to expand $f(\theta)$ in a Fourier series.

Proceeding, the $R$ and $\Theta$ equations are

$$r^2R'' + rR' - \kappa^2R = 0, \tag{5}$$

$$\Theta'' + \kappa^2\Theta = 0. \tag{6}$$

Although (5) has nonconstant coefficients (so the solution form $R = e^{\lambda r}$ will not work), it is elementary because it is of Cauchy–Euler form. Accordingly, seek $R = r^\lambda$ and obtain $\lambda(\lambda - 1) + \lambda - \kappa^2 = 0$ or $\lambda = \pm\kappa$. Thus, (5) admits two linearly independent solutions, $R = r^\kappa$ and $R = r^{-\kappa}$, unless $\kappa = 0$ in which case the two solutions coalesce into the one solution $R = \text{constant}$. To find the missing solution, for the case $\kappa = 0$, put $\kappa = 0$ into (5) and obtain $r^2R'' + rR' = 0$. The latter can be reduced to the first-order equation $r\,dp/dr + p = 0$ by the substitution $R' = p$ and integrated to give $p(r) = C_1/r$. Thus, $R(r) = \int p\,dr = C_1\ln r + C_2$, so we have these general solutions for $R$ and $\Theta$:

$$R(r) = \begin{cases} A + B\ln r, & \kappa = 0 \\ Cr^\kappa + Dr^{-\kappa}, & \kappa \neq 0 \end{cases} \tag{7}$$

$$\Theta(\theta) = \begin{cases} E + F\theta, & \kappa = 0 \\ G\cos\kappa\theta + H\sin\kappa\theta, & \kappa \neq 0 \end{cases} \tag{8}$$

Then, with the help of superposition, we have

$$u(r,\theta) = (A + B\ln r)(E + F\theta) + (Cr^\kappa + Dr^{-\kappa})(G\cos\kappa\theta + H\sin\kappa\theta). \tag{9}$$

Saving the boundary condition on $r = b$ for last, we first apply the boundary conditions on the adjacent edges $\theta = 0$ and $\theta = \alpha$:

$$u(r,0) = u_1 = (A + B\ln r)E + (Cr^\kappa + Dr^{-\kappa})G, \tag{10}$$

so we set $AE = u_1$, $B = 0$, and $G = 0$. Updating (9) accordingly,

$$u(r,\theta) = u_1 + I\theta + (Pr^\kappa + Qr^{-\kappa})\sin\kappa\theta, \tag{11}$$

where we have combined $AF$ as $I$, $CH$ as $P$, and $DH$ as $Q$. Next,

$$u(r,\alpha) = u_2 = u_1 + I\alpha + (Pr^\kappa + Qr^{-\kappa})\sin\kappa\alpha, \tag{12}$$

so $u_1 + I\alpha = u_2$ and $\sin\kappa\alpha = 0$, hence $I = (u_2 - u_1)/\alpha$ and $\kappa = n\pi/\alpha$ $(n = 1, 2, \ldots)$. Thus,

$$u(r,\theta) = u_1 + (u_2 - u_1)\frac{\theta}{\alpha} + \sum_{n=1}^{\infty}\left(P_nr^{n\pi/\alpha} + Q_nr^{-n\pi/\alpha}\right)\sin\frac{n\pi\theta}{\alpha}. \tag{13}$$

Then

$$u(a,\theta) = 0 = u_1 + (u_2 - u_1)\frac{\theta}{\alpha} + \sum_{n=1}^{\infty}\left(P_n a^{n\pi/\alpha} + Q_n a^{-n\pi/\alpha}\right)\sin\frac{n\pi\theta}{\alpha}.$$

or, moving the known terms on the right to the left-hand side,

$$-u_1 - (u_2 - u_1)\frac{\theta}{\alpha} = \sum_{n=1}^{\infty}\left(P_n a^{n\pi/\alpha} + Q_n a^{-n\pi/\alpha}\right)\sin\frac{n\pi\theta}{\alpha}. \tag{14}$$

The latter is a half-range sine expansion of $-u_1 - (u_2 - u_1)(\theta/\alpha)$ so we can compute the coefficients $P_n a^{n\pi/\alpha} + Q_n a^{-n\pi/\alpha}$ as

$$P_n a^{n\pi/\alpha} + Q_n a^{-n\pi/\alpha} = \frac{2}{\alpha}\int_0^{\alpha}\left[-u_1 - (u_2 - u_1)\frac{\theta}{\alpha}\right]\sin\frac{n\pi\theta}{\alpha}\,d\theta. \tag{15}$$

Finally,

$$u(b,\theta) = f(\theta) = u_1 + (u_2 - u_1)\frac{\theta}{\alpha} + \sum_{n=1}^{\infty}\left(P_n b^{n\pi/\alpha} + Q_n b^{-n\pi/\alpha}\right)\sin\frac{n\pi\theta}{\alpha},$$

or

$$f(\theta) - u_1 - (u_2 - u_1)\frac{\theta}{\alpha} = \sum_{n=1}^{\infty}\left(P_n b^{n\pi/\alpha} + Q_n b^{-n\pi/\alpha}\right)\sin\frac{n\pi\theta}{\alpha}, \tag{16}$$

so

$$P_n b^{n\pi/\alpha} + Q_n b^{-n\pi/\alpha} = \frac{2}{\alpha}\int_0^{\alpha}\left[f(\theta) - u_1 - (u_2 - u_1)\frac{\theta}{\alpha}\right]\sin\frac{n\pi\theta}{\alpha}\,d\theta. \tag{17}$$

We can evaluate the integrals in (15) and (17), once $f(\theta)$ is specified, so (15) and (17) amount to two linear algebraic equations in the unknown $P_n$'s and $Q_n$'s. They have a unique solution because the determinant of the coefficient matrix is

$$\begin{vmatrix} a^{n\pi/\alpha} & a^{-n\pi/\alpha} \\ b^{n\pi/\alpha} & b^{-n\pi/\alpha} \end{vmatrix} = \left(\frac{a}{b}\right)^{n\pi/\alpha} - \left(\frac{b}{a}\right)^{n\pi/\alpha} \neq 0 \tag{18}$$

since $b \neq a$. Thus, the solution to (2) is given by (13), with $P_n$ and $Q_n$ determined from (15) and (17). For instance, if $u_1 = u_2 = 0$ and $f(\theta) = 100$, then we obtain (Exercise 1)

$$u(r,\theta) = \frac{400}{\pi}\sum_{1,3,\ldots}^{\infty}\frac{1}{n}\,\frac{\left(\frac{r}{a}\right)^{n\pi/\alpha} - \left(\frac{a}{r}\right)^{n\pi/\alpha}}{\left(\frac{b}{a}\right)^{n\pi/\alpha} - \left(\frac{a}{b}\right)^{n\pi/\alpha}}\sin\frac{n\pi\theta}{\alpha}. \tag{19}$$

COMMENT 1. The Laplace equation in polar coordinates, (2a), did indeed prove to be separable. That is, putting $u(r,\theta) = R(r)\Theta(\theta)$ into (2a) we were able to get all of the $r$ dependence on one side of the equation and all of the $\theta$ dependence on the other [in (4)], and hence to infer the ODE's (5) and (6) on $R$ and $\Theta$. In fact, the diffusion, wave, and Laplace equations are all separable in Cartesian, polar, cylindrical, and spherical coordinates, for

which we can be grateful because *not all PDE's are separable*; see, for instance, Exercise 14 in Section 20.2.

COMMENT 2. Given that the Fourier expansion will take place on the $r = b$ edge, there is no extra difficulty in admitting $u =$ constant boundary conditions on the adjacent edges, namely, the conditions $u = u_1$ on $\theta = 0$ and $u = u_2$ on $\theta = \alpha$. The reason is that the $A(E + F\theta)$ part of the solution (9), which comes from $\kappa = 0$, is able to handle those boundary conditions; it gives the $u_1 + (u_2 - u_1)\theta/\alpha$ part of the final solution (13). If we spoke of the analogous $(A + Bx)(E + Fy)$ term in Section 20.2 as being a "ramp" function (if $B$ or $F$ is zero), we might call the $A(E + F\theta)$ term a "fan" function since its graph "fans" from the value $u_1$ at one value of $\theta$ to $u_2$ at another value of $\theta$ (Fig. 2).

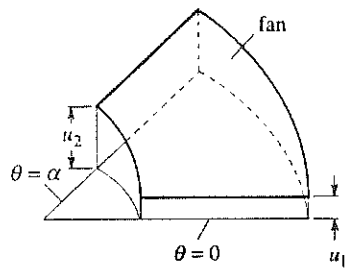



**Figure 2.** Fan from $u_1$ to $u_2$.

COMMENT 3. If the boundary condition along either radial edge ($\theta = 0$ or $\theta = \alpha$) is nonconstant, so that a Fourier expansion is needed along that edge, then the solution is more difficult because the needed expansion turns out not to be a familiar half- or quarter-range Fourier series but rather a Sturm–Liouville eigenfunction expansion. The solution for that case is outlined in Exercise 13.

COMMENT 4. The boundary conditions (2b)–(2e) are of Dirichlet type. What if they were of Neumann type ($\partial u/\partial n$ prescribed)? For example, if (2b) were replaced by the Neumann condition

$$\frac{\partial u}{\partial n}(r, 0) = g(r), \qquad (a < r < b) \tag{20}$$

where $g(r)$ is prescribed, then what is $\partial u/\partial n$ in terms of $r$ and $\theta$? The key is to use the directional derivative formula $du/ds = \nabla u \cdot \hat{\mathbf{s}}$ in Section 16.4, which gives

$$\frac{\partial u}{\partial n} = \nabla u \cdot \hat{\mathbf{n}} = \left(\frac{\partial u}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\hat{\mathbf{e}}_\theta\right) \cdot (-\hat{\mathbf{e}}_\theta) = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

on the $\theta = 0$ edge, so (20) can be expressed in terms of $r, \theta$ as $-(1/r)\partial u/\partial\theta = g(r)$ or

$$\frac{\partial u}{\partial \theta}(r, 0) = -rg(r). \qquad (a < r < b) \tag{21}$$

Similarly, a Neumann condition

$$\frac{\partial u}{\partial n}(a, \theta) = h(\theta) \qquad (0 < \theta < \alpha) \tag{22}$$

becomes

$$\frac{\partial u}{\partial r}(a, \theta) = -h(\theta) \qquad (0 < \theta < \alpha) \tag{23}$$

because

$$\frac{\partial u}{\partial n} = \nabla u \cdot \hat{\mathbf{n}} = \left(\frac{\partial u}{\partial r}\hat{\mathbf{e}}_r + \frac{1}{r}\frac{\partial u}{\partial \theta}\hat{\mathbf{e}}_\theta\right) \cdot (-\hat{\mathbf{e}}_r) = -\frac{\partial u}{\partial r}$$

on $r = a$. Physically, remember that if $u$ is a temperature field then $\partial u/\partial n$ is proportional to the *heat flux across* that boundary. ∎

**EXAMPLE 2.** *Dirichlet Problem for Circular Disk.* Next, consider the Dirichlet problem

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \qquad (0 \le r < b) \tag{24a}$$

$$u(b,\theta) = f(\theta) \qquad (-\infty < \theta < \infty) \tag{24b}$$

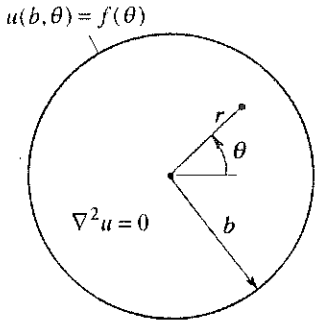



**Figure 3.** Dirichlet problem for disk.

shown in Fig. 3. We begin with (9) and write

$$u(r,\theta) = (A + B\ln r)(E + F\theta) + (Cr^{\kappa} + Dr^{-\kappa})(G\cos\kappa\theta + H\sin\kappa\theta), \tag{25}$$

but comparing (24) with (2) we seem to be missing some boundary conditions. After all, the PDE is of second order in $r$ and in $\theta$, so we expect to need two $r$ boundary conditions and two $\theta$ boundary conditions, as were present in (2b)–(2e). For insight, it is useful to imagine obtaining the disk in Fig. 3 as the limiting case of the region in Fig. 1 as $a \to 0$ and $\alpha \to 2\pi$. Letting $a \to 0$ first, observe that the $r = a$ boundary curve shrinks to a point; when that happens we lose that boundary curve and corresponding boundary condition and have a pie-shaped region. Next, let $\alpha \to 2\pi$. The moment $\alpha$ becomes $2\pi$ we lose the two boundary conditions on the edges $\theta = 0$ and $\theta = \alpha$ because those edges disappear as boundary edges and become interior to the region.

To compensate for these losses we do the following. First, we adjoin to (24a,b), in lieu of the missing $r$ boundary condition, a *boundedness condition* at $r = 0$, namely,

$$u(r,\theta) \quad \text{bounded as} \quad r \to 0. \tag{26}$$

To apply this condition, observe that both the $\ln r$ and $r^{-\kappa}$ terms in (25) are unbounded as $r \to 0$: $\ln r \to -\infty$ and $r^{-\kappa} \to \infty$ as $r \to 0$. Thus, (26) requires us to remove those terms by setting $B = D = 0$, in which case (25) reduces to

$$u(r,\theta) = I + J\theta + r^{\kappa}(P\cos\kappa\theta + Q\sin\kappa\theta). \tag{27}$$

To remedy the situation regarding the missing $\theta$ boundary condition we begin by observing a key difference between Example 1 and the present example: the $\theta$ domain in Example 1 was finite ($0 < \theta < \alpha$) whereas here it is infinite ($-\infty < \theta < \infty$), for we see from Fig. 3 that there is nothing to prevent the representative point from encircling the origin repeatedly, clockwise or counterclockwise. Thus, if $u(r,\theta)$ is to be a single-valued function of $\theta$ and hence uniquely defined at each point within the disk, then it needs to be $2\pi$-periodic in $\theta$:

$$u(r,\theta + 2\pi) = u(r,\theta). \tag{28}$$

This periodicity will compensate for the two missing $\theta$ boundary conditions (see also Exercise 8) so the full problem is given by (24a), (24b), (26), and (28).

Let us impose (28) on each term in (27). First, $I$ is a constant and is therefore $2\pi$-periodic; hence, retain that term. Next, $J\theta$ is *not* periodic (as can be seen from its linear graph), so we must set $J = 0$ to remove that term. Finally, the $\cos\kappa\theta$ and $\sin\kappa\theta$ terms are periodic, but we need to determine the allowable $\kappa$'s so that they are **$2\pi$-periodic**. According to the definition of periodicity, we need

$$\cos\kappa(\theta + 2\pi) = \cos\kappa\theta \tag{29}$$

for all $\theta$ (and similarly for the sine term) or, since $\cos(A + B) = \cos A\cos B - \sin A\sin B$, we need

$$\cos\kappa\theta\cos 2\pi\kappa - \sin\kappa\theta\sin 2\pi\kappa = \cos\kappa\theta.$$

Equating coefficients of the linearly independent $\cos\kappa\theta$ and $\sin\kappa\theta$ terms gives $\cos 2\pi\kappa = 1$ and $\sin 2\pi\kappa = 0$, with the roots*

$$\cos 2\pi\kappa = 1: \qquad \kappa = 1, 2, 3, \ldots, \tag{30a}$$

$$\sin 2\pi\kappa = 0: \qquad \kappa = \frac{1}{2}, 1, \frac{3}{2}, 2, \ldots. \tag{30b}$$

Since both conditions need to hold, we accept only $\kappa$'s that are in both lists, namely, $\kappa = 1, 2, 3, \ldots$. The same result is obtained when we enforce the $2\pi$-periodicity of the $\sin\kappa\theta$ term.

Thus, with $J = 0$, $\kappa = n$, and with the help of superposition, (26) gives

$$u(r,\theta) = I + \sum_{n=1}^{\infty} r^n (P_n \cos n\theta + Q_n \sin n\theta), \tag{31}$$

We are ready for the final boundary condition:

$$u(b,\theta) = f(\theta) = I + \sum_{n=1}^{\infty} b^n (P_n \cos n\theta + Q_n \sin n\theta), \tag{32}$$

which holds on $-\infty < \theta < \infty$. Notice carefully that whereas (14) and (16) are half-range sine expansions on the finite interval $0 < \theta < \alpha$, (32) is the full Fourier series expansion of the $2\pi$-periodic function $f(\theta)$ on $-\infty < \theta < \infty$. Accordingly [see (5) in Section 17.3, with $\ell = \pi$],

$$I = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta, \qquad P_n = \frac{1}{\pi b^n}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta,$$
$$Q_n = \frac{1}{\pi b^n}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\,d\theta, \tag{33}$$

and the solution is given by (31) and (33).

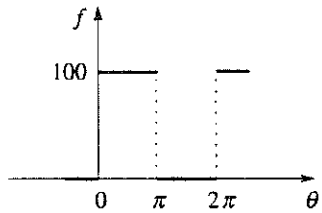



**Figure 4.** Square wave $f$.

To illustrate, let the boundary temperature $f(\theta)$ be 100 on the upper half of the circle and 0 on the lower half, in which case $f$ is actually the $2\pi$-periodic square wave shown in Fig. 4. Using (33), the result is

$$u(r,\theta) = 50 + \frac{200}{\pi}\sum_{1,3,\ldots}^{\infty}\left(\frac{r}{b}\right)^n \frac{\sin n\theta}{n}, \tag{34}$$

and representative isotherms are shown in Fig. 5.

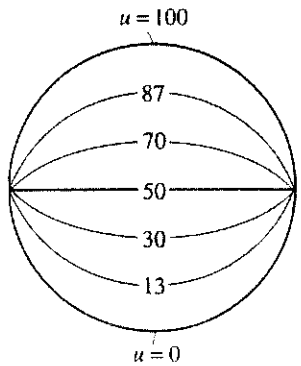



**Figure 5.** Isotherms corresponding to (34).

COMMENT 1. Setting $r = 0$ in (31), observe that $u$ at the center of the disk equals $I$, and $I$, according to (33), is the average value of the boundary temperature . For the example shown in Fig. 5, for instance, $f(\theta)$ is 100 on the upper half of the circumference and 0 on

*As usual, we exclude negative $\kappa$ values since they contribute nothing new. For instance, if we change $\kappa$ to $-\kappa$ in the last term in (25) then that term takes the equivalent form $(Cr^{-\kappa} + Dr^{\kappa})(G\cos\kappa\theta - H\sin\kappa\theta)$. This result is not surprising since the $\kappa^2$ in (4) cannot distinguish between positive $\kappa$'s and negative $\kappa$'s. Further, we disallow $\kappa = 0$ in (30) because the $\kappa = 0$ case is handled separately, in (25), by the $(A + B\ln r)(E + F\theta)$ term.

the lower half so the average value is 50. Sure enough, the isotherm $u = 50$ does pass through the origin in Fig. 5.

COMMENT 2. Let us put (33) into (31). First changing the dummy variable of integration to $\vartheta$, say, to avoid confusion with the $\theta$'s in (31), the result is

$$u(r,\theta) = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{b}\right)^n \cos n(\vartheta - \theta)\right] f(\vartheta)\, d\vartheta, \tag{35}$$

where we have formally interchanged the order of integration and summation. It is striking that the infinite series in (35) can be summed, that is, gotten into closed form. The result (Exercise 9) is

$$\boxed{\begin{aligned} u(r,\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{b^2 - r^2}{b^2 - 2br\cos(\vartheta - \theta) + r^2} f(\vartheta)\, d\vartheta \\ &\equiv \textstyle\int_{-\pi}^{\pi} P(r, \vartheta - \theta) f(\vartheta)\, d\vartheta; \end{aligned}} \tag{36}$$

(36) is the **Poisson integral formula for the circular disk** and $P(r, \vartheta - \theta)$ is the corresponding Poisson kernel. ∎

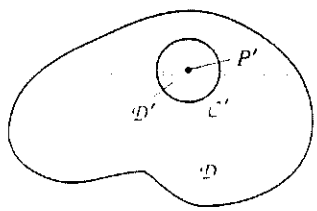



**Figure 6.** Average value theorem.

In Comment 1, above, we wrote that the temperature $u$ at the center of the disk equals the average of the boundary temperatures $f(\theta)$ around the circumference. This result can be generalized as follows. Within an arbitrary domain $\mathcal{D}$, not necessarily circular (Fig. 6), consider any point $P'$ and any circular domain $\mathcal{D}'$ that is centered at $P'$ and that lies entirely within $\mathcal{D}$, and suppose that $\nabla^2 u = 0$ in $\mathcal{D}$. Then, whatever the temperatures are on the boundary $\mathcal{C}'$ of $\mathcal{D}'$ we can consider them as boundary conditions for the sub-problem $\nabla^2 u = 0$ in $\mathcal{D}'$, which problem was the subject of Example 2. From the average value result found in Example 2 we know that $u$ at $P'$ is the average of the $u$ values around $\mathcal{C}'$. Thus, *if $\nabla^2 u = 0$ in a two-dimensional domain $\mathcal{D}$, then the temperature $u$ at any point $P'$ within $\mathcal{D}$ is equal to the average temperature around any circle centered at $P'$ and lying within $\mathcal{D}$.* This result, known as the **average value property** of the Laplace equation, is said to be a "local" result since it holds for an arbitrarily small circle $\mathcal{C}'$ and is insensitive to the shape of $\mathcal{D}$. It holds in one dimension (Exercise 10) and in three dimensions as well (as we will see in Example 5).

The average value property enables us to prove the **maximum principle** for the Laplace equation, which is as follows: *Let $u$ be the steady-state temperature field within a two-dimensional domain $\mathcal{D}$, so $u$ satisfies the Laplace equation $\nabla^2 u = 0$. Then $u$ cannot attain its maximum value in $\mathcal{D}$ (unless $u$ is a constant everywhere): it must attain its maximum on the boundary of $\mathcal{D}$.* For suppose that $u$ does have a maximum value $M$, say, at a point $P$ within $\mathcal{D}$. Since $u$ at $P$ equals the average value of $u$ around any circle centered at $P$, $u$ must achieve values less than $M$ and greater than $M$ at points within $\mathcal{D}$ if it is not simply a constant everywhere. But that result contradicts our assumption that the maximum value of $u$ is $M$, hence $u$ cannot have a maximum value within $\mathcal{D}$. By virtually the same argument we can show

that $u$ cannot attain its minimum value in $\mathcal{D}$, and we obtain the **minimum principle**, which is identical to the maximum principle (italicized above) with the two "maximums" changed to "minimums." It follows from these maximum and minimum principles that the values of $u$ within $\mathcal{D}$ necessarily lie between the minimum and maximum values of $u$ on the boundary of $\mathcal{D}$.

These results make sense physically, for suppose that the temperature maintained on the boundary of $\mathcal{D}$ lies between 50°C and 70°C and let $u = 2,000$°C at some point inside $\mathcal{D}$. Surely that "hot spot" will cool down, with time, and the surrounding material will heat up. But that is impossible for we have assumed steady-state heat conduction: that is, the Laplace equation $\nabla^2 u = 0$ is the steady-state version of the heat conduction equation $\alpha^2 \nabla^2 u = u_t$.

**20.3.2. Cylindrical coordinates. (Optional)** The steady-state temperature field within a cylindrical rod (Fig. 7) is governed by the Laplace equation in cylindrical coordinates,

$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u_{zz} = 0. \tag{37}$$

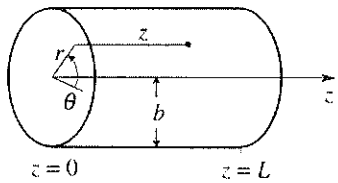



**Figure 7.** Cylindrical rod.

Suppose there is axisymmetry so that $u$ does not vary with $\theta$. Then (37) reduces to

$$\nabla^2 u = u_{rr} + \frac{1}{r} u_r + u_{zz} = 0, \tag{38}$$

which case we consider here. Specifically, we consider the problem shown in the left-hand member of Fig. 8, and we begin by breaking it into the two problems shown in the figure. We will solve the $u_1$ problem as Example 3 and the $u_2$ problem as Example 4.

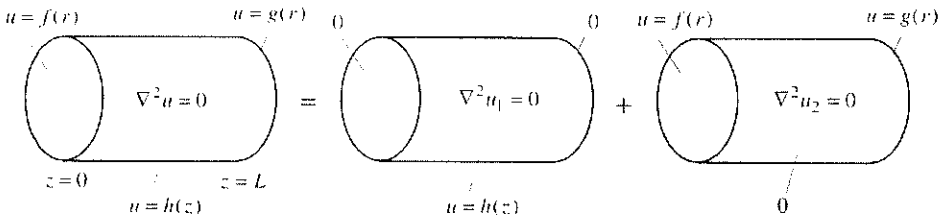



**Figure 8.** Breakdown by superposition.

**EXAMPLE 3.** *The $u_1$ Problem.* To solve (38), seek

$$u_1(r, z) = R(r)Z(z). \tag{39}$$

Putting the latter into (38) and separating variables gives

$$\frac{R'' + \frac{1}{r}R'}{R} = -\frac{Z''}{Z} = \text{constant} = \kappa^2 \tag{40}$$

and the ODE's

$$R'' + \frac{1}{r}R' - \kappa^2 R = 0, \tag{41}$$

$$Z'' + \kappa^2 Z = 0. \tag{42}$$

We chose the $+\kappa^2$ in (40) so as to obtain $\cos\kappa z$ and $\sin\kappa z$ (rather than $\cosh\kappa z$ and $\sinh\kappa z$) solutions of the $Z$ equation since we look ahead to a Fourier series expansion of $h(z)$. Distinguishing the cases $\kappa = 0$ and $\kappa \neq 0$, the general solutions of (41) and (42) are (Exercise 14):

$$R = \begin{cases} A + B\ln r, & \kappa = 0 \\ CI_0(\kappa r) + DK_0(\kappa r), & \kappa \neq 0 \end{cases} \tag{43}$$

$$Z = \begin{cases} E + Fz, & \kappa = 0 \\ G\cos\kappa z + H\sin\kappa z, & \kappa \neq 0 \end{cases} \tag{44}$$

where $I_0, K_0$ are modified Bessel functions of the first and second kind, respectively, of order zero. Thus,

$$u_1(r,z) = (A + B\ln r)(E + Fz) + [CI_0(\kappa r) + DK_0(\kappa r)](G\cos\kappa z + H\sin\kappa z). \tag{45}$$

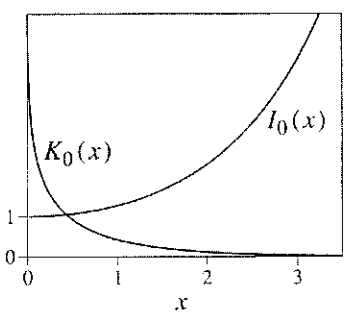



**Figure 9.** $I_0$ and $K_0$.

In the $z$ variable we have the two boundary conditions at $z = 0$ and $z = L$, but in $r$ we have only the boundary condition at $r = b$ so in lieu of a second $r$ boundary condition we require that $u$ be bounded as $r \to 0$ (i.e., all along the $z$ axis). Since $\ln r \to -\infty$ we set $B = 0$, and since $K_0(\kappa r) \sim -\ln r \to \infty$ as $r \to 0$ (Fig. 9) we set $D = 0$, so (45) becomes

$$u_1(r,z) = P + Qz + I_0(\kappa r)(S\cos\kappa z + T\sin\kappa z), \tag{46}$$

where we have combined $AE$ as $P$, $AF$ as $Q$, $CG$ as $S$, and $CH$ as $T$, for brevity.

Next, we apply the conditions at $z = 0$, $z = L$, and $r = b$, in turn:

$$u_1(r,0) = 0 = P + I_0(\kappa r)S, \tag{47}$$

so $P = 0$ and $S = 0$. Updating (46) accordingly,

$$u_1(r,z) = Qz + TI_0(\kappa r)\sin\kappa z. \tag{48}$$

Next,

$$u_1(r,L) = 0 = QL + TI_0(\kappa r)\sin\kappa L, \tag{49}$$

so $Q = 0$ and $\kappa = n\pi/L$ for $n = 1, 2, \ldots$. Updating (48),

$$u_1(r,z) = \sum_{n=1}^{\infty} T_n I_0\left(\frac{n\pi r}{L}\right)\sin\frac{n\pi z}{L}. \tag{50}$$

Finally,

$$u_1(b,z) = h(z) = \sum_{n=1}^{\infty} T_n I_0\left(\frac{n\pi b}{L}\right)\sin\frac{n\pi z}{L}, \qquad (0 < z < L) \tag{51}$$

which is a half-range sine series so

$$T_n I_0\left(\frac{n\pi b}{L}\right) = \frac{2}{L}\int_0^L h(z)\sin\frac{n\pi z}{L}\,dz,$$

or

$$T_n = \frac{2}{LI_0(n\pi b/L)}\int_0^L h(z)\sin\frac{n\pi z}{L}\,dz. \tag{52}$$

The desired solution is given by (51) and (52).

COMMENT. As in Example 2 we applied the boundedness condition first. Indeed, whenever there is a boundedness condition we suggest that you apply it first because it will eliminate one or more terms, thereby giving an immediate simplification of the solution form. ∎

**EXAMPLE 4.** *The $u_2$ Problem.* To solve for $u_2$ (see Fig. 8) we again seek

$$u_2(r, z) = R(r)Z(z) \tag{53}$$

and obtain

$$\frac{R'' + \frac{1}{r}R'}{R} = -\frac{Z''}{Z} = \text{constant} = -\kappa^2 \tag{54}$$

and the ODE's

$$R'' + \frac{1}{r}R' + \kappa^2 R = 0, \tag{55}$$

$$Z'' - \kappa^2 Z = 0, \tag{56}$$

where this time we chose $-\kappa^2$, so as to obtain oscillatory solutions of the $R$ equation (rather than the $I_0$, $K_0$ pair obtained in Example 3). Specifically (Exercise 15),

$$R = \begin{cases} A + B\ln r, & \kappa = 0 \\ CJ_0(\kappa r) + DY_0(\kappa r), & \kappa \neq 0 \end{cases} \tag{57}$$

$$Z = \begin{cases} E + Fz, & \kappa = 0 \\ G\cosh\kappa z + H\sinh\kappa z, & \kappa \neq 0 \end{cases} \tag{58}$$

where $J_0, Y_0$ are Bessel functions of the first and second kind, respectively, of order zero. Thus,

$$u_2(r, z) = (A + B\ln r)(E + Fz) + [CJ_0(\kappa r) + DY_0(\kappa r)](G\cosh\kappa z + H\sinh\kappa z). \tag{59}$$

Boundedness as $r \to 0$ requires that $B = 0$ and $D = 0$, since $Y_0(\kappa r) \sim (2/\pi)\ln r \to -\infty$ as $r \to 0$ (Fig. 10), so (59) reduces to

$$u_2(r, z) = P + Qz + J_0(\kappa r)(S\cosh\kappa z + T\sinh\kappa z). \tag{60}$$

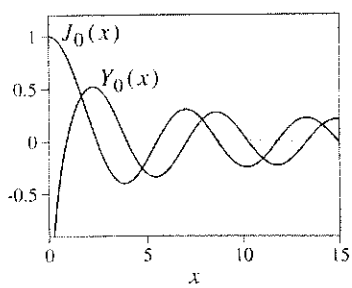



**Figure 10.** $J_0$ and $Y_0$.

Since we look ahead to expanding $f(r)$ and $g(r)$, we must apply both $r$ boundary conditions first, before attempting either of the end conditions at $z = 0$ and $L$. Having already applied the boundedness condition at $r = 0$, we next write

$$u_2(b, z) = 0 = P + Qz + J_0(\kappa b)(S\cosh\kappa z + T\sinh\kappa z),$$

which requires that $P = 0$, $Q = 0$, and

$$J_0(\kappa b) = 0 \tag{61}$$

[since we cannot afford to set $S = T = 0$ and lose the entire $J_0(\kappa r)(S\cosh\kappa z + T\sinh\kappa z)$ term in (60)]. Denoting the roots of $J_0(x) = 0$ as $z_1, z_2, \ldots$ ($z$'s because they give the *zeros* of $J_0$), we learn from (61) that $\kappa = z_n/b$ ($n = 1, 2, \ldots$). Updating (60) accordingly,

$$u_2(r,z) = \sum_{n=1}^{\infty} J_0\left(z_n \frac{r}{b}\right)\left[S_n \cosh\left(z_n \frac{z}{b}\right) + T_n \sinh\left(z_n \frac{z}{b}\right)\right]. \tag{62}$$

Finally, the end conditions give

$$u_2(r,0) = f(r) = \sum_{n=1}^{\infty} S_n J_0\left(z_n \frac{r}{b}\right), \tag{63}$$

and

$$u_2(r,L) = g(r) = \sum_{n=1}^{\infty}\left[S_n \cosh\left(z_n \frac{L}{b}\right) + T_n \sinh\left(z_n \frac{L}{b}\right)\right] J_0\left(z_n \frac{r}{b}\right) \tag{64}$$

on $0 < r < b$. How are we to solve (63) and (64) for $S_n$ and $T_n$? There are two questions raised by (63): first, is it *possible* to expand a given function $f(r)$ on the interval $0 < r < b$ in the form of an infinite linear combination of $J_0(z_n r/b)$ terms and, second, if so, how do we compute the $S_n$ coefficients? Similarly for (64). Both questions are answered by the Sturm–Liouville theory, for the problem

$$(rR')' + \kappa^2 rR = 0, \qquad (0 < r < b) \tag{65a}$$

$$R(0) \text{ bounded}, \quad R(b) = 0 \tag{65b}$$

governing $R$ is a Sturm–Liouville problem, where "$\lambda$" $= \kappa^2$. This problem is studied in Example 2 of Section 17.8 so, referring you to that example for the details, we can conclude from (63) that

$$S_n = \frac{2}{b^2[J_1(z_n)]^2}\int_0^b f(r) J_0\left(z_n \frac{r}{b}\right) r\,dr, \tag{66}$$

and from (64) that

$$S_n \cosh\left(z_n \frac{L}{b}\right) + T_n \sinh\left(z_n \frac{L}{b}\right) = \frac{2}{b^2[J_1(z_n)]^2}\int_0^b g(r) J_0\left(z_n \frac{r}{b}\right) r\,dr. \tag{67}$$

Thus, the solution is obtained by solving (66) and (67) for $S_n$ and $T_n$ (once $f$ and $g$ are specified), and putting these values into (62).

COMMENT. Observe how the problem is "self-contained": how to carry out the necessary expansions (63) and (64) is fully explained by the Sturm–Liouville problem on $R$ that is "built right in." Likewise in Example 3, although we did not mention it because we merely noticed that (51) is a half-range sine expansion. Alternatively, we could have used the Sturm–Liouville theory there too. Specifically, the relevant Sturm–Liouville problem there is

$$Z'' + \kappa^2 Z = 0, \qquad (0 < z < L) \tag{68a}$$

$$Z(0) = 0, \quad Z(L) = 0, \tag{68b}$$

with the eigenfunctions $\sin(n\pi z/L)$, so (51) gives

$$T_n I_0\left(\frac{n\pi b}{L}\right) = \frac{\langle h(z), \sin\frac{n\pi z}{L}\rangle}{\langle \sin\frac{n\pi z}{L}, \sin\frac{n\pi z}{L}\rangle}$$
$$= \frac{\int_0^L h(z)\sin\frac{n\pi z}{L}\,dz}{\int_0^L \sin^2\frac{n\pi z}{L}\,dz} = \frac{2}{L}\int_0^L h(z)\sin\frac{n\pi z}{L}\,dz,$$

which result agrees with (52). Note that the inner product weight function is $r$ in (65) and 1 in (68).

Likewise in Example 1. We deduced (15) from (14) and (17) from (16) by noticing that (14) and (16) were half-range sine expansions of their left-hand sides, over $0 < \theta < \alpha$. Alternatively, we could have used the Sturm–Liouville theory. Since the expansions were in the $\theta$ variable, look to the $\Theta$ problem for the Sturm–Liouville problem, namely,

$$\Theta'' + \kappa^2\Theta = 0, \qquad (0 < \theta < \alpha) \tag{69a}$$
$$\Theta(0) = 0, \quad \Theta(\alpha) = 0, \tag{69b}$$

with eigenfunctions $\sin(n\pi\theta/\alpha)$ and weight function 1. Where do we get the homogeneous boundary conditions (69b) from, considering that $u(r,0) = u_1$ and $u(r,\alpha) = u_2$ are, in general, nonzero? If we retrace the solution steps, beginning with (9), we find that the $G\cos\kappa\theta + H\sin\kappa\theta$ factor in (9), which contributes the $\sin(n\pi\theta/\alpha)$ eigenfunctions, satisfies the homogeneous boundary conditions (69b), with the $(A + B\ln r)(E + F\theta)$ term handling the $u_1$ and $u_2$ values. And, of course, we can see directly that $\sin(n\pi\theta/\alpha)$ does indeed vanish at $\theta = 0$ and at $\theta = \alpha$. ∎

Then the solution to the problem on $u$ that is shown in the left-hand part of Fig. 8 is $u(r,z) = u_1(r,z) + u_2(r,z)$.

**20.3.3. Spherical coordinates. (Optional)** If the domain under consideration is bounded by constant $\rho, \phi, \theta$ surfaces, then we need to work with the Laplace equation in spherical coordinates,

$$\nabla^2 u = \frac{1}{\rho^2}\left[\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial u}{\partial\rho}\right) + \frac{1}{\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial u}{\partial\phi}\right) + \frac{1}{\sin^2\phi}\frac{\partial^2 u}{\partial\theta^2}\right] = 0. \tag{70}$$

Let us restrict our attention to cases where $u$ is axisymmetric about the polar axis $z$, that is, where $u$ does not vary with $\theta$. Then the $u_{\theta\theta}$ term in (70) is zero and (70) reduces to the PDE

$$\nabla^2 u = u_{\rho\rho} + \frac{2}{\rho}u_\rho + \frac{1}{\rho^2}u_{\phi\phi} + \frac{\cot\phi}{\rho^2}u_\phi = 0 \tag{71}$$

on $u(\rho,\phi)$.

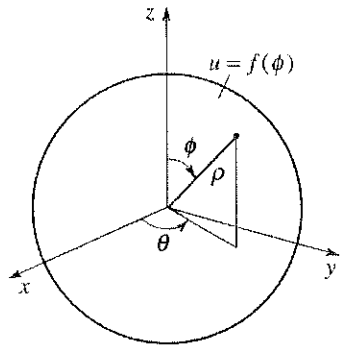



**Figure 11.** Dirichlet problem for sphere.

**EXAMPLE 5.** *Dirichlet Problem for Sphere.* Consider the Dirichlet problem consisting of the PDE (71) in the sphere $0 \le \rho < c$, with the boundary condition

$$u(c, \phi) = f(\phi) \qquad (0 \le \phi \le \pi) \tag{72}$$

and the stipulation that $u$ be bounded in the given domain (Fig. 11). This problem is the three-dimensional analog of the Dirichlet problem for a circular disk, which was the subject of Example 2.

To solve, seek

$$u(\rho, \phi) = R(\rho)\Phi(\phi) \tag{73}$$

and obtain, from (71),

$$\frac{\rho^2 R'' + 2\rho R'}{R} = -\frac{\Phi'' + \cot\phi\Phi'}{\Phi} = \text{constant} = \kappa^2, \tag{74}$$

and the ODE's

$$\rho^2 R'' + 2\rho R' - \kappa^2 R = 0, \tag{75}$$

$$\Phi'' + \cot\phi\,\Phi' + \kappa^2\Phi = 0. \tag{76}$$

The change of variables

$$\mu = \cos\phi$$

in (76) reduces that equation (Exercise 17) to the Legendre equation

$$(1 - \mu^2)\frac{d^2\Phi}{d\mu^2} - 2\mu\frac{d\Phi}{d\mu} + \kappa^2\Phi = 0. \tag{77}$$

From our study of the Legendre equation in Section 4.4, we know that to obtain solutions of (77) that are bounded on $-1 \le \mu \le 1$ we need

$$\kappa^2 = n(n+1), \qquad (n = 0, 1, 2, \ldots) \tag{78}$$

in which case the corresponding bounded solutions are the Legendre polynomials

$$\Phi = P_n(\mu) = P_n(\cos\phi). \tag{79}$$

In terms of the physical domain, $\mu = 1$ corresponds to $\phi = 0$ so that unboundedness of $\Phi$ at $\mu = 1$ would mean unboundedness of the solution $u$ all along the $z$ axis from the "north pole" to the origin. Similarly, unboundedness of $\Phi$ at $\mu = -1$ would mean unboundedness of $u$ along the $z$ axis from the origin to the "south pole."

Turning to the $R$ equation, with $\kappa^2 = n(n+1)$, the general solution of (75), which is a Cauchy–Euler equation, is

$$R(\rho) = A\rho^n + \frac{B}{\rho^{n+1}}. \qquad (n = 0, 1, 2, \ldots) \tag{80}$$

Here, the stipulated boundedness of $u$ requires that $B = 0$. Putting (79) and (80) together and using superposition we have, thus far,

$$u(\rho, \phi) = \sum_{n=0}^{\infty} A_n \rho^n P_n(\cos\phi). \tag{81}$$

Finally, the boundary condition (72) requires that

$$u(c,\phi) = f(\phi) = \sum_{n=0}^{\infty} A_n c^n P_n(\cos\phi). \qquad (0 \le \phi \le \pi) \tag{82}$$

Since (82) involves an expansion in the $\phi$ variable, let us examine the boundary-value problem on $\Phi$, namely,*

$$((1-\mu^2)\Phi')' + \kappa^2\Phi = 0, \qquad (-1 \le \mu \le 1) \tag{83a}$$

$$\Phi(-1) \text{ and } \Phi(1) \text{ finite.} \tag{83b}$$

Thus, we see that (82) amounts to a Fourier–Legendre expansion of the given function $f$ in terms of the orthogonal eigenfunctions $P_n(\mu)$ or $P_n(\cos\phi)$, as was illustrated in Example 3 of Section 17.8. Accordingly,

$$A_n c^n = \frac{\langle f, P_n(\mu)\rangle}{\langle P_n(\mu), P_n(\mu)\rangle} = \frac{\displaystyle\int_{-1}^{1} f P_n\, d\mu}{\displaystyle\int_{-1}^{1} P_n^2\, d\mu} = \frac{2n+1}{2}\int_{-1}^{1} f P_n\, d\mu, \tag{84}$$

or

$$A_n = \frac{2n+1}{2c^n}\int_0^{\pi} f(\phi) P_n(\cos\phi)\sin\phi\, d\phi. \tag{85}$$

Hence, the solution is given by (81) and (85).

COMMENT. Observe that the value of $u$ at the center of the sphere is

$$\begin{aligned} u(0,\phi) &= A_0 \qquad \text{[from (81)]} \\ &= \frac{1}{2}\int_{-1}^{1} f\, d\mu, \qquad \text{[from (84)]} \end{aligned}$$

which is the average value of the boundary temperature $f$. Thus, the average value property of the Laplace equation, discussed above, holds in three dimensions as well as two, and similarly for the maximum principle. ∎

**Closure.** We see, in Section 20.3.1, that the Laplace equation in plane polar coordinates is successfully separated and that although the ODE on $R(r)$ has nonconstant coefficients it is nevertheless an elementary equation, a Cauchy–Euler equation. We derive, as a result of Example 2, the average value property of the Laplace equation in two dimensions and the maximum principle as well. In Sections 20.3.2 and 20.3.3 we find that the Laplace equation can be separated in cylindrical and spherical coordinates as well, but that not all of the resulting ODE's are elementary: in cylindrical coordinates the $R(r)$ equation gives Bessel functions, and in spherical coordinates the $\Phi(\phi)$ equation gives Legendre polynomials.

*Strictly speaking, we should use a new name, such as $\Phi(\phi) = \Phi(\phi(\mu)) \equiv \Psi(\mu)$, say, but for economy of notation we use $\Phi$ whether the independent variable is $\phi$ or $\mu$.

Expansions in Examples 4 and 5 involve the Bessel and Legendre functions, and they are carried out by relying on the Sturm–Liouville theory. If the expansion is on the $r$ variable, for example, then we bring to light the Sturm–Liouville problem on $R(r)$ and use that problem and the Sturm–Liouville theory to guide our expansion.

Until now, our rule of thumb for choosing the sign of the $\kappa^2$ separation constant has been to choose the sign that makes oscillatory functions available for the eventual Fourier series expansion. If you are familiar with the Sturm–Liouville theory it would be helpful to make that rule more explicit as follows. Choose the sign of $\kappa^2$ so that the sign of the last term in the Sturm–Liouville ODE is positive because in the Sturm–Liouville equation

$$(py')' + qy + \lambda ry = 0$$

the eigenvalues $\lambda$ are generally nonnegative. For instance, in Example 5 we anticipate (from the boundary conditions) that the expansion will be on the $\phi$ variable, so the Sturm–Liouville ODE is (76), not (75). Thus, we choose the $+\kappa^2$ in (74), so that the last term on the left side of (76) is $+\kappa^2\Phi$, not $-\kappa^2\Phi$.

## EXERCISES 20.3

**1.** Show that if $u_1 = u_2 = 0$ and $f(\theta) = 100$, then (13), (15), and (17) give the result (19).

**2.** Solve for $u(r,\theta)$ and sketch, based on intuition, the $u = 25, 50, 75$ isotherms: $\nabla^2 u = 0$

(a) in $1 < r < 2$, $0 < \theta < \pi$; $u(r,0) = u(2,\theta) = 0$, $u(r,\pi) = u(1,\theta) = 100$
(b) in $1 < r < 2$, $0 < \theta < \pi$; $u_\theta(r,0) = u(2,\theta) = 0$, $u(r,\pi) = u(1,\theta) = 100$
(c) in $1 < r < 2$, $0 < \theta < \pi$; $u(r,\pi) = 100$, $u(r,0) = u_r(1,\theta) = u(2,\theta) = 0$
(d) in $1 < r < 2$, $-\infty < \theta < \infty$; $u(1,\theta) = 0$, $u(2,\theta) = 100$
(e) in $1 < r < 2$, $0 < \theta < \pi$; $u(1,\theta) = u_\theta(r,0) = u_\theta(r,\pi) = 0$, $u(2,\theta) = 100$
(f) in $0 < r < 3$, $0 < \theta < 3\pi/2$; $u(r,0) = u(r,3\pi/2) = 100$, $u(3,\theta) = 0$, $u$ bounded
(g) in $0 < r < 3$, $0 < \theta < 3\pi/2$; $u_\theta(r,0) = u(3,\theta) = 0$, $u(r,3\pi/2) = 100$, $u$ bounded
(h) in $0 < r < 3$, $0 < \theta < 3\pi/2$; $u(r,0) = u_r(3,\theta) = 0$, $u(r,3\pi/2) = 100$, $u$ bounded
(i) in $0 < r < 3$, $0 < \theta < 3\pi/2$; $u(r,0) = u(r,3\pi/2) = 0$, $u(2,\theta) = 100$ on $0 < \theta < \pi/2$ and 0 on $\pi/2 < \theta < 3\pi/2$, $u$ bounded
(j) in $3 < r < \infty$, $0 < \theta < \pi/2$; $u(r,0) = u(r,\pi/2) = 0$, $u(3,\theta) = 100$, $u$ bounded
(k) in $3 < r < \infty$, $0 < \theta < \pi$; $u(r,0) = u_\theta(r,\pi) = 0$, $u(3,\theta) = 100$
(l) in $0 < r < 2$, $0 < \theta < \pi$; $u(r,0) = u(r,\pi) = 0$, $u(2,\theta) = 100$

**3.** Solve for $u(r,\theta)$ and give a labeled plot (by computer if necessary) of representative isotherms, as many as it takes to give a clear picture of the temperature field: $\nabla^2 u = 0$ in $r < 1$, $u$ bounded, $u(1,0) = f(\theta)$.

(a) $f(\theta) = 50 + 20\cos\theta$
(b) $f(\theta) = 50 + 50(\cos\theta + \sin\theta)$
(c) $f(\theta) = 20\cos 2\theta$
(d) $f(\theta) = 20\sin 2\theta$
(e) $f(\theta) = 20\cos 3\theta$
(f) $f(\theta) = 20\sin 3\theta$
(g) $f(\theta) = 20\cos 4\theta$
(h) $f(\theta) = 20\cos 5\theta$

**4.** Consider a thin flat circular plate of radius $b$, that is thermally insulated on its two flat faces. With a hacksaw we make a radial cut along $\theta = 0$, say, from $r = b$ to $r = 0$. The small gap, due to the cut, may be approximated as a thermal insulator, so that $\partial u/\partial n = 0$ on the edges $\theta = 0$ and $\theta = 2\pi$. If the circumference of the plate is held at the temperature $50(1+\sin\theta)$ for a long time, the steady-state temperature field

$u(r,\theta)$ is governed by the boundary-value problem

$$\nabla^2 u = 0, \qquad (0 < r < b,\ 0 < \theta < 2\pi)$$
$$\frac{\partial u}{\partial n}(r,0) = \frac{\partial u}{\partial n}(r,2\pi) = 0,$$
$$u(b,\theta) = 50(1+\sin\theta), \quad u \text{ bounded.}$$

Solve for, and plot, the temperature distributions $u(r,0)$ and $u(r,2\pi)$ along the two edges of the cut. Does the answer depend in any way on whether the plate is steel or brass or whatever? Explain.

**5.** (*Plane with circular hole*) As a sort of "inverse" of Example 2, consider the domain to be the whole plane, with a circular *hole* of radius $a$:

$$\nabla^2 u = 0, \qquad (a < r < \infty)$$
$$u(a,\theta) = f(\theta), \quad u \text{ bounded as } r \to \infty.$$

Solve for $u(r,\theta)$, leaving expansion coefficients in integral form. What is the value of $u$ at $r = \infty$?

**6.** (*Plane flow over a circular bump*) First, read Example 3 of Section 16.10.

(a) Then, solve (38) in Section 16.10 and derive the solution

$$\Phi(r,\theta) = U\left(r + \frac{a^2}{r}\right)\cos\theta + C, \tag{6.1}$$

where $C$ is an arbitrary constant that can be set equal to zero without loss. NOTE: The velocity field $\mathbf{v}$ is then available as $\mathbf{v} = \nabla\Phi$. Knowing $\mathbf{v}$, one could use the Bernoulli equation of fluid mechanics (which is derived in Exercise 12 of Section 16.10) to determine the pressure field and, in particular, the resulting aerodynamic force on the semicircular bump (which might, for instance, be the roof of a building). Observe that in this problem *a boundedness condition on* $\Phi$ *would be inappropriate* since $\Phi \sim Ur\cos\theta$ as $r \to \infty$. Nevertheless, the physical quantity $\mathbf{v} = \nabla\Phi$ is bounded: $\mathbf{v} \sim U\hat{\mathbf{i}}$ as $r \to \infty$. Finally, observe that (6.1) can be expressed as the superposition $\Phi = \Phi_1 + \Phi_2$, where $\Phi_1 = Ur\cos\theta = Ux$ is the potential of the "free stream," and $\Phi_2 = U(a^2/r)\cos\theta$ accounts for the disturbance caused by the presence of the bump (indeed, $\Phi_2 \to 0$ as $a \to 0$).

(b) By way of graphics, the most interesting display is not a display of constant $\Phi$ curves but a display of representative streamlines, as in Fig. 7 of Section 16.10. By streamlines we mean the constant $\Psi$ curves, where $\Psi$ is the stream function introduced in Exercise 8 of Section 16.10. From (8.3) therein, $\Psi$ is related to $\Phi$ according to

$$\frac{\partial\Psi}{\partial x} = -\frac{\partial\Phi}{\partial y}, \qquad \frac{\partial\Psi}{\partial y} = \frac{\partial\Phi}{\partial x}. \tag{6.2a,b}$$

Use (6.2) and (6.1) to derive the result

$$\Psi(x,y) = U\left(y - \frac{a^2 y}{x^2+y^2}\right). \tag{6.3}$$

(c) Then (with $U = a = 1$, say) use computer software such as the *Maple* implicitplot command to generate the streamline pattern that we sketched in Fig. 7 (Section 16.10). Choose the streamlines through the points $(x,y) = (-4,0.2), (-4,0.8), (-4,1.4)$, and $(-4,2)$, say.

**7.** (*Flow past a circular cylinder; nonuniqueness*) In Exercise 6 we consider the flow of a free stream past a semicircular bump. Here, we consider the flow of a free stream past a circular cylinder. The boundary-value problem is

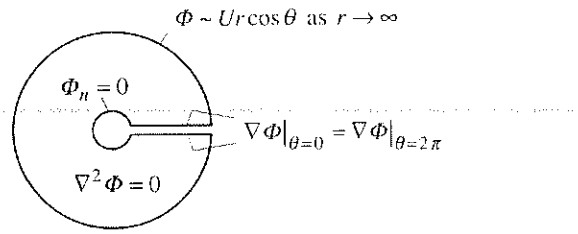
$$\nabla^2\Phi = \Phi_{rr} + \frac{1}{r}\Phi_r + \frac{1}{r^2}\Phi_{\theta\theta} = 0,$$
$$(a < r < \infty,\ 0 < \theta < 2\pi)$$
$$\Phi_n(a,\theta) = 0,$$
$$\nabla\Phi\Big|_{\theta=0} = \nabla\Phi\Big|_{\theta=2\pi} \quad \text{over } a < r < \infty,$$
$$\Phi \sim Ur\cos\theta \quad \text{as } r \to \infty.$$

(7.1a,b,c,d)

That is, if we specify that $0 < \theta < 2\pi$, the radial lines $\theta = 0$ and $\theta = 2\pi$ $(a < r < \infty)$ become part of the boundary (see the accompanying sketch), so that boundary conditions are needed along these lines. The appropriate boundary condition is that physical velocity $\mathbf{v}$ be the same at $\theta = 0$ and $\theta = 2\pi$ (for all $a < r < \infty$), which condition is expressed as (7.1c) or

$$\Phi_r(r,0) = \Phi_r(r,2\pi), \qquad (a < r < \infty)$$
$$\frac{1}{r}\Phi_\theta(r,0) = \frac{1}{r}\Phi_\theta(r,2\pi). \qquad (a < r < \infty)$$

(7.1e,f)




Integrating (7.1e) with respect to $r$ and cancelling $r$'s in (7.1f), we obtain

$$\Phi(r,0) = \Phi(r,2\pi) + \Gamma, \quad \Phi_\theta(r,0) = \Phi_\theta(r,2\pi) \qquad (7.1g,h)$$

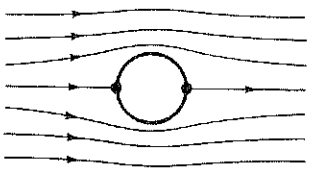
on $a < r < \infty$, where $\Gamma$ is an arbitrary constant.

(a) Solve the resulting boundary-value problem (7.1a,b,g,h,d) and show that

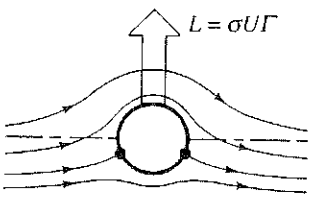
$$\Phi(r,\theta) = U\left(r + \frac{a^2}{r}\right)\cos\theta - \frac{\Gamma}{2\pi}\theta, \qquad (7.2)$$

which solution differs from (7.1) (with $C = 0$, say) only by the $-\Gamma\theta/2\pi$ term. NOTE: In physical terms, the $U(r + a^2r^{-1})\cos\theta$ term in (7.2) corresponds to a flow that is symmetric about the $x$ axis (as sketched in the figure below), and which has *stagnation points* (i.e., where $\mathbf{v} = \mathbf{0}$) at $r = a$ and $\theta = 0, \pi$. The $-\Gamma\theta/2\pi$ term contributes the additional velocity

$$\mathbf{v} = \nabla\left(-\frac{\Gamma}{2\pi}\theta\right) = -\frac{\Gamma}{2\pi r}\hat{\mathbf{e}}_\theta,$$



which is a clockwise circular vortex flow (see Exercise 3 of Section 16.5) induced by a fictitious clockwise *vortex*, of strength $\Gamma$, at the origin – "fictitious" because there is no fluid inside the circle $r = a$. The vector superposition of these two velocity contributions gives a flow somewhat as we have sketched in the next figure, namely, the symmetric flow (the preceding figure) plus some clockwise "swirl" that is proportional to $\Gamma$.




The two flows add on the upper part of the cylinder and subtract on the lower part, so there are higher velocities on the upper surface of the cylinder and lower velocities on the lower surface. Since Bernoulli's equation $\sigma v^2/2 + p =$ constant ($\sigma =$ mass density, $p =$ pressure) tells us that the higher the velocity the lower the pressure, and vice versa, it follows that a *lift force* is generated on the cylinder, $L = \sigma U\Gamma$ force per unit length of the cylinder (i.e., per unit $z$ length).

(b) Show that the two stagnation points (in the preceding figure) are located on the cylinder surface by the equation $\sin\theta = -\Gamma/(4\pi Ua)$; e.g., if $\Gamma = 0$ then $\theta = \pi$ and $2\pi$, as in the symmetric-flow figure. What happens regarding the existence and location of stagnation points if $\Gamma > 4\pi Ua$? Explain.

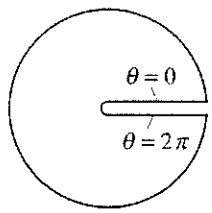
(c) From (7.2), $\mathbf{v} = \nabla\Phi$, and the Bernoulli equation, obtain the pressure distribution on the cylinder, integrate it, and thus derive the famous **Kutta–Joukowski lift formula**

$$L = \sigma U\Gamma \qquad (7.3)$$

stated above.

**8.** (*Alternative approach to Example 2*) In Example 2 we observe that $-\infty < \theta < \infty$ and impose the periodicity condition (28). Alternatively, we can consider that $0 < \theta < 2\pi$, in which case the lines $\theta = 0$ and $\theta = 2\pi$ occur as boundary edges of the domain. That is, we make an infinitely thin slit in the region as shown in the figure below. Since $0 < \theta < 2\pi$, we discard the $2\pi$-periodicity condition, but we now have two artificially created boundaries, $\theta = 0$ and $\theta = 2\pi$, along which to specify boundary conditions. In particular, we impose the two conditions

$$\begin{aligned} u(r,0) &= u(r,2\pi), \\ u_\theta(r,0) &= u_\theta(r,2\pi) \end{aligned} \qquad (8.1,2)$$




over $0 < r < b$, so that both the temperature and heat flux are continuous across the slit. With the problem reformulated in this manner, solve for $u(r,\theta)$ and show that the solution obtained is the same as in Example 2. NOTE: The boundedness condition at $r = 0$ is still needed.

**9.** Derive the result

$$\frac{1}{2} + \sum_{n=1}^{\infty}\left(\frac{r}{b}\right)^n \cos n(\vartheta - \theta) = \frac{1}{2}\,\frac{b^2 - r^2}{b^2 - 2br\cos(\vartheta - \theta) + r^2} \qquad (9.1)$$

stated in Comment 2 of Example 2. HINT: Write

$$\sum_{n=1}^{\infty}\left(\frac{r}{b}\right)^n \cos n(\vartheta-\theta) = \operatorname{Re}\sum_{n=1}^{\infty}\left(\frac{r}{b}\right)^n e^{in(\vartheta-\theta)}, \quad (9.2)$$

and use the *geometric series*

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}, \quad (9.3)$$

which holds even if $z$ is complex, provided that $|z| < 1$. As usual, Re denotes the real part of the quantity.

**10.** Below Example 2 we show that the average value property of the Laplace equation holds in two dimensions, and at the end of Example 5 we showed that it holds in three dimensions. Here, we ask you to show that it holds for the *one*-dimensional Laplace equation $d^2u/dx^2 = 0$ as well.

**11.** Use the maximum and minimum principles to show, for the Laplace equation (in two or three dimensions), that if we change the boundary values only slightly, then the solution values (i.e., within the solution domain) change only slightly too. Specifically, show that if $\nabla^2 u_1 = 0$ in $\mathcal{D}$ with boundary values $u_1 = f_1$, and $\nabla^2 u_2 = 0$ in $\mathcal{D}$ with boundary values $u_2 = f_2$, then $\min(f_1 - f_2) \le u_1 - u_2 \le \max(f_1 - f_2)$.

**12.** (*Delta function behavior of Poisson kernel*) With $b = 1$ and $\theta = 1$, say, obtain computer plots of the Poisson kernel $P(r, \vartheta - \theta)$ versus $\vartheta$, from $\vartheta = -\pi$ to $\vartheta = +\pi$, for these $r$ values: $r = 0, 0.5, 0.8$, and $0.9$. Using *Maple*, for instance, you can use the plot command, which can be accessed by first typing the with(plots): command. NOTE: Surely, if the boundary temperature in (36) is $f(\vartheta) = \text{constant} = 1$, then the solution will be $u(r,\theta) = \text{constant} = 1$ as well. It follows that $\int_{-\pi}^{\pi} P(r; \vartheta - \theta)\, d\vartheta = 1$ for all $0 \le r < b$. Thus, each of your plots, for different $r$ values, will have unit area. Further, they become increasingly focused at $\vartheta = \theta$ as $r \to b$. Thus, it appears that $P(r, \vartheta - \theta)$ is a *delta sequence* at $\theta$ as $r \to b$. In fact, the boundary condition requires of (36) that

$$f(\theta) = \lim_{r\to b}\int_{-\pi}^{\pi} P(r, \vartheta - \theta) f(\vartheta)\, d\vartheta,$$

which result confirms our suspicion: as $r \to b$, the Poisson kernel becomes a delta function at $\vartheta = \theta$, which picks out the value $f(\theta)$ and satisfies the boundary condition. This result is typical of linear PDE's: The solution due to Dirichlet boundary data $f$ can be expressed as an integration, over the boundary, of a kernel times the boundary values. As a boundary point is approached from within the domain, the kernel becomes a delta function and picks out the value of $f$ at that boundary point, thereby satisfying the boundary condition.

**13.** Consider the Dirichlet problem

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0 \quad (13.1)$$

in $a < r < b$, $0 < \theta < \alpha$, with boundary conditions

$$\begin{aligned} u(r,0) = u(a,\theta) = u(b,\theta) = 0, \\ u(r,\alpha) = f(r). \end{aligned} \quad (13.2,3)$$

(a) Seeking $u(r,\theta) = R(r)\Theta(\theta)$ and anticipating the Fourier expansion along the $\theta = \alpha$ edge, obtain

$$\begin{aligned} u(r,\theta) &= (A + B\ln r)(E + F\theta) \\ &+ [C\cos(\kappa \ln r) + D\sin(\kappa \ln r)](G\cosh\kappa\theta + H\sinh\kappa\theta). \end{aligned}$$

(b) Applying the boundary conditions (13.2), arrive at

$$u(r,\theta) = \sum_{n=1}^{\infty} I_n \sin\left(\kappa_n \ln\frac{r}{a}\right)\sinh\kappa_n\theta, \quad (13.4)$$

where

$$\kappa_n = n\pi/\ln\left(\frac{b}{a}\right). \quad (13.5)$$

(c) Applying the boundary condition (13.3), show that

$$\begin{aligned} I_n &= \frac{1}{\sinh\kappa_n\alpha}\,\frac{\displaystyle\int_a^b f(r)\phi_n(r)\frac{1}{r}\,dr}{\displaystyle\int_a^b \phi_n^2(r)\frac{1}{r}\,dr} \\ &= \frac{2}{\ln\left(\dfrac{b}{a}\right)\sinh\kappa_n\alpha}\int_a^b f(r)\phi_n(r)\frac{1}{r}\,dr, \end{aligned} \quad (13.6)$$

where $\phi_n(r) = \sinh[\kappa_n \ln(r/a)]$. HINT: The $\phi_n$'s are the eigenfunctions of the Sturm – Liouville problem on $R(r)$. Identify that problem (i.e., the ODE, the $r$ interval, the boundary conditions on $R$, and the weight function).

EXERCISES FOR THE OPTIONAL SECTIONS 20.3.2, 20.3.3

**14.** Derive the $\kappa \ne 0$ part of the general solution (43) using equation (50) in Section 4.6.

**15.** Derive the $\kappa \ne 0$ part of the general solution (57) using equation (50) in Section 4.6.

**16.** Solve by separation of variables, leaving expansion coefficients in integral form; $u$ is to be bounded, and

$$\nabla^2 u = u_{rr} + \frac{1}{r}u_r + u_{zz} = 0$$

(a) in $0 \le r < b$, $0 < z < \infty$, with $u(b,z) = 0$, $u(r,0) = f(r)$
(b) in $0 \le r < b$, $-\infty < z < \infty$, with $u(b,z) = f(z)$, where $f(z)$ is a $2L$-periodic square wave defined over one period as 100 over $0 < z < L$ and 0 over $L < z < 2L$
(c) in $0 \le r < b$, $0 < z < L$, with $u_z(r,0) = u_z(r,L) = 0$, $u(b,z) = 50$
(d) in $0 \le r < b$, $0 < z < L$, with $u_z(r,0) = u(r,L) = 0$, $u(b,z) = 50$
(e) in $0 \le r < b$, $0 < z < L$, with $u_z(r,0) = f(r)$, $u(r,L) = u(b,z) = 0$
(f) in $a \le r < \infty$, $0 < z < L$, with $u(r,0) = u_1$, $u(r,L) = u_2$, $u(a,z) = u_3$
(g) in $a \le r < \infty$, $0 < z < \infty$, with $u(r,0) = 0$, $u(a,z) = 25\sin(3z/2)$

**17.** Show that the change of variables $\mu = \cos\phi$ does, indeed, change (76) to the Legendre equation (77).

**18.** In Example 5, let $f(\phi)$ be 0 on the bottom half of the sphere ($\pi/2 < \phi < \pi$, $-1 < \mu < 0$) and 100 on the top half ($0 < \phi < \pi/2$, $0 < \mu < 1$). Evaluating the first several $A_n$'s, show that (81) becomes

$$u(\rho,\phi) = 50\left[P_0(\cos\phi) + \frac{3}{2}\left(\frac{\rho}{c}\right)P_1(\cos\phi) - \frac{7}{8}\left(\frac{\rho}{c}\right)^3 P_3(\cos\phi) + \frac{11}{16}\left(\frac{\rho}{c}\right)^5 P_5(\cos\phi) - \frac{75}{128}\left(\frac{\rho}{c}\right)^7 P_7(\cos\phi) + \cdots\right].$$

HINT: It is simplest to evaluate the integral $\int_{-1}^{1} f P_n\, d\mu = 100\int_0^1 P_n(\mu)\, d\mu$ using computer software. Using *Maple*, for example, first enter with(orthopoly): to access the Legendre polynomials. Then, with $n = 5$ say, enter int($P(5,x), x = 0..1$); to evaluate $\int_0^1 P_5(\mu)\, d\mu$.

**19.** (*Variations on Example 5*) Solve the Laplace equation (71) in spherical coordinates, with symmetry about the $z$ axis, and evaluate the expansion coefficients so as to obtain the first five (if there are that many) nonvanishing terms of the series solutions, as we did in Exercise 18:

(a) in $0 \le \rho < c$, $0 < \phi < \pi/2$, with $u(\rho,\pi/2) = 0$, $u(c,\phi) = 100$
(b) in $0 \le \rho < c$, $0 < \phi < \pi/2$, with (the normal derivative) $u_n(\rho,\pi/2) = 0$, $u(c,\phi) = 100$
(c) in $c < \rho < \infty$, $0 < \phi < \pi/2$, with $u(\rho,\pi/2) = 0$, $u(c,\phi) = 100$
(d) in $c < \rho < \infty$, $0 < \phi < \pi/2$, with $u_\rho(c,\phi) = 0$, $u(\rho,\pi/2) = 100$

## 20.4 Fourier Transform (Optional)

In Section 18.4 we studied the solution of diffusion problems by the Fourier and Laplace transforms. Laplace transforming on the $t$ variable is always an option for the diffusion equation because $0 < t < \infty$, and the problem is of initial-value type with respect to $t$. However, it is of boundary-value type with respect to $x$ (i.e., there is a boundary condition at each end) so, alternatively, we can use a Fourier transform on $x$ if the $x$ domain is $-\infty < x < \infty$ or a Fourier cosine or sine transform on $x$ if the domain is $0 < x < \infty$.

In contrast, the Laplace equation is of boundary-value type in both independent variables, so the Laplace transform is not helpful. Still, if the domain is infinite in one of the independent variables then we can employ a Fourier transform on that variable; if it is semi-infinite then we can employ a Fourier cosine or sine transform.

**EXAMPLE 1.** *Dirichlet Problem for Half Plane.* Consider the half-plane problem

$$\nabla^2 u = u_{xx} + u_{yy} = 0, \qquad (-\infty < x < \infty,\ 0 < y < \infty) \tag{1a}$$

$$u(x,0) = f(x) \qquad (-\infty < x < \infty) \tag{1b}$$

depicted in Fig. 1, realizing that we may be stipulating additional boundary conditions as we proceed.

Fourier transform (1a) with respect to $x$:

$$F\{u_{xx} + u_{yy}\} = F\{0\}, \tag{2a}$$

$$F\{u_{xx}\} + F\{u_{yy}\} = 0, \tag{2b}$$

$$(i\omega)^2\hat{u} + \int_{-\infty}^{\infty} \frac{\partial^2 u}{\partial y^2}\, e^{-i\omega x}\, dx = 0, \tag{2c}$$

$$-\omega^2\hat{u} + \frac{d^2}{dy^2}\int_{-\infty}^{\infty} u(x,y)e^{-i\omega x}\, dx = 0, \tag{2d}$$

or

$$\frac{d^2\hat{u}}{dy^2} - \omega^2\hat{u} = 0, \tag{3}$$

with general solution

$$\hat{u}(\omega, y) = Ae^{|\omega|y} + Be^{-|\omega|y}. \tag{4}$$

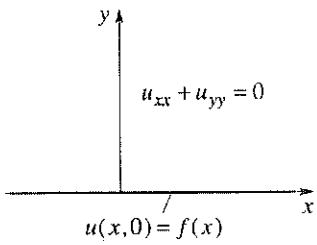



**Figure 1.** Problem (1).

(Let us defer discussion of the absolute value signs to Comment 1 below.) Recall from our study of the Fourier transform that for $F\{u_{xx}\}$ to equal $(i\omega)^2\hat{u}$ we need

$$u \to 0 \text{ and } u_x \to 0 \text{ as } x \to \pm\infty, \tag{5}$$

so let us suppose that $u$ does satisfy the boundary conditions (5) to the east ($x \to +\infty$) and to the west ($x \to -\infty$). Condition (1b) is our southern boundary condition, but we are still lacking a second $y$ boundary condition, to the north as $y \to \infty$. If we assume that

$$u(x,y) \to 0 \text{ as } y \to \infty, \tag{6}$$

then we formally obtain

$$\begin{aligned}\lim_{y\to\infty} \hat{u}(\omega, y) &= \lim_{y\to\infty}\int_{-\infty}^{\infty} u(x,y)e^{-i\omega x}\, dx \\ &= \int_{-\infty}^{\infty}\left[\lim_{y\to\infty} u(x,y)\right] e^{-i\omega x}\, dx \\ &= \int_{-\infty}^{\infty} 0\, e^{-i\omega x}\, dx = 0.\end{aligned} \tag{7}$$

Applying the latter result to (4) reveals that we need $A = 0$, so

$$\hat{u}(\omega, y) = Be^{-|\omega|y}. \tag{8}$$

To evaluate $B$ take the transform of (1b),

$$\hat{u}\Big|_{y=0} = \hat{f}(\omega), \tag{9}$$

and impose that condition on (8):

$$\hat{u}\Big|_{y=0} = \hat{f}(\omega) = B. \tag{10}$$

Thus,

$$\hat{u}(\omega, y) = \hat{f}(\omega)e^{-|\omega|y}.$$

If we use entry 1 in Appendix D, along with the Fourier convolution property (entry 21), we obtain the final result

$$u(x, y) = f(x) * \frac{y}{\pi}\frac{1}{x^2 + y^2}, \tag{11}$$

or

$$\boxed{\begin{aligned} u(x, y) &= \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{f(\xi)}{(x-\xi)^2 + y^2}\, d\xi \\ &\equiv \int_{-\infty}^{\infty} P(\xi - x, y) f(\xi)\, d\xi; \end{aligned}} \tag{12}$$

(12) is the **Poisson integral formula for the half plane**, and $P(\xi - x, y)$ is the corresponding **Poisson kernel**. The analogous formula for the circular disk is given in (36) of Section 20.3.

COMMENT 1. Why did we express the solution of (3) as $Ae^{|\omega|y} + Be^{-|\omega|y}$ rather than as

$$\hat{u}(\omega, y) = Ce^{\omega y} + De^{-\omega y}? \tag{13}$$

The forms (4) and (13) are indeed equivalent, but (4) is more convenient for applying the northern boundary condition (7) (namely, that $\hat{u} \to 0$ as $y \to \infty$). For remember that the Fourier inversion formula involves an integral on $\omega$ from $\omega = -\infty$ to $\omega = +\infty$. Thus, we need to allow for $\omega > 0$ and $\omega < 0$ in (13) and conclude, from (7), that $C(\omega) = 0$ for $\omega > 0$ and $D(\omega) = 0$ for $\omega < 0$, which story is more complicated than observing, in (4), that $e^{|\omega|y}$ is the "bad" term and $e^{-|\omega|y}$ is the "good" term so that we need to set $A = 0$.

COMMENT 2. Let us focus our attention on the kernel $P$. Observe that $P$ has unit area, for each $y > 0$, since

$$\int_{-\infty}^{\infty} P(\xi - x, y)\, d\xi = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\xi}{(\xi - x)^2 + y^2} = 1, \tag{14}$$

and that its graph becomes more and more sharply focused (at $\xi = x$) as $y \to 0$, as seen in Fig. 2. Thus, $P(\xi - x, y)$ looks like a delta sequence at $\xi = x$, as indeed must be true since the boundary condition (1b) really means that $\lim_{y\to 0} u(x, y) = f(x)$ or, since $u(x, y)$ is given by (12),

$$\lim_{y\to 0}\int_{-\infty}^{\infty} P(\xi - x, y) f(\xi)\, d\xi = f(x), \tag{15}$$

which, by definition, means that $P$ becomes a delta function at $x$ as $y \to 0$.

COMMENT 3. As a check case, let us use (12) for the case where $u(x, 0) = f(x) =$ constant $= f_0$, since then the solution should, by inspection, be $u(x, y) = f_0$ everywhere. In fact, (12) does give that correct result – even though the assumed conditions at infinity ($u \to 0$ as $x \to \pm\infty$ and as $y \to \infty$) are not satisfied. That is, (12) is even more robust than anticipated. ∎

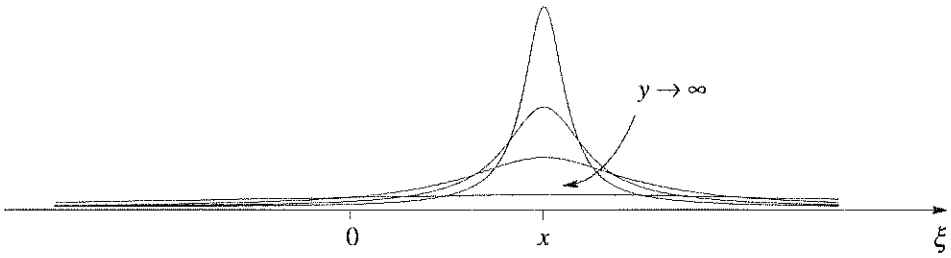



**Figure 2.** Poisson kernel $P$.

**Closure.** Besides illustrating the use of the Fourier transform in solving the Laplace equation on an infinite domain, we also obtain an important specific solution, the Poisson integral formula (12) for the half plane. As usual (see Exercise 12 of Section 20.3), the solution due to Dirichlet boundary data $f$ can be expressed as an integration, over the boundary, of a kernel times the boundary values. As a boundary point is approached from within the domain, the kernel becomes a delta function and picks out the value of $f$ at that point, thereby satisfying the boundary condition.

## EXERCISES 20.4

**1.** (a) Use (12) to evaluate $u(x, y)$ if $f(x) = 100H(x)$ (where $H$ is the Heaviside function).
(b) Draw the isotherms $u = 25, 50, 75$.
(c) Plot $u(x, y)$ versus $x$ at $y = 0, 1, 3$.

**2.** (a) Use (12) to evaluate $u(x, y)$ if $f(x) = 100[H(x+1) - H(x-1)]$.
(b) Plot $u(x, y)$ versus $x$ at $y = 0$ and at $y = 2$.
(c) Show that $u(x, y) \sim 200/(\pi y)$ as $y \to \infty$. HINT: Note that

$$\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$

for $|x| < 1$, and

$$\tan^{-1} x = \frac{\pi}{2} - \frac{1}{x} + \frac{1}{3x^3} - \frac{1}{5x^5} + \cdots$$

for $|x| < 1$ where, in each case, $\tan^{-1}$ denotes the choice (of the multivalued $\tan^{-1}$ function) lying between $-\pi/2$ and $+\pi/2$.

**3.** (a) Show, from (12), that if $f(x)$ is an even function of $x$ then so is $u(x, y)$.
(b) Show, from (12), that if $f(x)$ is an odd function of $x$ then so is $u(x, y)$.

**4.** Use (12) and the method of images (explained in the optional Section 18.5) to solve $\nabla^2 u = u_{xx} + u_{yy} = 0$ in the first quadrant ($x > 0, y > 0$), with $u(x, 0) = f(x)$, with suitable conditions at $x = \infty$ and at $y = \infty$, and with

(a) $u(0, y) = 0$ (b) $u_x(0, y) = 0$

**5.** Consider the infinite strip problem

$$\nabla^2 u = u_{xx} + u_{yy} = 0, \qquad (|x| < \infty,\ 0 < y < a)$$
$$u(x, 0) = f(x), \quad u(x, a) = g(x).$$

(a) Show that

$$u(x, y) = F^{-1}\left\{\hat{f}(\omega)\,\frac{\sinh \omega(a-y)}{\sinh \omega a} + \hat{g}(\omega)\,\frac{\sinh \omega y}{\sinh \omega a}\right\},$$

but do not try to evaluate that Fourier inverse.
(b) With $f(x) = g(x) = 100H(x)$, use intuition to sketch the isotherms, say $u = 10, 25, 50, 75, 90$.

**6.** In Comment 2 we discuss the delta function behavior of the Poisson kernel $P$ as $y \to 0$. In fact, letting $f(x) = \delta(x - x_0)$, show that $P(x_0 - x, y) = P(x - x_0, y)$ is itself the solution or "response" due to a boundary temperature that is a delta function at $x_0$. NOTE: With this result in mind, we can interpret (12) as a superposition principle. For let us break $f$ into narrow vertical rectangles. What is the response $du(x, y)$ due to

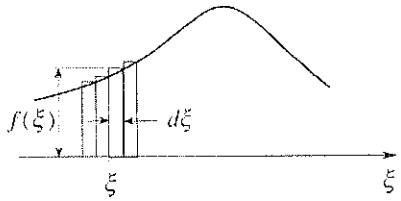
the single shaded rectangular pulse (shown in the figure)? The pulse is a delta function at $\xi$ (as $d\xi \to 0$), so (as noted above) its response at $x, y$ is $P(\xi - x, y)$; actually, the pulse is a delta function *scaled* by $f(\xi)\,d\xi$ because its area is $f(\xi)\,d\xi$ rather than unity, so its response is likewise scaled,

$$du(x,y) = P(x-\xi, y) f(\xi)\,d\xi$$

or, since $P$ is an even function of its first argument,

$$du(x,y) = P(\xi - x, y) f(\xi)\,d\xi. \tag{6.1}$$

And superimposing these infinitesimal responses (by integration) gives (12). Thus, we can understand (12) as a *superposition principle*.




## 20.5 Numerical Solution

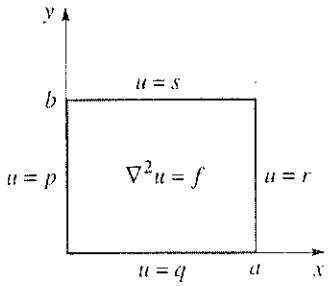



**Figure 1.** The problem (1).

**20.5.1. Rectangular domains.** Following the same general lines as in Section 18.6, where we develop the finite-difference solution technique for the diffusion equation, here we do the same for the Laplace equation or, more generally and with no additional difficulty, for the Poisson equation $\nabla^2 u = f$.

Limiting our attention to the two-dimensional case, we begin with the problem

$$\nabla^2 u = u_{xx} + u_{yy} = f(x,y), \qquad (0 < x < a,\ 0 < y < b) \tag{1a}$$

$$u(0,y) = p(y),\ \ u(x,0) = q(x),\ \ u(a,y) = r(y),\ \ u(x,b) = s(x) \tag{1b}$$

depicted in Fig. 1, and generalize to nonrectangular domains in Section 20.5.2. Seeking an approximate numerical solution, we discretize the problem by dividing $a$ into $M$ equal parts of dimension $\Delta x = a/M$, dividing $b$ into $N$ equal parts of dimension $\Delta y = b/N$, and defining nodal points $P_{jk} = (x_j, y_k) = (j\Delta x, k\Delta y)$ for $j = 0, 1, 2, \ldots, M$ and $k = 0, 1, 2, \ldots, N$. Accordingly, we seek $u$ not everywhere in the domain but only at the nodal points – more specifically, at the interior nodal points since $u$ is prescribed at the boundary nodal points by the Dirichlet boundary conditions (1b). The grid is shown in Fig. 2 for the choice $M = N = 3$.

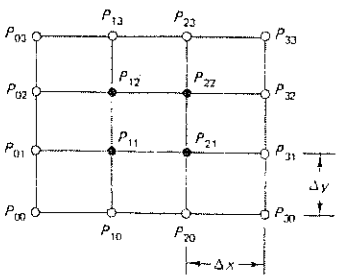



**Figure 2.** Finite-difference mesh for $M = N = 3$.

Next, we replace the PDE (1a) by a finite-difference approximation that will lead to a set of linear algebraic equations in the unknown nodal values of $u$. As in Section 18.6, we adopt the approximations

$$\frac{\partial^2 u}{\partial x^2} \approx \frac{u(x-\Delta x, y) - 2u(x,y) + u(x+\Delta x, y)}{(\Delta x)^2}, \tag{2a}$$

$$\frac{\partial^2 u}{\partial y^2} \approx \frac{u(x, y-\Delta y) - 2u(x,y) + u(x, y+\Delta y)}{(\Delta y)^2}, \tag{2b}$$

Putting (2a, b) into the PDE, with $x = x_j$, $x - \Delta x = x_{j-1}$, $x + \Delta x = x_{j+1}$ (and similarly for $y$) gives

$$\frac{u(x_{j-1}, y_k) - 2u(x_j, y_k) + u(x_{j+1}, y_k)}{(\Delta x)^2} + \frac{u(x_j, y_{k-1}) - 2u(x_j, y_k) + u(x_j, y_{k+1})}{(\Delta y)^2} \approx f(x_j, y_k). \quad (3)$$

Thus, we adopt the algebraic equation

$$\boxed{\frac{U_{j-1,k} - 2U_{j,k} + U_{j+1,k}}{(\Delta x)^2} + \frac{U_{j,k-1} - 2U_{j,k} + U_{j,k+1}}{(\Delta y)^2} = f_{j,k}} \quad (4)$$

as our finite-difference approximation of (1a), where $f_{j,k}$ is shorthand for $f(x_j, y_k)$. As in Section 18.6, we use different letters (lowercase and uppercase) to distinguish between the exact solution $u(x, y)$ of (1) and the approximating solution $U_{j,k}$ generated by the finite-difference equation (4). We call $u(x_j, y_k) - U_{j,k}$ the **truncation error** at $P_{j,k}$, namely, the error incurred by replacing $u_{xx}$ and $u_{yy}$ in (1a) by the finite-difference approximations (2). Observe that whereas in Section 18.6 we distinguish between the *local* truncation error (incurred in carrying out a single time step) and the *accumulated* truncation error (incurred in carrying out all the time steps up until the time in question) – here we do not – because *there are no time steps*. Thus, there is simply "the truncation error."

Suppose that we compute $U_{j,k}$ at a particular point $P$ in the domain, then again using a finer mesh, again using a finer mesh, and so on. If, as the mesh becomes infinitely fine (i.e., as $\Delta x$ and $\Delta y$ both tend to zero) the computed values converge to the exact solution at $P$, then the finite-difference scheme is **convergent**. Recall that in our study of the diffusion equation (Section 18.6) we pay comparable attention to the companion questions of convergence and stability; the difference scheme is said to be stable if the accumulated roundoff error remained small. For the Poisson and Laplace equations, however, we do not "march out" a solution in time, so the issue of stability is not relevant. In fact, roundoff error should be quite negligible compared with the truncation error for the methods considered in this section. If we choose $\Delta x = \Delta y \equiv h$, say, then (4) becomes

$$\boxed{U_{j-1,k} + U_{j,k-1} + U_{j+1,k} + U_{j,k+1} - 4U_{j,k} = h^2 f_{j,k},} \quad (5)$$

which is often expressed, schematically, in the form

$$\begin{bmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{bmatrix} U = h^2 f.$$

If, in addition, $f(x, y) = 0$, so that (1a) reduces to the Laplace equation, then (5) gives

$$U_{j,k} = \frac{1}{4}\left(U_{j-1,k} + U_{j,k-1} + U_{j+1,k} + U_{j,k+1}\right). \quad (6)$$

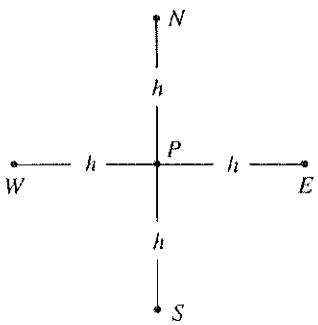



**Figure 3.** The five points.

We call these **five-point formulas** because they involve five grid points, which we denote as $P$, $W$(est), $S$(outh), $E$(ast), and $N$(orth), respectively, in Fig. 3. It is striking that (6) is a discrete and approximate version of the average value property of the two-dimensional Laplace equation, namely, that $u$ at any given point $P$ in the solution domain is the average value of $u$ over any circle centered at $P$ and lying entirely within the domain.

**EXAMPLE 1.** To illustrate the use of (5), let $a = b = 1$ and $f(x, y) = p(y) = q(x) = s(x) = 0$, and $r(y) = 100 \sin \pi y$ so that, by separation of variables, we have the simple exact solution

$$u(x, y) = 100 \frac{\sinh \pi x}{\sinh \pi} \sin \pi y \tag{7}$$

available for comparison with our computed approximate solution. As the simplest (and crudest) case, let $M = N = 2$. Then $\Delta x = \Delta y = 0.5$ and there is only one internal node, (Fig. 4). Writing out (5) for that point (i.e., with $j = k = 1$) gives

$$P_{11}: \quad U_{01} + U_{10} + U_{21} + U_{12} - 4U_{11} = 0 \tag{8}$$

or, recalling the given boundary conditions, $0 + 0 + 100 \sin(\pi/2) - 4U_{11} = 0$. Solving, $U_{11} = 25$ and we have the following comparison.

$$\text{Computed:} \qquad U_{11} = 25 \tag{9a}$$

$$\text{Exact:} \qquad u_{11} = 100 \frac{\sinh(\pi/2)}{\sinh \pi} \sin \frac{\pi}{2} = 19.9, \tag{9b}$$

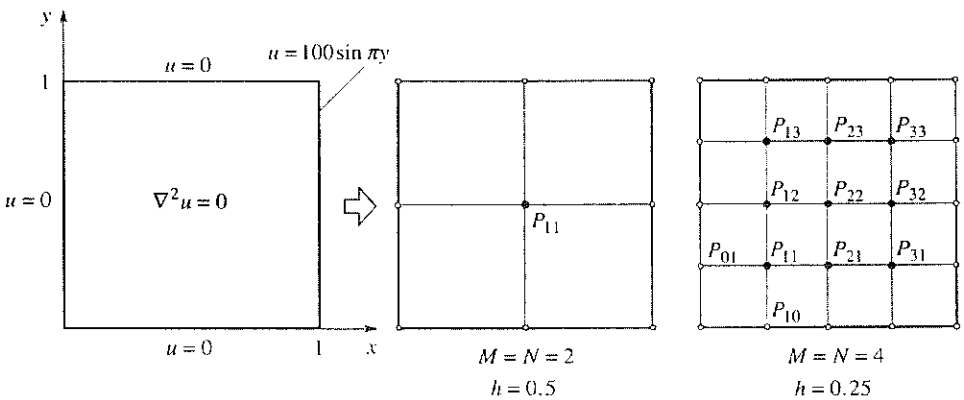



**Figure 4.** Example 1.

where $u_{jk}$ means the exact solution $u(x, y)$ evaluated at $P_{jk}$; note that we will generally omit the comma between the two subscripted indices, for brevity. It is not surprising that the error is so great because the grid is so coarse; that is, $h = 0.5$ is not small compared to $a = b = 1$.

Next, let $M = N = 4$ (right-hand member of Fig. 4). Write out (5) for the nine internal grid points $P_{11}, P_{21}, \ldots, P_{23}, P_{33}$:

$$
\begin{aligned}
P_{11} \qquad & 0 + 0 + U_{21} + U_{12} - 4U_{11} &= 0, \\
P_{21} \qquad & U_{11} + 0 + U_{31} + U_{22} - 4U_{21} &= 0, \\
& \vdots \\
P_{23} \qquad & U_{13} + U_{22} + U_{33} + 0 - 4U_{23} &= 0, \\
P_{33} \qquad & U_{23} + U_{32} + 100 \sin \frac{3\pi}{4} + 0 - 4U_{33} &= 0
\end{aligned}
\tag{10}
$$

or, in matrix form,

$$
\left[\begin{array}{ccc|ccc|ccc}
-4 & 1 & 0 & 1 & & & & \cdots & 0 \\
1 & -4 & 1 & 0 & 1 & & & & \vdots \\
0 & 1 & -4 & 0 & 0 & 1 & & & \\
\hline
1 & 0 & 0 & -4 & 1 & 0 & 1 & & \\
 & 1 & 0 & 1 & -4 & 1 & 0 & 1 & \\
 & & 1 & 0 & 1 & -4 & 0 & 0 & 1 \\
\hline
 & & & 1 & 0 & 0 & -4 & 1 & 0 \\
\vdots & & & & 1 & 0 & 1 & -4 & 1 \\
0 & \cdots & & & & 1 & 0 & 1 & -4
\end{array}\right]
\left[\begin{array}{c}
U_{11} \\ U_{21} \\ U_{31} \\ \hline U_{12} \\ U_{22} \\ U_{32} \\ \hline U_{13} \\ U_{23} \\ U_{33}
\end{array}\right]
=
\left[\begin{array}{c}
0 \\ 0 \\ -100 \sin \frac{\pi}{4} \\ \hline 0 \\ 0 \\ -100 \sin \frac{\pi}{2} \\ \hline 0 \\ 0 \\ -100 \sin \frac{3\pi}{4}
\end{array}\right],
\tag{11}
$$

where matrix elements not shown are zeros and the partitioning lines are to be ignored for the moment. Roughly speaking, the first equation in (10) ensures the satisfaction of the Laplace equation in the neighborhood of $P_{11}$, the second equation in (10) ensures the satisfaction of the Laplace equation in the neighborhood of $P_{21}$, and so on, so that the satisfaction of (10) is equivalent to the approximate satisfaction of the Laplace equation in the entire domain (as well as the Dirichlet boundary conditions).

Solving (11) by a computer algebra system (e.g., using the *Maple* linsolve command described at the end of Section 8.3), we obtain these values.

$$
\begin{aligned}
\text{Computed:} \qquad & U_{11} = 5.8, \quad U_{21} = 15.1, \quad U_{31} = 33.2, \\
& U_{12} = 8.3, \quad U_{22} = 21.3, \quad U_{32} = 46.9, \\
& U_{13} = 5.8, \quad U_{23} = 15.1, \quad U_{33} = 33.2
\end{aligned}
\tag{12a}
$$

$$
\begin{aligned}
\text{Exact:} \qquad & u_{11} = 5.3, \quad u_{21} = 14.1, \quad u_{31} = 32.0, \\
& u_{12} = 7.5, \quad u_{22} = 19.9, \quad u_{32} = 45.3, \\
& u_{13} = 5.3, \quad u_{23} = 14.1, \quad u_{33} = 32.0.
\end{aligned}
\tag{12b}
$$

These results are seen to be in better agreement than (9a) and (9b) but are still quite crude. If this were a realistic application, rather than only an illustration of the method, we might choose $h$ to be 0.05 or smaller.

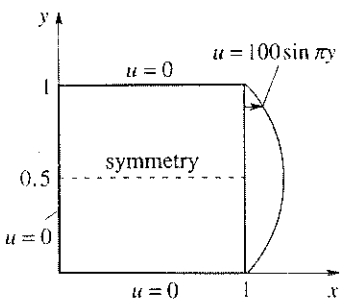



**Figure 5.** Symmetry.

COMMENT 1. Observe from Fig. 5 that both the domain and the boundary conditions are symmetric about the mid-line $y = 0.5$, so it is evident that the solution $u(x, y)$ should,

likewise, be symmetric about that line.* Thus, our numerical solution, for the case where $M = N = 4$, is wasteful because we know in advance that

$$U_{13} = U_{11}, \quad U_{23} = U_{21}, \quad U_{33} = U_{31}, \tag{13}$$

so there are really only six unknowns rather than nine. It suffices to apply (5) to the six points $P_{11}, P_{21}, P_{31}, P_{12}, P_{22}, P_{33}$, and to use (13). Thus, the reduced system is as follows:

$$\begin{aligned}
P_{11}: &\quad 0 + 0 + U_{21} + U_{12} - 4U_{11} = 0,\\
P_{21}: &\quad U_{11} + 0 + U_{31} + U_{22} - 4U_{21} = 0,\\
P_{31}: &\quad U_{21} + 0 + 100\sin\tfrac{\pi}{4} + U_{32} - 4U_{31} = 0,\\
P_{12}: &\quad 0 + U_{11} + U_{22} + \underline{U}_{11} - 4U_{12} = 0,\\
P_{22}: &\quad U_{12} + U_{21} + U_{32} + \underline{U}_{21} - 4U_{22} = 0,\\
P_{32}: &\quad U_{22} + U_{31} + 100\sin\tfrac{\pi}{2} + \underline{U}_{31} - 4U_{32} = 0,
\end{aligned} \tag{14}$$

where the underlined terms are those that result from the symmetry relations (13).

COMMENT 2. Observe that (11) may be partitioned, according to the thin lines in (11), as

$$\begin{bmatrix} \mathbf{B} & \mathbf{I} & & \cdots & \mathbf{0} \\ \mathbf{I} & \mathbf{B} & \mathbf{I} & & \vdots \\ & & \ddots & & \\ \vdots & & \mathbf{I} & \mathbf{B} & \mathbf{I} \\ \mathbf{0} & \cdots & & \mathbf{I} & \mathbf{B} \end{bmatrix} \begin{bmatrix} \mathbf{U}_1 \\ \mathbf{U}_2 \\ \vdots \\ \mathbf{U}_{N-2} \\ \mathbf{U}_{N-1} \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 \\ \mathbf{c}_2 \\ \vdots \\ \mathbf{c}_{N-2} \\ \mathbf{c}_{N-1} \end{bmatrix}, \tag{15}$$

where $\mathbf{B}$ is the $(N-1) \times (N-1)$ matrix

$$\mathbf{B} = \begin{bmatrix} -4 & 1 & & \cdots & 0 \\ 1 & -4 & 1 & & \vdots \\ & & \ddots & & \\ \vdots & & 1 & -4 & 1 \\ 0 & \cdots & & 1 & -4 \end{bmatrix}, \tag{16}$$

and $\mathbf{I}$ is an identity matrix of order $N-1$. In (11), for instance, $N = 4$ so $\mathbf{B}$ and $\mathbf{I}$ are $3 \times 3$. Each vector is comprised of the unknown nodal values across the $j$th row of the mesh. Whereas $\mathbf{B}$ is tridiagonal, $\mathbf{A}$ is not tridiagonal; it is **block** tridiagonal. ∎

The method is powerful because it enables us to obtain solutions even if the inputs $p(y), q(x), r(y), s(x)$, and $f(x, y)$ are nonconstant functions, in which case

*It surely seems clear, if only intuitively, that the solution is symmetric about the line $y = 0.5$ as claimed, but to put that claim on solid ground we can put forward arguments similar to those given in Section 18.5 on the method of images. We will leave that point for the exercises.

analytical solution becomes extremely laborious. However, it is to be appreciated that the computer calculation is not trivial if we seek good accuracy because (in the absence of helpful symmetries) we need to solve a system of $(M-1)\times(N-1)$ linear algebraic equations. If, for instance, we choose $M = N = 50$, for the sake of accuracy, then we have a system of 2,401 equations! In such cases one concern is how to solve that large system of equations *efficiently*, to which topic we return in Section 20.5.3. First, in Section 20.5.2, we indicate how to extend the method to handle domains of essentially arbitrary shape.

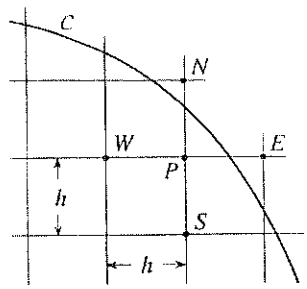



**Figure 6.** Nonrectangular domain.

**20.5.2. Nonrectangular domains.** Thus far we have dealt with cases where the boundary curve is rectangular so that grid lines can coincide with the edges of the domain as in Fig. 4, that is, where the mesh "fits" the domain. What happens if the mesh does not fit, as illustrated in Fig. 6? We cannot apply the finite-difference scheme (5) at grid points such as $P$ because the points $N$ and $E$ do not fall on the boundary curve $C$; they fall outside the domain. To handle this case we slide $N$ and $E$ so that they do fall on $C$, as shown in Fig. 7, and revise the difference quotient approximations (2a,b) accordingly. [We need to modify (2a), for instance, because it gives $u_{xx}$ at $P$ as a linear combination of the values of $u$ at $W$, $P$, and $E$. If we move $E$, then the weighting of those values will change; we can expect $U_E$ to be weighted more heavily than $U_W$ because $E$ is closer to $P$ than $W$. Similarly for (2b).] For the geometry shown in Fig. 7 we need to slide $N$ and $E$, but in other cases we may need to slide $W$ and/or $S$ as well, so let us consider the most general case shown in Fig. 8, where $0 < \alpha \le 1$, $0 < \beta \le 1$, $0 < \gamma \le 1$, and $0 < \delta \le 1$.

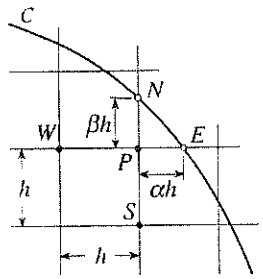



**Figure 7.** Adjusting the mesh near $C$.

We begin with Taylor expansions about $P$ (namely, the point $x_j, y_k$) in the eastern and western directions, respectively,

$$u(x_j+\alpha h, y_k) = u(x_j, y_k) + u_x(x_j, y_k)\alpha h + \frac{1}{2!}u_{xx}(x_j, y_k)(\alpha h)^2 + \cdots,$$

$$u(x_j-\gamma h, y_k) = u(x_j, y_k) + u_x(x_j, y_k)(-\gamma h) + \frac{1}{2!}u_{xx}(x_j, y_k)(-\delta h)^2 + \cdots,$$

or, using $N, E, S, W, P$ subscript notation instead,

$$u_E = u_P + u_x|_P\,\alpha h + \frac{1}{2}\,u_{xx}|_P\,\alpha^2 h^2 + \cdots, \tag{17a}$$

$$u_W = u_P - u_x|_P\,\gamma h + \frac{1}{2}\,u_{xx}|_P\,\gamma^2 h^2 - \cdots. \tag{17b}$$

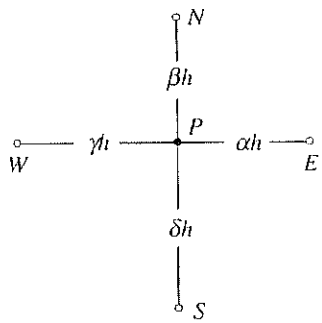



**Figure 8.** Most general case.

Multiplying (17a) by $\gamma$ and (17b) by $\alpha$ and adding, to cancel the $u_x$ terms, gives

$$\gamma u_E + \alpha u_W = (\gamma+\alpha)u_P + \frac{1}{2}(\alpha^2\gamma + \alpha\gamma^2)h^2\,u_{xx}|_P + \cdots, \tag{18}$$

so that if we neglect terms of order $h^3$ and higher in (18) then we obtain

$$u_{xx}|_P \approx \frac{2}{\alpha\gamma(\alpha+\gamma)h^2}\left[\gamma u_E + \alpha u_W - (\gamma+\alpha)u_P\right] \tag{19a}$$

in place of (2a). Similarly, Taylor expansions about $P$ in the northern and southern directions give the result

$$u_{yy}|_P \approx \frac{2}{\beta\delta(\beta+\delta)h^2}\left[\delta u_N + \beta u_S - (\delta+\beta)u_P\right] \tag{19b}$$

in place of (2b). Using (19a,b), our finite-difference approximation to the Poisson equation $u_{xx} + u_{yy} = f(x, y)$, at $P$, becomes

$$\boxed{\begin{aligned}&\frac{2}{\gamma(\gamma+\alpha)}U_W + \frac{2}{\delta(\delta+\beta)}U_S + \frac{2}{\alpha(\alpha+\gamma)}U_E \\ &\qquad + \frac{2}{\beta(\beta+\delta)}U_N - 2\frac{\alpha\gamma+\beta\delta}{\alpha\beta\gamma\delta}U_P = h^2 f_P.\end{aligned}} \tag{20}$$

Naturally, if $\alpha = \beta = \gamma = \delta = 1$, then (20) reduces to (5).

**EXAMPLE 2.** To illustrate the use of (20), let us solve the Poisson problem shown in Fig. 9a, using the grid shown in Fig. 9b. Obviously, the grid is quite coarse, but it should suffice for the purpose of illustration. In this case the source term is $(f(x, y) = 5x - y$ and we have chosen $h = 2$. Since there are not many grid points it will be more convenient to denote the points as $a, b, \ldots, r, s$ rather than with the double subscript notation. We need to write (20) for each of the internal grid points $o, p, q, r, s$. Thus, we need to determine $\alpha, \beta, \gamma, \delta$ for each of these points. At $r$, for instance, we compute $\alpha$ from $\overline{rg} \equiv \alpha h$ (where $\overline{rg}$ denotes the length of the line $rg$). Thus, $\alpha = \overline{rg}/h = (x_g - x_r)/h = (5.5-4)/2 = 0.75$. Next, $\overline{rg} \equiv \beta h$ gives $\beta = \overline{rg}/h = h/h = 1$, $\overline{rk} \equiv \gamma h$ gives $\gamma = (x_r - x_k)/h = (4 - \sqrt{4^2 - 2^2})/2 \approx 0.27$, and $\overline{rj} \equiv \delta h$ gives $\delta = 1$. Similarly at $o, p$, and $s$, so we have these values:

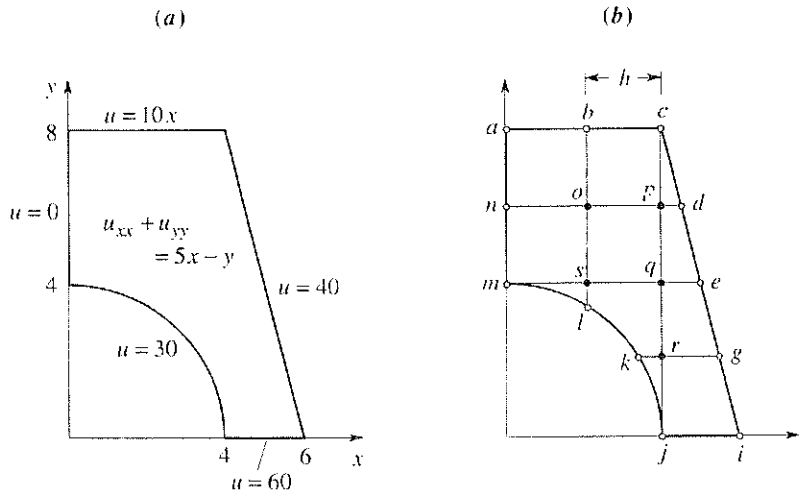



**Figure 9.** Example 2.

$$
\begin{aligned}
o: &\quad \alpha = \beta = \gamma = \delta = 1, \\
p: &\quad \alpha = 0.25, \ \ \beta = \gamma = \delta = 1, \\
q: &\quad \alpha = 0.5, \ \ \beta = \gamma = \delta = 1, \\
r: &\quad \alpha = 0.75, \ \ \beta = 1, \ \ \gamma \approx 0.27, \ \ \delta = 1, \\
s: &\quad \alpha = \beta = \gamma = 1, \ \ \delta \approx 0.27.
\end{aligned}
$$

Thus, writing (20) at these points gives the equations

$$
\begin{aligned}
o: &\quad U_n + U_s + U_p + U_b - 4U_o = 2^2[5(2) - 6], && \text{(21a)} \\
p: &\quad \frac{2}{1.25}U_o + U_q + \frac{2}{0.25(1.25)}U_d + U_c - 2\frac{1.25}{0.25}U_p \\
&\qquad = 2^2[5(4) - 6], && \text{(21b)} \\
q: &\quad \frac{2}{1.5}U_s + U_r + \frac{2}{0.5(1.5)}U_e + U_p - 2\frac{1.5}{0.5}U_q \\
&\qquad = 2^2[5(4) - 4], && \text{(21c)} \\
r: &\quad \frac{2}{0.27(1.02)}U_k + U_j + \frac{2}{0.75(1.02)}U_g + U_q - 2\frac{1.20}{0.20}U_r \\
&\qquad = 2^2[5(4) - 2], && \text{(21d)} \\
s: &\quad U_m + \frac{2}{0.27(1.27)}U_l + U_q + \frac{2}{1.27}U_o - 2\frac{1.27}{0.27}U_s \\
&\qquad = 2^2[5(2) - 4]. && \text{(21e)}
\end{aligned}
$$

With $U_n = 0$, $U_b = 20$, $U_c = U_d = U_e = U_g = 40$, $U_j = (60 + 30)/2 = 45$, $U_k = U_l = 30$, and $U_k = U_l = 30$, and $U_m = (30 + 0)/2 = 15$ from the boundary conditions (where we've used average values at the corners $j$ and $m$, as suggested in Section 18.6), (21) becomes

$$
\begin{aligned}
-4U_o + U_p + U_s &= -4, && \text{(22a)} \\
1.6U_o - 10U_p + U_q &= -240, && \text{(22b)} \\
U_p - 6U_q + U_r + 1.33U_s &= -42.7, && \text{(22c)} \\
U_q - 12U_r &= -295.5, && \text{(22d)} \\
1.57U_o + U_q - 9.41U_s &= -166, && \text{(22e)}
\end{aligned}
$$

with the solution

$$
U_o = 13.6, \ \ U_p = 28.3, \ \ U_q = 21.1, \ \ U_r = 26.4, \ \ U_s = 22.2. \qquad \text{(23)}
$$

COMMENT. If these values don't look correct, relative to the given boundary values, don't forget that besides the boundary condition inputs there is also the internal source distribution $f(x, y) = 5x - y$. Mathematically, we call the forcing term $f$ in $\nabla^2 u = f$ a "source" term. However, if we think of this problem in terms of steady-state heat conduction, then we need to recall, from (39) in Section 16.8, that the heat source term there has a minus sign in front of it. Therefore, the $f(x, y) = 5x - y$ term in our PDE is, in physical terms,

a heat *sink*, and it is the presence of that heat sink distribution that causes the interior temperatures to be lower than we would expect if we considered only the boundary conditions. For instance, if there were no sink distribution ($f = 0$) then we would expect $U_p$ to be around 35, rather than the value 28.3 given in (23). ∎

**20.5.3. Iterative algorithms. (Optional)** The resulting systems of linear algebraic equations on the $U_{jk}$'s are of the form $\mathbf{AU} = \mathbf{c}$, where $\mathbf{A}$, typically, is quite large. To illustrate, suppose that the domain is square (as in Example 1) and that $j = 0, 1, 2, \ldots, N$ and $k = 0, 1, 2, \ldots, N$. Then there are $(N-1)^2$ unknown $U_{jk}$'s and $\mathbf{A}$ is $(N-1)^2 \times (N-1)^2$. For instance, in Example 1, $N = 4$ so $\mathbf{A}$ is $9 \times 9$. If, for greater accuracy, we choose $N = 25$, say, then $\mathbf{A}$ is $576 \times 576$. Thus, it is important to develop *efficient* methods of solution of the typically large systems of linear algebraic equations that arise.

Recall that a similar difficulty arose in Section 18.6, where the choice of an implicit (rather than explicit) scheme led to coupled (tridiagonal) systems of linear algebraic equations. However, that situation was not nearly as difficult since the problem was of initial-value type and we merely needed to solve for one time-line of unknowns at a time. Thus, with $N = 25$, say, the $\mathbf{A}$ matrix was only $24 \times 24$ rather than $576 \times 576$! It's true that we need to solve these 24th-order systems for each time step, but (supposing that there are 24 time steps, say) it is much easier to solve twenty four 24th-order systems than to solve one 576th-order system.*

Fortunately, the $\mathbf{A}$ matrix is strongly diagonal, so that we can use the same iterative techniques that were described in Section 18.6. For instance, suppose that the rectangular grid fits the domain so that we can use the scheme (5) rather than its generalization (20). Because the $-4$ coefficient of $U_{jk}$ dominates the other coefficients on the left-hand side of (5) we can, to a first approximation, write $-4U_{jk} \approx h^2 f_{jk}$. Solving the latter for the $U_{jk}$'s and calling them $U_{jk}^{(0)}$ gives

$$U_{jk}^{(0)} = -\frac{h^2}{4} f_{jk}. \tag{24}$$

Next, we put those values into the thus-far-neglected first four terms on the left-hand side of (5), transpose them to the right, and obtain the improved values

$$U_{jk}^{(1)} = \frac{1}{4}\left(U_{j-1,k}^{(0)} + U_{j,k-1}^{(0)} + U_{j+1,k}^{(0)} + U_{j,k+1}^{(0)} - h^2 f_{jk}\right).$$

Repeating this process gives the **Jacobi** iterative scheme

$$\boxed{U_{jk}^{(n+1)} = \frac{1}{4}\left(U_{j-1,k}^{(n)} + U_{j,k-1}^{(n)} + U_{j+1,k}^{(n)} + U_{j,k+1}^{(n)} - h^2 f_{jk}\right)} \tag{25}$$

*If the truth behind this claim is not obvious to you, try changing the numbers: wouldn't you rather solve twenty 2nd-order systems than one 40th-order system?

for $n = 0, 1, 2, \ldots$, with the "starting values" given by (24).† As discussed in Section 18.6 for the diffusion version of (25), we can improve on (25) by using the latest iterates as soon as they become available and moving systematically across the first row (left to right), then the second, and so on. Known as the **Gauss–Seidel** or **Liebmann** method, it is expressed as

$$\boxed{U_{jk}^{(n+1)} = \frac{1}{4}\left(U_{j-1,k}^{(n+1)} + U_{j,k-1}^{(n+1)} + U_{j+1,k}^{(n)} + U_{j,k+1}^{(n)} - h^2 f_{jk}\right).} \tag{26}$$

The latter converges more rapidly than the Jacobi method and is more readily programmed.

Re-expressing (26) as

$$\begin{aligned} U_{jk}^{(n+1)} &= U_{jk}^{(n)} + \frac{1}{4}\left(U_{j-1,k}^{(n+1)} + U_{j,k-1}^{(n+1)} + U_{j+1,k}^{(n)} + U_{j,k+1}^{(n)} + U_{jk}^{(n)} - h^2 f_{jk}\right) \\ &\equiv U_{jk}^{(n)} + \Delta U_{jk}^{(n)}, \end{aligned} \tag{27}$$

we can insert a numerical control parameter $\omega$ as follows,

$$\boxed{U_{jk}^{(n+1)} = U_{jk}^{(n)} + \omega \Delta U_{jk}^{(n)},} \tag{28}$$

and choose $\omega$ so as to speed the convergence. It has been shown that the optimal value* of $\omega$ is

$$\omega_{\text{opt}} = 2\frac{1 - \sin(\pi/N)}{\cos^2(\pi/N)}. \tag{29}$$

For large $N$, $\omega_{\text{opt}} \sim 2$. Since $\omega > 1$ amounts to an overcorrection, in (28), the method (28) is called **successive overrelaxation**, or SOR, for brevity. We will omit a numerical illustration of these methods because the ideas are the same as for the diffusion equation; see Example 3 of Section 18.6.

One final point. We stated that the **A** matrix is strongly diagonal, but that situation will be obtained only if we write the scalar equations on the $U_{jk}$'s in the correct sequence, which we clarify by means of the following example.

**EXAMPLE 3.** Applying (5) to the Laplace problem shown in Fig. 10 gives the scalar equations

$$\begin{aligned} a: \quad -4U_a + U_b &= -100, && \text{(30a)} \\ b: \quad U_a - 4U_b + U_c &= -50, && \text{(30b)} \\ c: \quad U_b - 4U_c + U_d &= -30, && \text{(30c)} \\ d: \quad U_c - 4U_d &= -90 && \text{(30d)} \end{aligned}$$

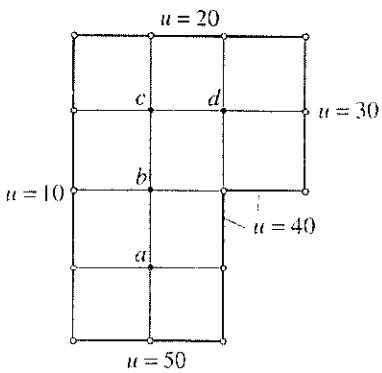



**Figure 10.** Example 3.

†Or, equivalently and more easily programmed, we can take $U_{j,k}^{(0)} = 0$. Then (25) gives $U_{j,k}^{(1)} = -(h^2/4)f_{jk}$, which is identical to the right-hand side of (24), so the subsequent iterates are the same as before.

*S. P. Frankel, "Convergence Rates of Iterative Treatments of Partial Differential Equations," *Mathematical Tables and other Aids to Computation*, Vol. 4, 1950, pp. 65–75.

or, in matrix form,

$$\begin{bmatrix} -4 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 \\ 0 & 1 & -4 & 1 \\ 0 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} U_a \\ U_b \\ U_c \\ U_d \end{bmatrix} = \begin{bmatrix} -100 \\ -50 \\ -30 \\ -90 \end{bmatrix}. \tag{31}$$

However, if we interchange the scalar equations (30a) and (30c), say, then we obtain, in place of (31), the equivalent system

$$\begin{bmatrix} 0 & 1 & -4 & 1 \\ 1 & -4 & 1 & 0 \\ -4 & 1 & 0 & 0 \\ 0 & 0 & 1 & -4 \end{bmatrix} \begin{bmatrix} U_a \\ U_b \\ U_c \\ U_d \end{bmatrix} = \begin{bmatrix} -30 \\ -50 \\ -100 \\ -90 \end{bmatrix}. \tag{32}$$

Thus, whereas the matrix in (31) is strongly diagonal, the one in (32) is not.

To obtain the strongly diagonal form, in any given example, proceed as follows. Order the elements of the $\mathbf{U}$ vector (in $\mathbf{AU} = \mathbf{c}$) any way you like, but then be sure to write the scalar equations in the same order, as we did in (30) and (31). ∎

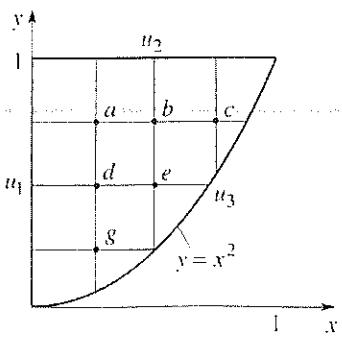
**Closure.** We derive the finite-difference schemes (4) and (if $\Delta x = \Delta y = h$) (5), and the generalization (20) for curvilinear boundaries. We illustrate their implementation in Examples 1 and 2, respectively, using coarse grids for simplicity. Efficient iterative solution techniques, that are needed for fine grids (i.e., for large $\mathbf{A}$ matrices), are described in the optional Section 20.5.3. Dirichlet boundary conditions are the simplest and are used throughout, although other boundary conditions are considered in the exercises.

## EXERCISES 20.5

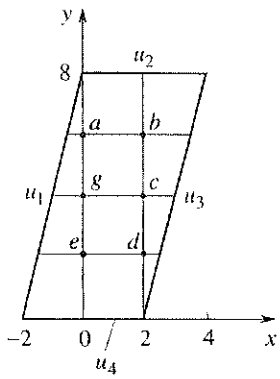
**1.** Consider the Poisson problems $u_{xx}+u_{yy} = f(x,y)$ labeled A, B, C, D in the accompanying figures, each with Dirichlet boundary conditions. Write the linear algebraic equations governing the unknown nodal values and solve for those values, either by hand or by computer.

(a) Problem A with $u_1(y) = 10y$, $u_2 = u_3 = 20$, $f(x,y) = -20xy$.
(b) Problem A with $u_1 = u_2 = 0$, $u_3 = 50$, $f(x,y) = 0$.
(c) Problem A with $u_1 = u_2 = u_3 = 0$, $f(x,y) = -100$.
(d) Problem B with $u_1 = u_2 = u_3 = u_4 = 0$, $f(x,y) = -50$.
(e) Problem B with $u_2 = 100$, $u_1 = u_3 = u_4 = f(x,y) = 0$.
(f) Problem B with $u_1 = u_2 = u_3 = 0$, $u_4 = 200$, $f(x,y) = x^2 + y^2$.
(g) Problem C with $u_1 = u_2 = u_3 = u_4 = 0$, $u_5 = 50$, $f(x,y) = -20$.
(h) Problem C with $u_1 = u_2 = u_3 = u_4 = u_5 = 100$, $f(x,y) = 30$.
(i) Problem C with $u_1 = u_3 = u_4 = u_5 = 0$, $u_2(x) = 10x$, $f(x,y) = 50(x^2 - y^2)$.
(j) Problem D with $u_1 = u_3 = f(x,y) = 0$, $u_2 = u_4 = u_5 = 100$. HINT: Note the symmetry about $y = 4$.
(k) Problem D with $u_1 = u_3 = u_4 = f(x,y) = 0$, $u_2 = 50$, $u_5 = -50$. HINT: Note the antisymmetry about $y = 4$, so $u(x,4) = 0$. You may wish to work the problem twice: first, using the noted antisymmetry and then again, this time not using it (in case you have any doubts).
(l) Problem D with $u_1 = u_3 = u_4 = 0$, $u_2 = u_5 = 50$, $f(x,y) = 10(x^2 + y^2)$.

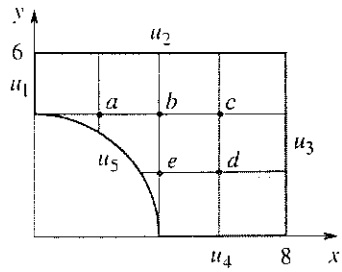
*A.*



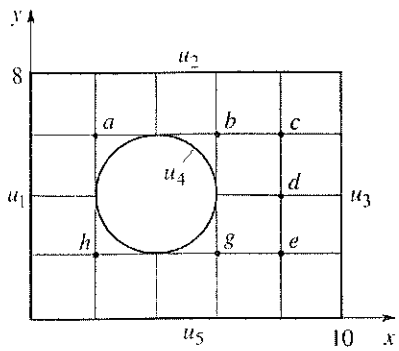

*B.*




*C.*




*D.*

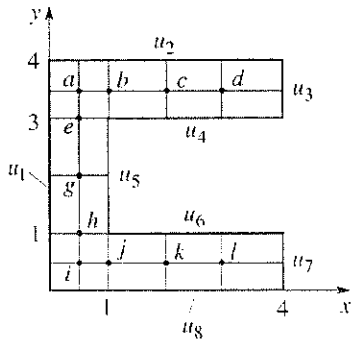



**2.** Consider the Poisson equation $u_{xx} + u_{yy} = f(x,y)$ on the domain between two nested squares, one with corners at $(1,1), (-1,1), (-1,-1), (1,-1)$, and the other with corners at $(0.6, 0.6), (-0.6, 0.6), (-0.6, -0.6), (0.6, -0.6)$. Let $\Delta x = \Delta y = h = 0.2$. Given $f(x,y)$ and the boundary conditions, use (5) to solve for $u$ at each of the 32 nodal points using any symmetries or antisymmetries that are present to re-

duce the number of unknowns.

(a) $f(x,y) = -100$, $u = 0$ on inner and outer boundaries
(b) $f(x,y) = 0$, $u = 0$ on inner boundary, $u = 100$ on outer boundary
(c) $f(x,y) = 0$, $u(0.6,y) = u(1,y) = u(x,0.6) = u(x,1) = 50, u(-0.6,y) = u(-1,y) = u(x,-0.6) = u(x,-1) = -50$
(d) $f(x,y) = 0, u(-1,y) = u(-0.6,y) = u(0.6,y) = u(1,y) = 0, u(x,-1) = u(x,-0.6) = -20, u(x,0.6) = u(x,1) = 20$

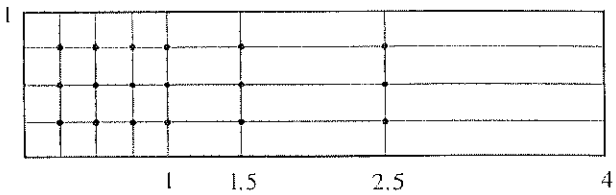
**3.** We developed equation (20), based on the pattern in Fig. 8, to enable us to handle the domains with irregular boundaries. However, we can also use (20) [or, equivalently, (4) with $\Delta x \neq \Delta y$] to increase the nodal point density in regions where greater resolution is needed, as indicated in the figure below. Solve for $U_a, U_b, \ldots, U_l$ using any symmetries or antisymmetries that are present to reduce the number of unknowns. HINT: At $a$ use $h = 0.5$, at $b$ use $h = 1$ and so on.




(a) $f(x,y) = -10$, $u_1 = \cdots = u_8 = 0$
(b) $f(x,y) = 0$, $u_1 = u_2 = u_4 = u_5 = u_6 = u_8 = 0$, $u_3 = u_7 = 50$
(c) Same as (b), but with $u_7$ changed to $-50$.

**4.** First, read Exercise 3. For the Laplace equation on the domain shown, with the Dirichlet boundary conditions $u(0,y) = 100$, $u(x,0) = u(x,1) = u(4,y) = 0$, $u(x,y)$ will vary rapidly only near the left end. Thus, let us bunch the nodal points as shown.

(a) Solve for $u$ at the 21 nodal points using the finite-difference method. HINT: Use symmetry to reduce the number of unknowns to 12.
(b) Compare your computed values at the six nodal points on the horizontal centerline $y = 0.5$ with the exact values, obtained by separation of variables.




**5.** Show that (20) agrees with (4), as it should, if we set $\alpha h = \gamma h = \Delta x$ and $\beta h = \delta h = \Delta y$.

**6.** (*Empirical estimate of the order of the method*) (a) Show that the truncation error in (5) is $O(h^2)$ so the method is of second order.
(b) To test the assertion in part (a), note that if the method is of order $p$ that means that

$$u(x_j, y_k) - U_{j,k} \sim Ch^p \tag{6.1}$$

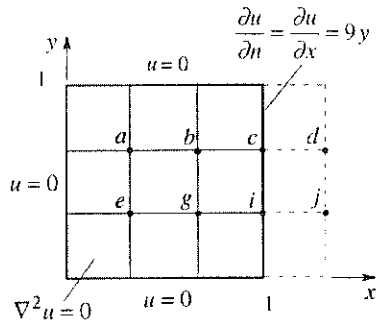
as $h \to 0$, where $x_j, y_k$ is any fixed field point within the domain, $u(x_j, y_k)$ is the exact solution there, $U_{j,k}$ is the computed solution there, and $C$ is some constant. It suffices to use a concrete example. For that purpose, use the point $x = y = 0.5$ in Example 1. We found that $u(0.5, 0.5) = 19.9$, with $h = 0.5$ we obtained $U_{1,1} = 25$ there, and with $h = 0.25$ we obtained $U_{2,2} = 21.3$ there. Writing (6.1) for each of these two cases gives two equations in the unknowns $C$ and $p$. Solving for $p$, show that $p \approx 1.87$. NOTE: In (6.1) $U_{j,k}$ denotes the value obtained using a perfect computer, one with no roundoff error, and that is virtually the case since the roundoff error should be extremely small compared to the truncation error.
(c) In obtaining $p \approx 1.87$, in part (b), we treated (6.1) as an equation (i.e., with an equal sign), whereas it is only true as $h \to 0$. Thus, we can expect a better empirical estimate of $p$ if we use two successive small values of $h$ such as 1/4 and 1/6, rather than 1/2 and 1/4. Here, we ask you to compute $U$ at $x = 0.5$, $y = 0.5$ using $h = 1/6$ (in which case there will be 25 equations in 25 unknowns, which can be reduced to 15 equations in 15 unknowns by using symmetry) and to use the $h = 1/4$ and $h = 1/6$ results to obtain a more accurate estimate of $p$. NOTE: Remember that when we say $h$ is small we mean small relative to the size of the domain. In this case the domain is a unit square so the values $h = 1/4$ and $h = 1/6$ are not especially small. If, instead, the domain were of dimension $50 \times 50$, then these $h$ values would be quite small.

**7.** (*Neumann and Robin boundary conditions*) First, recall the forward, backward and central difference quotients.

$$f'(x) \approx \frac{f(x+h) - f(x)}{h}, \qquad \text{(forward)}$$

$$f'(x) \approx \frac{f(x) - f(x-h)}{h}, \qquad \text{(backward)}$$
$$f'(x) \approx \frac{f(x+h) - f(x-h)}{2h}, \qquad \text{(central)}$$
(7.1,2,3)

with truncation errors that are $O(h)$, $O(h)$, and $O(h^2)$, respectively. Thus far in this section we have considered boundary conditions only of Dirichlet type. To see how to handle a Neumann condition consider the representative problem shown in the figure.




(a) Writing out equation (5) at $a, b, e, g$ gives four equations in six unknowns $U_a, U_b, U_c, U_e, U_g$, and $U_i$. To apply the Neumann condition $u_x(1,y) = 9y$ at $c$ and $i$, use the backward difference quotient (7.2), show that the resulting system is

$$\begin{bmatrix} -4 & 1 & 0 & 1 & 0 & 0 \\ 1 & -4 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & -4 & 1 & 0 \\ 0 & 1 & 0 & 1 & -4 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} U_a \\ U_b \\ U_c \\ U_e \\ U_g \\ U_i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad (7.4)$$

and solve for $U_a, \ldots, U_i$.

(b) However, whereas the first, second, fourth, and fifth scalar equations in (7.4) [namely, those resulting from the application of (5)] have a truncation error that is $O(h^2)$ the third and sixth [those resulting from the application of (7.2) to the Neumann boundary condition] have a truncation error that is $O(h)$. Just as "the chain is as weak as the weakest link," these $O(h)$ errors contaminate the whole system (7.4) and cause an overall truncation error that is $O(h)$. To prevent this reduction in accuracy we can use the central difference quotient (7.3) instead of the backward difference quotient (7.2). The idea is to extend the domain as indicated by the dashed lines. Applying (5) at $a, b, c, e, g, i$ gives six equations in the eight unknowns. To apply the Neumann condition at $c$ and $i$ use (7.3), thereby obtaining two more equations. The result is eight linear algebraic equations on $U_a, \ldots, U_j$. Obtain that system and solve it for $U_a, \ldots, U_j$. NOTE: Of course, in the end you can discard the auxiliary values $U_d$ and $U_g$.

(c) Solve the problem exactly, by separation of variables, and compare your values of $U_c$, say, from parts (a) and (b), with the exact values at $a, b, e$, and $g$.

(d) With the Neumann boundary condition $u_x(1,y) = 9y$ changed to the Robin boundary condition

$$u_x(1,y) + 3u(1,y) = 9y,$$

modify (7.4) accordingly.

**8.** (*Other elliptic PDE's*) The Laplace and Poisson equations are *elliptic*, and the methods developed in this section are applicable to other elliptic PDE's as well. In each case, first verify that the PDE is elliptic. Then derive a finite-difference scheme (with $\Delta x = \Delta y = h$) analogous to (5), using (2a), (2b), and central difference quotients (see Exercise 7) for first-order derivatives. Then, apply the finite-difference scheme at each interior node for the case where the domain is $0 < x < 1$, $0 < y < 1$ with $u(x,0) = 0$, $u(0,y) = 20$, $u(x,1) = 50$, $u(1,y) = 10y$, and with $h = 1/3$. (You need not solve the resulting system of equations.)

(a) $(1 + x^2)u_{xx} + u_{yy} = 0$
(b) $u_{xx} + 2u_{yy} - u_x = 4$
(c) $u_{xx} + u_{yy} - u = 20x - 5y$
(d) $u_{xx} + (1 + x^2 + y^2)u_{yy} + 2u_y = 15(x^2 + y^2)$

# Chapter 20 Review

Recall that the PDE

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = f, \tag{1}$$

where $A, \ldots, F, f$ may be functions of $x$ and $y$, is classified as hyperbolic, parabolic, or elliptic as follows:

$$\begin{array}{ll} \text{hyperbolic} & \text{if } B^2 - AC > 0, \\ \text{parabolic} & \text{if } B^2 - AC = 0, \\ \text{elliptic} & \text{if } B^2 - AC < 0, \end{array} \tag{2}$$

the classification depending only on the coefficients $A, B, C$ of the second-order derivatives. The prototypical equations considered in these chapters are the hyperbolic wave equation $c^2u_{xx} - u_{tt} = 0$ (with $t$ in place of $y$), the parabolic diffusion equation $\alpha^2 u_{xx} - u_t = 0$, and the elliptic Laplace equation $u_{xx} + u_{yy} = 0$.

Observe from (2) that the parabolic case is on the borderline between the hyperbolic and elliptic cases, so it is not surprising to learn, in these chapters, that parabolic equations share some properties with hyperbolic equations and some with elliptic equations. For example:

(1) *Both the hyperbolic wave equation and the parabolic diffusion equation are of initial-value type (in the t variable), whereas the elliptic Laplace equation is of boundary-value type.* For instance for a wave or diffusion problem defined on $0 < x < L$ and $0 < t < \infty$ the solution at any $x, t$ point depends on the initial data (at $t = 0$) and on the boundary data up to time $t$ but not beyond, whereas for a Laplace problem defined on $0 < x < L$ and $0 < y < \infty$ the solution at any $x, y$ point depends on the data at $y = 0$ and on the data at $x = 0$ and $x = L$ over the entire interval $0 < y < \infty$. The initial-value/boundary-value distinction is especially important in numerical solution by the finite-difference method. For instance, contrast the finite-difference solutions of a diffusion problem on $0 < x < L$, $0 < t < T$ with $\Delta x = L/N$ and $\Delta t = T/M$, and a Laplace problem on $0 < x < L$, $0 < y < Y$ with $\Delta x = L/N$ and $\Delta y = Y/M$. Because of its initial-value type, the diffusion problem admits an explicit solution or, at worst, requires the solution of an $(N-1)$th-order matrix equation for each time-line of nodal values. In contrast, the boundary-value type Laplace problem requires us to solve for all the unknown nodal values at once; that is, we need to solve an $[(M-1)(N-1)]$th-order matrix equation.

(2) *For the hyperbolic wave equation discontinuities or kinks in initial or boundary data propagate into the solution domain, whereas for the parabolic diffusion equation and the elliptic Laplace equation they do not; they are smoothed upon "entering" the solution domain.* For instance, contrast Fig. 4 in Section 18.4, where the initial condition is a Heaviside step function and where the resulting solution is a smooth (even infinitely differentiable) function of

$x$ for all $t > 0$, and Fig. 4 in Section 19.4, where the initial deflection is "kinky" and where those kinks propagate into the solution domain.

We can summarize these comparisons, for mnemonic purposes, in the following tabular form.

| | | |
|---|---|---|
| Wave Equation: | kinks propagate, | initial-value type |
| Diffusion Equation: | smooth, | initial-value type |
| Laplace equation: | smooth, | boundary-value type |

Let us return from this comparison of the wave, diffusion, and Laplace equations to the present chapter on the Laplace equation. Included are the solutions to particularly well known Dirichlet problems, namely, for the rectangle (Section 20.2), circular disk (Section 20.3.1), circular cylinder with axisymmetry (Section 20.3.2), sphere with axisymmetry (Section 20.3.3), and half plane (Section 20.4).

Prominent theoretical results include the average value property and maximum principle. The *average value property* is that if $\nabla^2 u = 0$ in a two-dimensional (or three-dimensional) domain $\mathcal{D}$, then $u$ at any point $P'$ within $\mathcal{D}$ is equal to the average value of $u$ around any circle (or on any sphere) centered at $P'$ and lying entirely within $\mathcal{D}$. And the **maximum (minimum) principle** is that $u$ cannot attain its maximum (or minimum) value in $\mathcal{D}$ (unless $u$ is a constant everywhere); it must attain its maximum (or minimum) on the boundary of $\mathcal{D}$.

# Chapter 21

# Functions of a Complex Variable

## 21.1 Introduction

There is much to recommend the study of complex variable theory. In this text on applied mathematics, we should state first that the subject is of great importance in applications. For example, it may be recalled from our discussion of the Laplace transform that the inverse Laplace transformation is given as a contour integral in a complex $s$ plane. Not yet having studied complex variable theory, we were forced to avoid the inversion formula and to rely, instead, on the use of transform tables. More generally, the evaluation of a wide class of definite integrals (even along the real axis) is facilitated by use of the complex integral calculus. Another particularly important application is in the use of conformal mapping to solve boundary-value problems in two-dimensional potential theory (i.e., governed by the two-dimensional Laplace equation).

Also important about complex variable theory is that it serves to "complete" our understanding, from the calculus, of real-valued functions of a single real variable. For example, in studying Taylor series one finds that the expansion

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots$$

holds only for $|x| < 1$. Yet the function $1/(1+x^2)$ is infinitely differentiable for *all* $x$, and offers no clue as to why the interval of convergence should be anything less than infinite. It is only when we consider instead the function $1/(1+z^2)$, where $z = x + iy$, that the source of the difficulty comes into view, namely, singularities at $z = \pm i$, off of the real axis, in the complex $z$ plane.

Our sequence of topics echoes the usual format of the real variable calculus. Beginning in Chapter 21 with a discussion of *complex numbers* and the complex plane, we next introduce the notion of a complex valued *function* of a (single) complex variable and then define and discuss a number of *elementary functions*, such as the exponential, trigonometric, and hyperbolic functions. Defining a *limit concept* and *continuity*, we are able in the final section to define a particularly important

limit, the *derivative*. Thus, in this chapter, we get as far as the differential calculus of functions of a complex variable.

The complex integral calculus and series expansions are then developed in Chapters 23 and 24. In between is Chapter 22 on conformal mapping. That chapter is not a prerequisite for Chapters 23 or 24. Thus, in a shorter course one might cover only Chapters 21 and 22, or, instead, Chapters 21, 23, and 24.

## 21.2 Complex Numbers and the Complex Plane

Historically, complex numbers were created several hundred years ago, within the context of the theory of equations. For if one allowed only real numbers, then equations such as $x^2 + 1 = 0$ and $x^2 + 2x + 4 = 0$ had no solution. Thus, in a step that was slow to gain general acceptance, a broader number system was devised so that the equations given above, and indeed every polynomial equation, possess solutions within that number system.* Eventually named **complex numbers** by *Carl Friedrich Gauss* (1777–1855), these new numbers were of the form $a + ib$ where $a$ and $b$ are real and where $i$ satisfies the equation $i^2 = -1$. It is important to understand that the plus sign in $a + ib$ does not denote addition; rather, $a + ib$ is a single number, not the sum of $a$ and $ib$.

The (real) numbers $a, b$ are called the **real part** and **imaginary part** of $a + ib$, respectively:

$$\mathrm{Re}(a + ib) \equiv a, \qquad \mathrm{Im}(a + ib) \equiv b \quad (\text{not } ib). \tag{1}$$

We do not distinguish between $a + ib$ and $a + bi$, and we generally write $a + i0$ as $a$, and $0 + ib$ as $ib$, for brevity. The former complex number is said to be *purely real* and the latter is said to be *purely imaginary*. Finally, two complex numbers are said to be *equal* if their real and imaginary parts, respectively, are equal; that is,

$$a_1 + ib_1 = a_2 + ib_2 \tag{2}$$

holds if and only if $a_1 = a_2$ and $b_1 = b_2$.

Beyond introducing complex numbers, we will need an algebra for their manipulation. The idea will be to stay as close as possible to the rules of ordinary arithmetic (i.e., governing real numbers). For example, the rules of ordinary arithmetic would seem to dictate that

$$(a_1 + ib_1) + (a_2 + ib_2) = (a_1 + a_2) + i(b_1 + b_2)$$

and that

$$\begin{aligned}(a_1 + ib_1)(a_2 + ib_2) &= a_1a_2 + ia_1b_2 + ib_1a_2 + i^2b_1b_2 \\ &= (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2).\end{aligned}$$

*Perhaps the reluctance to accept complex numbers can be better appreciated if we mention that, before complex numbers, there had even been reluctance to accept *negative* numbers. After all, how could one have a negative amount of something. (Of course, that was before the invention of the credit card.)

But, be sure to see that these equalities have no logical support since, as noted above, $a_1 + ib_1$ is a single complex number, not $a_1$ plus $ib_1$; similarly for $a_2 + ib_2$. Rather, we have merely been trying to motivate reasonable definitions for the addition and multiplication of complex numbers. Accordingly, we now *define*

$$(a_1 + ib_1) + (a_2 + ib_2) \equiv (a_1 + a_2) + i(b_1 + b_2) \tag{3}$$

and

$$(a_1 + ib_1)(a_2 + ib_2) = (a_1a_2 - b_1b_2) + i(a_1b_2 + b_1a_2). \tag{4}$$

For instance, $(3 - 2i) + (5 + i) = 8 - i$ and $(3 - 2i)(5 + i) = 17 - 7i$.

Before proceeding, let us agree to denote complex numbers by a single letter, usually $z$, for brevity. Thus, we write $z = a + ib$. With the definitions above, it is readily verified that the familiar rules of algebra hold for complex numbers. For instance, if we denote any three complex numbers as $z_1 = a_1 + ib_1$, $z_2 = a_2 + ib_2$, and $z_3 = a_3 + ib_3$, then

$$z_1 + z_2 = z_2 + z_1, \qquad z_1z_2 = z_2z_1 \qquad \text{(commutative)} \tag{5}$$

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3), \qquad (z_1z_2)z_3 = z_1(z_2z_3) \qquad \text{(associative)} \tag{6}$$

$$z_1(z_2 + z_3) = z_1z_2 + z_1z_3 \qquad \text{(distributive).} \tag{7}$$

Further, there are zero and unit complex numbers since, from (3),

$$z + (0 + i0) = z$$

and, from (4),

$$(1 + i0)z = z$$

for all $z$. Thus, for brevity, we write $0 + i0 = 0$ and $1 + i0 = 1$. Finally, $-z \equiv (-1)z$, and the *subtraction* of complex numbers is defined, in terms of the already-defined operations of addition and multiplication, by $z_1 - z_2 \equiv z_1 + (-z_2) = z_1 + (-1)z_2$.

It is to be stressed that mathematical notation is important, and that the foregoing discussion can be reorganized by introducing complex numbers as **ordered pairs** of real numbers. Thus, in place of $z = a + ib$ we could write

$$z = (a, b),$$

where $a$ and $b$ are called the real and imaginary parts, respectively, of the complex number $z$. If $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$ are any two complex numbers, we define their sum and product as

$$z_1 + z_2 = (a_1, b_1) + (a_2, b_2) \equiv (a_1 + a_2, b_1 + b_2) \tag{8}$$

and

$$z_1z_2 = (a_1, b_1)(a_2, b_2) \equiv (a_1a_2 - b_1b_2, a_1b_2 + a_2b_1), \tag{9}$$

respectively. These definitions are equivalent to (3) and (4). Although the form $z = a + ib$ is more convenient, and will be used almost exclusively in the sequel, the ordered-pair approach, created by the great Irish mathematician *William*

*R. Hamilton* (1805–1865), is important. First, it is based entirely on real numbers and completely avoids the mysterious "imaginary" number $i = \sqrt{-1}$. Since the forms $z = a + ib$ and $z = (a, b)$ are found to be equivalent, we can thus put to rest apprehensions about the quantity $i$ and accept it as a computational convenience. Second, the ordered-pair format $z = (a, b)$ suggests a graphical representation of $z$ as a point in a Cartesian $a, b$ plane. Since Cartesian axes are more usually denoted by $x$ and $y$ rather than by $a$ and $b$, let us write $z = x + iy = (x, y)$ and represent $z$ as a point in a so-called **complex $z$ plane**, as shown in Fig. 1.* The $x$ axis is called the **real axis**, and the $y$ axis is called the **imaginary axis**.†

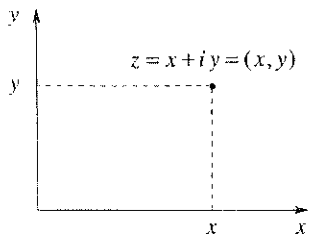



**Figure 1.** Complex $z$ plane.

Observe from Fig. 2 that the addition of complex numbers defined by (3) [or, equivalently, by (8)] satisfies the parallelogram law for the addition of vectors so that it is often convenient to think of complex numbers as *vectors*. That is, a $z$ vector is the vector from the origin to the point $z$.

The distance from the origin to the point $z$ (i.e., the "length of the $z$ vector") is called the **modulus** of $z$ and, by analogy with the absolute value of a real number, is denoted as $|z|$, or as mod($z$). Clearly (Fig. 1),

$$|z| = \sqrt{x^2 + y^2}. \tag{10}$$

For example, $|2 - i| = \sqrt{5}$. It is easy to verify (Exercise 2) that

$$|z_1 z_2| = |z_1|\,|z_2|\,; \tag{11}$$

that is, the modulus of a product equals the product of the moduli of the factors. Further (Fig. 3), the inequality

$$|z_1 + z_2| \le |z_1| + |z_2| \tag{12}$$

follows from the Euclidean proposition that the length of any one side of a triangle is less than or equal to the sum of the lengths of the other two sides. Hence (12) is known as the **triangle inequality**.

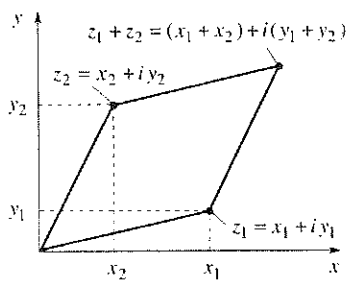



**Figure 2.** Addition in the $z$ plane.

Note carefully that the complex numbers are not ordered as real numbers are. For example, whereas $-6 < 2$, $10 > 7$, and so on, analogous statements such as $z < 0$, $z > 3$, and $4 + 3i > 1 + 2i$ are not meaningful! Of course, statements such as Re $z < 6$, Im $z < 6$, $|z| > 4$, and $|1 - i| < |3 - i|$ *do* make sense because Re $z$, Im $z$, and $|z|$ are real numbers.

Besides $z = x + iy$, it is useful to define the **complex conjugate** of $z$ as

$$\bar{z} \equiv x - iy. \tag{13}$$

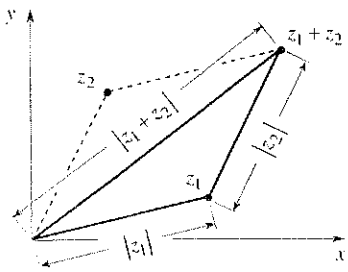



**Figure 3.** Triangle inequality.

*The complex plane is sometimes called the *Argand diagram* after the French mathematician *Jean Robert Argand* (1768–1822), who was one of the first to propose such representation of complex numbers. A bookkeeper by profession, Argand was a self-taught mathematician.

†One might be concerned that we have labeled the point on the $y$ axis as "$y$." Shouldn't it be "$iy$?" The answer is that either the real or the complex label is correct, depending on whether we are regarding the point as a point on the real $y$ axis or as a point in the complex $z$ plane.

Thus, if $z_1 = 8 + 3i$ and $z_2 = -4i$, then $\overline{z}_1 = 8 - 3i$ and $\overline{z}_2 = 4i$. It is readily shown (Exercise 3) that

$$\overline{z_1 + z_2} = \overline{z}_1 + \overline{z}_2, \tag{14a}$$

$$\overline{z_1 z_2} = \overline{z}_1 \overline{z}_2, \tag{14b}$$

and that

$$|z| = \sqrt{z\overline{z}}. \tag{15}$$

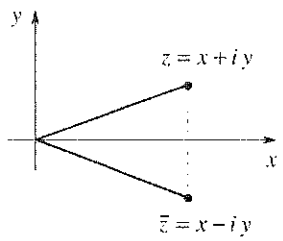



**Figure 4.** Complex conjugate.

Graphically, $\overline{z}$ is simply the reflection of $z$ about the real axis (Fig. 4).

Finally, the complex conjugate is useful in defining the *division* of two complex numbers. Division is defined as the inverse of multiplication. That is, the quotient $z = z_1/z_2$ is the complex number $z = x + iy$ such that $z_2 z = z_1$. Writing out the latter as

$$(x_2 + iy_2)(x + iy) = x_1 + iy_1, \tag{16}$$

expanding the left-hand side, and equating real and imaginary parts on the left- and right-hand sides, we obtain

$$x = \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}, \qquad y = \frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}. \tag{17}$$

Practically speaking, however, it is simpler to multiply numerator and denominator by the complex conjugate of the denominator,

$$z = \frac{z_1}{z_2} = \frac{z_1}{z_2}\frac{\overline{z}_2}{\overline{z}_2} \tag{18}$$

because the denominator $z_2\overline{z}_2 = |z_2|^2$ is now real. Thus, writing out (18), we find that

$$\begin{aligned} z = \frac{z_1}{z_2} = \frac{z_1\overline{z}_2}{z_2\overline{z}_2} &= \frac{(x_1 + iy_1)(x_2 - iy_2)}{x_2^2 + y_2^2} \\ &= \frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2} + i\,\frac{x_2 y_1 - x_1 y_2}{x_2^2 + y_2^2}, \end{aligned} \tag{19}$$

which result agrees with (17).

**EXAMPLE 1.**

$$\frac{2+i}{3-4i} = \frac{2+i}{3-4i}\,\frac{3+4i}{3+4i} = \frac{(6-4)+(8+3)i}{9+16} = \frac{2}{25} + \frac{11}{25}i,$$

which result can be checked by showing that $3 - 4i$ times $\frac{2}{25} + \frac{11}{25}i$ gives $2 + i$. ∎

**Closure.** In closing this section, we note that besides the *Cartesian* notation $z = x + iy$, for complex numbers, there is also a *polar* notation "$z = re^{i\theta}$" that is often

convenient. However, because it contains the complex exponential $e^{i\theta}$, we delay our introduction of the polar notation until we discuss the complex exponential function.

## EXERCISES 21.2

**1.** Using the definitions (3) and (4), verify

(a) the commutative properties (5)
(b) the associative properties (6)
(c) the distributive property (7)

**2.** Verify (11).

**3.** Verify (14a), (14b), and (15).

**4.** Recall that the definitions of addition and multiplication, (3) and (4), were rigged so that the familiar commutative, associative, and distributive rules hold. Thus, one can manipulate complex numbers easily and with little fuss. Nevertheless, it may be valuable to work an example or two, paying careful attention to the definitions and properties (3) to (7). In this spirit we ask you to solve the following equations for $z$ and then to verify, carefully, that the root(s) do indeed satisfy the given equation. In each step of the verification, identify the definition or property [(3) to (7)] being used. Naturally, $z^2$ means $zz$.

(a) $z^2 - 2z + 2 = 0$ (b) $2z^2 + 2z + 1 = 0$
(c) $3z^2 + 2z + 1 = 0$ (d) $z^2 + 2iz + 1 = 0$
(e) $10z^2 - 6z + 1 = 0$ (f) $z^2 - 3iz - 2 = 0$

**5.** Show that

(a) $|z^3| = |z|^3$
(b) $|z^n| = |z|^n$ and $|1/z^n| = 1/|z|^n$ for $n = 1, 2, \ldots$
(c) $|z_1 z_2 z_3| = |z_1||z_2||z_3|$
(d) $|z_1 z_2 \cdots z_n| = |z_1||z_2|\cdots|z_n|$ for $n = 1, 2, \ldots$
(e) $|z_1 + z_2 + z_3| \le |z_1| + |z_2| + |z_3|$
(f) $|z_1 + z_2 + \cdots + z_n| \le |z_1| + |z_2| + \cdots + |z_n|$
for $n = 1, 2, \ldots$

**6.** Show that

(a) $\overline{\overline{z}} = z$
(b) $\overline{\left(\dfrac{1}{z}\right)} = \dfrac{1}{\overline{z}}$
(c) $z$ is real if and only if $z = \overline{z}$
(d) $\overline{z^3} = \overline{z}^3$
(e) $\overline{z^n} = \overline{z}^n$ for $n = 1, 2, \ldots$

**7.** Show that if $z = x + iy$, then $x = (z + \overline{z})/2$ and $y = (z - \overline{z})/2i$.

**8.** Show that if $z_1 z_2 = 0$, then at least one of the two factors must be zero.

**9.** Evaluate each of the following. That is, express each in standard Cartesian form $x + iy$.

(a) $(2 - i)^3$ (b) $\dfrac{1}{1 - 2i}$
(c) $\dfrac{i}{2 + 5i}$ (d) $\dfrac{1 + i}{1 - i}$
(e) $\left(\dfrac{1 + i}{2 - i}\right)^3$ (f) $\operatorname{Re} \dfrac{2 + 3i}{4 + 5i}$
(g) $\operatorname{Im} (1 + i)^3$ (h) $\left(\operatorname{Re} \dfrac{1}{1 + i}\right)^3$
(i) $\operatorname{Im} \dfrac{a + ib}{c + id}$ (j) $\operatorname{Re} \dfrac{1}{1 - i}$

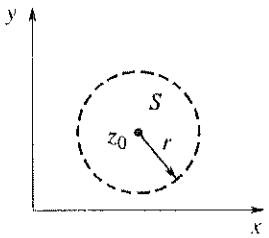
**10.** Evaluate

(a) $\left|\dfrac{1 - i}{1 + i}\right|$ (b) $\left|\dfrac{(2 - i)^3}{(1 + 3i)^2}\right|$
(c) $\left|(1 - 2i)^2 + (1 + i)^2\right|$

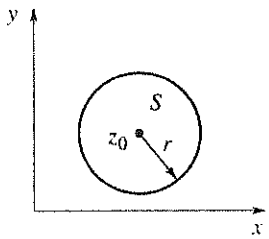
**11.** Verify the triangle inequality (12) for each of the following cases by working out the left- and right-hand sides.

(a) $z_1 = 2 + 3i,\ z_2 = 4 - i$
(b) $z_1 = 1 + i,\ z_2 = 7i$
(c) $z_1 = 5,\ z_2 = 4i$
(d) $z_1 = 3 + 4i,\ z_2 = 2 + i$
(e) $z_1 = 1 + i,\ z_2 = 1 - i$

(a) Open disk $|z-z_0|<r$




(b) Closed disk $|z-z_0|\le r$




(c) Open half-plane $\operatorname{Re} z > 2$

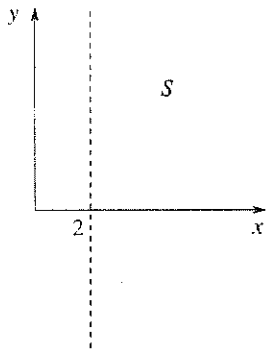



**Figure 1.** The regions $|z-z_0| < r$, $|z-z_0| \le r$, $\operatorname{Re} z > 2$.

## 21.3 Elementary Functions

**21.3.1. Preliminary ideas.** Having introduced complex numbers $z = x + iy$, we next introduce *functions* of a complex variable. Since functions will be defined on sets of complex numbers, we need to distinguish various topological features of these sets.

First, we define the **distance** between any two points $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ as

$$d(z_1, z_2) \equiv |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}. \tag{1}$$

Next, we define a **neighborhood** $N(z_0; r)$ of a point $z_0$ in the $z$ plane as the set of points $z$ closer to $z_0$ than $r$, that is, all $z$'s such that

$$|z - z_0| < r \qquad (r > 0). \tag{2}$$

A set $S$ is **connected** if each pair of points in $S$ can be joined by an unbroken line consisting of a finite number of straight segments, each contained entirely within $S$. A point $P$ is a **boundary point** of $S$ if every neighborhood of $P$ (i.e., no matter how small) contains points in $S$ as well as points not in $S$, and a point $P$ in $S$ is an **interior point** of $S$ if there exists some neighborhood of $P$ lying entirely within $S$. Finally, a connected set $S$ is a **region**; $S$ is an **open region** or **domain** if it contains *none* of its boundary points, and is a **closed region** if it contains *all* of its boundary points.

To illustrate, $|z - z_0| < r$ defines an open region (Fig. 1a), $|z - z_0| \le r$ defines a closed region (Fig. 1b), and $\operatorname{Re} z > 2$ defines an open region (Fig. 1c).

Turning to functions, recall from the elementary calculus that a function $f$ of a real variable $x$ is a rule that assigns a unique value $f(x)$ to each point $x$ in some set on an $x$-axis, as illustrated in Fig. 2. One also calls $f$ a **mapping**. Following *Descartes* (1596–1650), a useful graphical display of $f$ (the "graph" of $f$) can be obtained by arranging the $x$ and $f$ axes at right angles to each other and plotting the set of points $x, f(x)$, as illustrated in Fig. 3. Analogously, we define a **function $w$ of a complex variable $z$** as a rule that assigns a unique value $w(z)$ to each point $z$ in some set $D$ in the complex plane.*

In general, $w(z)$ is complex, so the mapping is as depicted in Fig. 4 from the $z$ plane to a $w$ plane. The set $D$ on which $w$ is defined is called the **domain** of definition† of $w$, and the set $R$ of $w(z)$ values is called the **range** of values of $w$. Strictly speaking, one distinguishes between the function (or mapping) and the *values* of the function by using the notation $w$ for the former and $w(z)$ for the latter. However, we plead guilty of occasionally writing "the function

*Just as the letter $f$ is traditionally, although by no means exclusively, used for functions of a real variable, it is traditional to use either $w$ or $f$ for functions of a complex variable. We plan to use both extensively, though not exclusively.

†The domain of definition of a function is not necessarily an open set and is thus not necessarily a domain in the sense defined above.

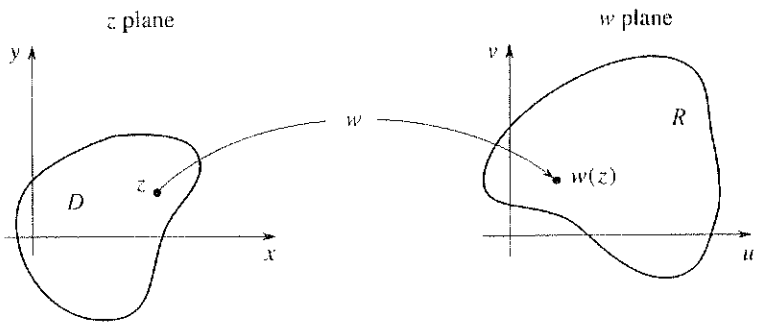



**Figure 4.** The mapping $w$.

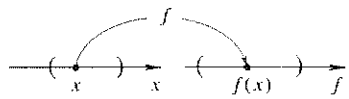



**Figure 2.** The mapping $f$.

$w(z)$" as a shorthand way of writing "the function $w$ whose values are $w(z)$."

Notice carefully that Descartes's method of graphical display (Fig. 3) is not available to us in the complex case because both $D$ and $R$ are two-dimensional so that the "graph of $w$" would require a plot in four dimensions. *Thus, graphs like the one shown in Fig. 3 will not be possible for a function of a complex variable.* Of course, the real and imaginary parts of $w(z)$, say $u$ and $v$, respectively, must be functions of $x$ and $y$. Thus, we can express

$$w(z) = u(x, y) + i\,v(x, y), \tag{3}$$

and although we cannot plot "the graph of $w$," we can do three-dimensional plots of $u$ and $v$ as functions of $x$ and $y$, but this form of display is seldom used.

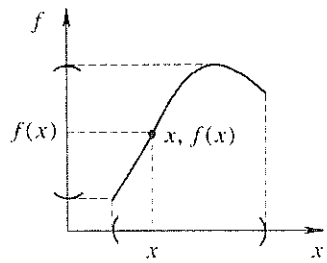



**Figure 3.** The graph of $f$.

**EXAMPLE 1.** To illustrate some of these ideas, consider $w(z) = z^2$, defined on the first quadrant of the $z$ plane: $0 < x < \infty$, $0 < y < \infty$. Then $w(z) = (x + iy)^2 = x^2 - y^2 + i2xy$ so that $u(x, y) = x^2 - y^2$ and $v(x, y) = 2xy$. Since $0 < x < \infty$ and $0 < y < \infty$, it follows from the form of $u$ and $v$ that $-\infty < u < \infty$ and $0 < v < \infty$ so that the range of $w$ is the entire upper half plane $v > 0$ as shown in Fig. 5.

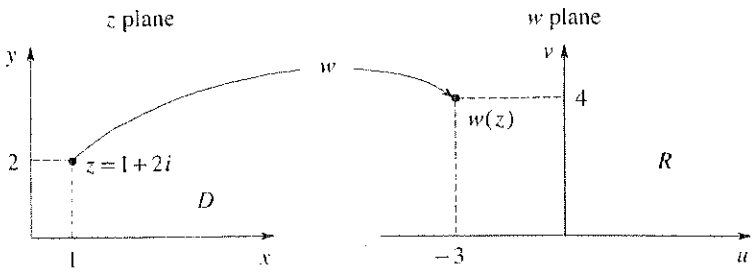



**Figure 5.** Mapping defined in Example 1.

If, for example, $z = 1 + 2i$, then $w(z) = (1 - 4) + i2(1)(2) = -3 + 4i$ as shown in the figure.

Since we cannot draw a graph of $w(z) = z^2$ [as we could for the real function $w(x) = x^2$], and we do not wish to plot the surfaces $u(x, y) = x^2 - y^2$, $v(x, y) = 2xy$

(although this could be done using computer graphics), the question remains: what *can* be done, easily, to provide some form of graphical display of $w$? We could, of course, select a number of *points* in $D$ and display each, together with its image in $R$, as we have done in Fig. 5 for the single point $z = 1 + 2i$. However, this seems hardly an attractive idea. A more appealing and commonly used device is to display the images of representative *curves*. For instance, the image of the straight line $x = 1$ $(0 < y < \infty)$ is given parametrically by $u = 1 - y^2$, $v = 2y$ or, eliminating the parameter $y$, by the parabola $u = 1 - v^2/4$ $(0 < v < \infty)$. Similarly, the image of $y = 1$ $(0 < x < \infty)$ is given parametrically by $u = x^2 - 1$, $v = 2x$ $(0 < x < \infty)$ or, equivalently, by the parabola $u = (v/2)^2 - 1$. And so on. The representative curves $x = 1, 2, 3, 4$, $y = 1, 2, 3, 4$, and their images are shown

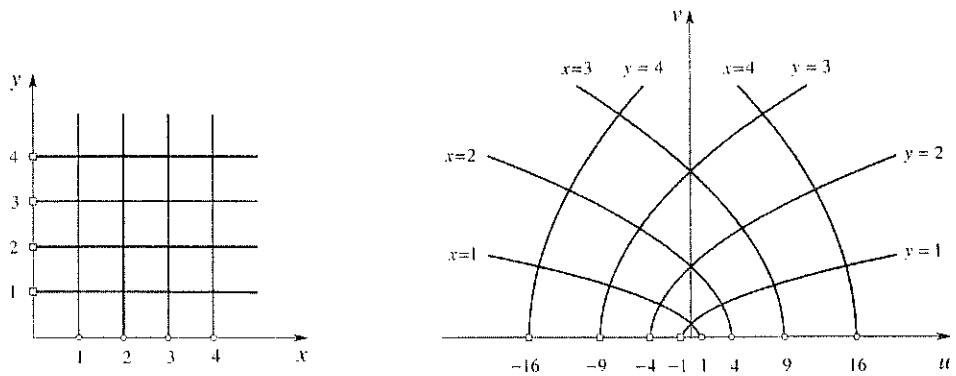



**Figure 6.** Representative curves and their images, under the mapping $w = z^2$.

in Fig. 6. This type of display is of importance in Chapter 23, where we study conformal mapping. ∎

Next, we introduce the important elementary functions $e^z$, $\sin z$, $\cos z$, $\sinh z$, and $\cosh z$.

**21.3.2. Exponential function.** Beginning with the **exponential function** $e^z$ [also written as $\exp(z)$], it is essential to see, first, that we cannot "figure out" how to compute $e^z = e^{x+iy}$; the latter is a new quantity for us and we render it meaningful by *defining* it. As a rule of thumb, it is generally fruitful to define new objects in terms of, or as extensions of, old ones. For example, in the present case we could recall that for real $x$ we have the familiar Taylor series

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \qquad (|x| < \infty) \tag{4}$$

Thus, it seems reasonable to *define*

$$e^z \equiv 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \tag{5}$$

since (5) reduces to (4) in the event that $z$ is purely real. In fact, this series approach was adopted by *Karl Weierstrass* (1815–1897) in his development of the complex

variable theory. This approach would be awkward here since it would confront us with complex infinite series early in our development of the subject. Thus, we prefer the more traditional approach, wherein the introduction of series is delayed until after we have developed both the differential and integral calculus of functions of a complex variable.

Recalling the real variable formula $e^{x_1+x_2} = e^{x_1}e^{x_2}$, let us tentatively write

$$e^z = e^{x+iy} = e^x e^{iy}. \tag{6}$$

The latter falls short of defining $e^z$ because $e^{iy}$ is still undefined. At this point we do turn to Taylor series, but only to motivate our final definition of $e^z$. For if we treat $iy$ as real, then (4) gives

$$\begin{aligned} e^{iy} &= 1 + (iy) + \frac{1}{2!}(iy)^2 + \frac{1}{3!}(iy)^3 + \frac{1}{4!}(iy)^4 + \cdots \\ &= \left(1 - \frac{y^2}{2!} + \frac{y^4}{4!} - \cdots\right) + i\left(y - \frac{y^3}{3!} + \frac{y^5}{5!} - \cdots\right) \\ &= \cos y + i \sin y. \end{aligned}$$

From this result it seems reasonable to *define*

$$\boxed{e^{iy} \equiv \cos y + i \sin y} \tag{7}$$

and combining (7) with (6) to *define*

$$\boxed{e^z \equiv e^x (\cos y + i \sin y)\,.} \tag{8}$$

Due to Euler, and known as **Euler's formula**, (7) is only a special case of (8) (for $z = iy$), but it is so important in the history of mathematics that we have framed it, together with (8). Naturally, the name of the variable in (7) is immaterial. Thus, it is equally true that $e^{i\theta} = \cos\theta + i\sin\theta$, $e^{it} = \cos t + i \sin t$, and so on.

Note carefully from (7) that

$$\left|e^{iy}\right| = |\cos y + i \sin y| = \sqrt{\cos^2 y + \sin^2 y} = 1 \tag{9}$$

for all $y$, and from (8) that

$$|e^z| = |e^x (\cos y + i \sin y)| = |e^x|\,|\cos y + i \sin y| = e^x \tag{10}$$

for all $z$. For example, $\left|e^{3-5i}\right| = e^3$ and $\left|e^{-4+i}\right| = e^{-4}$. Finally, observe from (8) that $e^z \neq 0$ for all (finite) $z$'s because $e^x \neq 0$ for all (finite) $x$'s, and $\cos y$ and $\sin y$ do not simultaneously vanish for any value of $y$. Thus, we say that $e^z$ "has no zeros" in the finite $z$ plane.

**21.3.3. Trigonometric and hyperbolic functions.** Changing $y$ to $-y$ in (7), and recalling that the cosine is even and the sine is odd, we obtain the companion formula

$$e^{-iy} = \cos y - i \sin y, \tag{11}$$

and solving (7) and (11) for $\cos y$, $\sin y$, we obtain

$$\cos y = \frac{e^{iy} + e^{-iy}}{2}, \tag{12}$$

$$\sin y = \frac{e^{iy} - e^{-iy}}{2i}. \tag{13}$$

Equations (12) and (13) suggest that we now define the **cosine** and **sine functions** as

$$\boxed{\begin{aligned} \cos z &\equiv \frac{e^{iz} + e^{-iz}}{2}, \\ \sin z &\equiv \frac{e^{iz} - e^{-iz}}{2i}. \end{aligned}} \tag{14,15}$$

The right-hand sides are meaningful by virtue of (8). For example, $e^{iz} = e^{i(x+iy)} = e^{-y+ix} = e^{-y}(\cos x + i \sin x)$.

Furthermore, the formulas

$$\cosh x = \frac{e^{x} + e^{-x}}{2}, \tag{16}$$

$$\sinh x = \frac{e^{x} - e^{-x}}{2} \tag{17}$$

suggest that we define the **hyperbolic cosine** and **hyperbolic sine functions** as

$$\boxed{\begin{aligned} \cosh z &\equiv \frac{e^{z} + e^{-z}}{2}, \\ \sinh z &\equiv \frac{e^{z} - e^{-z}}{2}. \end{aligned}} \tag{18,19}$$

From (14), (15), (18), and (19) follow the connections

$$\cos(iz) = \cosh z, \tag{20a}$$

$$\sin(iz) = i \sinh z, \tag{20b}$$

$$\cosh(iz) = \cos z, \tag{20c}$$

$$\sinh(iz) = i \sin z \tag{20d}$$

between the trigonometric and hyperbolic functions.

Out of habit, it is natural to want to know what $\cos z$, $\sin z$, $\cosh z$, $\sinh z$ "look like." But whereas the *real* functions $\cos x$, $\sin x$, $\cosh x$, $\sinh x$ admit the familiar

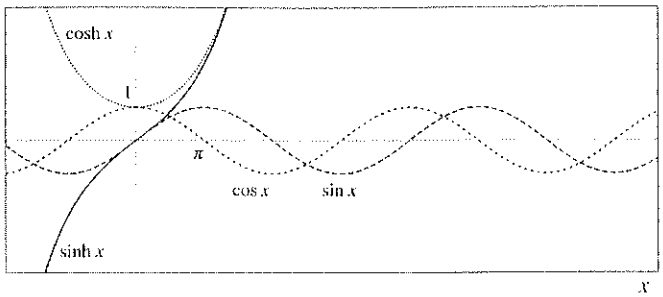



**Figure 7.** Graphs of $\cos x$, $\sin x$, $\cosh x$, $\sinh x$.

graphs shown in Fig. 7, recall that we cannot draw analogous graphs for the complex functions $\cos z$, $\sin z$, $\cosh z$, and $\sinh z$ because we would need four dimensions to do so. Nevertheless, a limited visual image of these functions is readily available by looking only along the real and imaginary axes. Considering $\sin z$, for example, observe that on the real axis, $\sin z = \sin x$ is the familiar oscillatory function whose graph is shown in Fig. 7. On the imaginary axis, however, $\sin z = \sin(iy) = i \sinh y$ is not oscillatory, and it is not even bounded! Incidentally, it is only by coincidence that $\sin z$ happens to be purely real along the real axis and purely imaginary along the imaginary axis. For instance, $(z + i)\sin z$ is neither purely real nor purely imaginary along either axis.

Because we have based our definitions of $e^z$, $\cos z$, $\sin z$, $\cosh z$, and $\sinh z$ on real variable formulas [for example, compare (18) and (19) with (16) and (17)], *it now turns out that the familiar formulas for real exponential, trigonometric, and hyperbolic functions are true for the complex functions as well.* For example,

$$\sin^2 z + \cos^2 z = 1, \qquad \sin(-z) = -\sin z, \qquad \cos(-z) = \cos z \tag{21a}$$

hold for any $z$, and

$$\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2, \tag{21b}$$

$$\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \sin z_2 \cos z_1 \tag{21c}$$

hold for any $z_1, z_2$.

**EXAMPLE 2.** As representative, let us prove that $\sin^2 z + \cos^2 z = 1$:

$$\begin{aligned}\sin^2 z + \cos^2 z &= -\tfrac{1}{4}\left(e^{iz} - e^{-iz}\right)^2 + \tfrac{1}{4}\left(e^{iz} + e^{-iz}\right)^2 \\ &= \tfrac{1}{4}\left(-e^{i2z} + 2 - e^{-i2z} + e^{i2z} + 2 + e^{-i2z}\right) \\ &= 1,\end{aligned} \tag{22}$$

as claimed. ∎

Additional identities are listed in the exercises.

Finally, besides $\sin z, \cos z, \sinh z, \cosh z$, we define

$$\tan z \equiv \frac{\sin z}{\cos z}, \qquad \cot z \equiv \frac{\cos z}{\sin z}, \tag{23a}$$

$$\sec z \equiv \frac{1}{\cos z}, \qquad \csc z \equiv \frac{1}{\sin z}, \tag{23b}$$

$$\tanh z \equiv \frac{\sinh z}{\cosh z}, \qquad \coth z \equiv \frac{\cosh z}{\sinh z}, \tag{23c}$$

$$\operatorname{sech} z \equiv \frac{1}{\cosh z}, \qquad \operatorname{csch} z \equiv \frac{1}{\sinh z}. \tag{23d}$$

**21.3.4. Application of complex numbers to integration and the solution of differential equations.** If $f(x)$ is a complex-valued function of $x$ such as $e^{ix}$, then

$$\begin{aligned}\int f(x)\,dx &= \int [\mathrm{Re} f(x) + i\,\mathrm{Im} f(x)]\,dx \\ &= \int \mathrm{Re} f(x)\,dx + i\int \mathrm{Im} f(x)\,dx,\end{aligned} \tag{24}$$

where the second equality follows from the linearity of integration. Since $\int \mathrm{Re} f(x)\,dx$ and $\int \mathrm{Im} f(x)\,dx$ are real, it follows from (24) that

$$\boxed{\mathrm{Re}\int f(x)\,dx = \int \mathrm{Re} f(x)\,dx} \tag{25}$$

and

$$\boxed{\mathrm{Im}\int f(x)\,dx = \int \mathrm{Im} f(x)\,dx.} \tag{26}$$

Equations (25) and (26) can be used to simplify the evaluation of certain integrals.

**EXAMPLE 3.** To illustrate, let us evaluate the integral

$$I = \int_0^\infty e^{-x}\cos ax\,dx. \tag{27}$$

We can evaluate $I$ by applying integration by parts twice, but it is simpler to proceed as follows:

$$\begin{aligned} I = \int_0^\infty e^{-x}\,\mathrm{Re}\,e^{iax}\,dx &= \int_0^\infty \mathrm{Re}(e^{-x}e^{iax})\,dx \\ &= \mathrm{Re}\int_0^\infty e^{-(1-ia)x}\,dx \quad [\text{by (25)}] \\ &= \mathrm{Re}\left(\frac{e^{-(1-ia)x}}{-(1-ia)}\right)\Bigg|_{x=0}^{x=\infty} \\ &= \mathrm{Re}\,\frac{1}{1-ia} = \frac{1}{1+a^2}. \end{aligned} \tag{28}$$

The advantage of using (25) is that the integral of $e^{-(1-ia)x}$ is simply $-e^{-(1-ia)x}/(1-ia)$, whereas the integral of $e^{-x}\cos ax$ is more difficult, requiring two applications of integration by parts.

COMMENT. Be sure to understand the reasoning behind the fifth equality:

$$\begin{aligned}\left.\frac{e^{-(1-ia)x}}{-(1-ia)}\right|_{x=0}^{x=\infty} &= \lim_{x\to\infty}\frac{e^{-(1-ia)x}}{-(1-ia)} - \frac{e^0}{-(1-ia)} \\ &= \frac{\lim_{x\to\infty}(e^{-x}e^{iax})}{-(1-ia)} + \frac{1}{1-ia}.\end{aligned} \tag{29}$$

Further,

$$\left|e^{-x}e^{iax}\right| = \left|e^{-x}\right|\left|e^{iax}\right| = (e^{-x})(1) = e^{-x} \to 0 \tag{30}$$

as $x \to \infty$. Since the modulus of the complex number $e^{-x}e^{iax}$ tends to zero, the complex number itself must tend to zero. Thus, the first term on the right-hand side of (29) is zero and we are left with $1/(1-ia)$. ∎

Another application in which it is useful to introduce complex numbers is the determination of particular solutions of linear differential equations with cosinusoidal or sinusoidal forcing functions. For instance, consider the task of finding a particular solution $x_p(t)$ of the differential equation

$$L[x] = F\cos\omega t, \tag{31}$$

where $L$ is a linear differential operator with constant coefficients, and $F$ and $\omega$ are constants. According to the method of undetermined coefficients, we can obtain $x_p(t)$ in the form $C_1\cos\omega t + C_2\sin\omega t$. However, it is simpler to solve the problem

$$L[v] = Fe^{i\omega t} \tag{32}$$

for a particular solution $v_p(t)$, and to obtain $x_p$ from $v_p$ as

$$x_p(t) = \operatorname{Re} v_p(t). \tag{33}$$

The reason the $v$ problem is simpler is that by the method of undetermined coefficients a particular solution $v_p(t)$of (32) can be found in the form $Ae^{i\omega t}$, which form is simpler than the two-term form $C_1\cos\omega t + C_2\sin\omega t$ needed for equation (31). To verify the truth of (33), write (32) as

$$L[\operatorname{Re} v + i\operatorname{Im} v] = F\cos\omega t + iF\sin\omega t. \tag{34}$$

Since $L$ is linear, by assumption, $L[\operatorname{Re} v + i\operatorname{Im} v] = L[\operatorname{Re} v] + iL[\operatorname{Im} v]$. Putting the latter expression into (34) and equating real and imaginary parts on the left- and right-hand sides gives

$$L[\operatorname{Re} v] = F\cos\omega t \tag{35a}$$

and

$$L[\operatorname{Im} v] = F\sin\omega t. \tag{35b}$$

Comparing (35a) with (31), the truth of (33) follows. [Of course, if the right-hand side of (31) were $F \sin \omega t$ instead, then in place of (33) we would use $x_p(t) = \operatorname{Im} v(t)$.]

**EXAMPLE 4.** If the applied voltage is $E(t) = E_0 \sin \omega t$, then the current $i(t)$ in the electrical circuit shown in Fig. 8 is governed by the differential equation

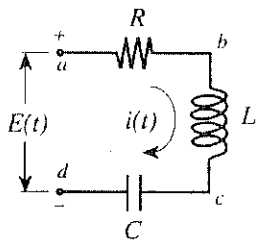



**Figure 8.** $RLC$ circuit.

$$\begin{aligned} Li'' + Ri' + \frac{1}{C} i &= \frac{dE(t)}{dt} \\ &= \omega E_0 \cos \omega t, \end{aligned} \tag{36}$$

where $L, R, C, E_0, \omega$ are constants, and the inductance $L$ is not to be confused with the operator $L$ in (31)–(35). According to the **complex function method** described above, to find a particular solution $i_p(t)$ of (36), consider instead the simpler equation

$$Lv'' + Rv' + \frac{1}{C} v = \omega E_0 e^{i\omega t} \tag{37}$$

and seek

$$v_p(t) = Ae^{i\omega t}. \tag{38}$$

Putting (38) into (37) gives

$$\left(-L\omega^2 + iR\omega + \frac{1}{C}\right) Ae^{i\omega t} = \omega E_0 e^{i\omega t}. \tag{39}$$

Thus, $A = \omega E_0 / \left(-L\omega^2 + iR\omega + \frac{1}{C}\right)$,

$$v_p(t) = \frac{\omega E_0 C}{(1 - LC\omega^2) + iRC\omega} e^{i\omega t}, \tag{40}$$

and $i_p(t) = \operatorname{Re} v_p(t)$ gives

$$\begin{aligned} i_p(t) &= \omega E_0 C \operatorname{Re} \left( \frac{\cos \omega t + i \sin \omega t}{(1 - LC\omega^2) + iRC\omega} \frac{(1 - LC\omega^2) - iRC\omega}{(1 - LC\omega^2) - iRC\omega} \right) \\ &= \frac{\omega E_0 C}{(1 - LC\omega^2)^2 + R^2C^2\omega^2} [(1 - LC\omega^2) \cos \omega t - RC\omega \sin \omega t]. \end{aligned} \tag{41}$$

COMMENT 1. According to the discussion in Section 3.5, we can also express (41) in the form

$$i_p(t) = \frac{\omega E_0 C}{\sqrt{(1 - LC\omega^2)^2 + R^2C^2\omega^2}} \sin(\omega t + \phi), \tag{42a}$$

if we prefer, where the phase angle $\phi$ is given by

$$\phi = \tan^{-1} \left( \frac{LC\omega^2 - 1}{RC\omega} \right). \tag{42b}$$

COMMENT 2. If the inductor and capacitor were removed from the circuit, then the parentheses in (39) would contain only $iR\omega$, where $R$ is the resistance. With the inductor and capacitor present, we can express the terms within those parentheses as

$$-L\omega^2 + iR\omega + \frac{1}{C} = i\omega\left[R + i\left(L\omega - \frac{1}{\omega C}\right)\right] \tag{43}$$

so we can think of

$$Z \equiv R + i\left(L\omega - \frac{1}{\omega C}\right) \tag{44}$$

as a sort of equivalent or generalized resistance. In electrical engineering terminology, $Z$ is called the **complex impedance** of the circuit.

COMMENT 3. Note that because of the $Ri'$ term in (36) the homogeneous solution of (36) will inevitably tend to zero as $t \to \infty$. Thus, (41) [or (42)] is not only a particular solution of (36), it is the **steady-state solution**. The complex function method is commonly used, in engineering, as a convenient method for obtaining the steady-state response. ∎

**Closure.** In this section we introduce the notion of a function $w$ of a single complex variable $z$ in general, and a number of elementary functions in particular. Additional elementary functions are presented in Sections 21.4 and 21.5. It would be reasonable to expect that we will subsequently turn to functions of *several* complex variables $w(z_1, \ldots, z_n)$, as one does in studying *real* variable theory. Yet it is curious and interesting that virtually the entire body of complex variable theory deals only with functions of a single complex variable. To be sure, functions of several complex variables are considered,* but this case is relatively obscure both in theory and applications.

**Computer software.** Graphs of the sort shown in Fig. 6 can be obtained, using *Maple*, from the **conformal** command. For instance, to obtain the image curves shown in Fig. 6, type

with(plots):

and enter, to gain access to the conformal command. Then enter

conformal($z$^2, $z = 0..4 + 4 * \mathrm{I}, w = -20..20 + 26 * \mathrm{I}$,
grid = [5, 5], numxy = [20, 20]);

The $z = 0..4 + 4 * \mathrm{I}$ gives the lower left corner ($z = 0$) and upper right corner ($z = 4 + 4i$) of a rectangular region of interest in the $z$ plane; the $w = -20..20 + 26 * \mathrm{I}$ indicates the lower left corner ($w = -20$) and upper right corner ($w = 20 + 26i$) of a rectangular region in the $w$ plane within which the image curves are to be plotted; grid $= [m, n]$ indicates that we seek the image of $m$ constant-$x$ lines (the five lines $x = 0, 1, 2, 3, 4$ in this case) and $n$ constant-$y$ lines (the five lines $y = 0, 1, 2, 3, 4$

*See, for example, V. Vladimirov's *Methods of the Theory of Functions of Many Complex Variables* (Cambridge, MA: MIT Press, 1966), which includes applications to quantum field theory, function theory, and the theory of differential equations with constant coefficients.

in this case); and numxy $= [p, q]$ indicates the number of points to be used for each image curve. If $-20..20 + 26 * 1$ were omitted, the default would be to choose the smallest rectangle, in the $w$ plane, that contains the image. If numxy $= [20, 20]$ were omitted, the default would be numxy $= [11, 11]$, which might be too coarse to give sufficiently smooth image curves.

## EXERCISES 21.3

**1.** With a labeled sketch, show the point sets defined by the following.

(a) $|z - 1| \le 4$ (b) $|z + 1| < 3$
(c) $|z + 2 - i| < 2$ (d) $|z + 2 - i| = 2$
(e) $|z| < |z - 4|$ (f) $|z + 1| \le |z|$
(g) $|z - z_1| < |z - z_2|$ (h) $\text{Re}\,(z - i) > 3$
(i) $|z + 1| = |z| + 1$ (j) $\text{Re}\,(z + i) < 2$
(k) $2 \le |z + i| \le 5$ (l) $\text{Im}\,(z - i) > 1$

**2.** Determine the range $R$ for the given function. Include sketches of both the domain $D$ and the range $R$, and give the equations of any curved parts of the boundary of $R$.

(a) $w(z) = z + 2 + i$ on $0 < x < 1, 0 < y < 1$
(b) $w(z) = 2iz$ on $0 < x < 1, 0 < y < \infty$
(c) $w(z) = iz + 3$ on $0 < x < \infty, 0 < y < \infty$
(d) $w(z) = z^2$ on $-\infty < x < 0, 0 < y < \infty$
(e) $w(z) = z^2$ on $1 < x < 2, 1 < y < 3$
(f) $w(z) = iz^2$ on $0 < x < 1, 0 < y < 1$
(g) $w(z) = z^3$ on $0 < x < \infty, 0 < y < \infty$

**3.** Show whether or not $|e^z| = e^{|z|}$. More generally, is $|w(z)| = w(|z|)$?

**4.** Show whether or not $\overline{e^z} = e^{\overline{z}}$. More generally, is $\overline{w(z)} = w(\overline{z})$?

**5.** Show that (20a) to (20d) follow from (14), (15), (18), and (19).

**6.** Show that

(a) $e^{z_1}e^{z_2} = e^{z_1+z_2}$
(b) $(e^z)^n = e^{nz}$ for any integer $n$ [More generally, it is true that $(e^{z_1})^{z_2} = e^{z_1 z_2}$, but proof of that fact needs to await our introduction of the logarithmic function in Section 21.4.]

**7.** Show that

(a) $\sin(-z) = -\sin z$ and $\cos(-z) = \cos z$
(b) $\cos(z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2$
(c) $\sin(z_1 + z_2) = \sin z_1 \cos z_2 + \sin z_2 \cos z_1$
(d) $\cos(z + 2\pi) = \cos z$ and $\sin(z + 2\pi) = \sin z$
(e) $\cos(x + iy) = \cos x \cosh y - i \sin x \sinh y$
(f) $\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$

**8.** Show that

(a) $\sinh(-z) = -\sinh z$ and $\cosh(-z) = \cosh z$
(b) $\cosh(z_1 + z_2) = \cosh z_1 \cosh z_2 + \sinh z_1 \sinh z_2$
(c) $\sinh(z_1 + z_2) = \sinh z_1 \cosh z_2 + \cosh z_1 \sinh z_2$
(d) $\cosh(z + 2\pi i) = \cosh z$ and $\sinh(z + 2\pi i) = \sinh z$
(e) $\cosh(x + iy) = \cosh x \cos y + i \sinh x \sin y$
(f) $\sinh(x + iy) = \sinh x \cos y + i \cosh x \sin y$
(g) $\cosh^2 z - \sinh^2 z = 1$

**9.** Evaluate each of the following in standard Cartesian form.

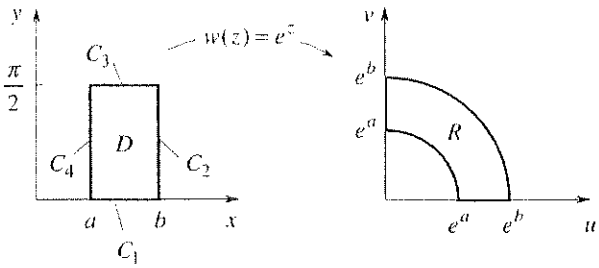
(a) $e^{2+\pi i}$ (b) $e^{1-i}$
(c) $e^{-\pi i/4}$ (d) $\sin(3 + \pi i)$
(e) $\cos(-2 + 3\pi i)$ (f) $\sec(1 + i)$
(g) $\csc(1 - i)$ (h) $\tan\left(-\dfrac{3\pi}{4}i\right)$
(i) $\cot\left(\dfrac{\pi i}{4}\right)$ (j) $\sinh(3 + \pi i)$
(k) $\cosh(1 - \pi i)$ (l) $\tanh(2 + 4\pi i)$

**10.** Consider this reasoning: $|e^{iz}| = |\cos z + i \sin z| = \sqrt{\cos^2 z + \sin^2 z} = 1$ for all $z$, yet $z = -2i$, say, gives $|e^{iz}| = e^2 \ne 1$. Explain the flaw(s) in that reasoning.

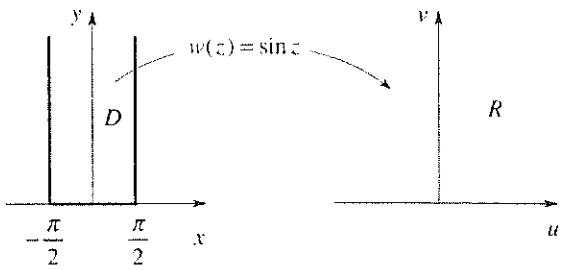
**11.** Prove that

(a) $e^z = 1$ if and only if $z = 2n\pi i$, where $n$ is any integer.
(b) $e^{z_1} = e^{z_2}$ if and only if $z_1 = z_2 + 2n\pi i$, where $n$ is any integer.

**12.** Show that the range $R$ of the function $e^z$ on the rectangular domain $a < x < b$, $0 < y < \pi/2$ is as shown in the accompanying figure. HINT: At first glance it is tempting to take a number of points on the boundary of $D$ [such as $a, (a + b)/2, b, b + (\pi/4)i, b + (\pi/2)i$, etc.] and find their images in the $f$ plane. That procedure can only give *points* on the boundary of $R$, not the boundary *curve*. Thus one should find the image of the entire line segment $C_1$, then $C_2, C_3$, and $C_4$.




**13.** Show that the range $R$ of the function $\sin z$ on the semi-infinite strip $-\pi/2 < x < \pi/2$, $0 < y < \infty$ is shown in the accompanying figure. (See the hint in Exercise 11.)




**14.** Use the *Maple* conformal command, or similar software, to obtain the image of the given rectangle under the mapping $w = z^2$. Specifically, map the straight line segments comprising the border of the rectangle, as well as ten equally spaced constant-$x$ and constant-$y$ line segments within its interior. Label the image curves as $x =$ etc. or $y =$ etc., as we have in Fig. 6.

(a) $0 \le x \le 4,\ 0 \le y \le 4$ (b) $-4 \le x \le 4,\ 0 \le y \le 4$
(c) $2 \le x \le 4,\ 0 \le y \le 5$ (d) $2 \le x \le 4,\ 2 \le y \le 4$

**15.** Same as Exercise 14, but for the mapping $w = z^3$.

(a) $0 \le x \le 4,\ 0 \le y \le 4$ (b) $2 \le x \le 4,\ 0 \le y \le 4$
(c) $2 \le x \le 4,\ 2 \le y \le 6$ (d) $1 \le x \le 4,\ -1 \le y \le 0$

**16.** Evaluate the given integral using the complex method that is illustrated in Example 3.

(a) $\int_0^\infty e^{-x} \sin \omega x \, dx$

(b) $\int_0^\infty e^{-st} \cos \omega t \, dt \quad (\text{Re}\, s > 0)$

(c) $\int_0^\infty e^{-st} \sin \omega t \, dt \quad (\text{Re}\, s > 0)$

**17.** Show that the following statements are *incorrect*, in general.

(a) $\int_a^b \cos mx \cos nx \, dx = \text{Re} \int_a^b e^{i(m+n)x} \, dx$

(b) $\int_a^b \sin mx \sin nx \, dx = \text{Im} \int_a^b e^{i(m+n)x} \, dx$

**18.** Find a particular solution $x_p(t)$ of the given differential equation, using the complex function method that is illustrated in Example 4.

(a) $mx'' + cx' + kx = F_0 \sin \omega t$
(b) $x' + 2x = 10 \sin 3t$
(c) $x' + 3x = 20 \cos 5t$
(d) $x'' + x' = 100 \sin 5t$
(e) $x'''' + 2x' + x = 10 \sin t$
(f) $x'''' - x' + 5x = 20 \cos 2t$
(g) $x'''' - 2x' - 3x = 60 \sin 3t$
(h) $x'''' - x''' + x'' - x' + x = 25 \cos 3t$

**19.** The same as Exercise 12 of Section 18.4. NOTE: This exercise illustrates the use of the complex function method in solving *partial* differential equations with cosinusoidal or sinusoidal forcing functions.

## 21.4 Polar Form, Additional Elementary Functions, and Multi-valuedness

Having introduced the complex exponential function in the preceding section, we can now present the polar representation of complex numbers. This representation will be especially useful when, later in this section, we discuss powers of $z$ and the logarithmic function.

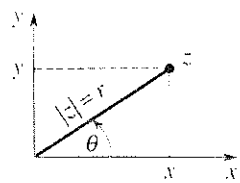



**Figure 1.** Polar coordinates.

**21.4.1. Polar form.** If $r, \theta$ are the usual polar coordinates with $r \ge 0$ and $\theta$ measured in radians counterclockwise from the positive $x$ axis, then (Fig. 1)

$x = r\cos\theta$ and $y = r\sin\theta$, so

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta). \tag{1}$$

Recalling Euler's formula, $e^{i\theta} = \cos\theta + i\sin\theta$, we see that (1) can be re-expressed as

$$\boxed{z = re^{i\theta},} \tag{2}$$

which is the **polar** form of the complex number $z$. Since

$$|z| = \sqrt{x^2 + y^2} = r,$$

we see that $r$ is the **modulus** of $z$:

$$r = \text{mod}\,(z) = \sqrt{x^2 + y^2}. \tag{3}$$

The angle $\theta$, called the **argument** of $z$ and denoted as $\arg(z)$, can be determined from the formula

$$\theta = \arg(z) = \tan^{-1}\frac{y}{x} \tag{4}$$

for $z \neq 0$; for $z = 0$ the angle $\theta$ is undefined, as is evident from Fig. 1. (Subsequently, we will omit the parentheses around $z$ and will write mod $z$ and arg $z$.)

Observe from Fig. 1 that given any point $z$ ($\neq 0$) the angle $\theta$ can be determined only to within an arbitrary integer multiple of $2\pi$. As a useful reference, the value of $\theta$ satisfying of $-\pi < \theta \leq \pi$ will be called the **principal argument** of $z$ and will be written as Arg $z$. Denoting the value of Arg $z$ as $\theta_0$, we have

$$\theta = \arg z = \text{Arg}\, z + 2k\pi = \theta_0 + 2k\pi \tag{5}$$

for $k = 0, \pm 1, \pm 2, \ldots$. This infinite set of $\theta$ values is indeed obtained using (4) because $\tan^{-1}(\,)$ is multi-valued. However, observe carefully that (4) gives additional values as well, which are spurious and must be discarded. Let us illustrate with an example.

**EXAMPLE 1.** Let $z = 1 + i$, so that $x = 1$ and $y = 1$. Then (3) gives $r = \sqrt{2}$, and (4) gives $\theta = \tan^{-1}(1/1) = \tan^{-1}(1)$ which, correctly, gives $\pi/4$ plus any integer multiple of $2\pi$. However, (4) also gives, incorrectly, $5\pi/4$ plus any integer multiple of $2\pi$. Where do these incorrect values come from? They arise because the $\tan^{-1}(y/x)$ in (4) cannot distinguish between the given point $z = 1 + i$ ($x = y = +1$) and the point $z = -1 - i$ ($x = y = -1$), since both points give $\theta = \tan^{-1}(1)$. ∎

Observe that whereas the Cartesian form $x + iy$ is especially convenient for the addition and subtraction of complex numbers, the polar form is especially convenient for their multiplication and division. Specifically,

$$z_1 z_2 = (r_1 e^{i\theta_1})(r_2 e^{i\theta_2}) = r_1 r_2 e^{i(\theta_1 + \theta_2)}, \tag{6}$$

so that *the modulus of the product is the product of the moduli*, and *the argument of the product is the sum of the arguments*. Further, for nonzero complex numbers $z_1$ and $z_2$,

$$\frac{z_1}{z_2} = \frac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)} \tag{7}$$

so *the modulus of the quotient is the quotient of the moduli*, and *the argument of the quotient is the difference of the arguments*.

**21.4.2. Integral powers of $z$ and de Moivre's formula.** Introducing the function $z^n$, where $n$ is any integer (positive, negative, or zero), we find that its calculation is particularly convenient using the polar form of $z$ since

$$z^n = (re^{i\theta})^n = r^n e^{in\theta}. \tag{8}$$

Equation (8) leaves the answer in polar form; if we want it re-expressed in the Cartesian form $a + ib$ we simply use the Euler formula $e^{in\theta} = \cos n\theta + i \sin n\theta$, so that

$$\boxed{z^n = r^n(\cos n\theta + i \sin n\theta),} \tag{9}$$

which result is well known as **de Moivre's formula**. There is a nagging question: Since $\theta$ is not uniquely determined, at any given point $z$ in the complex plane, we wonder if the nonuniqueness of $\theta$ carries over to a nonuniqueness in the value of $z^n$. Let us see. Putting $\theta = \theta_0 + 2k\pi$ into (8) [or (9)] gives

$$z^n = r^n e^{in(\theta_0 + 2k\pi)} = r^n e^{in\theta_0} e^{i2nk\pi}. \tag{10}$$

Since $2nk$ is necessarily an even integer, $\exp(i2nk\pi) = \cos(2nk\pi) + i \sin(2nk\pi) = 1 + i0 = 1$ for all integers $k$. Thus, the nonuniqueness in $\theta$ does *not* carry over to the function $z^n$; $z^n$ has the unique value $r^n \exp(in\theta_0)$.

**EXAMPLE 2.** Compute $(1+i)^3$. First, re-express $z = 1 + i$ in polar form: $r = \sqrt{2}$ and $\theta_0 = \pi/4$, so

$$(1+i)^3 = \left(\sqrt{2}e^{i\pi/4}\right)^3 = 2^{3/2} e^{i3\pi/4} \tag{11}$$

gives the result in polar form. As displayed in Fig. 2, cubing $z$ cubes its modulus and triples its argument. If desired, we can now return to Cartesian form and express

$$(1+i)^3 = 2^{3/2}\left(\cos\frac{3\pi}{4} + i \sin\frac{3\pi}{4}\right) = 2^{3/2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = -2 + 2i.$$

Of course, the latter result could also have been obtained by the Cartesian calculation $(1+i)^3 = (1+i)(1+i)(1+i) = (1+i)(2i) = 2i - 2$, but it is to be appreciated that the Cartesian calculation of $z^n$ becomes quite unwieldy if $n$ is large, whereas the polar calculation is no more difficult for large $n$ than for small $n$. ∎

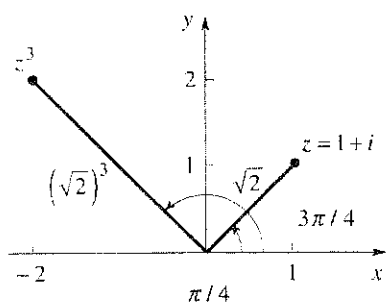



**Figure 2.** $1 + i$ and $(1+i)^3$.

A small technical point: in what should be the simplest case, $z = 0$, $z^n$ is not well defined through (8) because $\theta$ is not defined at the origin. However, the

Cartesian calculation shows that $z^n$ is zero at $z = 0$, provided that the integer $n$ is positive.

**21.4.3. Fractional powers.** Next, we consider the function $z^{1/n}$, called the **$n$th root** of $z$. Proceeding as above, we write

$$z^{1/n} = \left(re^{i(\theta_0+2k\pi)}\right)^{1/n} = r^{1/n}e^{i(\theta_0/n)}e^{i(2k\pi/n)}, \tag{12}$$

where $k = 0, \pm 1, \pm 2, \ldots$. Now, the $\exp(i2nk\pi)$ factor in (10) takes on the unique value 1 for any choice of $k$. Hence, as noted above, $z^n$ takes on a unique value, namely, $r^{1/n}\exp(in\theta_0)$; that is, $z^n$ is single-valued. However, the $\exp(i2k\pi/n)$ factor in (12) takes on $n$ distinct values for various choices of $k$ so that factor, and hence $z^{1/n}$ is $n$-valued rather than single-valued!

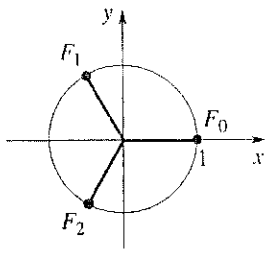



**Figure 3.** The three values of $\exp(i2k\pi/3)$.

**EXAMPLE 3.** To illustrate, consider the calculation of $(1+i)^{1/3}$. Then $r = \sqrt{2}$ and $\theta_0 = \pi/4$ so (12) gives

$$(1+i)^{1/3} = 2^{1/6}e^{i\pi/12}e^{ik(2\pi/3)}. \tag{13}$$

For $k = 0, 1, 2$, the last factor, which we will denote as $F_k$, takes on the values

$$F_k = e^{ik(2\pi/3)} = 1, e^{i2\pi/3}, e^{i4\pi/3}, \tag{14}$$

which correspond to the points $F_0, F_1, F_2$ on the unit circle shown in Fig. 3. As $k$ increases beyond 2, the values of $F_k$ simply repeat the values $F_0, F_1, F_2$ over and over; for instance, $F_3$ and $F_4$ fall upon $F_0$ and $F_1$, respectively, and $F_{-1}$ falls upon $F_2$.

Putting (14) into (13) gives the three cube roots of $1+i$ as

$$(1+i)^{1/3} = 2^{1/6}e^{i\pi/12},\ 2^{1/6}e^{i9\pi/12},\ 2^{1/6}e^{i17\pi/12} \tag{15}$$

and these are depicted in Fig. 4a and denoted there as $f_1, f_2, f_3$. Of course, we could re-express these in Cartesian form using Euler's formula if we wish.

It is worth taking a moment to interpret this result geometrically. Consider the root $f_1$, for instance. Squaring $f_1$ squares its modulus (from $2^{1/6}$ to $2^{2/6}$) and doubles its argument (from $\theta_0/3 = \pi/12$ to $2\pi/12$), and cubing it cubes its modulus (from $2^{1/6}$ to $2^{3/6} = \sqrt{2}$) and triples its argument (from $\pi/12$ to $3\pi/12 = \pi/4$) so that $f_1^3$ ends up at $1+i$ as shown in Fig. 4b. Similar diagrams can be drawn for $f_2$ and $f_3$. ∎

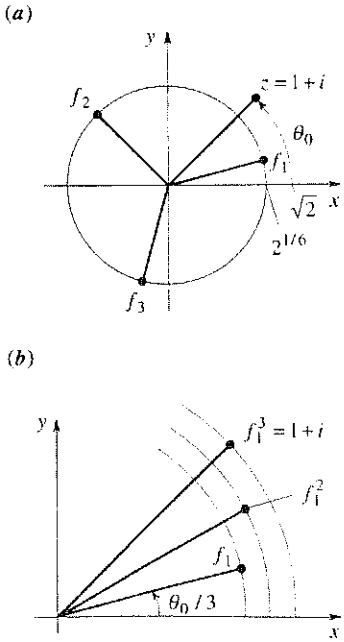



**Figure 4.** The three cube roots of $1+i$.

Example 3 reveals the following general pattern. The $n$ $n$th roots of $z$ (i.e., the $n$ distinct values of $z^{1/n}$) fall on a circle of radius $|z|^{1/n}$, centered at the origin, and are equally spaced on that circle. Thus, if we can find any one root by inspection then the others fall into place on that circle. For instance, to find the three cube roots of $-8$ observe by inspection that one of them is $-2$. Then Fig. 5 shows that the other roots, equally spaced on the circle of radius 2, are $2\exp(i\pi/3)$ and $2\exp(-i\pi/3)$. You might wonder how our discussion of $z^{1/n}$ applies to $(-8)^{1/3}$ since $-8$ is *real*. No, we must treat it as a point in the complex plane – namely, the point $8\exp[i(\pi + 2k\pi)]$ for $k = 0, \pm 1, \pm 2, \ldots$.

If we cannot find one root by inspection, then we compute $r$ and $\theta_0$ and use (12) with $k = 0$, say, to obtain the root $r^{1/n}e^{i(\theta_0/n)}$. With that root located, the other $n-1$ roots are equally spaced on the circle of radius $r^{1/n}$ centered at the origin. For instance, to compute the two square roots of $1+i$, observe that $r = \sqrt{2}$ and $\theta_0 = \pi/4$ so one root is $\sqrt{1+i} = (\sqrt{2})^{1/2}e^{i\pi/8}$, or $\left(2^{1/4}\cos\frac{\pi}{8}\right) + i\left(2^{1/4}\sin\frac{\pi}{8}\right)$. The other, being diametrically opposite this one (Fig. 6), is simply the negative of it. Similarly, $\sqrt{4} = \pm 2$ (of course) and $\sqrt{-4} = \pm 2i$.

Now that we know how to compute $z^n$ and $z^{1/n}$, $z^{m/n}$ (where the fraction $m/n$ is reduced to simplest form, such as $3/2$ rather than $6/4$) offers nothing new since $z^{m/n} = (z^m)^{1/n}$ is just a combination of the two and is $n$-valued.

**EXAMPLE 4.** Compute $(1-2i)^{7/3}$. For $z = 1-2i$ we have $r = \sqrt{5}$ and $\theta_0 = -1.107$ radians. Since we know that $(1-2i)^{7/3}$ is triple-valued, it suffices to work with the three arguments $\theta_0 = -1.107$, $\theta_0 + 2\pi = 5.176$, and $\theta_0 + 4\pi = 11.459$. Thus,

$$\begin{aligned}(1-2i)^{7/3} &= \left(\sqrt{5}e^{-1.107i}\right)^{7/3}, \quad \left(\sqrt{5}e^{5.176i}\right)^{7/3}, \quad \left(\sqrt{5}e^{11.459i}\right)^{7/3} \\ &= 6.538e^{-2.583i}, \qquad 6.538e^{12.077i}, \qquad 6.538e^{26.738i},\end{aligned}$$

which are equally spaced on the circle of radius 6.538. Naturally, these could be expressed in Cartesian form if we wish. For instance, the first is $-5.544 - 3.465i$. ∎

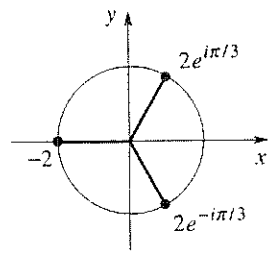



**Figure 5.** The cube roots of $-8$.

**21.4.4. The logarithm of $z$.** Next, we introduce the logarithmic function, $\log z$. To define this function, express $z$ in polar form and write

$$\begin{aligned}\log z &= \log\left(re^{i\theta}\right) = \ln r + \log\left(e^{i\theta}\right) \\ &= \ln r + i\theta \ln e = \ln r + i\theta \\ &= \ln r + i(\theta_0 + 2\pi k),\end{aligned} \tag{16}$$

for $k = 0, \pm 1, \pm 2, \ldots$. Here, and in what follows, we distinguish the notations ln and log: we use ln to denote the ordinary logarithmic function of real variable theory (e.g., $\ln 2 = 0.693$ and $\ln e = 1$), and we use log to denote the logarithmic function of a complex variable, hereby being defined. Be sure to understand that we cannot justify the second and third equalities in (16) because they simply mimic the *real* variable theory properties $\ln(xy) = \ln x + \ln y$ and $\ln(x^y) = y\ln x$. Rather, we write (16) heuristically, and then use the result to *define*

$$\boxed{\log z \equiv \ln r + i(\theta_0 + 2\pi k)} \tag{17}$$

for $k = 0, \pm 1, \pm 2, \ldots$. Observe that just as $z^n$ is single-valued and $z^{1/n}$ is $n$-valued, $\log z$ is *infinite*-valued because each choice of $k$ produces a distinct value of $\log z$.

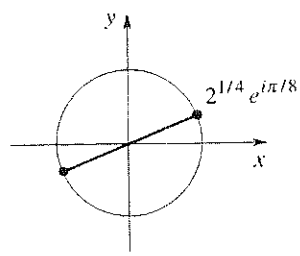



**Figure 6.** The square roots of $1+i$.

**EXAMPLE 5.** Compute $\log(1+i)$. With $r = \sqrt{2}$ and $\theta_0 = \pi/4$, (17) gives

$$\log(1+i) = \ln\sqrt{2} + i\left(\frac{\pi}{4} + 2\pi k\right)$$

$$= \frac{\ln 2}{2} + i\frac{\pi}{4}, \quad \frac{\ln 2}{2} + i\frac{9\pi}{4}, \quad \frac{\ln 2}{2} - i\frac{7\pi}{4}, \ \ldots$$

for $k = 0, +1, -1$, and so on. ∎

**21.4.5. General powers of $z$.** Though we have studied $z^n$ and $z^{1/n}$ (and $z^{m/n}$ as well, where $m$ and $n$ are integers and $m/n$ is reduced to simplest form), what about $z^c$, where $c$ is irrational or complex? Since the polar form is so convenient for exponentiation, let us try evaluating $z^c$ as

$$z^c = (re^{i\theta})^c = r^c e^{i(\theta_0 + 2\pi k)c}. \tag{18}$$

Let us not yet worry about the case where $c$ is complex yet; consider $c$ to be real and irrational. Then (18) does serve to give the values of $z^c$, and we can see that $z^c$ is, like $\log z$, *infinite*-valued. To see that it is infinite-valued note that the factor $F_k = e^{i2\pi kc}$ starts out as $F_0 = 1$ and then takes on different values as $k$ increases. For $z^c$ to be finite-valued we would need $F_k$ to return to the value 1 for *some* integer $k = K$. That will happen if and only if $2\pi Kc$ is an integer multiple of $2\pi$, say $2\pi M$. But then $2\pi Kc = 2\pi M$ gives $c = M/K$, which is rational, whereas $c$ was to be irrational. Thus, $z^c$ cannot be finite-valued; it must be infinite-valued.

**EXAMPLE 6.** Compute $(1+i)^{\sqrt{3}}$. With $r = \sqrt{2}$ and $\theta_0 = \pi/4$, (18) gives the infinite set of distinct values

$$(1+i)^{\sqrt{3}} = (\sqrt{2})^{\sqrt{3}} e^{i(\frac{\pi}{4} + 2\pi k)\sqrt{3}}$$

for $k = 0, \pm 1, \pm 2, \ldots$. For $k = 5$, for instance, we obtain $1.823e^{i41\pi\sqrt{3}/4}$ or, in Cartesian form, $1.303 - 1.275i$. ∎

Finally, consider the case where $c$ is complex: $c = a + ib$, where $b \neq 0$. In that case (18) does not help because we do not know how to evaluate the $r^c = r^{a+ib}$ factor. Thus, in place of (18) let us mimic the real variable formula $x^a = e^{\ln x^a} = e^{a \ln x}$, and *define*

$$\boxed{z^c \equiv e^{c \log z} = e^{c[\ln r + i(\theta_0 + 2\pi k)]}} \tag{19}$$

for $k = 0, \pm 1, \pm 2, \ldots$. Clearly, $z^c$ is infinite-valued in this case as well since the factor $F_k = e^{i2\pi kc}$ occurs in (19), just as in (18).

**EXAMPLE 7.** For instance, evaluate (i.e., find all possible values of) $1^i$. Here, $r = 1$ and $\theta_0 = 0$, so (19) gives

$$1^i = e^{i(0 + i2\pi k)} = e^{-2\pi k},$$

for $k = 0, \pm 1, \pm 2, \ldots$. In case you thought (and not unreasonably) that the modulus of $1^i$ is unity, note that for $k = -20$, for instance, $1^i = e^{40\pi}$, which is enormous! ∎

**21.4.6. Obtaining single-valued functions by branch cuts.** We have seen that functions may be single-valued (e.g., $z^n$), multiple-valued (e.g., $z^{1/n}$ is $n$-valued), or even infinite-valued (e.g., $\log z$). In some circumstances multi-valuedness is perfectly acceptable. For example, in integrating $\int dx/(x^4+1)$ by partial fractions we need to know all possible solutions of the equation $x^4+1=0$, that is, all four values of $(-1)^{1/4}$, in order to factor the $x^4+1$. In other cases multi-valuedness is unacceptable. For example, in the Pythagorean formula $c=\sqrt{a^2+b^2}$ we must adopt the positive square root and discard the negative square root if $c$ is to be the *length* of the hypotenuse.

In fact, when one says that $f(z)$ is a *function* it is understood that for a given input $z$ there is a unique output $f(z)$. Thus, when we call $z^{1/n}$ (for instance) a "multi-valued function" we abuse the understanding that functions are to be single-valued. Yet, that terminology is standard so we will not deviate from it here.

In this subsection we introduce a concept of "branch cuts" by means of which we can take multi-valued functions and render them single-valued. Consider $\log z$, for instance. We can make that function single-valued by specifying the integer $k$ in (17). If, for instance, we choose $k=0$, then that decision amounts to restricting $\theta$ (i.e., $\arg z$) so that $-\pi<\theta\le\pi$. Graphically, we can imagine a "slit" or "cut" in the $z$ plane, along the negative real axis, from $x=-\infty$ all the way in to the origin, as depicted in Fig. 7a. The slit is of zero thickness; in the figure we have separated the upper and lower edges only for clarity. It is useful to regard the slit as a physical barrier that cannot be crossed. Within the *cut* plane $\log z$ is now single-valued and hence a legitimate function.

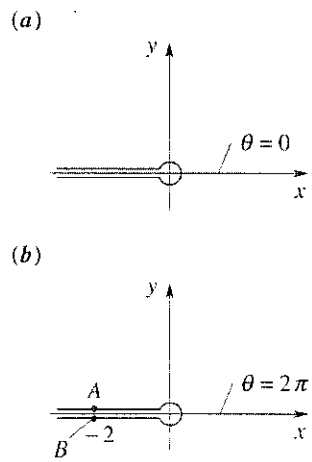



**Figure 7.** Two possible branch cuts for $\log z$.

The cut shown in Fig. 7a is by no means the only one possible. For instance, another suitable cut is shown in Fig. 7b, and restricts $\theta$ to the interval $\pi<\theta\le 3\pi$. The values of the $\log z$ functions defined by Fig. 7a and b are different (at any given $z$ point), and we can think of these two different $\log z$ functions as members, or *branches*, of a whole family of such $\log z$ functions. To illustrate, observe that if we denote them as $\log^{(a)} z$ and $\log^{(b)} z$, respectively, and use $z=i$ as representative, then $\log^{(a)} i=\pi i/2$, whereas $\log^{(b)} i=5\pi i/2$.

Suppose we are asked to compute $\log^{(b)} z$ at $z=-2$. We need to respond as follows. "Do you mean $-2$ on the top of the cut (point $A$ in Fig. 7b) or on the bottom of the cut (point $B$)?" For the values at $A$ and $B$ are different: $\log^{(b)}(z_A)=\ln 2+3\pi i$, whereas $\log^{(b)}(z_B)=\ln 2+\pi i$. Understand that even though the points $z_A$ and $z_B$ look like "neighbors," they are actually quite far apart since to travel from $z_A$ to $z_B$ we need to go around the origin, since the cut is a barrier that cannot be crossed.

The arrangements shown in Fig. 7 are examples of **branch cuts**. *The role of a branch cut is to render the function single-valued.* To render $\log z=\ln r+i\theta$ single-valued we need to make $\theta$ single-valued, and we do that by introducing a cut from infinity to the origin (as a barrier against complete encirclements of the origin, which are the source of the multi-valuedness) and then defining $\theta$ at *some* point in the cut plane. For instance, in Fig. 7a we defined $\theta$ at an arbitrary point on the positive $x$ axis. The branch of $\log z$ defined by the cut shown in Fig. 7a is known as the **principal value** of $\log z$.

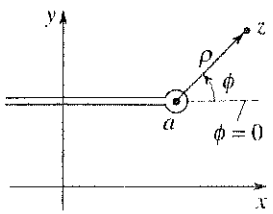



**Figure 8.** A branch cut for $\log(z-a)$.

How about the function $\log(z-a)$, where $a$ is some point in the plane? The idea is to introduce a new variable, say $\zeta$, according to $\zeta = z - a$, and to express $\zeta$ in terms of polar coordinates $\rho, \phi$ at that location as $\zeta = \rho e^{i\phi}$. ($\zeta$ is the "vector" from $a$ to $z$.) Then

$$\log(z-a) = \log\zeta = \log(\rho e^{i\phi}) = \ln\rho + i\phi, \tag{20}$$

and if we wish to render this function single-valued then we need to introduce a branch cut such as the one shown in Fig. 8, from infinity to $\zeta = 0$ (not $z = 0$), so as to prevent encirclements of $\zeta = 0$ and the multi-valuedness that would result.

We say that $\log(z-a)$ has a **branch point** at $z = a$ because it is encirclement of that point that causes the multi-valuedness, and $\log z$ has a branch point at $z = 0$.

Similarly for other multi-valued functions. For instance, the branch cuts shown in Fig. 7 would likewise be suitable for the functions $\sqrt{z}$ and $z^i$, and the one shown in Fig. 8 would be suitable for $\sqrt{z-a}$ and $(z-a)^i$.

**21.4.7. More about branch cuts. (Optional)** To deepen our understanding of branch cuts, consider two more examples.

**EXAMPLE 8.** It would be easy to conclude from Fig. 7 and 8, that a branch cut needs to limit $\theta$ to a $2\pi$ interval. No. To illustrate, consider the more exotic branch cut shown in Fig. 9, for the function $f(z) = z^{2/3}$. It extends straight down to $8 - i\infty$, and $\theta$ is defined as zero at the point shown. Let us evaluate $f(z)$ at $z = 2$ on the bottom of the cut. Beginning at the point where $\theta$ is zero, make a legitimate trip (i.e., one that does not cross the cut) to the point in question, as sketched in the figure. Keeping track of $\theta$ on that trip, we see that $\theta = 4\pi$ at the terminal point. Further, $r = 2$ there so

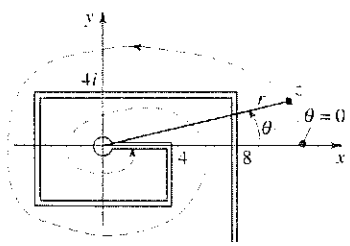



**Figure 9.** A more exotic branch cut.

$$f(z) = z^{2/3} = (2e^{i4\pi})^{2/3} = 2^{2/3}e^{i8\pi/3}. \tag{21}$$

Making a similar trip to $z = 2$ on the *top* of the cut, from the initial point where $\theta = 0$, gives the (different) value

$$f(z) = z^{2/3} = (2e^{i2\pi})^{2/3} = 2^{2/3}e^{i4\pi/3} \tag{22}$$

there. (Do you see that $\theta = 2\pi$ there?)

The cut shown is perfectly legitimate in that it renders $z^{2/3}$ single-valued, yet it does not restrict $\theta$ to a $2\pi$ interval. Rather, $-\pi/2 < \theta \le 4\pi$, the value $-\pi/2$ being approached as $y \to -\infty$ at any fixed $x$ greater than 8. ∎

**EXAMPLE 9.** Consider one last example, the function

$$f(z) = \sqrt{z^2 - 4}. \tag{23}$$

If we write $f$ as $\sqrt{(z-2)(z+2)}$ we see that it contains two functions, $\sqrt{z-2}$ and $\sqrt{z+2}$, each requiring its own branch cut. That is, $f(z)$ has two branch points, one at 2 and one at $-2$. If we wish to render $f$ single-valued, one possible branch cut is as shown in Fig. 10a. To keep the two cuts out of each other's way we have brought one in from the right

and one from the left. Each branch point serves as the origin of a polar coordinate system. To illustrate the use of this cut, let us compute $f$ at $z = -4$ on the bottom of the cut (point $c$). Beginning at $a$, where $\phi_1$ and $\phi_2$ are defined to be 0, let us make the trip $abc$ shown,

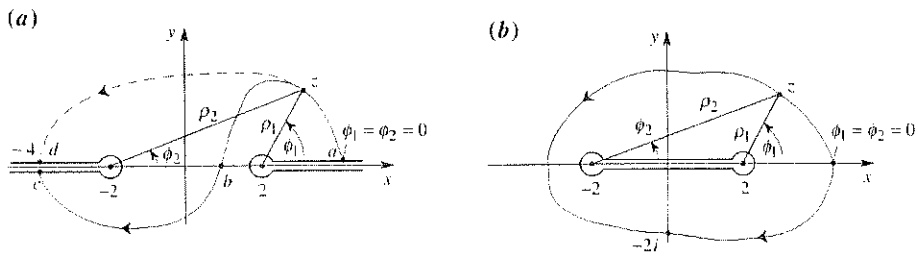



**Figure 10.** Branch cuts for $\sqrt{z^2 - 4}$.

keeping track of $\phi_1$ and $\phi_2$. Proceeding from $a$ to $b$, $\phi_1$ increases to $\pi$, but $\phi_2$ increases and then decreases again, and equals 0 at $b$. Over the $bc$ part of the trip, $\phi_1$ increases a bit, but then decreases again so that $\phi_1 = \pi$ at $c$. Meanwhile, $\phi_2$ changes from 0 at $b$ to $-\pi$ at $c$. Thus, designating $z$ at $c$ as $z_c$,

$$f(z) = \left[(\rho_1 e^{i\phi_1})(\rho_2 e^{i\phi_2})\right]^{1/2} = \sqrt{\rho_1\rho_2}\, e^{i(\phi_1+\phi_2)/2} \tag{24}$$

gives

$$f(z_c) = \sqrt{(6)(2)}\, e^{i(\pi-\pi)/2} = \sqrt{12}\, e^{i0} = \sqrt{12}. \tag{25}$$

On the *top* of the cut, at $d$, however, the dashed path $ad$ gives

$$f(z_d) = \sqrt{(6)(2)}\, e^{i(\pi+\pi)/2} = \sqrt{12}\, e^{\pi i} = -\sqrt{12}. \tag{26}$$

As noted earlier, there are an infinite number of branch cuts possible for $f$. For instance, we could use the same cuts as in Fig. 10a, but define $\phi_1 = 4\pi$ and $\phi_2 = -26\pi$ at $a$. An interesting branch cut is the one shown in Fig. 10b, a *finite* cut extending from $-2$ to 2. This cut does *not* render $\phi_1$ and $\phi_2$ single-valued, yet it *does* render $f$ single-valued, and is therefore an acceptable choice! To illustrate, let us compute $f(-2i)$. If we get to $-2i$, from the initial point (at which $\phi_1$ and $\phi_2$ are defined to be 0), by the counterclockwise path shown, then at $-2i$ we have $\phi_1 = 5\pi/4$ and $\phi_2 = 7\pi/4$, so (24) gives

$$f(-2i) = \sqrt{(\sqrt{8})(\sqrt{8})}\, e^{i(\frac{5\pi}{4}+\frac{7\pi}{4})/2} = -\sqrt{8}\, i. \tag{27}$$

Alternatively, if we take the clockwise path to $-2i$, shown in the figure, then $\phi_1 = -3\pi/4$ and $\phi_2 = -\pi/4$ there, so

$$f(-2i) = \sqrt{(\sqrt{8})(\sqrt{8})}\, e^{i(-\frac{3\pi}{4}-\frac{\pi}{4})/2} = -\sqrt{8}\, i$$

once again. The upshot, general proof of which is left for the exercises, is that the finite cut does indeed render $f$ single-valued and is therefore an acceptable branch cut, even though it does not render $\phi_1$ and $\phi_2$ single-valued. The underlying idea is that although the finite cut permits encirclements of the branch points, it forces us to encircle *both* of them. With each encirclement of 2, $\sqrt{z-2}$ undergoes a sign change. But when we encircle 2 we also encircle $-2$, and the $\sqrt{z+2}$ gives another sign change, and the sign changes cancel.

Finally, there is a question as to which branch cut to choose. In the absence of any context, one branch cut is as good as another – it's your choice. However, in the context of a physical application we will see that the branch cut choice is dictated by that context. In fact, in Chapter 23 the function (23) arises in the context of a fluid mechanics application (flow over a flat plate) and we will see that the appropriate branch cut, in that case, is the one shown in Fig. 10b. See Exercise 18. ∎

**Closure.** The common thread in this section is the polar form of complex numbers, and its use, especially insofar as the calculation of powers of $z$ and $\log z$. Indeed, try to evaluate $\sqrt{1+i}$, say, with*out* first reexpressing $1+i$ in polar form.

We found that $z^k$ is single-valued only if the exponent $k$ is an integer. If $k$ is a rational number $m/n$ (reduced to simplest form) it is $n$-valued. If $k$ is irrational, then $z^k$ is infinite-valued, as makes sense inasmuch as an irrational number can be expressed as the limit of a sequence of rational numbers. For instance, $\pi = 3.14159\ldots$ is the limit of the sequence $3, 31/10, 314/100 = 157/50, \ldots$ and $z^3$ is single-valued, $z^{31/10}$ is ten-valued, $z^{157/50}$ is 50-valued, and so on. Finally, $z^k$ is infinite-valued also if $k$ is complex, with $\operatorname{Im} k \neq 0$.

We note that sometimes we are interested in finding all the values of a multi-valued function, and that at other times it is necessary to render the function single-valued, as we shall see in subsequent chapters. To do so, we introduce a branch cut and define the polar angle at any specific point in the cut plane. Typically, the cut extends from the branch point to infinity, and limits the polar angle to a $2\pi$ interval. In the optional Section 21.4.7, we see that a cut of finite length can sometimes be used, and that the polar angle need not be limited to a $2\pi$ interval, nor even be made single-valued by the cut. What must always be true, however, is that the cut emanates from the branch point and renders the *function* single-valued in the cut plane.

## EXERCISES 21.4

**1.** Determine $r$ and the principal argument $\theta_0$ (in radians and in degrees) for each of the following values of $z$.

(a) $-3i$ (b) $8i$ (c) $-6$
(d) $1+5i$ (e) $-4-3i$ (f) $2-12i$
(g) $-1+i$ (h) $-1-i$ (i) $0.2+i$

**2.** Express $\operatorname{Re}(re^{i\theta})$ and $\operatorname{Im}(re^{i\theta})$ in terms of $r$ and $\theta$.

**3.** Consider formulas (6) and (7) for the product and quotient $z_1 z_2$ and $z_1/z_2$. Show that these quantities are uniquely determined even though $\arg z_1$, and $\arg z_2$ are, according to (5), multi-valued.

**4.** Obtain $z^{10}$ and $z^{20}$, in both polar and Cartesian form, for each given $z$.

(a) $-1+i$ (b) $1+i$ (c) $1+2i$
(d) $2-i$ (e) $3-i$ (f) $3+4i$
(g) $5-12i$ (h) $12+5i$ (i) $2-2i$

**5.** Find all values of $z^{1/2}$ and $z^{1/5}$ for each given $z$. Express those values in polar form, and show their location in the $z$ plane, as we have done in Fig. 5.

(a) $i$ (b) $1$ (c) $2$
(d) $-i$ (e) $3-2i$ (f) $-1+2i$
(g) $3+4i$ (h) $1-4i$ (i) $-32$

**6.** Obtain, in Cartesian form, all values of $\log z$ for each given $z$.

(a) $-2$ (b) $1$ (c) $i$
(d) $-5i$ (e) $2-i$ (f) $3-4i$
(g) $13-5i$ (h) $1+6i$ (i) $-3-2i$

**7.** Recall that, by definition, two complex numbers $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$ are equal if $x_1 = x_2$ and $y_1 = y_2$. Show that it follows from this definition that $r_1 e^{i\theta_1} = r_2 e^{i\theta_2}$ (where $r_1 \neq 0$ and $r_2 \neq 0$) if $r_1 = r_2$ and $\theta_1 = \theta_2 +$ arbitrary integer multiple of $2\pi$.

**8.** Obtain, in polar form, all values of $z^{2/3}, z^{3/2}$, and $z^{\pi}$ for each given $z$.

| | | |
|---|---|---|
| (a) $2i$ | (b) $3$ | (c) $-4$ |
| (d) $-6i$ | (e) $1-i$ | (f) $2+i$ |
| (g) $1-4i$ | (h) $2+2i$ | (i) $-2-2i$ |

**9.** (a)–(i) Obtain, in Cartesian form, all values of $z^i$ and $z^{1-i}$ for each $z$ given in Exercise 8.

**10.** Let $c$ be any real or complex number other than 0. Then for each integer value of $k$,

$$c^z = e^{z \log c} = e^{z[\ln|c| + i(\operatorname{Arg} c + 2k\pi)]} \tag{10.1}$$

defines a distinct single-valued function of $z$. Suppose that we set $k = 0$, because then if $c$ is real and equal to $e$, $c^z$ reduces to the familiar exponential function $e^z$. Thus let us define

$$c^z \equiv e^{z(\ln|c| + i\operatorname{Arg} c)}. \tag{10.2}$$

We call it the **generalized exponential function** because it allows for any value of $c$ ($\neq 0$) and reduces to the exponential function $e^z$ for the case where $c = e$. We now state the problem. With $c^z$ defined by (10.2), evaluate $c^z$ for $c = 1 + \sqrt{3}\,i$ and $z = 2 - 5i$.

**11.** Obtain, in Cartesian form, the principal values of $\log z$ and $\sqrt{z}$ for each given $z$.

| | | |
|---|---|---|
| (a) $-3i$ | (b) $2$ | (c) $-4$ |
| (d) $2-i$ | (e) $1+\sqrt{3}\,i$ | (f) $-1-i$ |
| (g) $-5i$ | (h) $4-2i$ | (i) $-2+4i$ |

**12.** Now that we have seen how to render $\log z$ single-valued, we can complete our discussion of the logarithmic function by stating that the familiar properties of the real logarithmic function do carry over to the complex case. For example,

$$\log(z_1 z_2) = \log z_1 + \log z_2, \tag{12.1}$$

$$\log\left(\frac{z_1}{z_2}\right) = \log z_1 - \log z_2, \tag{12.2}$$

and

$$\log z^c = c \log z \qquad (c \text{ real or complex}), \tag{12.3}$$

provided that all of the log functions in the equation are defined by the same branch cut. For example, (12.3) is correct if we use the principal value definition of both log functions, provided that $\arg z = \theta$ satisfies $-\pi < \theta \leq \pi$ and $\arg(z^c) = c\theta$ satisfies $-\pi < c\theta \leq \pi$.

(a) Derive (12.1). (b) Derive (12.2). (c) Derive (12.3).

**13.** (*Inverse of sine function*) We define the inverse of the sine function

$$w(z) = \sin^{-1} z \tag{13.1}$$

such that $z = \sin w$.

(a) Writing the latter as

$$z = \left(e^{iw} - e^{-iw}\right)/2i,$$

show that $e^{iw} = iz + (1 - z^2)^{1/2}$, and hence that

$$\sin^{-1} z = -i \log\left[iz + \sqrt{1 - z^2}\right]. \tag{13.2}$$

(b) Observe that $\sin^{-1} z$ is multi-valued because of the $(1 - z^2)^{1/2}$ and also because of the $\log[\,]$. Specifically, for each value of $z$ ($\neq \pm 1$), the $(1 - z^2)^{1/2}$ gives two values. Then, for each of these values the log gives an infinite set of values. To illustrate this point, show that

$$\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6} + 2k\pi \quad \text{or} \quad \frac{5\pi}{6} + 2k\pi$$

for $k = 0, \pm 1, \pm 2, \ldots$.

(c) Determine all possible values of $\sin^{-1} 2$.

(d) Determine all possible values of $\sin^{-1}(2i)$.

**14.** (*Inverses of other trigonometric functions*) Proceeding as in Exercise 13, derive these formulas.

(a) $\cos^{-1} z = -i \log\left[z + \sqrt{z^2 - 1}\right]$

(b) $\tan^{-1} z = -\dfrac{i}{2} \log \dfrac{i - z}{i + z}$

(c) $\cot^{-1} z = -\dfrac{i}{2} \log \dfrac{z + i}{z - i}$

**15.** (*Inverses of hyperbolic functions*) Proceeding as in Exercise 13, derive these formulas.

(a) $\sinh^{-1} z = \log\left[z + \sqrt{1 + z^2}\right]$

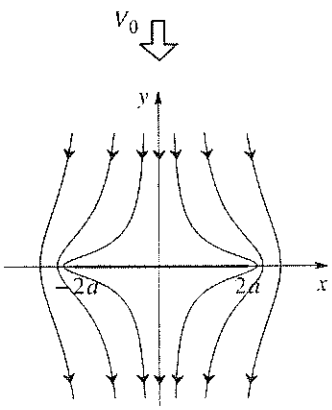
(b) $\cosh^{-1} z = \log\left[z + \sqrt{z^2 - 1}\right]$

(c) $\tanh^{-1} z = \dfrac{1}{2}\log\dfrac{1+z}{1-z}$

(d) $\coth^{-1} z = \dfrac{1}{2}\log\dfrac{z+1}{z-1}$

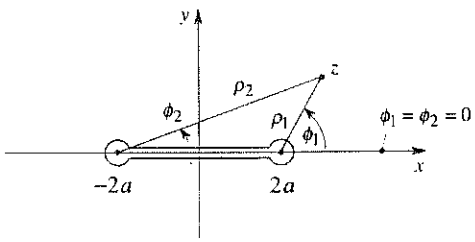
**16.** In Section 22.6 we will find that the plane irrotational incompressible flow of a downward free stream $V_0$ over a flat plate that extends from $x = -2a$ to $x = +2a$ (see the figure) is given by

$$u - iv = \frac{iV_0 z}{\sqrt{z^2 - 4a^2}}, \tag{16.1}$$




where $u(x, y)$ and $v(x, y)$ are the $x$ and $y$ velocity components, respectively. The context dictates using a branch cut, for $\sqrt{z^2 - 4a^2}$, extending from $-2a$ to $+2a$ (as in Example 9) as shown below. One reason for choosing that cut is that the mathematical barrier presented by the cut corresponds to the physical barrier presented by the plate. Another is that the mathematical discontinuities that arise across the cut will correspond to physically anticipated flow discontinuities across the plate. In fact, if we use a cut like the one in Fig. 10a, then those discontinuities would occur within the flow field, which would be unacceptable. The problem that we pose is for you to use (16.1) and the cut shown above to show that the velocity components on the top and bottom of the plate are

$$u(x, 0+) = V_0 x/\sqrt{4a^2 - x^2}, \qquad v(x, 0+) = 0,$$
$$u(x, 0-) = -V_0 x/\sqrt{4a^2 - x^2}, \qquad v(x, 0-) = 0.$$




## 21.5 The Differential Calculus and Analyticity

Continuing to parallel the development of real variable theory, having introduced the complex number system and then a number of elementary functions defined on that number system, we next define the *limit* of a function, *continuity*, and the *derivative*. In fact, differentiability (more precisely, the closely related concept of analyticity) lies at the heart of complex variable theory and will be crucial throughout our subsequent study of conformal mapping, the complex integral calculus, and series expansions.

Let $z_0$ be an interior point in the domain of definition of $f(z)$. We say that *the* **limit** *of $f(z)$, as $z$ approaches $z_0$, is $L$,*

$$\lim_{z \to z_0} f(z) = L \tag{1}$$

*[or, equivalently, we write $f(z) \to L$ as $z \to z_0$] if to each $\epsilon > 0$ (i.e., no matter how small) there corresponds a $\delta > 0$ such that*

$$|f(z) - L| < \epsilon \tag{2}$$

*for all z's satisfying* $0 < |z - z_0| < \delta$. That is (Fig. 1), we can keep the function values $f(z)$ arbitrarily close to $L$ (namely, within the $\epsilon$-disk) by keeping $z$ sufficiently close to $z_0$ (namely, within $0 < |z - z_0| < \delta$). Observe carefully that because of the first inequality in $0 < |z - z_0| < \delta$, the value of $f$ *at* $z = z_0$ is not

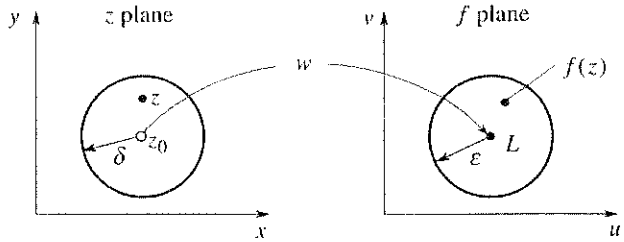



**Figure 1.** Limit of $f(z)$ as $z \to z_0$.

at all germane. Specifically, it need not be true that $f(z_0) = L$ for (1) to hold; (1) merely states that $f(z)$ approaches $L$ as $z$ approaches $z_0$. However, if in addition to having $\lim_{z \to z_0} f(z) = L$ we also have $f(z_0) = L$, then we say that the function $f(z)$ is **continuous** at $z_0$.

**EXAMPLE 1.** Show that

$$\lim_{z \to i} (z^2 + iz) = -2. \tag{3}$$

To do so, we need to show that to each $\epsilon > 0$ (no matter how small) there corresponds a $\delta > 0$ such that $|(z^2 + iz) - (-2)| < \epsilon$ for all $z$'s in $0 < |z - i| < \delta$. No doubt, the smaller we choose $\epsilon$, the smaller $\delta$ will need to be. Thus, $\delta = \delta(\epsilon)$, and our objective is to put forward a suitable $\delta(\epsilon)$ in order to verify (3). Since

$$\begin{aligned} |z^2 + iz + 2| &= |(z - i)(z + 2i)| = |z - i|\,|z + 2i| \\ &= |z - i|\,|(z - i) + 3i| \le |z - i|\,(|z - i| + 3), \end{aligned} \tag{4}$$

where the last step follows from the triangle inequality, $|z^2 + iz + 2| < \epsilon$ will be satisfied if

$$|z - i|\,(|z - i| + 3) < \epsilon. \tag{5}$$

Now, solving $x(x + 3) = \epsilon$ we obtain the positive solution $x = \left(-3 + \sqrt{9 + 4\epsilon}\right)/2$ so evidently (5) will be satisfied if

$$|z - i| < \frac{-3 + \sqrt{9 + 4\epsilon}}{2}. \tag{6}$$

In summary, we have shown that given any $\epsilon > 0$ we will have $|(z^2 + iz) - (-2)| < \epsilon$ for all $z$'s in $0 < |z - i| < \delta(\epsilon)$ if we choose

$$\delta(\epsilon) = \frac{-3 + \sqrt{9 + 4\epsilon}}{2}, \tag{7}$$

or smaller. Hence, the truth of (3) is established. In fact, it is also true that the value of $z^2 + iz$ at $z = i$ is $-2$ so the function $z^2 + iz$ is continuous at $z = i$. Similarly, one can show that it is continuous everywhere in the $z$ plane. ∎

We do not wish to dwell on limits and continuity too long, since the focus of this section is differentiability. Thus, let us note merely that the functions $z^n$ $(n = 0, 1, 2, \ldots)$, $e^z, \sin z, \cos z, \sinh z$, and $\cosh z$, for example, are continuous everywhere in the $z$ plane and the function $1/z$, for instance, is continuous for all $z$ except $z = 0$ because $\lim_{z\to 0} 1/z$ does not exist. We are now in a position to define the derivative $df/dz$, which we usually denote as $f'(z)$. In doing so, we stay as close as possible to the real-variable definition. Thus, if $z_0$ is an interior point of the domain of definition of $f$, we define the **derivative** of $f$, at $z_0$, as

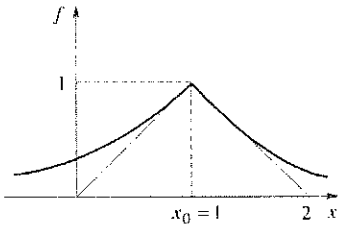



**Figure 2.** Approach to $x_0$.

$$\boxed{f'(z_0) \equiv \lim_{\Delta z\to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z},} \tag{8}$$

provided, of course, that the limit exists. Equivalently, and sometimes more conveniently,

$$f'(z_0) \equiv \lim_{z\to z_0} \frac{f(z) - f(z_0)}{z - z_0}. \tag{9}$$

Remember that in the real-variable case, where

$$f'(x_0) \equiv \lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0}, \tag{10}$$

the limit value needs to be independent of the way in which $x$ approaches $x_0$. For example, for the case shown in Fig. 2, $f'(x_0)$ does not exist for $x_0 = 1$ because the limiting value obtained when $x \to x_0$ from the left (namely, the slope 1) is different from the limiting value obtained when $x \to x_0$ from the right (namely, the slope $-1$). In fact, $x$ does not need to approach $x_0$ from one side or the other. For example, the sequence $x_n = x_0 + (-1)^n e^{-n}$ $(n = 0, 1, 2, \ldots)$ approaches $x_0$ as $n \to \infty$, but not from one side.

Analogously, in the complex case (9), we insist that the limit exist uniquely, independent of the way in which $z$ approaches $z_0$. (Note that we avoid saying "independent of the direction of approach" because that wording seems to limit the mode of approach to linear as in Fig. 3a, whereas it may be spiral, for example, as in Fig. 3b. Even more so, we avoid saying "as $z$ approaches $z_0$ from all directions."*

Let us illustrate the definition (8).

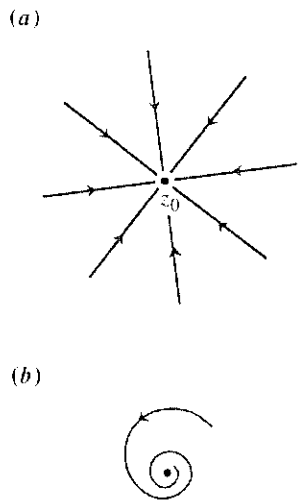



**Figure 3.** Approach to $z_0$.

**EXAMPLE 2.** If $f(z) = z^2$, then [dropping the subscripted zero in (8)]

$$f'(z) = \lim_{\Delta z\to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} = \lim_{\Delta z\to 0} (2z + \Delta z) = 2z \tag{11}$$

*If this point is unclear, it should suffice to imagine Stephen Leacock's Lord Ronald, who "... said nothing; he flung himself from the room, flung himself upon his horse and rode madly off in all directions."

so that $f(z) = z^2$ is differentiable for all $z$, and $f'(z) = 2z$. [The last step, in (11), should be clear even if we do not go to the trouble of putting forward a $\delta(\epsilon)$ relation as we did in Example 1.] ∎

Recalling the real-variable result $d(x^2)/dx = 2x$, and the similarity between (9) and (10), it is not surprising that $d(z^2)/dz$ turned out to be $2z$. Similarly, one finds that $d(e^z)/dz = e^z$, $d(\sin z)/dz = \cos z$, and so on (Exercise 10). Furthermore, the various familiar rules of differentiation carry over to the complex case. For example, we have

$$[f(z) + g(z)]' = f'(z) + g'(z), \tag{12}$$

$$[f(z)g(z)]' = f'(z)g(z) + f(z)g'(z), \tag{13}$$

and the chain rule

$$\frac{d}{dz} f(g(z)) = f'(g(z))g'(z), \tag{14}$$

where $f'(g(z))$ denotes $df/dg$. Furthermore, differentiability implies continuity, and l'Hôpital's rule holds, as in the real case (Exercises 7 and 8).

From the examples cited above, one might wonder how $f(z)$ can *fail* to be differentiable.

**EXAMPLE 3.** $f(z) = 1/z$ is not differentiable at $z = 0$ because it is not continuous there. That is, since differentiability implies continuity, continuity is a necessary condition for differentiability. For all $z \neq 0$, however, $f'(z)$ exists (i.e., $f$ is differentiable) and equals $-1/z^2$. ∎

**EXAMPLE 4.** Whereas $f(z) = 1/z$ fails to be differentiable only at a single point,

$$f(z) = \overline{z}$$

is not differentiable *anywhere*. For observe that

$$\frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{\overline{z + \Delta z} - \overline{z}}{\Delta z} = \frac{\overline{\Delta z}}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y}. \tag{15}$$

For a horizontal approach to $z$ we have $\Delta y = 0$, and the limit of the difference quotient (15), as $\Delta x \to 0$, is $+1$. For a vertical approach, on the other hand, $\Delta x = 0$, and the limit of the difference quotient (15), as $\Delta y \to 0$, is $-1$. Since a unique limit (of $[f(z + \Delta z) - f(z)]/\Delta z$, as $\Delta z \to 0$) does not exist, it follows that $f(z) = \overline{z}$ is not differentiable anywhere in the $z$ plane. ∎

Because differentiability turns out to be crucial in the study of functions of a complex variable, it will be important to find a test that can be applied to a given function $f(z) = u(x, y) + iv(x, y)$ to see if it is differentiable. Toward that end,

let us first set $\Delta y = 0$ in $\Delta z = \Delta x + i\Delta y$ so that $\Delta z = \Delta x$. That is, let $z_0 + \Delta z$ approach $z_0$ parallel to the $x$ axis (Fig. 4). Then the limit of the difference quotient in (8) becomes

$$\begin{aligned}
&\lim_{\Delta x \to 0} \frac{[u(x_0 + \Delta x, y_0) + iv(x_0 + \Delta x, y_0)] - [u(x_0, y_0) + iv(x_0, y_0)]}{\Delta x} \\
&\quad = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x} \\
&\quad = \frac{\partial u}{\partial x} + i\,\frac{\partial v}{\partial x}.
\end{aligned} \tag{16}$$

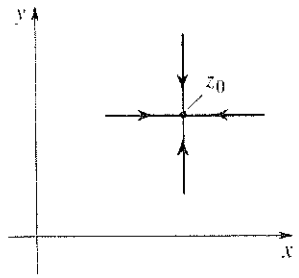



**Figure 4.** Horizontal and vertical approaches.

Alternatively, let us set $\Delta x = 0$ in $\Delta z = \Delta x + i\Delta y$ so that $\Delta z = i\Delta y$. That is, let $z_0 + \Delta z$ approach $z_0$ parallel to the $y$ axis (Fig. 4). This time, the limit of the difference quotient in (8) becomes

$$\begin{aligned}
&\lim_{\Delta y \to 0} \frac{[u(x_0, y_0 + \Delta y) + iv(x_0, y_0 + \Delta y)] - [u(x_0, y_0) + iv(x_0, y_0)]}{i\Delta y} \\
&\quad = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}.
\end{aligned} \tag{17}$$

Since the value of the limit in (8) is to be independent of the path of approach, as emphasized above, the results in (16) and (17) must agree if $f$ is to be differentiable at $z_0$, that is, if $f'(z_0)$ is to exist. Thus, equating the right-hand sides of (16) and (17) shows that $u(x, y)$ and $v(x, y)$ must satisfy the relations

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}} \tag{18}$$

at the point in question. The latter are known as the **Cauchy–Riemann equations.***

Clearly, satisfaction of these equations is necessary for differentiability, but may not be sufficient since they have been obtained by considering only two possible paths of approach to the point $z_0$. (In fact, they are *not* sufficient, as follows from the example in Exercise 9.) We have the following theorem.

*Principal, in the development of complex variable theory were *Augustin–Louis Cauchy* (1789–1857), pronounced *ko-she*, *Georg Friedrich Bernhard Riemann* (1826–1866), and *Karl Weierstrass* (1815–1897). One of the most prolific mathematicians, Cauchy entered the Ecole Polytechnique to study engineering but, because of poor health, was advised by Lagrange and Laplace to study mathematics instead. His works span almost all branches of mathematics. Entering Göttingen to study theology, Riemann turned to mathematics, studied under Gauss, and became a professor of mathematics there in 1859. After studying law for four years at Bonn, Weierstrass likewise turned to mathematics. Weierstrass developed complex variable theory based upon the power series representation of functions, which approach failed to attract a significant following. Rather, the traditional approach, presented in this text, follows Cauchy, who began by laying a groundwork of complex differential and integral calculus. As we shall see, only after that foundation is securely laid will we be in a position to develop power series representations of functions. Thus, Weierstrass's approach was somewhat the reverse of that developed by Cauchy.

**THEOREM 21.5.1** *Necessary, Sufficient Conditions for Differentiability*
Let $f(z) = u(x, y) + iv(x, y)$ be defined throughout some neighborhood of a point $z_0 = x_0 + iy_0$. For $f$ to be differentiable at $z_0$,

**(i)** it is **necessary** that the Cauchy–Riemann equations (18) be satisfied at $x_0, y_0$;

**(ii)** it is **sufficient** that the Cauchy–Riemann equations (18) be satisfied at $z_0$, and that $u$ and $v$ be $C^1$ in some neighborhood of $z_0$.

If $f$ is differentiable, then $f'$ is given by any of these four equivalent expressions:

$$\boxed{f' = u_x + iv_x = v_y - iu_y = u_x - iu_y = v_y + iv_x.} \tag{19}$$

*Proof*: We have already proved item (i). To prove (ii), let us assume that besides the Cauchy–Riemann equations (18) being staisfied at $z_0$ the partial derivatives $u_x, u_y, v_x, v_y$ are continuous in some neighborhood of $z_0$; in terms of the $C^n$ notation introduced at the end of Section 13.5 we say that $u$ and $v$ are $C^1$ in some neighborhood of $z_0$. Consider an *arbitrary* approach to the point $z_0$, as indicated schematically in Fig. 5 by the curve $C$, and let us denote $(x_0, y_0)$ and $(x, y)$ as $P_0$ and $P$, respectively, for brevity. From (8),

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{[u(P) + iv(P)] - [u(P_0) + iv(P_0)]}{\Delta z}, \tag{20}$$

where $\Delta z$ is the "vector" from $P_0$ to $P$. Now, since $u$ and $v$ have been assumed to be $C^1$ in some neighborhood of $P_0$, we can use the mean value theorem (Section 13.5) to express

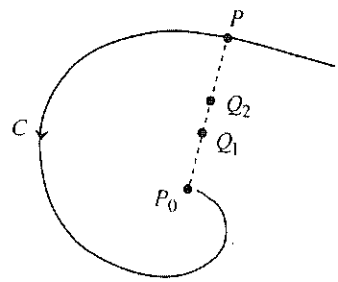



**Figure 5.** Arbitrary approach to $z_0$.

$$u(P) = u(P_0) + u_x(Q_1)\Delta x + u_y(Q_1)\Delta y \tag{21a}$$
$$v(P) = v(P_0) + v_x(Q_2)\Delta x + v_y(Q_2)\Delta y, \tag{21b}$$

where the points $Q_1$, $Q_2$ lie somewhere on the straight line between $P_0$ and $P$, as we have illustrated in Fig. 5. Using (21), we may restate (20) as

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{u_x(Q_1)\Delta x + u_y(Q_1)\Delta y + i[v_x(Q_2)\Delta x + v_y(Q_2)\Delta y]}{\Delta z}.$$

Finally, using the fact that $Q_1$ and $Q_2$ approach $P_0$ as $\Delta z \to 0$, let us rewrite the last result as

$$\begin{aligned} f'(z_0) &= \lim_{\Delta z \to 0} \left\{ \frac{u_x(P_0)\Delta x + u_y(P_0)\Delta y + i[v_x(P_0)\Delta x + v_y(P_0)\Delta y]}{\Delta z} + \delta \right\} \\ &= \lim_{\Delta z \to 0} \left\{ \frac{u_x(P_0)(\Delta x + i\Delta y) - iu_y(P_0)(\Delta x + i\Delta y)}{\Delta z} + \delta \right\} \\ &= u_x(P_0) - iu_y(P_0) + \lim_{\Delta z \to 0} \delta, \end{aligned} \tag{22}$$

where we have used the Cauchy–Riemann relations to obtain the second equality, and where

$$\delta = [u_x(Q_1) - u_x(P_0)]\frac{\Delta x}{\Delta z} + [u_y(Q_1) - u_y(P_0)]\frac{\Delta y}{\Delta z}$$
$$+i[v_x(Q_2) - v_x(P_0)]\frac{\Delta x}{\Delta z} + i[v_y(Q_2) - v_y(P_0)]\frac{\Delta y}{\Delta z}. \tag{23}$$

From the continuity of the four partial derivatives, it follows that each bracketed quantity in (23) tends to zero as $\Delta z \to 0$. Furthermore, $|\Delta x| \le |\Delta z|$ and $|\Delta y| \le |\Delta z|$ give $|\Delta x/\Delta z| \le 1$ and $|\Delta y/\Delta z| \le 1$. Thus, $\lim_{\Delta z \to 0} \delta = 0$, and (22) gives the result

$$f'(z_0) = u_x(P_0) - iu_y(P_0) = u_x(x_0, y_0) - iu_y(x_0, y_0), \tag{24}$$

independent of the manner of approach of $P$ to $P_0$. ■

Before continuing, observe that the additional conditions in (ii), that $u_x, u_y, v_x, v_y$ be continuous in some neighborhood of $z_0$, are quite reasonable since the difference quotient limits obtained when approaching $z_0$ along any straight line, say, can be obtained by interpolating the limits obtained by horizontal and vertical approaches, and the validity of interpolation depends upon the continuity of the quantities being interpolated.

Let us continue. Suppose that $f(z)$ is differentiable at $z_0$. If, in addition, it is differentiable throughout some neighborhood of $z_0$, then we say that it is **analytic** at $z_0$. If it is not analytic at $z_0$, it is **singular** there, and if $f$ is analytic at each point of a region $D$ we say that $f$ is analytic in $D$. (The terms *regular* and *holomorphic* are used by some authors, in place of analytic.)

To see how a function $f(z)$ can be differentiable at a point and yet *fail* to be differentiable in some neighborhood of the point, consider the following example.

**EXAMPLE 5.** Consider

$$f(z) = |z|^2 = z\overline{z}. \tag{25}$$

Then $f = (x^2 + y^2) + 0i$, so $u = x^2 + y^2$ and $v = 0$. Since

$$u_x = 2x, \qquad u_y = 2y,$$
$$v_x = 0, \qquad v_y = 0$$

we see that $u, v, u_x, u_y, v_x, v_y$ are continuous everywhere so $u$ and $v$ are $C^1$ everywhere. Moreover, $u_x = v_y$ is satisfied at all points on $x = 0$ (the $y$ axis) and $u_y = -v_x$ is satisfied at all points on $y = 0$ (the $x$ axis) so both Cauchy–Riemann equations are satisfied only at the origin. Thus, $f(z) = |z|^2$ is differentiable only at $z = 0$ and hence it is analytic *nowhere*. ■

Such behavior, however, is hardly typical of the functions that we are apt to encounter, most of which are either analytic everywhere or almost everywhere.

Functions that are analytic everywhere (i.e., at each point $z$ in the $z$ plane) are said to be **entire**.

**EXAMPLE 6.** Consider

$$f(z) = \sin z = \sin x \cosh y + i \cos x \sinh y. \tag{26}$$

Then

$$\begin{aligned} u_x &= \cos x \cosh y, & u_y &= \sin x \sinh y, \\ v_x &= -\sin x \sinh y, & v_y &= \cos x \cosh y. \end{aligned}$$

Clearly, $u, v, u_x, u_y, v_x, v_y$ are continuous everywhere, and $u_x = v_y$ and $u_y = -v_x$ everywhere so that, according to Theorem 21.5.1, $f(z) = \sin z$ is analytic everywhere; it is an entire function. From (24),

$$f'(z) = \cos x \cosh y - i \sin x \sinh y. \tag{27}$$

To re-express the latter in terms of $z$ we could substitute $x = (z + \overline{z})/2$ and $y = (z - \overline{z})/2i$ and simplify. The $\overline{z}$'s would cancel and we would obtain $f'(z) = \cos z$. However, in the present case it is simpler to notice, from (27), that

$$f'(z) = \cos x \cosh y - \sin x \sin (iy) = \cos (x + iy) = \cos z. \tag{28}$$

Practically speaking, we could have depended on the familiar real variable result $d(\sin x)/dx = \cos x$ to tell us that $d(\sin z)/dz = \cos z$, but the point of this example was to illustrate Theorem 21.5.1 and the concept of analyticity. ∎

Similarly, $\cos z, e^z, \sinh z, \cosh z$, and polynomial functions of $z$, for example, are entire functions. More complicated cases can be built from these as composite functions. Consider, for example, $f(z) = e^{\sin z}$. Recalling that

$$\frac{d}{dz} f(g(z)) = \frac{df}{dg}\frac{dg}{dz}, \quad \text{we have} \quad \frac{d}{dz} f = \left(\frac{d}{dg} e^g\right)\left(\frac{d}{dz} \sin z\right).$$

Each derivative on the right-hand side exists because the exponential and sine functions are entire; thus $f(z) = e^{\sin z}$ is entire too, and $d(e^{\sin z})/dz = e^{\sin z} \cos z$.

The function $1/z$ is analytic for all $z \neq 0$. At $z = 0$ the derivative

$$\frac{d}{dz}\left(\frac{1}{z}\right) = -\frac{1}{z^2}$$

fails to exist so $z = 0$ is a singular point of $f(z) = 1/z$. Alternatively, $z = 0$ is necessarily a singular point of $1/z$ because the latter is not continuous there.

In some cases $f$ is expressed more conveniently in terms of the polar variables $r, \theta$ as

$$f(z) = u(r, \theta) + iv(r, \theta). \tag{29}$$

The Cauchy–Riemann equations are then found (Exercise 12) to be

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}, \qquad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta},} \tag{30}$$

and the derivative of $f$ is

$$\begin{aligned} f'(z) &= e^{-i\theta}(u_r + iv_r) = \frac{e^{-i\theta}}{r}(v_\theta - iu_\theta) \\ &= e^{-i\theta}\left(u_r - \frac{i}{r}u_\theta\right) = e^{-i\theta}\left(\frac{1}{r}v_\theta + iv_r\right). \end{aligned} \tag{31}$$

**EXAMPLE 7.** Consider the principal value of $\log z$, defined by

$$\log z = \ln r + i\theta, \tag{32}$$

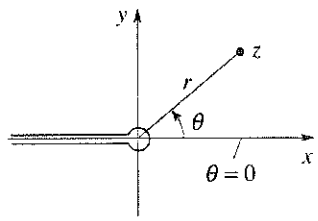



**Figure 6.** Domain of definition of principal value of $\log z$.

where $0 < r < \infty$ and $-\pi < \theta \le \pi$. That is, the domain of definition of $\log z$ is the cut plane shown in Fig. 6. In this case,

$$\begin{gathered} u(r,\theta) = \ln r, \qquad v(r,\theta) = \theta, \\ u_r = \frac{1}{r}, \qquad u_\theta = 0, \qquad v_r = 0, \qquad v_\theta = 1 \end{gathered} \tag{33}$$

so we see that $u, v, u_r, u_\theta, v_r, v_\theta$ are all continuous in the cut plane, and that the Cauchy–Riemann equations (30) are satisfied everywhere in the cut plane (which does not include the origin). Thus, $\log z$ is analytic everywhere in the cut plane, and (31) gives

$$\frac{d}{dz}(\log z) = e^{-i\theta}\left(\frac{1}{r} + i0\right) = \frac{1}{re^{i\theta}} = \frac{1}{z},$$

which result is no shock in view of the real-variable result $d(\ln x)/dx = 1/x$. ∎

As the final item in this section, we establish an important and interesting link between analytic function theory and partial differential equations – specifically, the Laplace equation in two dimensions. If $f(z) = u(x,y) + iv(x,y)$ is analytic in some domain $D$, then

$$u_x = v_y \quad \text{and} \quad u_y = -v_x \tag{34}$$

in $D$. Taking $\partial/\partial x$ of the first and $\partial/\partial y$ of the second,

$$u_{xx} = v_{yx} \quad \text{and} \quad u_{yy} = -v_{xy}. \tag{35}$$

We will find, in Section 23.5, that if $f$ is analytic in $D$, then the partial derivatives of $u$ and $v$ of *all* orders exist and are continuous in $D$. It then follows (Section 13.3) that $v_{yx} = v_{xy}$ so that (35) gives $u_{xx} + u_{yy} = 0$ in $D$. Similarly, taking $\partial/\partial y$ of the first equation in (34) and $\partial/\partial x$ of the second, leads to $v_{xx} + v_{yy} = 0$. Functions

that satisfy Laplace's equation and are $C^2$ (i.e., $u, v$ and their various first- and second-order partial derivatives, with respect to $x$ and $y$, are continuous functions of $x$ and $y$) are called **harmonic functions**, so we have the following result.

---

**THEOREM 21.5.2** *Harmonic Functions*
If $f(z) = u(x, y) + iv(x, y)$ is analytic in a domain $D$, then $u$ and $v$ are harmonic in $D$ that is, they are $C^2$ and they satisfy the Laplace equations

$$\nabla^2 u = u_{xx} + u_{yy} = 0, \tag{36a}$$

$$\nabla^2 v = v_{xx} + v_{yy} = 0 \tag{36b}$$

in $D$.

---

Observe that if $f(z) = u(x, y) + iv(x, y)$ is analytic, then the harmonic functions $u$ and $v$ are a related pair since they are related through the Cauchy–Riemann equations. To call attention to this relationship, we refer to them as **conjugate** harmonic functions. Given one harmonic function, conjugate harmonic functions can be inferred from the Cauchy–Riemann equations.

**EXAMPLE 8.** It is readily verified that the function

$$u(x, y) = 3xy^2 - x^3 \tag{37}$$

is harmonic in the entire $z$ plane. To find a conjugate harmonic function $v(x, y)$ we rely on the relations $u_x = v_y$ and $u_y = -v_x$. Thus,

$$v_x = -6xy, \tag{38a}$$

$$v_y = 3y^2 - 3x^2. \tag{38b}$$

Integrating (38a) partially, with respect to $x$, we obtain

$$v(x, y) = \int -6xy\, \partial x = -3x^2 y + A(y), \tag{39}$$

where $A(y)$ is arbitrary. Then, by putting (39) into (38b), we obtain

$$-3x^2 + A'(y) = 3y^2 - 3x^2,$$

whence $A(y) = y^3 + c$ for any constant $c$. Thus, conjugate harmonic functions corresponding to the given $u$ are

$$v(x, y) = y^3 - 3x^2 y + c, \tag{40}$$

for arbitrary $c$. Further, one finds that the analytic function

$$f(z) = u + iv = (3xy^2 - x^3) + i(y^3 - 3x^2 y + c) \tag{41}$$

is actually $f(z) = -z^3 + ic$. ∎

More generally, it can be shown (Exercise 17) that given one harmonic function ($u$ or $v$), a conjugate harmonic function ($v$ or $u$) can always be determined (to within an arbitrary additive constant) so that $f = u + iv$ is analytic.

**Closure.** We begin this section by defining and discussing the differentiability of a function $f(z)$. However, differentiability at a given point will prove to be much less significant than differentiability throughout a given region, so we proceed to introduce the notion of analyticity. In a sense, the remaining chapters amount to a study of the consequences of a function's analyticity or lack of analyticity. In our closing discussion of harmonic functions we discover a connection between analytic function theory and two-dimensional potential theory – that is, between analytic function theory and the two-dimensional Laplace equation. That connection is the focus of the next chapter.

## EXERCISES 21.5

**1.** Show that $\lim_{z\to 1} 3iz = 3i$, by referring to our definition of *limit* and actually putting forth a suitable $\delta$. Specifically, show that $\delta = \epsilon/3$ at most. Further, show that $f(z) = 3iz$ is continuous at $z = 1$.

**2.** (a) Show that $\lim_{z\to 0} z^2 = 0$ by referring to our definition of *limit* and actually putting forth a suitable $\delta$. Further, show that $f(z) = z^2$ is continuous at $z = 0$.
(b) Similarly, show that $\lim_{z\to z_0} z^2 = z_0^2$, and show that $f(z) = z^2$ is continuous for all $z$.

**3.** Prove that if $\lim_{z\to z_0} f(z) = A$ and $\lim_{z\to z_0} g(z) = B$, then

(a) $\lim_{z\to z_0}[f(z) + g(z)] = A + B$
(b) $\lim_{z\to z_0} f(z)g(z) = AB$
(c) $\lim_{z\to z_0} \dfrac{f(z)}{g(z)} = \dfrac{A}{B}$ if $B \neq 0$

**4.** If $f(x)$ is continuous for all (real) $x$, does it follow that $f(z)$ is continuous everywhere in the $z$ plane? Explain.

**5.** Use the difference quotient formula (8) to evaluate $f'(z)$, as we did in Example 2.

(a) $f(z) = z^3$
(b) $f(z) = 1/z \quad (z \neq 0)$
(c) $f(z) = 1/z^2 \quad (z \neq 0)$
(d) $f(z) = 1/(z+1) \quad (z \neq -1)$

**6.** Use the difference quotient formula (8) to prove
(a) equation (12) (b) equation (13) (c) equation (14)

**7.** Prove that if $f(z)$ is differentiable at $z_0$, it must be continuous there. HINT: Write

$$f(z) = \frac{f(z) - f(z_0)}{z - z_0}(z - z_0) + f(z_0).$$

**8.** (*l'Hôpital's rule*) Prove l'Hôpital's rule, namely, that if $f(z)$ and $g(z)$ are differentiable at $z_0$, with $f(z_0) = g(z_0) = 0$ and $g'(z_0) \neq 0$, then

$$\lim_{z\to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}. \tag{8.1}$$

**9.** (*Insufficiency of Cauchy–Riemann conditions*) We have seen that the Cauchy–Riemann conditions are necessary for differentiability. The example

$$f(z) = \begin{cases} \dfrac{(x^3 - y^3) + i(x^3 + y^3)}{x^2 + y^2}, & z \neq 0 \\ 0, & z = 0 \end{cases}$$

due to S. Pollard (1928), shows that they are not, however, sufficient. Verify that statement by showing that the Cauchy–Riemann conditions are satisfied at $z = 0$ but that $f$ is not differentiable at that point.

**10.** Given $f(z)$, use (19) to obtain $f'(z)$. Express the answer in terms of $z$.

(a) $\cos z$ (b) $e^z$
(c) $\sinh z$ (d) $\cosh z$
(e) $\dfrac{1}{z}$ $(z \neq 0)$ (f) $\dfrac{1}{z+2}$ $(z \neq -2)$
(g) $\dfrac{z+1}{z-1}$ $(z \neq 1)$

**11.** Given $f(z)$, determine $f'(z)$, where it exists, and state where $f$ is analytic and where it is not.

(a) $(1-2z^3)^5$ (b) $\dfrac{x+iy}{x^2+y^2}$
(c) $|z|\sin z$ (d) $\dfrac{1}{z^2+3iz-2}$
(e) $\dfrac{1}{z^3+1}$ (f) $x+i\sin y$

**12.** (*Cauchy–Riemann equations in polar coordinates*) Derive the Cauchy–Riemann equations (30) in the manner indicated.

(a) By carrying out the limit in (8). HINT: First let $\Delta z \to 0$ along the constant-$\theta$ line through $z_0$, and then let $\Delta z \to 0$ along the constant-$r$ line through $z_0$. Pay careful attention to your expression for $\Delta z$ in each of these cases because these cases are trickier than the cases of a horizontal approach ($\Delta z = \Delta x$) and a vertical approach ($\Delta z = i\Delta y$), used in (16) and (17).
(b) By making the change of variables $x = r\cos\theta$, $y = r\sin\theta$ in (18).

**13.** Determine where these functions are differentiable and where they are analytic, by checking for satisfaction of the Cauchy–Riemann equations and for continuity of $u, v$ and their first-order partial derivatives.

(a) $f(z) = z^{100}$
(b) $f(z) = \sqrt{z}$, defined by the branch cut shown in Fig. 6
(c) $f(z) = 1/\sqrt{z}$, where the $\sqrt{z}$ is defined by the branch cut shown in Fig. 6

**14.** Prove that:

(a) $f(z)$ and $\overline{f(z)}$ can both be analytic, in a given region, if and only if $f(z)$ is a constant.
(b) If $f(z)$ is analytic in a region $D$, and $f'(z) = 0$ in $D$, then $f(z)$ is a constant.
(c) $f$ cannot be analytic if it depends on $\overline{z}$. HINT: Since $f = u(x,y)+iv(x,y)$, where $x = (z+\overline{z})/2$ and $y = (z-\overline{z})/2i$, we can regard $f$ as a function of $z$ and $\overline{z}$. Show that the Cauchy–Riemann conditions imply that $\partial f/\partial\overline{z} = 0$, so that $f$ must be a function of $z$ only, if it is to be analytic. For instance, recall from Example 5 that $z\overline{z}$ is analytic nowhere.

**15.** Determine whether or not the given function $u$ is harmonic and, if so, in what region. If it is, find the most general conjugate function $v$ and corresponding analytic function $f(z)$. Express $f$ in terms of $z$.

(a) $e^x\cos y$ (b) $e^{2x}\sin 2y$
(c) $x^3 - 3xy^2$ (d) $r^3\sin 3\theta$
(e) $r^2\cos 2\theta + 4$ (f) $r$
(g) $x\cos 2x\cosh 2y + y\sin 2x\sinh 2y$

**16.** (*Orthogonality of $u$ = constant and $v$ = constant curves*)
(a) Prove that if $f(z) = u + iv$ is analytic in a region $D$, then the two families of level curves $u$ = constant and $v$ = constant are mutually orthogonal at all points in $D$ at which $f'(z) \neq 0$.
(b) Illustrate the idea contained in part (a) by sketching the $u$ and $v$ level curves for the case $f(z) = z = x + iy$.
(c) Repeat part (b) for the case $f(z) = z^2 = (x^2-y^2)+i2xy$.
(d) Repeat part (b) for the case

$$f(z) = \frac{1}{z} = \frac{x}{x^2+y^2} - i\frac{y}{x^2+y^2}.$$

**17.** (*Existence of conjugate harmonic function*) Let $u(x,y)$ be a given function that is harmonic in the rectangle $x_1 \le x \le x_2$, $y_1 \le y \le y_2$. Prove that there exists a conjugate harmonic function $v(x,y)$ in the rectangle $x_1 < x < x_2$, $y_1 < y < y_2$, say $D$, such that $f = u + iv$ is analytic in $D$. Show, further, that $v(x,y)$ is uniquely determined, to within an arbitrary additive constant. HINT: Infer from the Cauchy–Riemann equation $\partial u/\partial x = \partial v/\partial y$ that

$$v(x,y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x,y')\partial y' + A(x),$$

and use the second Cauchy–Riemann equation and the fact that $u$ is harmonic to obtain

$$A(x) = -\int_{x_0}^{x} \frac{\partial u}{\partial y}(x',y_0)dx',$$

where $(x_0, y_0)$ is any point in $D$. Verify, with the help of the Leibniz rule (in Section 13.8), that

$$v(x,y) = \int_{y_0}^{y} \frac{\partial u}{\partial x}(x,y')\partial y' - \int_{x_0}^{x} \frac{\partial u}{\partial y}(x',y_0)dx' \quad (17.1)$$

does indeed satisfy the Cauchy–Riemann equation and is harmonic in $D$. Next, show that (17.1) is equivalent to the line integral representation

$$v(x,y) = \int_{(x_0,y_0)}^{(x,y)} \left[-\frac{\partial u}{\partial y}(x',y')dx' + \frac{\partial u}{\partial x}(x',y')dy'\right] \quad (17.2)$$

over any path in $D$, and conclude from (17.2) that $v$ is determined only to within an arbitrary additive constant. NOTE:

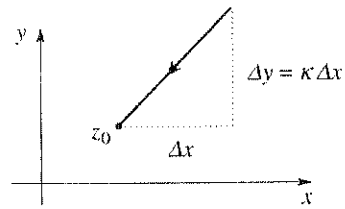
Similarly, one can show that given a harmonic function $D$, there exists a conjugate harmonic function $u$ that can be determined to within an arbitrary additive constant.

**18.** Consider the following proposed proof of part (ii) in Theorem 21.5.1. By chain differentiation

$$\frac{df}{dz} = \frac{\partial f}{\partial x}\frac{dx}{dz} + \frac{\partial f}{\partial y}\frac{dy}{dz}. \tag{18.1}$$

If we consider a linear approach to $z$ along a straight line with slope $dy/dx = \kappa$ (see figure), then $dz = dx + i\,dy = dx + i\kappa\,dx = (1 + i\kappa)\,dx$ so $dx/dz = 1/(1 + i\kappa)$ and $dy/dz = \kappa/(1 + i\kappa)$ in (18.1). Thus,

$$\frac{df}{dz} = (u_x + iv_x)\,\frac{1}{1 + i\kappa} + (u_y + iv_y)\,\frac{\kappa}{1 + i\kappa}. \tag{18.2}$$




Putting $u_y = -v_x$ and $v_y = u_x$ into (18.2), according to the Cauchy–Riemann conditions, show that (18.2) reduces to $df/dz = u_x + iv_x$ independent of $\kappa$. Then give a critical appraisal of the foregoing proposed proof, citing any weak points that you can find.

# Chapter 21 Review

In this chapter we begin by introducing the complex number system, then we define the various elementary functions, and get as far as the differential calculus and analyticity.

In defining the elementary functions it is stressed that one cannot "figure out" the values of $e^z = e^{x+iy}$, for instance; $e$ to a complex power is a new object and its values are a matter of definition. The guiding idea is to stay as close to real variable theory as possible. For instance, we write $e^z = e^{x+iy} = e^x e^{iy} = e^x[1 + iy + (iy)^2/2! + \cdots] = e^x[(1 - y^2/2! + \cdots) + i(y - y^3/3! + \cdots)] = e^x(\cos y + i\sin y)$ heuristically, and then *define* $e^z \equiv e^x(\cos y + i\sin y)$, which definition is due to Euler. Observe that on the $x$ axis ($y = 0$), the $e^z$ function defined in this way does agree identically with the real function $e^x$, and similarly for each of the other elementary functions.

Some of the elementary functions turn out to be multi-valued, namely $z^{m/n}$ ($m$ and $n$ integers, with $m$ not divisible by $n$), $\log z$, $z^c$ ($c$ a real or complex number not an integer or rational number), and the inverse trigonometric and inverse hyperbolic functions. Besides showing how to determine all of their possible values, we show how to specify single-valued branches by branch cuts. A branch cut is designed to limit the domain of the function so that the function thereby defined is single-valued. Usually, the relevant polar angle(s) will be made single-valued and limited to a $2\pi$ interval, but neither of these conditions is necessary.

In the final section we define the derivative of a function $f(z)$, again staying as close to real variable theory as possible. Consequently, just as $d(\sin x)/dx = \cos x$, for instance, so does it turn out that $d(\sin z)/dz = \cos z$. The essential idea is that

the limit of the difference quotient needs to exist, and have the same value, for any manner of approach to the point in question. Requiring that the limits of horizontal (parallel to the $x$ axis) and vertical (parallel to the $y$ axis) approach agree, gives the Cauchy–Riemann equations $u_x = v_y$ and $u_y = -v_x$ as necessary conditions for differentiability. Requiring also that the four partial derivatives be continuous in some neighborhood of the point then give sufficient conditions.

Besides differentiability at a point, we define a function to be *analytic* at $z_0$ if it is differentiable throughout some neighborhood of $z_0$. Typically, the functions that we deal with will either be analytic everywhere, or else everywhere except for one or more isolated points – singular points. Such functions will be found to be very "nice" indeed in that they are subject to powerful integral theorems, which is the subject of Chapters 23 and 24.

# Chapter 22

# Conformal Mapping

## 22.1 Introduction

In Chapter 21 we note that a function $w = f(z)$ is actually a mapping, from a given region in the $z$ plane to a corresponding one in the $w$ plane. However, we paid little attention to the geometrical aspects of the mapping and concentrated instead on analytical aspects such as the values of the function, differentiability, and analyticity. In the present chapter we complement that discussion by turning to the geometrical issues. Besides completing our introduction to functions of a complex variable by considering functions as mappings, the concept of conformal mapping will provide us with a powerful solution technique for problems in two-dimensional potential theory.

In Section 22.2 we lay the groundwork and in Section 22.3 we examine a particular important mapping in detail, the bilinear transformation, and explain its use in solving certain two-dimensional boundary-value problems. In Section 22.4 we give additional mappings and applications, in Section 22.5 we show how to handle more general boundary conditions, and in Section 22.6 we discuss applications specifically to fluid mechanics.

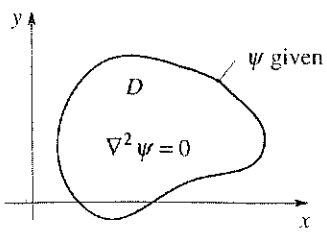



**Figure 1.** Generic two-dimensional potential problem with Dirichlet boundary conditions.

## 22.2 The Idea Behind Conformal Mapping

Suppose that we wish to solve the two-dimensional Laplace equation

$$\nabla^2\psi = \psi_{xx} + \psi_{yy} = 0 \tag{1}$$

in some domain $D$ of the $x, y$ plane. We ask $\psi$ to be $C^2$ in $D$, and to satisfy prescribed boundary conditions as well. For example, $\psi$ might be the steady-state temperature distribution within $D$ due to the maintaining of a certain temperature distribution along the boundary of $D$ (Fig. 1).

The degree of difficulty in solving for $\psi$ varies considerably with the shape of the domain $D$, and the methods developed in Chapter 19 fail if the shape is not

sufficiently simple, such as rectangular or circular. Thus, there is interest in seeking a change of variables

$$u = u(x, y), \qquad v = v(x, y), \tag{2}$$

from $x, y$ to $u, v$, in the hope that the new region (in the $u, v$ plane), say $D'$, will be simpler than $D$. Our plan, then, is to solve the new (and presumably simpler) problem in the $u, v$ plane, and to then "return to the $x, y$ plane" using the relations (2). Evidently, we will want a one-to-one correspondence between points in $D$ and points in $D'$ so that we can move back and forth unambiguously. Thus, besides asking $u(x, y)$ and $v(x, y)$ to be single valued in $D$, we also want single-valued *inverse* functions

$$x = x(u, v), \qquad y = y(u, v) \tag{3}$$

to exist in $D'$. According to the implicit function theorem (Section 13.6), we can ensure the existence of the desired inverse functions by requiring that $u$ and $v$ be $C^1$ in $D$ and that the Jacobian be nonzero in $D$:

$$\frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x \neq 0. \tag{4}$$

With these assumptions, we proceed to enter the change of variables into (1). Denoting $\psi(x(u, v), y(u, v)) \equiv \Psi(u, v)$, chain differentiation gives

$$\psi_x = \Psi_u u_x + \Psi_v v_x$$

and

$$\psi_{xx} = (\Psi_{uu} u_x + \Psi_{uv} v_x) u_x + \Psi_u u_{xx} + (\Psi_{vu} u_x + \Psi_{vv} v_x) v_x + \Psi_v v_{xx}. \tag{5}$$

Similarly,

$$\psi_{yy} = (\Psi_{uu} u_y + \Psi_{uv} v_y) u_y + \Psi_u u_{yy} + (\Psi_{vu} u_y + \Psi_{vv} v_y) v_y + \Psi_v v_{yy} \tag{6}$$

so equation (1) becomes

$$\begin{aligned} &(u_x^2 + u_y^2)\Psi_{uu} + (u_x v_x + u_y v_y)(\Psi_{uv} + \Psi_{vu}) \\ &\qquad + (v_x^2 + v_y^2)\Psi_{vv} + (u_{xx} + u_{yy})\Psi_u + (v_{xx} + v_{yy})\Psi_v = 0. \end{aligned} \tag{7}$$

Thus, in making the change of variables (2) in order to simplify the domain, we inadvertently render the governing PDE much more complicated, for in place of the familiar Laplace equation (1) we have the rather unwieldy and complicated looking equation (7). For instance, since $u$ and $v$ are functions of $x$ and $y$ it follows that the coefficients in (7) are nonconstant; they are functions of $x$ and $y$ which, through the inverse relations (3), are functions of $u$ and $v$.

The result seems fair enough: if we make a change of variables (2) to simplify the region $D$, we can expect to complicate the PDE. However, it is striking that

if we restrict the change of variables (2) so that $u(x, y)$ and $v(x, y)$ satisfy the relations

$$u_x = v_y \quad \text{and} \quad u_y = -v_x \tag{8}$$

in $D$, and ask $u$ and $v$ to be $C^2$ so that $u_{xy} = u_{yx}$ and $v_{xy} = v_{yx}$ in $D$, then (8) gives $u_x v_x + u_y v_y = (u_x)(-u_y) + (u_y)(u_x) = 0$, $u_{xx} + u_{yy} = v_{yx} - v_{xy} = 0$, and $v_{xx} + v_{yy} = -u_{yx} + u_{xy} = 0$ so (7) simplifies dramatically to

$$(u_x^2 + u_y^2)(\Psi_{uu} + \Psi_{vv}) = 0. \tag{9}$$

Using (8) again, we can write $u_x^2 + u_y^2 = u_x v_y - u_y v_x$, which Jacobian has already been assumed nonzero in $D$. Then it follows from (9) that

$$\boxed{\Psi_{uu} + \Psi_{vv} = 0} \tag{10}$$

everywhere in $D'$; that is, the Laplace equation is preserved under the change of variables!

Notice carefully that thus far there has been no mention, or use, of complex variable theory. However, the fact that equations (8) are the familiar Cauchy–Riemann conditions causes us to raise an eyebrow, and suggests that it might be helpful (though not essential) to regard the $x, y$ and $u, v$ planes as complex $z$ and $w$ planes, where $z = x + iy$ and $w = f(z) = u(x, y) + iv(x, y)$. (Whether the $x, y$ and $u, v$ planes are real planes or complex planes is merely a matter of viewpoint.) Then $u_x v_y - u_y v_x = u_x^2 + v_x^2 = |f'(z)|^2$, so the nonvanishing-Jacobian condition (4) can be stated more crisply in terms of complex variable theory as $f'(z) \neq 0$ in $D$, and we have established the following important result.

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**THEOREM 22.2.1** *Preservation of Laplace Equation*
Let $w = f(z) = u(x, y) + iv(x, y)$ be analytic everywhere in a domain $D$, with $f'(z) \neq 0$ everywhere in $D$. Denote the image of $D$ as $D'$, and let the mapping be one-to-one. If $\psi(x, y)$ is harmonic in $D$, then $\psi(x(u, v), y(u, v)) \equiv \Psi(u, v)$ is harmonic in $D'$.

---

The point, then, is that if we use an analytic function $f(z) = u(x, y) + iv(x, y)$ to give the change of variables (2), then the Laplace equation on $\psi(x, y)$ is guaranteed to carry over to a Laplace equation on $\Psi(u, v)$. At the same time, there is the possibility of selecting the analytic function $f(z)$ so as to simplify the region.

Let us make two remarks about the theorem. First, recall that $D$ is open because it is a domain. Thus, $f'(z) \neq 0$ in $D$ means at all interior points of $D$; $f'(z)$ need not be nonzero on the boundary of $D$. Put differently, we ask that the Laplace equation $\nabla^2 \psi = 0$ be preserved only on the interior of $D$ because (as noted in the chapters on PDE's) we require the satisfaction of that equation only on the interior of $D$. Second, why do we change notation, from $\psi$ to $\Psi$? Because $\psi$ and $\Psi$ are different functions. For instance, suppose that $\Psi(u, v) = u + 2v$

and $f(z) = z^2$. From $f(z) = z^2$ it follows that $u = x^2 - y^2$ and $v = 2xy$ so $\Psi(u, v) = u + 2v = x^2 - y^2 + 4xy = \psi(x, y)$, and surely $\Psi(u, v) = u + 2v$ and $\psi(x, y) = x^2 - y^2 + 4xy$ are different functions; for instance, $\Psi(2, 3) = 8$ whereas $\psi(2, 3) = 19$.

We call the function $f(z)$ a **conformal map**. The relevance of the adjective conformal will be clarified by the next theorem. However, before proceeding with that let us immediately give an application, to show how the method is used.

**EXAMPLE 1.** *Application of Theorem 22.2.1.* Consider the Dirchlet problem shown in Fig. 2: the Laplace equation on $\psi(x, y)$ together with Dirichlet boundary conditions

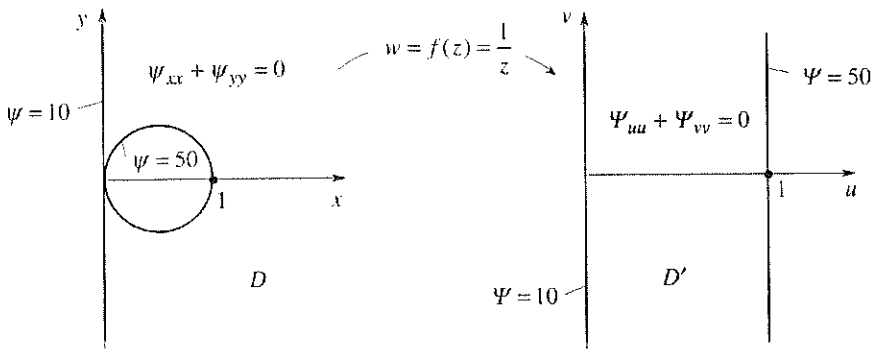


**Figure 2.** Steady-state temperature problem.

that $\psi = 50$ on the circular part of the boundary and $\psi = 10$ on the straight part, and that $\psi$ is bounded on $D$. Physically, $\psi(x, y)$ might be the steady-state temperature distribution in a large plate with a circular cutout near one edge.

First, observe that the given problem cannot be solved by separation of variables in Cartesian coordinates (because the boundary is not comprised of constant-$x$ and constant-$y$ curves), nor in plane polar coordinates (because no matter where the origin is located the boundary is not comprised of constant-$r$ and constant-$\theta$ curves). Nor do the Laplace and Fourier transforms provide any help.

Turning to Theorem 22.2.1, suppose we know that the mapping $f(z) = 1/z$ sends the given domain $D$ into the simpler strip $D'$ shown in Fig. 2 (Exercise 1). Furthermore, $f(z) = 1/z$ is analytic for all $z \neq 0$, and hence everywhere inside $D$; $f'(z) = -1/z^2 \neq 0$ everywhere in $D$, and the mapping is one-to-one since both $f(z) = 1/z$ and the inverse function $z = 1/f$ are, obviously, single-valued. Thus, it follows from Theorem 22.2.1 that

$$\nabla^2 \Psi = \Psi_{uu} + \Psi_{vv} = 0 \tag{11}$$

in $D'$. Finally, the images of the boundary curves $x = 0$ and $(x - \frac{1}{2})^2 + y^2 = \frac{1}{4}$ are $u = 0$ and $u = 1$ so that $\Psi(0, v) = 10$ and $\Psi(1, v) = 50$ as indicated in Fig. 2. And since $\psi$ is to be bounded on $D$, $\Psi$ must be bounded on $D'$.

Observe that whereas the $\psi$ problem is hard, the $\Psi$ problem is easy because (see Fig. 2) it is evidently one-dimensional. That is, $\Psi$ varies with $u$ but not $v$. Then the Laplace equation $\Psi_{uu} + \Psi_{vv} = 0$ reduces to the ordinary differential equation $d^2\Psi/du^2 = 0$, with

general solution $\Psi = Au + B$. The boundary conditions $\Psi(0, v) = 10$ and $\Psi(1, v) = 50$ give $A = 40$ and $B = 10$ so

$$\Psi(u, v) = 10 + 40u \tag{12}$$

in $D'$. To obtain the solution for $\psi(x, y)$ in $D$, we merely substitute $u = u(x, y)$ and $v = v(x, y)$ into the right-hand side of (12). (In this case there happen to be no $v$'s.) We get $u(x, y)$ and $v(x, y)$ from the mapping $f$. Specifically, since $f(z) = 1/z = u + iv$, we have

$$u + iv = \frac{1}{x + iy}\frac{x - iy}{x - iy} = \frac{x}{x^2 + y^2} - i\frac{y}{x^2 + y^2}$$

so

$$u = \frac{x}{x^2 + y^2}, \qquad v = -\frac{y}{x^2 + y^2}. \tag{13}$$

Thus,

$$\psi(x, y) = \Psi(u(x, y), v(x, y)) = 10 + \frac{40x}{x^2 + y^2}. \tag{14}$$

COMMENT 1. Notice that whereas the method of separation of variables, used extensively in Chapters 18–20, generally gives the solution as an infinite series, (14) is expressed, more conveniently, in closed form.

COMMENT 2. In case it was not clear that $\Psi(u, v)$ is independent of $v$, as claimed, observe a posteriori that (12) does indeed satisfy the full equation (11) as well as the boundary conditions. The only nagging question is whether or not the solution (12) is *unique*, for perhaps there are other solutions, as well, that do vary with $v$. This is a subtle point, and is discussed in Exercise 1.

COMMENT 3. We stated, without explanation, that the boundary conditions carry over, from the $z$ plane to the $w$ plane. For example, if $\psi = 20$ on some boundary curve in the $z$ plane, then $\Psi = 20$ on the image of that curve in the $w$ plane. That result follows immediately from the fact that $\psi(x, y) = \psi(x(u, v), y(u, v)) = \Psi(u, v)$. That is, if the point $(u, v)$ (or $u + iv$ in complex notation) is the image of $(x, y)$ (i.e., $x + iy$), then $\Psi$ at $(u, v)$ is equal to $\psi$ at $(x, y)$. ∎

Though the title of this chapter is Conformal Mapping, we have not yet introduced the notion of conformality. Recall that Theorem 22.2.1 requires that $f(z)$ be analytic and that $f'(z) \neq 0$ in the given domain. As we shall prove, such mappings are "conformal." That is, they preserve angles both in magnitude and in sense. To make this claim precise, consider two oriented smooth curves $C_1$ and $C_2$ that intersect at $z_0$ (Fig. 3). Let $C_1$ and $C_2$ be parametrized by $z_1(\tau) = x_1(\tau) + iy_1(\tau)$ and $z_2(\tau) = x_2(\tau) + iy_2(\tau)$, respectively, where $x_1, y_1, x_2, y_2$ are differentiable functions of the real parameter $\tau$, where $z_1(0) = z_2(0) = z_0$ and where $\dot{z}_1(0)$ and $\dot{z}_2(0)$ are both nonzero. (Dots will be used to denote $d/d\tau$.) Then the complex number

$$\dot{z}_1(0) = \lim_{\tau \to 0} \frac{z_1(\tau) - z_1(0)}{\tau} = \dot{x}_1(0) + i\dot{y}_1(0) \tag{15}$$

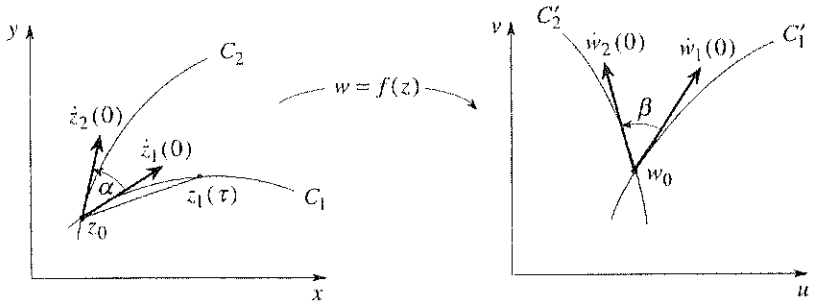


**Figure 3.** Conformality.

is a tangent "vector" to $C_1$ at $z_0$ and similarly for $\dot{z}_2(0)$. Let the angle measured counterclockwise from $\dot{z}_1(0)$ to $\dot{z}_2(0)$ be denoted as $\alpha$.

Now, consider a mapping $w = f(z)$ which is a nonconstant analytic function. Under this mapping, $z_0$ is sent into $w_0 = f(z_0)$ and the curves $C_1$, $C_2$ are sent into curves $C_1'$, $C_2'$ that are parametrized by $w_1(\tau) = f(z_1(\tau))$, $w_2(\tau) = f(z_2(\tau))$, respectively. By chain differentiation,

$$\dot{w}_1(0) = f'(z_0)\dot{z}_1(0), \qquad \dot{w}_2(0) = f'(z_0)\dot{z}_2(0) \tag{16}$$

so that if (and only if) $f'(z_0) \neq 0$, there exist unique tangent vectors $\dot{w}_1(0)$, $\dot{w}_2(0)$, at $w_0$, given by (16) and depicted in Fig. 3. From (16) it follows that

$$\begin{aligned} \arg \dot{w}_1(0) &= \arg f'(z_0) + \arg \dot{z}_1(0), \\ \arg \dot{w}_2(0) &= \arg f'(z_0) + \arg \dot{z}_2(0), \end{aligned} \tag{17}$$

and subtraction gives

$$\arg \dot{w}_1(0) - \arg \dot{w}_2(0) = \arg \dot{z}_1(0) - \arg \dot{z}_2(0). \tag{18}$$

If we denote the angle $\arg \dot{w}_1(0) - \arg \dot{w}_2(0)$ as $\beta$ (Fig. 3), (18) tells us that

$$\beta = \alpha, \tag{19}$$

and this result is the promised conformality. (See Exercise 8.) Observe that the result (19) holds for *any* two oriented smooth curves $C_1, C_2$ intersecting at $z_0$. Let us pull these results together now by defining conformality and then giving conditions that ensure that a mapping is conformal.

We say that a mapping $w = f(z)$ is **conformal** at $z_0$ if it preserves, both in magnitude and in sense, the angle between every pair of oriented smooth curves that intersect at $z_0$.

**THEOREM 22.2.2** *Conformality*
If $f(z)$ is analytic, then the mapping $w = f(z)$ is conformal, except at points where $f'(z) = 0$.

**EXAMPLE 2.** *Illustration of Conformality.* To emphasize the *local* nature of conformality, consider the mapping $w = z^2$. If the domain of definition $D$ is the first quadrant $0 < x < \infty$, $0 < y < \infty$, then the range $D'$ is the upper half plane $v > 0$ as shown in Fig. 4. In this case $f$ is analytic everywhere, and $f'(z) = 2z = 0$ only at the origin. Thus, the mapping $w = z^2$ is conformal everywhere in $D$. If angles are preserved, we may well wonder how the quarter plane manages to open out into the upper half plane. Thus, consider, as representative, the curve $ABC$ and its image $A'B'C'$. The angle of intersection at $B$, namely $\pi/2$, is indeed preserved, although *away from the neighborhood of* $B$ the curve $ABC$ opens out like the pages of an open book (Fig. 4). Thus, conformality is local, not global. At $z = 0$, the only point at which conformality breaks down, the angle $\pi/2$ is *not* preserved; it is doubled.

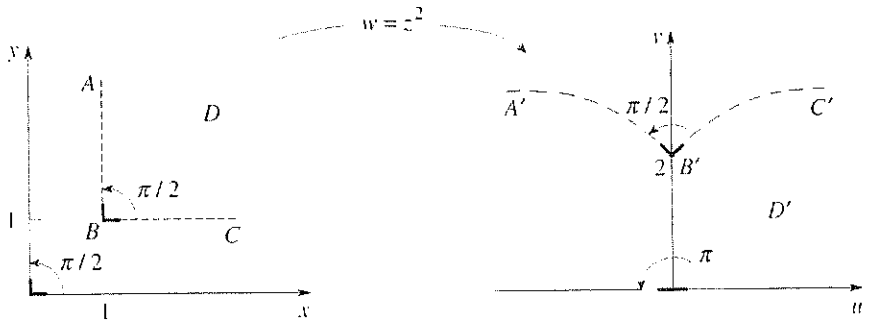


**Figure 4.** The mapping $w = z^2$.

COMMENT. We do not mean to imply that if conformality breaks down at a point, then angles are necessarily doubled there. That result is specific to the present example. ∎

**Closure.** We show first that under mappings $w = f(z)$, where $f$ is analytic and $f'(z) \neq 0$, the Laplace equation is preserved. This fact is of great importance in two-dimensional potential theory for it implies that given a particular problem in two-dimensional potential theory, one might be able to effect a simplification in the shape of the domain without disturbing the governing Laplace equation. We illustrate this line of approach in Example 1, and plan to develop it more fully in the sections to follow. Furthermore, we show that the above-noted conditions on $f$ guarantee conformality of the map.

Some additional results are included in the exercises, but important questions remain. For a given domain $D$ and a desired image $D'$, does a one-to-one conformal map *exist*? If so, is it *unique*? These questions are answered, for simply connected

domains (other than the entire $z$ plane), by the **Riemann mapping theorem**.* The existence part of the theorem states, in effect, that any simply connected domain $D$ (not the entire plane) can be mapped one-to-one and conformally onto any other simply connected domain $D'$ (not the entire plane).

## EXERCISES 22.2

**1.** (*More about Example 1*) (a) Show that the image of the domain $D$ shown in Fig. 2, under the mapping $w = f(z) = 1/z$, is the infinite strip $D'$ shown in the figure.
(b) Using separation of variables, show that the problem $\Psi_{uu} + \Psi_{vv} = 0$ in the strip $0 < u < 1$ and $-\infty < v < \infty$, with $\Psi(0, v) = 10$ and $\Psi(1, v) = 50$, admits the solution

$$\Psi(u, v) = 10 + 40u + (Ae^{\pi v} + Be^{-\pi v}) \sin \pi u, \qquad (1.1)$$

where $A, B$ are arbitrary constants. NOTE: Observe that if we fail to also require that $\psi$, and hence $\Psi$, be bounded, then the solution is nonunique because $A$ and $B$ are arbitrary in (1.1). If we do require that $\Psi$ be bounded, then we need $A = 0$ because $e^{\pi v} \to \infty$ as $v \to \infty$, and we need $B = 0$ because $e^{-\pi v} \to \infty$ as $v \to -\infty$, so (1.1) reduces to $\Psi(u, v) = 10 + 40u$ as given by (12). Although we see from (1.1) that the solution is not unique if a boundedness condition is not included, understand that we have not proved uniqueness if a boundedness condition *is* included because there might be solutions of $\nabla^2 \Psi = 0$, besides (1.1), that cannot be obtained by the method of separation of variables.
(c) If $A$ and $B$ are not both zero in (1.1), then $\Psi$ grows unboundedly as $|v| \to \infty$. What is the corresponding behavior of $\psi$ in the $x, y$ plane?

**2.** (*Analyticity of inverse mapping*) If $w = f(z)$ is a one-to-one mapping, there exists a single-valued inverse function, say $z = f^{-1}(w)$. Show that if $f$ is an analytic function of $z$, then $f^{-1}$ (i.e., the inverse function, not the numerical inverse $1/f$) is an analytic function of $w$, and

$$\frac{df^{-1}(w)}{dw} = \frac{1}{df(z)/dz}$$

or, equivalently,

$$\frac{dz}{dw} = \frac{1}{dw/dz}. \qquad (2.1)$$

HINT: Show that if $\Delta z \to 0$, then $\Delta w \to 0$, too.

**3.** Give a shorter proof of Theorem 22.2.1 using the fact that the inverse mapping $z(w)$ is analytic, too, as noted in the preceding exercise. HINT: There exists a conjugate harmonic function corresponding to $\psi(x, y)$, say $\zeta(x, y)$, such that

$$F(z) = \psi(x, y) + i\zeta(x, y) \qquad (3.1)$$

is an analytic function of $z$. Thus

$$\begin{aligned} F(z(w)) &= \psi(x(u, v), y(u, v)) + i\zeta(x(u, v), y(u, v)) \\ &= \Psi(u, v) + iZ(u, v). \end{aligned} \qquad (3.2)$$

Show why it follows from (3.2) that $\Psi(u, v)$ is harmonic.

**4.** (*Local magnification*) Let $w = f(z)$ be analytic at $z_0$, with $f'(z_0) \neq 0$. Denoting $\lim_{\Delta z \to 0} |\Delta w/\Delta z|$ as the *local magnification*, show that the local magnification is equal to $|f'(z_0)|$.

**5.** Are the following mappings conformal at $z = 0$? If not, why not?

| | | |
|---|---|---|
| (a) $e^z$ | (b) $ze^z$ | (c) $iz + 3$ |
| (d) $iz^2$ | (e) $\sin z$ | (f) $1/(z - 1)$ |
| (g) $z^2 + z$ | (h) $z^3 - 1 + 2i$ | |

**6.** (a) In Example 2 we observed that the mapping $w = z^2$ is conformal for all $z$ except $z = 0$. At $z = 0$ we noted that the "corner angle" $\pi/2$ gets doubled to $\pi$. Show that *all* angles are doubled at $z = 0$ (i.e., between every pair of oriented smooth curves that intersect at $z = 0$).
(b) More generally, show that for the mapping $w = z^n$ for any integer $n = 2, 3, 4, \ldots$, angles are multiplied by $n$ at $z = 0$.
(c) Still more generally, suppose that $w = f(z) = (z - z_0)^n g(z)$, where $g(z)$ is analytic, $g(z_0) \neq 0$, $g'(z_0) \neq 0$, and $n$ is any positive integer greater than 1. Show that angles are multiplied by $n$ at $z_0$; i.e., in place of (19) we have $\beta = n\alpha$.

*Riemann's celebrated mapping theorem appears near the end of his doctoral dissertation. For discussion of this theorem, see E. B. Saff and A. D. Snider, *Fundamentals of Complex Analysis* (Englewood Cliffs, NJ: Prentice Hall, 1976).

**7.** Let $w = f(z) = u(x, y) + iv(x, y)$ be analytic everywhere in a domain $D$, with $f'(z) \neq 0$ everywhere in $D$. Denote the image of $D$ as $D'$, and let the mapping be one-to-one. If $\psi(x, y)$ satisfies the *Poisson equation*

$$\psi_{xx} + \psi_{yy} = Q(x, y),$$

where $Q$ is prescribed, what equation does $\psi(x(u, v), y(u, v)) = \Psi(u, v)$ satisfy in $D'$?

**8.** (*A simpler proof of conformality*) In case our proof of conformality [equation (19)] is unclear, let us suggest a simplified version of that proof, even though it is only heuristic by virtue of its use of differentials as computable quantities. Let $dz_1$ and $dz_2$ be infinitesimal "vectors" springing from $z_0$, and let $dw_1$ and $dw_2$ be their respective image vectors springing from $w_0$. Writing $dw_2 = f'(z_0)dz_2$ and $dw_1 = f'(z_0)dz_1$, conclude that $\arg dw_2 = \arg f'(z_0) + \arg dz_2$, and that $\arg dw_1 = \arg f'(z_0) + \arg dz_1$. Show that (19) follows from these two equations. Include a labeled sketch in your solution.

**9.** Here is a subtle question regarding Theorem 22.2.1. If the condition $f'(z) \neq 0$ ensures the existence of single-valued inverse functions $x = x(u, v)$ and $y = y(u, v)$, and the "forward" mapping $f(z)$ is of course single valued, why do we bother including the additional condition that "the mapping be one-to-one" in the theorem?

**10.** What is equation (7) for the case where $x = u \cos v$ and $y = u \sin v$ (i.e., $u$ and $v$ are really the polar coordinates $r$ and $\theta$)? Is the map

$$w(z) = u + iv = \sqrt{x^2 + y^2} + i \tan^{-1} \frac{y}{x}$$

conformal? Explain.

## 22.3 The Bilinear Transformation

The transformation $w(z) = 1/z$ that was used in Example 1 of the preceding section is a special case of the **bilinear transformation***

$$\boxed{w = \frac{az + b}{cz + d} \qquad \text{for } ad - bc \neq 0,} \tag{1}$$

which is also known as the *Möbius transformation* or *linear fractional transformation*. The constants $a, b, c, d$ are allowed to be complex, and the condition $ad - bc \neq 0$ ensures that $w(z)$ is not merely a constant, or $0/0$.

Since

$$w'(z) = \frac{ad - bc}{(cz + d)^2} \tag{2}$$

exists for all $z$ except $z = -d/c$ (at which point $w$ is undefined as well) and is nonzero (since $ad - bc \neq 0$ by assumption), we see that *the bilinear transformation is conformal everywhere except at* $z = -d/c$, which point is called the **pole** of the transformation. Although $w$ is undefined at $z = -d/c$, it will be convenient to regard the image of that point as the **point at infinity** in the $w$ plane, denoted as $w = \infty$. To indicate that the $w$ plane has been augmented by the point at infinity, we call the result the **extended complex plane**. In contrast, the "usual" complex plane (i.e., not augmented by the point at infinity) will be called the **finite complex plane**. Observe that if we are to allow the values $z = \infty$ and $w = \infty$, we need to

*Cleared of fractions, (1) becomes $czw + dw - az = b$, which is linear in $z$ and linear in $w$; hence, it is *bilinear* in $z$ and $w$.

decide how to interpret $(az+b)/(cz+d)$, say, for $z=\infty$. We define the latter as $a/c$ if $c \neq 0$ and as $\infty$ if $c=0$.

Besides being conformal, the bilinear transformation is also *one-to-one* on the extended $z$ and $w$ planes, for (1) gives a unique $w$ for each $z$, and the inverse mapping

$$z = -\frac{dw-b}{cw-a} \tag{3}$$

[obtained by solving (1) for $z$] gives a unique $z$ for each $w$.

We will establish an important property of the bilinear transformation in a moment. To do so, we begin with the special case $w(z)=1/z$ and seek to determine the image (i.e., in the $w$ plane) of any circle in the $z$ plane. If the circle is centered at $(a,b)$ and has radius $c$, its equation is

$$(x-a)^2+(y-b)^2=c^2 \tag{4}$$

or

$$x^2+y^2-2ax-2by=c^2-a^2-b^2. \tag{5}$$

To express the latter in terms of $z$, and hence $w$, we use the relations $x=(z+\overline{z})/2$ and $y=(z-\overline{z})/2i$. Then (5) becomes

$$z\overline{z}-Az-\overline{A}\overline{z}=B, \tag{6}$$

where $A=a-ib$ and $B=c^2-a^2-b^2$. But $z=1/w$ and $\overline{z}=1/\overline{w}$ so (6) gives

$$\frac{1}{w\overline{w}}-\frac{A}{w}-\frac{\overline{A}}{\overline{w}}=B \tag{7}$$

as governing the image curve.

Consider separately the cases $B \neq 0$ and $B=0$. If $B \neq 0$, then we can divide both sides of (7) by $B$. Doing so, and multiplying by $w\overline{w}$ as well, gives

$$w\overline{w}+\frac{\overline{A}}{B}w+\frac{A}{B}\overline{w}=\frac{1}{B}. \tag{8}$$

Since $B$ is real, the coefficients of $w$ and $\overline{w}$ are complex conjugates. Thus, (8) is of the same form as (6) so (8) represents a circle in the $w$ plane.

Next, consider the case where $B=0$. Then (7) becomes

$$1-A\overline{w}-\overline{A}w=0 \tag{9}$$

or, with $A=a-ib$ and $w=u+iv$,

$$1-2au+2bv=0, \tag{10}$$

which is the equation of a straight line.

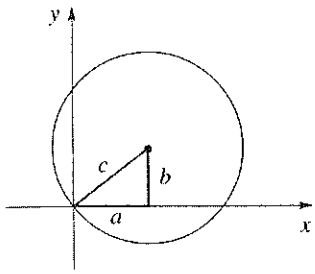


**Figure 1.** The case $B=0$.

We see from Fig. 1 that the geometrical significance of $B=0$ is that the circle in the $z$ plane passes through the origin, because $B$ is $c^2-a^2-b^2$. It makes sense

that the image of a circle through $z = 0$ is a straight line because the mapping $w = 1/z$ sends $z = 0$ to $w = \infty$, and the only circle that reaches $w = \infty$ is a straight line – that is, a circle with infinite radius.

Similarly, we see that a straight line $L$ in the $z$ plane is mapped to a circle if $L$ does not pass through $z = 0$, and to another straight line if $L$ does pass through $z = 0$.

The upshot is that $w(z) = 1/z$ *sends circles into circles, with a straight line considered as a circle of infinite radius.* That result is not at all surprising if the original curve is a circle centered at $z = 0$ since if $|z| = R$ then $|w| = |1/z| = 1/R$, so the image is a circle of radius $1/R$ centered at $w = 0$. But it is remarkable that all circles are sent into circles, even if they are not centered at $z = 0$!

Now we turn from the special case $w(z) = 1/z$ to the general bilinear transformation (1). If we call

$$w = Az, \tag{11a}$$

$$w = z + B, \tag{11b}$$

$$w = \frac{1}{z} \tag{11c}$$

*scaling and rotation*, *translation*, and *inversion* transformations, respectively, then (1) amounts to a sequence, or composition, of transformations of these three types. For if $c \neq 0$, then it is readily verified (by successive substitution) that

$$\begin{gathered} w_1 = cz, \qquad w_2 = w_1 + d, \qquad w_3 = \frac{1}{w_2}, \\ w_4 = \frac{bc - ad}{c}\, w_3, \qquad w = w_4 + \frac{a}{c} \end{gathered} \tag{12}$$

takes us from $z$ to $w_1$ to $w_2$ to $w_3$ to $w_4$ to $w$ by a finite sequence of scaling and rotation, translation, and inversion transformations. Similarly if $c = 0$, for then

$$w_1 = \frac{a}{d}\, z, \qquad w = w_1 + \frac{b}{d}. \tag{13}$$

We call (11a) a scaling and rotation transformation because each $z$ vector is scaled (by the modulus of the complex number $A$) and rotated (by the argument of $A$). It follows that the image region $D'$ in the $w$ plane will be a scaled and rotated version of the original region $D$ in the $z$ plane as illustrated in Fig. 2a.

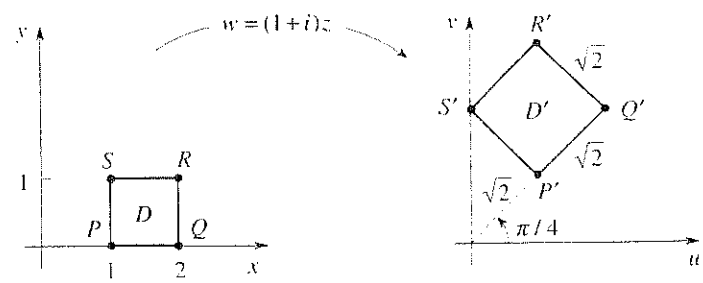


**Figure 2.** Scaling and rotation $w = Az$; $A = 1 + i$.

where $A = 1 + i$ and $D$ is taken to be a unit square.

We call (11b) a translation transformation because each $z$ point, and hence any given shape $D$, gets translated rightward by $\operatorname{Re} B$ and upward by $\operatorname{Im} B$. With $B = 1+2i$, for instance, the mapping $w = z+B = z+(1+2i)$ sends the region $D$ in Fig. 2 into a square with corners at $2+2i$, $3+2i$, $3+3i$, and $2+3i$, respectively. And we've already shown that the inversion (11c) sends circles into circles. It should be clear that (11a) and (11b) do too, and since (1) is a composition of these three kinds of transformation (1) sends circles into circles, with straight lines as special cases of circles. That result is the most important property of the bilinear transformation.

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**THEOREM 22.3.1** *Circles Into Circles*
The bilinear transformation (1) sends every circle or straight line into either a circle or a straight line.

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**EXAMPLE 1.** *Steady-State Temperature.* Find the steady-state temperature distribution $\psi(x, y)$ in the cut off disk shown in Fig. 3, subject to the boundary conditions shown there.

For the problem to be solvable by separation of variables we need the boundary of $D$ to be comprised of constant-coordinate curves. Cartesian coordinates don't work because of the circular part of the boundary. Polar coordinates don't work either for if we locate the origin at the center of the circle, then the straight edge is neither an $r$ = constant curve nor a $\theta$ = constant curve. And if we locate the origin as in Fig. 3 then although $BC$ and $AB$ are constant $\theta$ curves ($\theta = 0$ and $\theta = \pi$, respectively), the circular part $AEC$ is not an $r$ = constant curve.

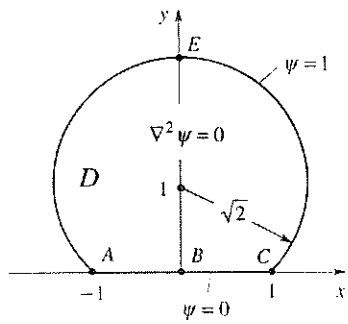



**Figure 3.** Steady-state temperature problem.

Thus, let us try solving by means of conformal mapping. To do so, we need to find a conformal map $w(z)$ that sends the problem shown in Fig. 3 into one that is simpler. There could be many such maps, we simply need one.

Since $ABC$ is straight and $AEC$ is circular, the bilinear transformation warrants consideration. But how do we "design" that transformation – that is, how do we choose the parameters $a, b, c, d$ in (1)? To begin, let us simply try the mapping $w = 1/z$ and see what happens. Since $AEC$ is a circular arc, we know that its image will be too. So, to find its image, it suffices to find the image of the three points $A, E, C$, and to fit a circular arc through them as shown in Fig. 4; three points suffice because three noncollinear points uniquely determine a circle. [Note that the image of $E$ is $1/(1+\sqrt{2})i = -(\sqrt{2}-1)i$.] The segment $ABC$ is also a circular arc (of infinite radius), but we need to be careful because $B$ is the pole of the transformation and gets sent to infinity. Thus, we need to treat $AB$ and $BC$ separately. Consider $AB$. Since $B$ goes to infinity, the image of $AB$ is a straight line that extends from the image of $A$ (namely, $A'$) to infinity. To determine that line we need one more point so consider $z = -1/2$: $A$ goes to $-1$ and $z = -1/2$ goes to $w = -2$ so the image of $AB$ is as shown in Fig. 4, extending from $u = -1$ to $u = -\infty$ on the $u$ axis. Similarly for $BC$: $B$ goes to infinity, $C$ goes to 1, and (since one more point is needed) $z = 1/2$ goes to $w = 2$ so the image of $BC$ is as shown in Fig. 4, extending from $u = 1$ to $u = \infty$ on the $u$ axis.

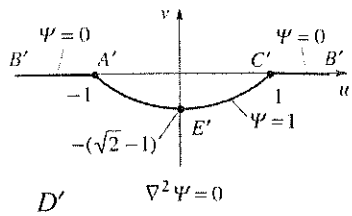



**Figure 4.** The mapped problem, using $w = 1/z$.

Do not be concerned that $B'$ shows up in two different places in Fig. 4 we simply call $B'$ "the point at infinity." It is approached from different directions, according to how $B$ is

approached in the $z$ plane.

Our conclusion is that the mapping $1/z$ is of no help because the problem in Fig. 4 is no easier than the one in Fig. 3. Trying again, let us design the bilinear transformation so as to put the pole at $A$ or $C$, say at $A$. If, for simplicity, we also send $C$ to $w = 0$, then we have the mapping

$$w(z) = \frac{z-1}{z+1}. \tag{14}$$

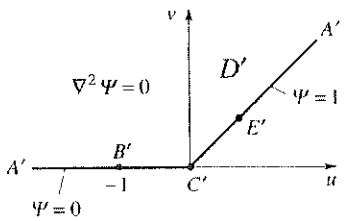



**Figure 5.** The mapped problem using $w = (z-1)/(z+1)$.

Then $A$ goes to infinity, $B$ goes to $w = -1$, $C$ goes to $w = 0$, and $E$ goes to $w = \frac{1+\sqrt{2}}{2+\sqrt{2}}(1+i)$, which has an argument of $\pi/4$. Thus, under the mapping (14) the $\Psi$ problem is as shown in Fig. 5. To solve, introduce polar coordinates $\rho, \phi$ with $u = \rho\cos\phi$, $v = \rho\sin\phi$. Then, expressing $\nabla^2\Psi = 0$ as $\Psi_{\rho\rho} + \frac{1}{\rho}\Psi_\rho + \frac{1}{\rho^2}\Psi_{\phi\phi} = 0$ and using separation of variables as in Section 20.3, we obtain

$$\Psi(\rho,\phi) = (A + B\ln\rho)(C + D\phi) + (E\rho^\kappa + F\rho^{-\kappa})(G\cos\kappa\phi + H\sin\kappa\phi). \tag{15}$$

To keep $\Psi$ bounded as $\rho \to \infty$ and as $\rho \to 0$ (and hence $\psi$ bounded as $z \to -1$ and $z \to 1$, respectively) set $B = E = F = 0$ so $\Psi$ is of the form

$$\Psi(\rho,\phi) = C_1 + C_2\phi. \tag{16}$$

The boundary conditions $\Psi(\rho,\pi/4) = 1$ and $\Psi(\rho,\pi) = 0$ give $C_1$ and $C_2$ so

$$\Psi(\rho,\phi) = \frac{4}{3}\left(1 - \frac{\phi}{\pi}\right).$$

To complete the solution we need to express $\rho$ and $\phi$ [actually, just $\phi$ because there are no $\rho$'s in (16)] in terms of $x$ and $y$. Equation (14) gives

$$u + iv = \frac{(x-1)+iy}{(x+1)+iy}\,\frac{(x+1)-iy}{(x+1)-iy} = \frac{(x^2+y^2-1)+2iy}{(x+1)^2+y^2} \tag{17}$$

so

$$u = \frac{x^2+y^2-1}{(x+1)^2+y^2}, \qquad v = \frac{2y}{(x+1)^2+y^2}. \tag{18}$$

Finally, since $\phi = \tan^{-1}(v/u)$, the desired solution is

$$\psi(x,y) = \frac{4}{3}\left[1 - \frac{1}{\pi}\tan^{-1}\left(\frac{2y}{x^2+y^2-1}\right)\right], \tag{19}$$

where the $\tan^{-1}$ lies between $\pi/4$ and $\pi$. For instance, at $x = y = 1$ we obtain $\psi(1,1) = 0.863$ which, from Fig. 3, seems reasonable.

COMMENT. Having found the boundary of $D'$, in Fig. 5, to be the bent line $A'B'C'E'A'$, how did we know that the region $D'$ was above it rather than below it? To answer that question it suffices to check a single point in the interior of $D$, say $z = i$. That point is mapped by (14), into $w = i$, so the region of $D'$ is above $A'B'C'E'A'$ as shown. ∎

**EXAMPLE 2.** Solve the potential problem shown in Fig. 6a. Since the boundary of $D$

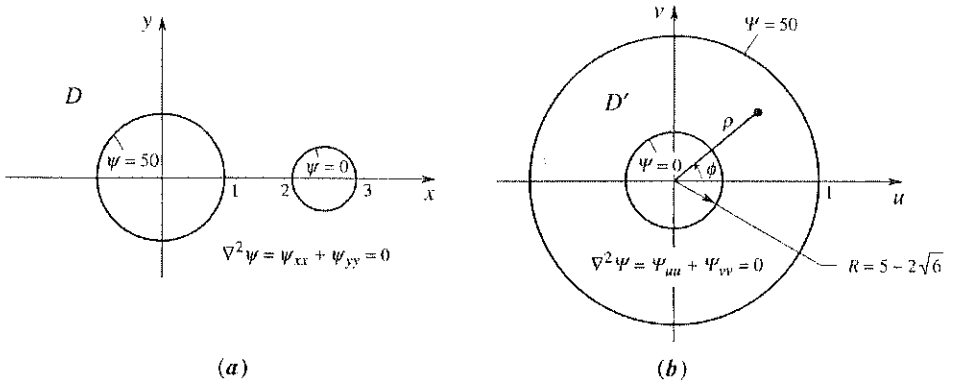



**Figure 6.** Region with two cutouts.

is comprised of circles, we consider the bilinear transformation once again. How are we to determine the parameters $a, b, c, d$ in (1) so that the region $D$ is sent into something simple, such as a region bounded by *concentric* circles? Since three noncollinear points determine a circle, we could seek $a, b, c, d$ so as to map three selected points on each of the two circles in the $z$ plane into three selected points on each of two concentric circles in the $w$ plane. That procedure would give six equations – actually twelve because in each equation we would equate real and imaginary parts on the left- and right-hand sides. How many parameters do we have? Each of $a, b, c, d$ has a real and an imaginary part so we have eight parameters. But we can divide both numerator and denominator in (1) by one of them, to normalize, so there are really seven "design parameters," and twelve linear algebraic equations for them. In general, such a system is overconstrained and there is no solution. However, the number of free parameters can be increased if we use the fact that we don't need to choose three specific points on each of the circles in the $z$ plane; *any* three points on each of those circles will do. Similarly for the points chosen on the image circles. Further, we don't care what the radii of the image circles are, so we can let them be free parameters.

The upshot is that it is possible to map any two distinct circles in the $z$ plane into two concentric circles in the $w$ plane, but design of such a map is tedious and best avoided. Thus, it is much more convenient to seek a suitable mapping in a table of mappings. A short table is given in Appendix F, and the desired mapping is given by entry 4,

$$w = \frac{z - a}{az - 1}, \tag{20}$$

where $a$ is defined in terms of the $x$-axis intercepts of the right-hand circle, $x_1 = 3$ and $x_2 = 2$. With these values $a = (7 + 2\sqrt{6})/5$, and the image is as shown in Fig. 6b.

To solve the problem on $\Psi$, introduce polar coordinates $\rho, \phi$ ($u = \rho\cos\phi, v = \rho\sin\phi$), so that

$$\nabla^2\Psi = \Psi_{\rho\rho} + \frac{1}{\rho}\Psi_\rho + \frac{1}{\rho^2}\Psi_{\phi\phi} = 0. \tag{21}$$

Because the annulus is axisymmetric and the boundary conditions do not vary with $\phi$, we can find a solution $\Psi(\rho)$ that varies only with $\rho$. Then the $\Psi_{\phi\phi}$ term in (21) drops out and (21) simplifies to the ordinary differential equation

$$\frac{d^2}{d\rho^2}\Psi + \frac{1}{\rho}\frac{d}{d\rho}\Psi = 0, \tag{22}$$

with general solution $\Psi(\rho) = A + B \ln \rho$. [Or, we could obtain the latter by setting the coefficients of all $\phi$-dependent terms in (15) equal to zero.] Using the boundary conditions $\Psi(R) = 0$ and $\Psi(1) = 50$ to evaluate $A$ and $B$ gives

$$\Psi(\rho) = 50\left(1 - \frac{\ln \rho}{\ln R}\right). \tag{23}$$

Finally, obtaining $\rho$ in terms of $x$ and $y$ (which step is left for the exercises) gives

$$\begin{aligned}\psi(x,y) &= \Psi(\rho(x,y)) \\ &= \frac{50}{\ln R}\left[\ln R - \frac{1}{2}\ln\frac{(x-a)^2+y^2}{(ax-1)^2+a^2y^2}\right]\end{aligned} \tag{24}$$

as the desired solution. Representative equipotentials are plotted in Fig. 7. ∎

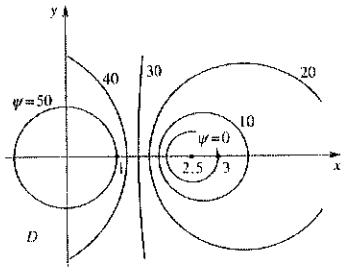



**Figure 7.** Representative equipotentials, from (24).

**Closure.** In this section we study only the bilinear transformation (1), the most important property of which is that it sends circles into circles (with straight lines as special cases of circles). That property makes the bilinear transformation useful for problems in which the domain $D$ is bounded by circles or circular arcs. That is not to say that we cannot use a bilinear transformation for other types of regions, but it is unlikely that a simplification of the domain can thereby be accomplished. Nor is it true that every domain $D$ that is bounded by circular arcs can be handled with a bilinear transformation since the seven design parameters might not be enough. For instance, a region with *three* or more circular cutouts is simply more than can be handled.

In any given application, be sure to verify that the mapping function $w(z)$ is analytic and that $w'(z) \neq 0$ everywhere within the interior of $D$. For the mapping given by (20), for instance, $w(z)$ is analytic for all $z$ except $z = 1/a \approx 0.42$, which falls within one of the cutouts rather than in $D$, and $w'(z) = (a^2-1)/(az-1)^2 \neq 0$ everywhere.

Finally, we suggest that simplified problems to aim at, in the $w$ plane, are ones in which $D'$ is either an annulus bounded by concentric circles (Fig. 6b) or an infinite wedge (Fig. 5) or strip, with constant boundary conditions.

## EXERCISES 22.3

**1.** We stated that (8) represents a circle in the $w$ plane. Give the equation of that circle in the form $(u-\alpha)^2 + (v-\beta)^2 = \gamma^2$, and give $\alpha, \beta, \gamma$ in terms of the quantities $a, b, c$ that appear in (4).

**2.** Following essentially the same steps as we used in equations (4)–(8) to show that $w(z) = 1/z$ sends circles into circles, show that the following transformation also sends circles into circles.

(a) $w(z) = Pz$, where $P$ is nonzero and, in general, complex.
(b) $w(z) = z + Q$, where $Q$ is nonzero and, in general, complex.

**3.** (a) Derive the expression

$$\rho(x,y) = \ln\sqrt{\frac{(x-a)^2+y^2}{(ax-1)^2+a^2y^2}}$$

that was used in (24).

(b) Use computer software, such as the *Maple* implicitplot command, to obtain the isotherms shown in Fig. 7.

**4.** (*Composition of two bilinear transformations*) Show that the composition of two bilinear transformations is also a bilinear transformation. That is, show that if $f(z)$ and $g(z)$ are bilinear transformations so is $f(g(z))$.

**5.** (*Fixed points*) A **fixed point** of a mapping $w = f(z)$ is a point $z_0$ that is mapped into itself. That is, $z_0$ is a fixed point of $w = f(z)$ if $f(z_0) = z_0$. Prove that a *bilinear transformation* $w = (az + b)/(cz + d)$ *has at most two fixed points, unless it is simply the identity transformation* $w = z$.

**6.** (*Three points into three points*) (a) Prove that *any three given distinct points* $z_1, z_2, z_3$ *in the extended* $z$ *plane can be mapped into any three given distinct points* $w_1, w_2, w_3$, *respectively, in the extended* $w$ *plane by a bilinear transformation* $w = f(z)$ *which is given, implicitly, by the relation*

$$\frac{w - w_1}{w - w_3}\,\frac{w_2 - w_3}{w_2 - w_1} = \frac{z - z_1}{z - z_3}\,\frac{z_2 - z_3}{z_2 - z_1}. \tag{6.1}$$

HINT: Show first that the transformation $w = f(z)$ implied by (6.1) is indeed bilinear. Then verify that (6.1) guarantees that $f(z_1) = w_1$, $f(z_2) = w_2$, and $f(z_3) = w_3$.
(b) Prove that the bilinear transformation $w = f(z)$ in part (a) is unique. HINT: Suppose that $w = g(z)$ is another bilinear transformation which maps the three given $z$ points into the three given $w$ points. With the help of the results stated in Exercises 4 and 5, show that the composite transformation $g^{-1}(f(z))$ has three fixed points so it must be the identity transformation.

**7.** Verify that if $a$ is a complex number with $|a| < 1$, then $w = (z - a)/(1 - \bar{a}z)$ maps $|z| < 1$ onto $|w| < 1$, with $a$ being sent to the origin.

**8.** (*On which side of the boundary curve is* $D'$*?*) First, review the COMMENT at the end of Example 1. Rather than rely on a pointwise check, one can determine whether $D'$ is on one side of the mapped boundary curve or the other, as follows. *Let three points on the boundary curve of* $D$ *be labeled consecutively as* $A, B, C$, *and imagine walking along that curve from* $A$ *to* $B$ *to* $C$. *If the region is on our left (right), then likewise the region* $D'$ *is on our left (right) when we walk along the corresponding curve in the* $w$ *plane, from* $A'$ *to* $B'$ *to* $C'$. For instance, when we walk along $ABC$ in Fig. 3 the region $D$ is on our left, so when we walk along $A'B'C'$ in Fig. 5 the region $D'$ must, likewise, be on our left. The problem is this. Show that the truth of the italicized claim, above, follows from the conformality of the mapping.

**9.** (a) Use (6.1) in Exercise 6 to find two distinct bilinear transformations that map $x < 0$ onto $|w| < 2$, and express them in the form $w = (az + b)/(cz + d)$. HINT: Use the idea in Exercise 8 to be sure that $D'$ is $|w| < 2$ rather than $|w| > 2$.
(b) The same as (a), but this time map $x < 0$ onto $|w| > 2$.
(c) The same as (a), but this time map $x + y < 1$ onto $u - v > 3$.
(d) The same as (a), but this time map $x + y > 4$ onto $|w| < 1$.

**10.** Determine the image of the given region under the mapping $w = (z + 1)/(z - i)$. Label the various key points in the $z$ and $w$ planes.

(a) $y < 0$ (b) $0 < y < 1$
(c) $y > 1$ (d) $x > 0$
(e) the wedge $0 < \arg z < \pi/4$
(f) the quadrant $0 < \arg z < \pi/2$
(g) the annulus $1 < |z| < 2$

**11.** Determine the image of the region $0 < x < \infty$, $0 < y < \infty$ under the given mapping. Label any key points.

(a) $w = 1/(z - i)$ (b) $w = 1/(z + i)$
(c) $w = z/(z - 1)$ (d) $w = 2 - i/z$
(e) $w = (z - 1)/(z + 2)$ (f) $w = (z - 3)/z$

**12.** Putting the pole of the transformation at $A$, we used (14) to solve the problem shown in Fig. 3. Putting the pole at $C$ instead, use

$$w(z) = \frac{z + 1}{z - 1},$$

and show that your final result agrees with that given by (19).

**13.** Use computer software and equation (19) to plot the isothermal curves $\psi = 0, 0.2, 0.4, 0.6, 0.8$, and 1. Using *Maple*, for instance, this can be done using the implicitplot command.

**14.** Solve the given Dirichlet problem with the help of a bilinear transformation. Obtain that transformation yourself, or use the table in Appendix F. In each case the PDE is $\nabla^2\psi = \psi_{xx} + \psi_{yy} = 0$.

(a) Let $D$ be the infinite plane, with a cutout defined by $x^2 + (y - 3)^2 = 25$ for $y \geq 0$ and by $x^2 + (y + 3)^2 = 25$ for $y \leq 0$. Let $\psi = 100$ on the former, and $\psi = 0$ on the latter. HINT: $D$ can be mapped onto an infinite wedge.
(b) Let $D$ be the region $x < 2$ with the disk $x^2 + y^2 \leq 1$ cut out. Let $\psi = 30$ on $x = 2$ and $\psi = 20$ on $x^2 + y^2 = 1$. HINT: The desired mapping can be obtained as a limiting case of the one given by item 4 of Appendix F.
(c) Let $D$ be the disk $x^2 + y^2 < 9$, with the disk $(x - \frac{3}{2})^2 + y^2 \leq \frac{9}{4}$ cut out. Let $\psi = 1$ on $x^2 + y^2 = 9$ and $\psi = 0$ on $(x - \frac{3}{2})^2 + y^2 = \frac{9}{4}$. HINT: Locate the pole of the transformation at a suitable point on the boundary of $D$.

(d) Let $D$ be the "crescent" bounded above by the circle passing from $-3$ to $9i$ to 3, and below by the circle passing from $-3$ to $3i$ to 3. Let $\psi = 1$ on the upper arc and $\psi = 0$ on the lower arc. In particular, evaluate $\psi(0, 4)$. Also, explain why $D$ cannot be mapped, by a bilinear transformation, onto an infinite strip or the region between concentric circles. HINT: Map $D$ onto an infinite wedge.

(e) Let $D$ be the disk $|z| < 1$ with the disk $\left|z - \frac{1}{4}\right| \le \frac{1}{4}$ cut out. Let $\psi = 50$ on $|z| = 1$ and $\psi = 0$ on $\left|z - \frac{1}{4}\right| = \frac{1}{4}$.

## 22.4 Additional Mappings and Applications

The bilinear transformation, discussed in Section 22.3, is nontypical in the sense that it is the only transformation that is one-to-one. That is, $w = (az + b)/(cz + d)$ gives a unique $w$ for each $z$ (except for $z = -d/c$), and solving for $z$ gives the unique value $z = (-dw + b)/(cw - a)$ for each $w$ (except for $w = a/c$).

Thus, with any other mapping we will inevitably need to deal with multivaluedness. For brevity, we will consider just two representative examples, with other mappings and applications reserved for the exercises.

**EXAMPLE 1.** Solve for the potential $\psi(x, y)$ within the curved strip shown in Fig. 1, subject to the boundary conditions shown there, together with a condition that $\psi$ be bounded on $D$.

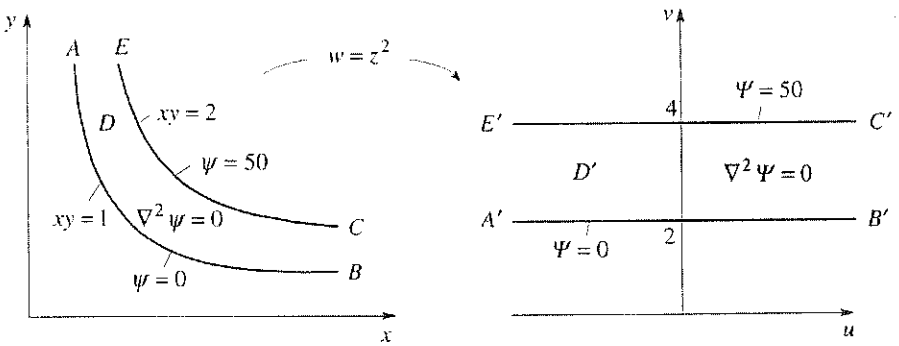



**Figure 1.** Potential in a curved strip.

A suitable mapping for the curved strip is not in the short table provided in Appendix F,* but is given by

$$w = z^2 = (x^2 - y^2) + i\,2xy \tag{1}$$

because we see from

$$u = x^2 - y^2, \qquad v = 2xy \tag{2}$$

that $AB$ and $EC$ map into the straight lines $v = 2$ and $v = 4$, over $-\infty < u < \infty$. Thus, the problem on $\Psi(u, v)$ shown in Fig. 1 is simple: we can seek $\Psi$ as a function of $v$ only so

*For a much more extensive table, see for example H. Kober, *Dictionary of Conformal Representation* (New York: Dover, 1952).

$\nabla^2\Psi = \Psi_{uu} + \Psi_{vv} = 0$ reduces to $d^2\Psi/dv^2 = 0$. The general solution is $\Psi = Av + B$, and the boundary conditions give

$$\Psi = 25(v - 2). \tag{3}$$

Finally, recalling from (2) that $v = 2xy$, we have the desired solution

$$\psi(x, y) = 50(xy - 1). \tag{4}$$

COMMENT. We mentioned above, that multi-valuedness was inevitable. Indeed, although $w = z^2$ is single-valued, the inverse transformation $z = \sqrt{w}$ is double-valued. That statement might be surprising inasmuch as when we returned from the $w$ plane [equation (3)] to the $z$ plane [equation (4)] we did not seem to have to make a choice between two values. Actually we did, but it might have gone unnoticed. Namely, the region $D'$ maps back into $D$ in the first quadrant but also into a similar curved strip between $xy = 1$ and $xy = 2$ in the third quadrant. Without fuss, we simply chose the one in the first quadrant. We could have introduced a branch cut for the $\sqrt{w}$ function, but it really wasn't necessary. (See Exercise 1.) ∎

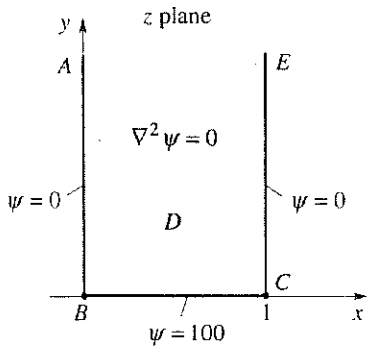



**Figure 2.** Electric potential in semi-infinite strip.

**EXAMPLE 2.** *Electric Potential in Semi-Infinite Strip.* We wish to solve for the electric potential (i.e., the voltage) $\psi(x, y)$ within the semi-infinite strip shown in Fig. 2. The edges $x = 0$ and $x = 1$ are "grounded" ($\psi = 0$), the edge $y = 0$ is maintained at 100 volts, and we ask $\psi$ to be bounded on $D$.

It's true that the boundary of $D$ is comprised of circular arcs (namely, straight lines), but the bilinear transformation is of no help (Exercise 2). Turning to Appendix F for help, let us try the mapping $w = -\cos \pi z$ (entry 10). Then the mapped problem is as shown in Fig. 3. That problem is easier in the sense that it is the Dirichlet problem for the upper half plane, which has already been solved using a Fourier transform in Section 20.4. Using the solution given in that section, we have

$$\Psi(u, v) = \frac{v}{\pi}\int_{-\infty}^{\infty} \frac{\Psi(u', v)du'}{(u' - u)^2 + v^2} = \frac{100v}{\pi}\int_{-1}^{1} \frac{du'}{(u' - u)^2 + v^2}$$

$$= \frac{100v}{\pi}\int_{-1-u}^{1-u} \frac{d\xi}{\xi^2 + v^2} = \frac{100}{\pi}\left[\tan^{-1}\left(\frac{1-u}{v}\right) - \tan^{-1}\left(\frac{-1-u}{v}\right)\right], \tag{5}$$

where $-\pi/2 < \tan^{-1}(\,) < \pi/2$ for each of the $\tan^{-1}(\,)$'s. [As a check, observe that for $u = 0$ and $v \to 0$ equation (1) gives $\Psi \to \frac{100}{\pi}(\frac{\pi}{2} + \frac{\pi}{2}) = 100$, for $u > 1$ and $v \to 0$ equation (5) gives $\Psi \to 0$, and for $u < -1$ and $v \to 0$ equation (5) gives $\Psi \to 0$, all of which are correct.]

Next, $w = -\cos \pi z = -\cos \pi(x + iy) = -\cos \pi x \cosh \pi y + i \sin \pi x \sinh \pi y$ gives

$$u = -\cos \pi x \cosh \pi y, \qquad v = \sin \pi x \sinh \pi y, \tag{6}$$

and putting these expressions into (5) gives the solution

$$\psi(x, y) = \frac{100}{\pi}\left[\tan^{-1}\left(\frac{\cos \pi x \cosh \pi y + 1}{\sin \pi x \sinh \pi y}\right) - \tan^{-1}\left(\frac{\cos \pi x \cosh \pi y - 1}{\sin \pi x \sinh \pi y}\right)\right], \tag{7}$$

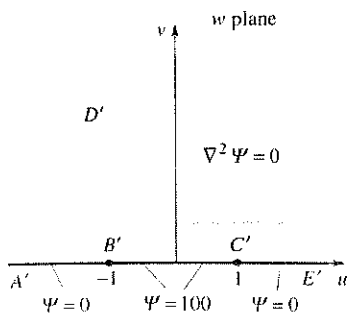



**Figure 3.** The mapped problem; $w = -\cos \pi z$.

where we recall that $-\pi/2 < \tan^{-1}(\,) < \pi/2$ for each $\tan^{-1}(\,)$.

If we do not notice that the $w$ plane problem can be solved by the result given in Section 20.4, we can make one more transformation, from the $w$ plane to a $\zeta$ plane (where $\zeta = \xi + i\eta$) using entry 8 of Appendix F. Denoting $\Psi(u(\xi,\eta), v(\xi,\eta)) \equiv \tilde{\Psi}(\xi,\eta)$, the result of this second mapping is indicated in Fig. 4. Finally, the problem in the $\zeta$ plane is

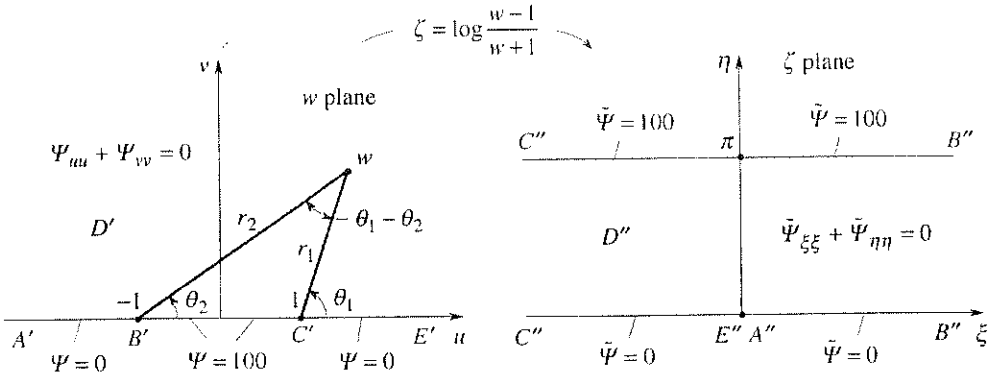



**Figure 4.** One more mapping.

simple (since $\tilde{\Psi}$ varies with $\eta$ but not $\xi$) and gives the solution

$$\tilde{\Psi}(\xi,\eta) = \frac{100}{\pi}\eta. \tag{8}$$

Retracing our steps, we need to express $\eta$ in terms of $u$ and $v$, and then $u$ and $v$ in terms of $x$ and $y$. First, we express $w - 1 = r_1 e^{i\theta_1}$ and $w + 1 = r_2 e^{i\theta_2}$ (Fig. 4) so that

$$\zeta = \xi + i\eta = \log\left(\frac{r_1}{r_2}e^{i(\theta_1-\theta_2)}\right) = \ln\frac{r_1}{r_2} + i(\theta_1 - \theta_2). \tag{9}$$

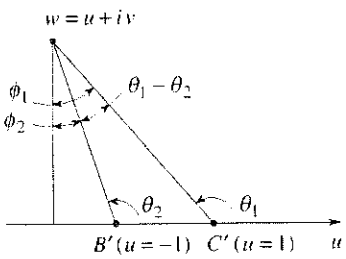



**Figure 5.** The angle $\theta_1 - \theta_2$.

It follows from (9) that $\eta = \theta_1 - \theta_2$ so

$$\Psi(u,v) = \frac{100}{\pi}(\theta_1 - \theta_2). \tag{10}$$

It is convenient to use the fact that the angle $\theta_1 - \theta_2$ in Fig. 4 is equivalent to $\phi_1 - \phi_2$ (Fig. 5) so

$$\Psi(u,v) = \frac{100}{\pi}(\phi_1 - \phi_2) = \frac{100}{\pi}\left[\tan^{-1}\left(\frac{1-u}{v}\right) - \tan^{-1}\left(\frac{-1-u}{v}\right)\right], \tag{11}$$

where $-\pi/2 < \tan^{-1}(\,) < \pi/2$ for each $\tan^{-1}(\,)$, and we see that this result is identical to (5).

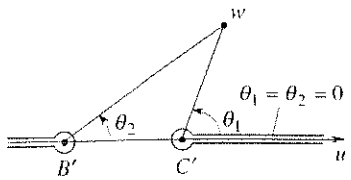



**Figure 6.** Branch cut chosen for $\log[(w-1)/(w+1)]$.

COMMENT 1. We didn't say so explicitly, but the branch cut that we used to make $\zeta = \log\frac{w-1}{w+1} = \log(w-1) - \log(w+1)$ single-valued is as shown in Fig. 6. Other choices could have been made, but the final solution would have been unaffected.

COMMENT 2. Observe that we mapped from the $z$ plane to the $w$ plane using $w = -\cos\pi z$, and then from the $w$ plane to the $\zeta$ plane using $\zeta = \log\frac{w-1}{w+1}$. The two steps

could have been combined into one as

$$\zeta = \log\left(\frac{\cos \pi z + 1}{\cos \pi z - 1}\right), \tag{12}$$

but it is much easier to proceed by a sequence of small steps – both in this problem and as a general rule.

COMMENT 3. In this example, the original semi-infinite strip problem could have been solved by separation of variables, which method gives (Exercise 3)

$$\psi(x, y) = \frac{400}{\pi} \sum_{n=1,3,\ldots}^{\infty} \frac{\sin n\pi x}{n} e^{-n\pi y}, \tag{13}$$

but surely the conformal mapping result (7) is nicer since it is in closed form, whereas the separation of variable result (13) is in the form of an infinite series. We should be able to obtain (13) from (7) by expressing

$$\cosh \pi y = \frac{e^{\pi y} + e^{-\pi y}}{2} = \left(\frac{1}{p} + p\right)\Big/2 = (1 + p^2)/2p,$$
$$\sinh \pi y = \frac{e^{\pi y} - e^{-\pi y}}{2} = \left(\frac{1}{p} - p\right)\Big/2 = (1 - p^2)/2p,$$

where $p = e^{-\pi y}$, and doing a Taylor expansion of the right-hand side of (7) in powers of $p$, but we will not pursue that point. ∎

**Closure.** Important points made in this section are as follows. First, in selecting a mapping, for a given application, it is reasonable to rely on conformal mapping tables. Second, one often employs not a single mapping but a sequence of mappings, as we illustrate in Example 2 and as are illustrated in the exercises of Section 22.6. Third, if we use a mapping other than the bilinear transformation, then it will inevitably involve multi-valuedness – that is, it will not be one-to-one. Finally, in seeking the image of a circular arc under a bilinear transformation, in Section 22.3, it sufficed to map only three points (two, in fact, if we know the image to be a straight line) because three noncollinear points uniquely determine the circular image curve. In general, however, the image of a given curve, under a given mapping, is not a circle, and three points by no means suffice. Rather, seek the image of the entire curve, not of individual points. For instance, if a given curve $C$ in the $z$ plane is given parametrically by $x = x(\tau)$ and $y = y(\tau)$, for $a < \tau < b$, then the image curve is given parametrically by $u = u(x(\tau), y(\tau))$ and $v = v(x(\tau), y(\tau))$, for $a < \tau < b$. As a simple illustration, let us take another look at finding the image of the curve $xy = 2$, in Example 1 (Fig. 1). If we parametrize that curve by

$$x = \tau, \quad y = 2/\tau, \qquad (0 < \tau < \infty)$$

then (2) gives

$$u = \tau^2 - 4/\tau^2, \quad v = 4, \qquad (0 < \tau < \infty)$$

which is the horizontal line $v = 4$, for $-\infty < u < \infty$.

**EXERCISES 22.4**

**1.** Show a branch cut for $z = \sqrt{w}$ that sends the infinite strip $D'$ (Fig. 1) back into the curved strip $1 < xy < 2$ in the first quadrant, rather than in the third quadrant.

**2.** Show that the region $D$ in Fig. 2 cannot be mapped into a circular annulus or an infinite strip by any bilinear transformation. HINT: Note that $D$ has two corners on its boundary.

**3.** Derive the separation of variables solution (13) to the problem on $\psi(x, y)$ in Example 2.

**4.** In each case, solve $\nabla^2\psi = 0$ for $\psi(x, y)$ for the specified region $D$ and the specified boundary conditions, together with the condition that $\psi$ be bounded on $D$. If useful, you may use the known solution of the Dirichlet problem for the half plane [(12) in Section 20.4], as we did in Example 2.

(a) Let $D$ be the strip $0 < y < \pi$, $-\infty < x < \infty$. Let $\psi = 0$ everywhere on the boundary except on $y = 0$ $(-\infty < x < 0)$, where $\psi = 20$. Also, evaluate $\psi(0, \pi/2)$. HINT: Use entry 6 in Appendix F.
(b) The same as part (a) but with $0 < y < 5$ instead of $0 < y < \pi$. HINT: Modify the mapping given in entry 6 slightly, so that $D'$ is once again the upper half plane.
(c) Let $D$ be the region $0 < x < \infty$, $0 < y < \infty$. Let $\psi = 0$ everywhere on the boundary except on $x = 0$ $(2 < y < \infty)$, where $\psi = 35$. Also, evaluate $\psi(2, 2)$. HINT: Use entry 5 in Appendix F.
(d) The same as part (c), but with different boundary conditions. This time, let $\psi = 0$ everywhere on the boundary except on $x = 0$ $(0 < y < 2)$, where $\psi = 35$.
(e) Let $D$ be the 45° wedge between the positive $x$ axis and the line $y = x$. Let $\psi = 0$ everywhere on the boundary except on the $x$ axis from $x = 5$ to $x = \infty$, where $\psi = 200$. Also evaluate $\psi(5, 1)$. HINT: Use entry 5 in Appendix F.
(f) The same as part (e) but with different boundary conditions. This time let $\psi = 10$ everywhere on the boundary except on the $x$ axis from $x = 0$ to $x = 10$, where $\psi = 0$.
(g) Let $D$ be the semi-infinite strip $0 < x < 1$, $0 < y < \infty$. Let $\psi = 0$ on $x = 0$ and on $y = 0$, and let $\psi = 400$ on $x = 1$. Also, evaluate $\psi(0.5, 0.1)$. HINT: Use entry 10 in Appendix F.
(h) Let $D$ be the semi-infinite strip $0 < x < 4$, $0 < y < \infty$. Let $\psi = 0$ on $y = 0$, and let $\psi = 200$ on $x = 0$ and $x = 4$. Also, evaluate $\psi(2, 2)$, $\psi(2, 4)$, $\psi(2, 6)$, $\psi(2, 8)$, $\psi(2, 10)$, and $\psi(2, 20)$. HINT: You can use entry 10 in Appendix F, but will need to modify it slightly.
(i) Let $D$ be the semi-infinite strip $-2 < x < 2$, $0 < y < \infty$. Let $\psi = 0$ on $x = -2$, and on $y = 0$ $(-2 < x < 0)$, and let $\psi = 50$ on $x = 2$ and $y = 0$ $(0 < x < 2)$. Also, evaluate $\psi(0, 1)$ and $\psi(0, 30)$. HINT: With a slight modification you can use entry 12 in Appendix F.
(j) Let $D$ be the region in the right half plane, between the lines $y = \pm x$ and the curve $x^2 - y^2 = 4$. Let $\psi = 30$ on $y = \pm x$ and let $\psi = 20$ on $x^2 - y^2 = 4$. Also, evaluate $\psi(1, 0)$ and $\psi(0.6, -0.4)$. HINT: Use the mapping $w(z) = z^2$.

## 22.5 More General Boundary Conditions

Thus far, all of the boundary conditions considered have been constant and of Dirichlet type, $\psi$ = constant. In this section we show how to handle nonconstant boundary conditions, Dirichlet conditions ($\psi$ given), and Neumann conditions ($\partial\psi/\partial n$ given) as well.

The fate of a **Dirichlet boundary condition**, under a mapping $w(z)$, is simple. Since

$$\boxed{\psi(x, y) = \psi(x(u, v), y(u, v)) \equiv \Psi(u, v),} \tag{1}$$

we see that if $\psi$ takes on the value 26, say, at a point $P$ on the boundary of $D$, then

$\Psi$ takes on that same value at the point $P'$ (the image of $P$) on the boundary $D'$.

**EXAMPLE 1.** *Nonconstant Dirichlet conditions.* Find the boundary conditions in the $w$ plane for the problem displayed in Fig. 1. Here, $D$ is the region between the hyperbola

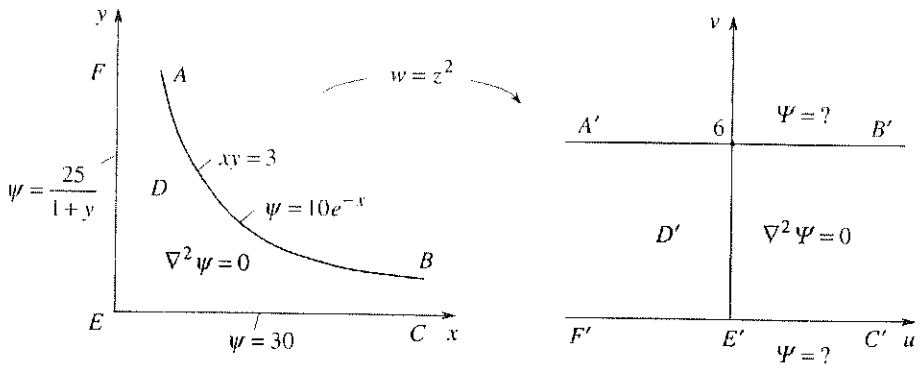



**Figure 1.** Nonconstant Dirichlet conditions.

$xy = 3$ and the positive $x$ and $y$ axes, and the $w = z^2$ mapping gives the relations

$$u = x^2 - y^2, \qquad v = 2xy. \tag{2}$$

The $\psi = 30$ boundary condition on $EC$ carries over to $\Psi = 30$ on its image $E'C'$. To determine the image of the $\Psi = 10e^{-x}$ condition on $AB$, we need to express $x$ (on $AB$) in terms of $u$ (on $A'B'$). To do so, put $y = 3/x$ into (2), obtaining

$$u = x^2 - \frac{9}{x^2}, \qquad v = 6. \tag{3}$$

Multiplying the first of these by $x^2$ gives the $x^4 - ux^2 - 9 = 0$, which is a quadratic equation in $x^2$, so that

$$x^2 = \frac{u \pm \sqrt{u^2 + 36}}{2}, \qquad x = \sqrt{\frac{u + \sqrt{u^2 + 36}}{2}}, \tag{4}$$

where, in the last step, we selected the $+$ sign since $x$ is to be real. Thus, the boundary condition on $A'B'$ is

$$\Psi(u, 6) = 10 \exp\left(-\sqrt{\frac{u + \sqrt{u^2 + 36}}{2}}\right). \qquad (-\infty < u < \infty) \tag{5}$$

On $EF$ we have $x = 0$ so (2) gives $u = -y^2$ and $v = 0$. Thus, $y = \sqrt{-u}$ (remember that $u < 0$ on $F'E'$) and the boundary condition on $F'E'$ is

$$\Psi(u, 0) = \frac{25}{1 + \sqrt{-u}}. \qquad (-\infty < u < 0) \tag{6}$$

We will not attempt to solve the $\Psi$ problem. Our purpose was simply to illustrate the transformation of the nonconstant Dirichlet boundary conditions. ∎

Next, we turn to **Neumann boundary conditions**, namely, where $\partial\psi/\partial n$ is prescribed on the boundary of $D$, $\partial\psi/\partial n$ being the directional derivative of $\psi$ in the outward normal direction. How does $\partial\psi/\partial n$ transform under the mapping $w = f(z)$?

Let $z$ be any point on the boundary $C$ of $D$, and let $z_0$ lie within $D$ on a straight line $L$ which is normal to $C$ at $z$ (Fig. 2). The image $L'$ of $L$ will, in general,

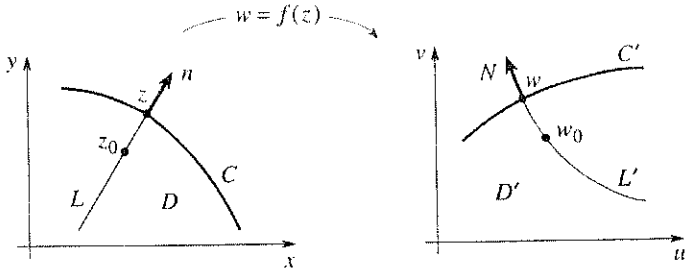



**Figure 2.** Transformation of Neumann condition.

be curved, but conformality guarantees that it be normal to $C'$ at $w$. Then, with $\Delta z = z - z_0$ and $\Delta w = w - w_0$ we have

$$\left.\frac{\partial\psi}{\partial n}\right|_z = \lim_{\Delta z\to 0} \frac{\psi|_z - \psi|_{z_0}}{|\Delta z|}. \tag{7}$$

But $\psi|_z = \Psi|_w$, $\psi|_{z_0} = \Psi|_{w_0}$, and $\lim_{\Delta z\to 0} \Delta w/\Delta z = f'(z)$ so

$$\left.\frac{\partial\psi}{\partial n}\right|_z = \lim_{\Delta z\to 0} \frac{\Psi|_w - \Psi|_{w_0}}{|\Delta w|}\,\frac{|\Delta w|}{|\Delta z|} = \left.\frac{\partial\Psi}{\partial N}\right|_w |f'(z)|, \tag{8}$$

where $\partial\Psi/\partial N$ denotes the directional derivative of $\Psi$ in the outward normal direction. In the final equality in (8) we used the fact that $\Delta z \to 0$ implies that $\Delta w \to 0$ for if $w = f(z)$ is a differentiable function of $z$, then it must surely be a continuous function of $z$. We conclude that Neumann boundary conditions transform, under a conformal map, according to

$$\boxed{\frac{\partial\Psi}{\partial N} = \frac{1}{|f'(z)|}\,\frac{\partial\psi}{\partial n},} \tag{9}$$

where $f'(z) \neq 0$ by the assumed conformality. As an important special case, we see from (9) that if $\partial\psi/\partial n = 0$ on $C$, then $\partial\Psi/\partial N = 0$ on $C'$. To remember (9), observe, heuristically, that $dN = |f'(z)|\,dn$ because $|f'(z)|$ is the local magnification (Exercise 1).

**EXAMPLE 2.** To illustrate (9), let us use the same problem as displayed in Fig. 1, but with the boundary conditions changed to Neumann boundary conditions: $\partial\psi/\partial n = 10e^{-x}$ on $AB$, $\partial\psi/\partial n = 30$ on $EC$, and $\partial\psi/\partial n = 25/(1+y)$ on $FE$.

Consider $AB$. With the help of (4), the $|f'(z)|$ in (9) is

$$\begin{aligned} |f'(z)| = |2z| = 2\sqrt{x^2+y^2} &= 2\sqrt{x^2+\frac{9}{x^2}} \\ &= 2\sqrt{\frac{u+\sqrt{u^2+36}}{2}+\frac{18}{u+\sqrt{u^2+36}}} \\ &= 2\sqrt[4]{u^2+36}. \end{aligned} \tag{10}$$

Further, the $\partial\psi/\partial n = 10e^{-x}$ needed in (9) is given by the right-hand side of (5), so

$$\frac{\partial\Psi}{\partial N} = \frac{1}{2\sqrt[4]{u^2+36}}\left[10\exp\left(-\sqrt{\frac{u+\sqrt{u^2+36}}{2}}\right)\right] \tag{11}$$

on $A'B'$. Next, consider $EC$. There, (2) gives $x = \sqrt{u}$, so

$$|f'(z)| = |2z| = 2x = 2\sqrt{u}$$

so (9) gives

$$\frac{\partial\Psi}{\partial N} = \frac{1}{2\sqrt{u}}(30) = \frac{15}{\sqrt{u}} \tag{12}$$

on $E'C'$. Finally, consider $FE$. We leave it for Exercise 2 to show that

$$\frac{\partial\Psi}{\partial N} = \frac{25}{2\left(\sqrt{-u}-u\right)} \tag{13}$$

on $F'E'$.

COMMENT. Simpler than (10) is the following approach:

$$|f'(z)| = |2z| = 2\,|z| = 2\sqrt{|w|} = 2\sqrt[4]{u^2+v^2}\Big|_{v=6} = 2\sqrt[4]{u^2+36}$$

on $A'B'$. ∎

**Closure.** The transformation of a Dirichlet boundary condition is simple: according to (1), $\Psi(u,v) = \psi(x,y)$. For instance, if $\psi = 47$ at a point $z$ on the boundary $C$ of the regon $D$, then $\Psi = 47$ at the image point $w$ on the boundary $C'$ of the region $D'$. Of course, one does need to determine the correspondence between points on $C$ and points on $C'$. In Example 1, for example, to transform the boundary condition $\psi = 10e^{-x}$ on $AB$ we need to solve for $x$ (on $AB$) as a function of $u$ (on $A'B'$).

In contrast, Neumann conditions do not carry over intact. For instance, if $\partial\psi/\partial n = 25$ at a point $z$ on $C$, then $\partial\Psi/\partial N$ does not equal 25 at the image point $w$ on $C'$, it equals 25 times the scale factor $1/|f'(z)|$ which, in general, is not unity. That scale factor is due to the local magnification of $dn$: $dN = |f'(z)|\,dn$, as can be seen in the denominators in (9).

## EXERCISES 22.5

**1.** In the sentence preceding Example 2, we stated that $|f'(z)|$ is the local amplification. That is, if $\Delta z = z - z_0$ is any vector springing from $z_0$, and $\Delta w = w - w_0$, where $w_0$ is the image of $z_0$ and $w$ is the image of $z$, then the "magnification" $|\Delta w|\,/\,|\Delta z|$ is equal to $|f'(z_0)|$ in the limit as $\Delta z \to 0$ ($z$ approaches $z_0$), if $w = f(z)$ is analytic at $z_0$ and $f'(z_0) \neq 0$ (i.e., if the map is conformal there). Why is that statement true?

**2.** Derive the expression (13) for $\partial\Psi/\partial N$ on $F'E'$.

**3.** In each case you are given the region $D$, Dirichlet and/or Neumann boundary conditions, and the mapping function $w(z)$. Determine the image $D'$ and transformed conditions on its boundary. You need *not* solve for $\Psi$.

(a) Region $D$: $-\infty < x < \infty$, $0 < y < 3\pi/2$. Boundary conditions: $\psi_y(x,0) = 5$, $\psi(x, 3\pi/2) = \sin x$. Map: $w(z) = e^z$.
(b) Region $D$: $-\infty < x < \infty$, $-\pi/2 < y < \pi/2$. Boundary conditions: $\psi_y(x, -\pi/2) = 20\cos x$, $\psi(x, \pi/2) = 3e^{-|x|}$. Map: $w(z) = e^z$.
(c) Region $D$: $0 < x < \infty$, $0 < y < \pi$. Boundary conditions: $\psi_y(x,0) = e^{-x}$, $\psi_x(0,y) = 3y$, $\psi(x,\pi) = e^{-4x}$. Map: $w(z) = e^z$.
(d) Region $D$: $-\infty < x < 0$, $0 < y < \pi/2$. Boundary conditions: $\psi(x,0) = 5e^x$, $\psi_x(0,y) = 2y$, $\psi_y(x,\pi/2) = 5$. Map: $w(z) = e^z$.
(e) Region $D$: $0 < x < 2$, $0 < y < \pi$. Boundary conditions: $\partial\psi/\partial n = 10$ on all four edges. Map: $w(z) = e^z$.
(f) Region $D$: $0 < x < 1$, $0 < y < \infty$. Boundary conditions: $\psi(0,y) = e^{-y}$, $\partial\psi/\partial n = 10x$ on $y = 0$, $\partial\psi/\partial n = 0$ on $x = 1$. Map: $w(z) = (z-1)/z$.
(g) Region $D$: $0 < x < \infty$, $-\infty < y < \infty$. Boundary conditions: $\psi(0,y) = 50$ on $-\infty < y \leq 0$, $\psi(0,y) = 50e^{-y}$ on $0 < y < \infty$. Map: $w(z) = (z-1)/(z+1)$.
(h) Region $D$: $0 < x < 1$, $0 < y < \infty$. Boundary conditions: $\psi(0,y) = 100$, $\partial\psi/\partial n = -2$ on $y = 0$, $\psi(1,y) = 3e^{-y}$. Map: $w(z) = 1/z$.
(i) Region $D$: $0 < x < 1$, $-\infty < y < 0$. Boundary conditions: $\psi(0,y) = 25\cos y$, $\partial\psi/\partial n = 5$ on $y = 0$ and on $x = 1$. Map: $w(z) = 1/z$.
(j) Region $D$: $0 < x < \sqrt{1+y^2}$, $0 < y < \infty$. Boundary conditions: $\psi(0,y) = \sin y$, $\partial\psi/\partial n = 1$ on $y = 0$, $\psi = 2e^{-y}$ on $x = \sqrt{1+y^2}$. Map: $w(z) = z^2$.
(k) Region $D$: $0 < x < \infty$, $0 < y < \infty$. Boundary conditions: $\psi(0,y) = 5/(y^2+1)$, $\psi(x,0) = 2 - 5e^{-x}$. Map: $w(z) = z^2$.

## 22.6 Applications to Fluid Mechanics

Let $\mathbf{q}(x,y,z)$ be a fluid velocity field. If the flow is irrotational (i.e., $\nabla \times \mathbf{q} = \mathbf{0}$), then there exists a scalar velocity potential $\phi$ such that $\mathbf{q} = \nabla\phi$. Further, if the flow is incompressible (i.e., $\nabla \cdot \mathbf{q} = 0$), then $\nabla \cdot \nabla\phi = \nabla^2\phi = 0$, so $\phi$ satisfies the Laplace equation. Thus, an irrotational incompressible flow is called a *potential flow*. These points are discussed in Section 16.10. (There, we used $\mathbf{v}$ for the velocity field; here, it is more convenient to use $\mathbf{q}$.)

In the present section we consider only two-dimensional potential flows governed by

$$\nabla^2\phi = \phi_{xx} + \phi_{yy} = 0. \tag{1}$$

Throughout this section we assume that the boundary condition is $\mathbf{q}\cdot\hat{\mathbf{n}} = 0$ along any rigid boundaries, where $\hat{\mathbf{n}}$ is the usual outward normal vector (to the flow field). That is, the flow neither separates from the wall nor penetrates it. Thus, $\mathbf{q}\cdot\hat{\mathbf{n}} = \nabla\phi\cdot\hat{\mathbf{n}} = \partial\phi/\partial n = 0$, where the second equality amounts to the directional derivative formula $d\phi/ds = \nabla\phi\cdot\hat{\mathbf{s}}$ in any $\hat{\mathbf{s}}$ direction, from Section 16.4.

**EXAMPLE 1.** *Flow in a Corner.* Determine the potential flow in a corner, sketched in Fig. 1. The function $\zeta = z^2$ maps the first quadrant of the $z$ plane, conformally, onto

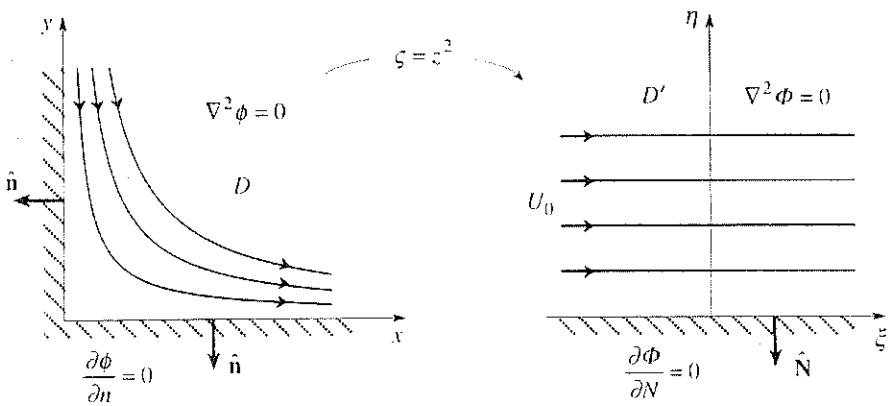



**Figure 1.** Corner flow.

the upper half of the $\zeta$ plane, with the boundary condition $\partial\phi/\partial n = 0$ carrying over to $\partial\Phi/\partial N = 0$. Note that we use a $\zeta = \xi + i\eta$ plane, rather than the usual $w = u + iv$ plane because $u, v$ designate the $x, y$ fluid velocity components, here, and $w$ will be used for the so-called complex velocity potential, defined below.

A simple flow in the $\zeta$ plane, which will give the desired corner flow in the $z$ plane, is the uniform stream shown in the figure, for any stream speed $U_0$. Then $\partial\Phi/\partial\xi = U_0$ and $\partial\Phi/\partial\eta = 0$, so $\Phi(\xi,\eta) = U_0\xi$. Since

$$\zeta = \xi + i\eta = (x+iy)^2 = x^2 - y^2 + i2xy, \tag{2}$$

we have $\xi = x^2 - y^2$ and $\eta = 2xy$ so

$$\phi(x,y) = U_0\xi(x,y) = U_0(x^2 - y^2). \tag{3}$$

Hence, the desired velocity field is given by

$$\mathbf{q} = \nabla\phi = 2U_0(x\mathbf{i} - y\mathbf{j}). \tag{4}$$

Recall, from our study of the gradient in Section 16.4, that at each point in the field the vector $\nabla\phi$ is normal to the $\phi =$ constant curve through that point. It is also true that if $\psi(x,y)$ is the conjugate harmonic function corresponding to $\phi(x,y)$ so that $w(z) =$

$\phi(x,y)+i\psi(x,y)$ is an analytic function of $z$, then the $\psi =$ constant curves are orthogonal to the $\phi =$ constant curves (as is discussed in Exercise 16 in Section 21.5). Thus, the fluid particles move along $\psi =$ constant curves. These curves are called *streamlines*, and $\psi(x,y)$ is called the *stream function*. (For additional discussion of the stream function, see the exercises in Section 16.10.)

Corresponding to $\phi(x,y) = U_0(x^2 - y^2)$, the conjugate harmonic function $\psi(x,y)$ is readily found (Exercise 1) to be

$$\psi(x,y) = 2U_0xy \tag{5}$$

so the streamlines $\psi(x,y) = 2U_0xy =$ constant are the hyperbolas $xy =$ constant as shown in Fig. 1. ∎

Observe that we solved the potential problem for $\phi(x,y)$. From $\phi$, we obtained the velocity field as $\mathbf{q} = \nabla\phi$, and we obtained the streamline pattern by finding the conjugate harmonic function $\psi(x,y)$ corresponding to $\phi(x,y)$. However, in the fluid mechanics literature it is more common to work with the **complex velocity potential**,

$$w(z) = \phi(x,y) + i\psi(x,y), \tag{6}$$

right from the start, rather than $\phi$. Then

$$\frac{dw}{dz} = \phi_x + i\psi_x = \phi_x - i\phi_y = u - iv, \tag{7}$$

where the first equality holds because $w(z)$ is analytic (see Theorem 22.5.1), and the second one holds because $\phi$ and $\psi$ satisfy the Cauchy–Riemann conditions. Thus, we recover the $x, y$ velocity components from $w(z)$ as $u = \operatorname{Re} w'(z)$ and $v = -\operatorname{Im} w'(z)$, respectively.

Let us denote

$$w(z(\zeta)) = W(\zeta) = \Phi(\xi,\eta) + i\Psi(\xi,\eta). \tag{8}$$

Then, by the same reasoning as used in (7),

$$\frac{dW}{d\zeta} = U - iV, \tag{9}$$

where $U(\xi,\eta)$ and $V(\xi,\eta)$ are the $\xi, \eta$ velocity components, respectively.

To illustrate the use of the complex potential $w$, let us rework Example 1 using $w$. In the $\zeta$ plane we have

$$\frac{dW}{d\zeta} = U(\xi,\eta) - iV(\xi,\eta) = U_0 - i0 \tag{10}$$

so $W = U_0\zeta$ (plus an arbitrary constant, which can be set equal to zero without loss). Returning to the $z$ plane via the mapping function $\zeta = z^2$, we have

$$w = U_0z^2 = U_0(x^2 - y^2) + i2U_0xy \tag{11}$$

and

$$\frac{dw}{dz} = 2U_0 z = 2U_0 x + i2U_0 y, \tag{12}$$

so the quantities of interest are $\phi(x,y) = U_0(x^2 - y^2)$, $\psi(x,y) = 2U_0 xy$, $u(x,y) = 2U_0 x$, and $v(x,y) = -2U_0 y$, as found in Example 1. Notice, in particular, that $u = v = 0$ at the corner ($x = y = 0$), which is therefore a *stagnation point*.

We consider just one more example, to further illustrate the use of the complex potential and to introduce the important Joukowski transformation.

**EXAMPLE 2.** *Flow Over Semicircular Bump.* We seek the plane potential flow over a semicircular bump of radius $a$, where the flow tends to a uniform stream $\mathbf{q} = U_0\hat{\mathbf{i}}$ as $r \to \infty$ along any $\theta =$ constant ray (Fig. 2). The boundary condition is that $\partial\phi/\partial n = 0$ all along the solid boundary $ABCEF$.

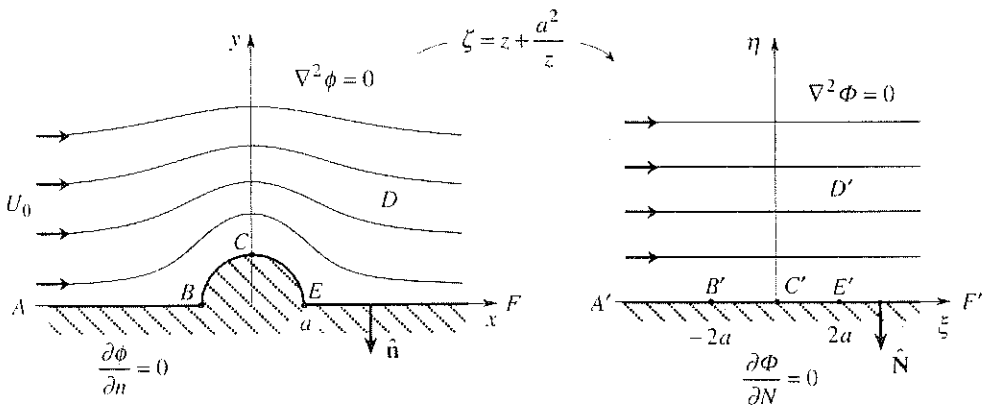



**Figure 2.** Flow over semicircular bump.

From Appendix F we find that the **Joukowski transformation**

$$\boxed{\zeta = z + \frac{a^2}{z}} \tag{13}$$

maps $D$, conformally, onto the upper half of the $\zeta$ plane (Fig. 2). Besides preserving the homogeneous Neumann boundary condition (because it is a conformal map), the Joukowski transformation also preserves the boundary condition at infinity,

$$\frac{dw}{dz} = u(x,y) - iv(x,y) \sim U_0 \qquad \text{as } z \to \infty, \tag{14}$$

because (13) gives $\zeta \sim z$ as $z \to \infty$. That is, it follows from

$$\frac{dw}{dz} = \frac{dW}{d\zeta}\frac{d\zeta}{dz} \sim \frac{dW}{d\zeta}, \tag{15}$$

and (14), that

$$\frac{dW}{d\zeta} \sim U_0 \qquad \text{as } \zeta \to \infty. \tag{16}$$

In fact, the flow in the $\zeta$ plane is simply a free stream

$$W = U_0\zeta, \tag{17}$$

not just at infinity but everywhere. Finally, putting (13) into (17) gives the desired complex potential

$$w = U_0\left(z + \frac{a^2}{z}\right). \tag{18}$$

Since the boundary of $D$ is made up of constant $r$ and constant $\theta$ curves it is convenient to put $z = re^{i\theta}$ (rather than $z = x + iy$) into (18) so

$$w = U_0\left(r + \frac{a^2}{r}\right)\cos\theta + iU_0\left(r - \frac{a^2}{r}\right)\sin\theta. \tag{19}$$

Thus,

$$\phi(r,\theta) = U_0\left(r + \frac{a^2}{r}\right)\cos\theta, \tag{20a}$$

$$\psi(r,\theta) = U_0\left(r - \frac{a^2}{r}\right)\sin\theta, \tag{20b}$$

and

$$\frac{dw}{dz} = u - iv = U_0\left(1 - \frac{a^2}{z^2}\right) = U_0\left(1 - \frac{a^2}{r^2}\cos 2\theta\right) + iU_0\frac{a^2}{r^2}\sin 2\theta \tag{21}$$

so the velocity components are

$$u(r,\theta) = U_0\left(1 - \frac{a^2}{r^2}\cos 2\theta\right), \tag{22a}$$

$$v(r,\theta) = -U_0\frac{a^2}{r^2}\sin 2\theta. \tag{22b}$$

COMMENT 1. Observe that (22) does satisfy the boundary condition at infinity: $u \sim U_0$ and $v \to 0$ as $r \to \infty$. And from (20b) we see that the solid boundary $ABCEF$ (Fig. 2) is indeed a streamline, namely $\psi = 0$: on $EF$ and $AB$ the $\sin\theta$ factor is zero, and on $BCE$ the $r - a^2/r$ is zero.

COMMENT 2. The closed form solution (20a) to the problem shown in Fig. 2 could also have been obtained by separation of variables.

COMMENT 3. We pointed out that the Joukowski transformation (13) preserves the flow at infinity, from the $z$ plane to the $\zeta$ plane, because (13) gives $\zeta \sim z$ as $z \to \infty$. It is interesting that we can think of (13) as the simplest (nontrivial) case of the more general transformation

$$\zeta = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$$

having that property. ∎

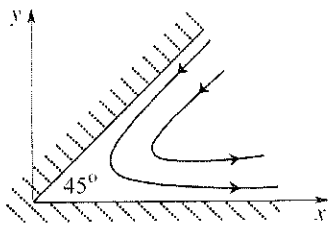
**Closure.** The key idea in this section is the use of the complex velocity potential $w(z) = \phi(x,y) + i\psi(x,y)$, rather than just $\phi(x,y)$. Once we find $w$, the $x$ and

$y$ velocity components, $u$ and $v$, are found from $dw/dz = \phi_x + i\psi_x = u - iv$, and the streamlines are found from $\operatorname{Im} w = \psi(x, y) =$ constant. We also meet the Joukowski transformation (13), which has the property of preserving the conditions at infinity because it simply gives $\zeta \sim z$ as $z \to \infty$. We will have more to say about that transformation in the exercises.

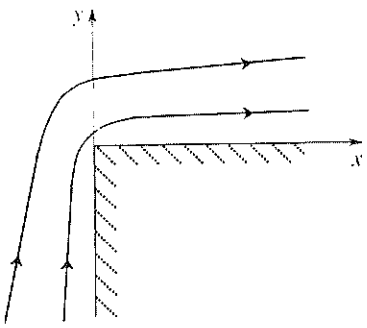
## EXERCISES 22.6

**1.** Given $\phi(x, y) = U_0(x^2 - y^2)$, derive the conjugate harmonic function given by (5).

**2.** (a) Instead of the 90° corner shown in Fig. 1, consider the 45° corner shown here. Solve for the corner flow sketched in the figure. That is, solve for $w(z)$, $\phi(x, y)$, $\psi(x, y)$, $u(x, y)$, and $v(x, y)$. Show whether or not the corner is a stagnation point, as it was for the 90° corner flow in Example 1.




(b) The same as part (a) but for the "outside corner" shown.




(c) The same as part (a) but for any corner angle $\alpha$ (radians). [In (a), for instance, $\alpha = \pi/4$.]
(d) In view of the results in part (a), would you say that the corner flow found in Example 1 is unique? Explain. HINT: A flow is possible for which the 45° ray $y = x$ is a streamline. Another flow is possible for which 30° and 60° rays are streamlines. And so on.

**3.** Use computer plotting software, such as the *Maple* implicitplot command, to obtain the $z$ plane flow pattern shown in Fig. 2, using four or five representative streamlines, somewhat as we have.

**4.** In each case find $w(z)$, $\phi(x, y)$, $\psi(x, y)$, $u(x, y)$, and $v(x, y)$.

(a) Solve for the downward and rightward potential flow between the positive $x$ and $y$ axes and the curve $xy = 6$ in the first quadrant.
(b) Solve for the downward potential flow in the upper half plane, between $xy = -1$, $xy = 4$, and the $x$ axis.

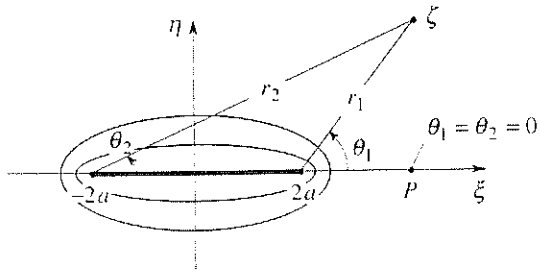
**5.** (*Joukowski transformation*) Show that the Joukowski transformation

$$\xi + i\eta = z + \frac{a^2}{z} \qquad (a > 0) \tag{5.1}$$

maps the family of circles $x^2 + y^2 = c^2$ onto the confocal ellipses

$$\frac{\xi^2}{(c + a^2/c)^2} + \frac{\eta^2}{(c - a^2/c)^2} = 1 \tag{5.2}$$

with foci at $\zeta = \pm 2a$ as shown in the figure. In particular,

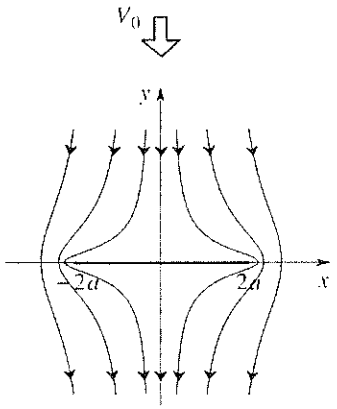



show that the circle $|z| = a$ maps onto the line segment $\eta = 0$, $|\xi| \le 2a$. Show, further, that the region $|z| > a$ maps onto the entire $\zeta$ plane minus that line segment, and that the region $|z| < a$ does too, so that the inverse transformation is double-valued. In fact, show that

$$z = \frac{\zeta + \sqrt{\zeta^2 - 4a^2}}{2}, \tag{5.3}$$

which is indeed double-valued because of the square root. Finally, show that with $\sqrt{\zeta^2 - 4a^2}$ defined by the branch cut shown in the figure (and extending from $-2a$ to $+2a$), the image of the slit $\zeta$ plane is $|z| > a$, whereas if we change the branch cut by defining $\theta_1 = 0$ and $\theta_2 = 2\pi$ at $P$, then the image is $|z| < a$. NOTE: For discussion of the finite branch cut, see the optional Section 21.4.7.

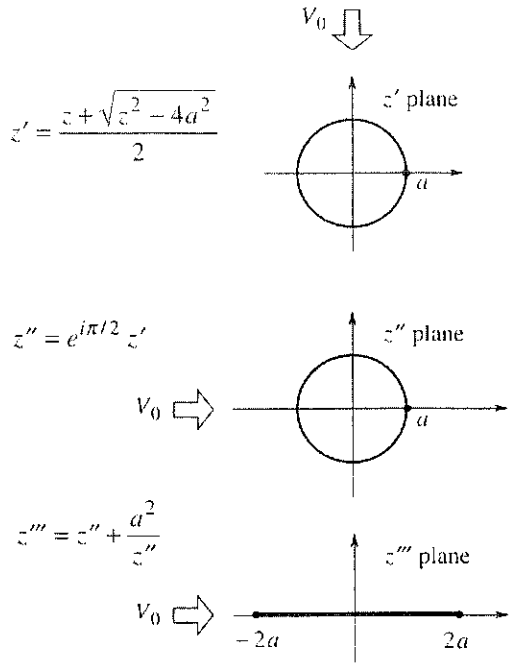
**6.** (*Flow perpendicular to flat plate*) (a) Solve for the potential flow around a thin flat plate that is perpendicular to a uniform stream $V_0$; that is, $dw/dz = u - iv \sim iV_0$ as $z \to \infty$. (Actually, that flow is not unique, but it is if we ask it to be symmetric about the $y$ axis, which symmetry we assume.) Specifically,




solve for $w(z)$. HINT: Use the mapping sequence suggested in the figure showing the $z'$, $z''$, and $z'''$ planes.
(b) Show that on the top of the plate $u = V_0 x/\sqrt{4a^2 - x^2}$ and $v = 0$, and on the bottom of the plate $u = -V_0 x/\sqrt{4a^2 - x^2}$ and $v = 0$.
(c) Show that $\psi(x, y)$ can be expressed in terms of $x$ and $y$ as

$$\psi(x,y) = \frac{V_0}{2}\left[\sqrt{(r_1 + r_1\cos\theta_1)(r_2 + r_2\cos\theta_2)} - \sqrt{(r_1 - r_1\cos\theta_1)(r_2 - r_2\cos\theta_2)}\right], \tag{6.1}$$

where $r_1 = \sqrt{(x-2a)^2 + y^2}$, $r_2 = \sqrt{(x+2a)^2 + y^2}$, $r_1\cos\theta_1 = x - 2a$, and $r_2\cos\theta_2 = x + 2a$. HINT: $\cos(A+B) = \cos A\cos B - \sin A\sin B$, $\cos\frac{A}{2} = \sqrt{(1+\cos A)/2}$, $\sin\frac{A}{2} = \sqrt{(1-\cos A)/2}$.




(d) Use computer plotting software, such as the *Maple* implicitplot command, to obtain the flow pattern in the $z$ plane. (Take $a = 1$, say.)

# Chapter 22 Review

A conformal map $w(z) = f(z) = u(x,y) + iv(x,y)$ is a change of variables from $x, y$ to $u, v$, that simplifies the given two-dimensional potential problem by simplifying the domain, while preserving the governing Laplace partial differential

equation. For $w = f(z)$ to be conformal, we need $f(z)$ to be analytic and $f'(z) \neq 0$ everywhere on the domain. Besides simplifying the domain while preserving the Laplace equation, such maps are conformal in the sense that they preserve angles, both in magnitude and sense, which property is often useful.

A virtue of conformal mapping is that it gives solutions in closed form. A disadvantage of the method, however, is that there does not exist a systematic procedure that inevitably leads us to a suitable mapping function $f(z)$. Rather, one needs to rely on some familiarity with various important maps such as the bilinear transformation, Joukowski transformation, some of the elementary functions, and so on. Fortunately, there exist extensive tables of conformal maps.

Thus, our program consists mainly of studying a number of different maps – the bilinear map in Section 22.3, and additional maps in Section 22.4. We noted that the bilinear map, which has the useful property of sending circles into circles, is the only map that is one-to-one so that, in general, one needs to be involved in selecting suitable branches of multi-valued functions.

Considering only the simplest boundary conditions through Section 22.4, constant boundary conditions of Dirichlet type, we show how to handle nonconstant Dirichlet conditions, and Neumann conditions as well, in Section 22.5.

Finally, in Section 22.6 we look specifically at applications of conformal mapping to problems in fluid mechanics, that case being just slightly different in that it is convenient, and traditional, to work with the complex potential $\phi(x, y) + i\psi(x, y)$, the $z$ derivative of which gives $u - iv$, where $u$ and $v$ are the $x, y$ velocity components, respectively.

# Chapter 23

# The Complex Integral Calculus

## 23.1 Introduction

In Chapters 21 and 22 we got as far as the differential calculus of functions of a complex variable, with special emphasis on the concept of analyticity. After applying the analytic function theory to the solution of two-dimensional potential problems by conformal mapping, in Chapter 22, we now pick up where we left off in Chapter 21 and turn to the complex integral calculus. We begin by defining the complex integral $\int_C f(z)\,dz$, and then proceed to derive three major integral theorems: the *Cauchy theorem*, the *fundamental theorem of the complex integral calculus*, and the *Cauchy integral formula*.

From the Cauchy integral formula we will derive *Taylor series* for functions of a complex variable, and a generalization of the latter known as *Laurent series*, in Chapter 24. Laurent series will enable us to categorize singularities into different types and to thereby understand them better. It also leads us, in the final section of Chapter 24, to the important *residue theorem* of the complex integral calculus.

It would be natural to expect the evaluation of complex integrals to be more difficult than the evaluation of real integrals. Thus, it is surprising to discover that it is often very simple, thanks to the extremely powerful integral theorems mentioned above.

## 23.2 Complex Integration

**23.2.1. Definition and properties.** Having studied the Riemann integral $\int_a^b f(x)\,dx$ of a real valued function $f$ on a real $x$ axis, in the calculus, we generalized that concept to line integrals in two or three dimensions, in Chapter 16. Here, we extend the concept once more and introduce the **complex integral**

$$I = \int_C f(z)\,dz \tag{1}$$

of a given function $f$ along a given oriented curve $C$ in the complex $z$ plane; $f$ may be analytic or not, and $C$ may be either a closed curve or an arc, but in either case

we shall assume that $C$ is piecewise smooth and simple.* The curve $C$ is called the *path of integration*, or *contour*. To define the integral (1), we stay as close as possible to the definition of line integrals developed in Chapter 16.

Suppose that $C$ has initial and final points $z = A$ and $z = B$, respectively. If $C$ is closed, then $B = A$. Divide $C$ into $n$ arcs by specifying points $z_0 = A$, $z_1, z_2, \ldots, z_{n-1}, z_n = B$ along $C$. Let the division be chosen arbitrarily, provided that the points $z_j$ are spaced and numbered so that the arc length from $A$ to $z_j$ is less than the arc length from $A$ to $z_k$ if $j < k$ (Fig. 1). Denote the $n$ arcs as $C_1, C_2, \ldots, C_n$, where the endpoints of $C_j$ are $z_{j-1}$ and $z_j$. On each $C_j$ choose some point $Q_j$ that is anywhere between the endpoints of $C_j$ or at one of the endpoints, and form the sum

$$J_n = \sum_{j=1}^{n} f(Q_j)\Delta z_j, \tag{2}$$

where $\Delta z_j = z_j - z_{j-1}$. The choice of the $z_j$'s and $Q_j$'s defines a *partition* of $C$, and we call the largest $|\Delta z_j|$ the *norm* of the partition. We introduce not just one partition but a sequence of them such that the norm of the $n$th partition tends to zero as $n \to \infty$. If the corresponding sequence of sums $J_1, J_2, \ldots$ converges to a limit, we call that limit the integral $\int_C f(z)\,dz$. We then say that the integral exists – that is, that it is *convergent*.

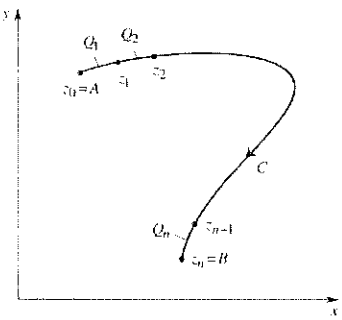



**Figure 1.** Partition of $C$.

From this definition it can be shown that complex integration is *linear* – that is,

$$\int_C [\alpha f(z) + \beta g(z)]\,dz = \alpha \int_C f(z)\,dz + \beta \int_C g(z)\,dz \tag{3}$$

for any scalars $\alpha$ and $\beta$ and for any functions $f$ and $g$ for which the three integrals exist, and that if we break $C$ into two parts, $C_1$ and $C_2$ (Fig. 2), then

$$\int_C f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{C_2} f(z)\,dz. \tag{4}$$

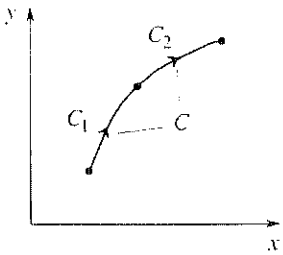



**Figure 2.** Breaking up the path.

Finally, if we reverse the orientation of $C$ and call the reverse path "$-C$," we have

$$\int_{-C} f(z)\,dz = -\int_C f(z)\,dz. \tag{5}$$

**EXAMPLE 1.** *Using the Limit Definition to Evaluate Integrals.* To illustrate the limit definition of the complex integral, consider the simple case where $f(z) = z$. Since we can choose each $Q_j$ point anywhere along its $C_j$ arc, let us choose each at the beginning of its arc: $Q_1$ at $z_0$, $Q_2$ at $z_1$, and so on. Then (2) becomes

$$J_n^b = z_0(z_1 - z_0) + z_1(z_2 - z_1) + \cdots + z_{n-2}(z_{n-1} - z_{n-2}) + z_{n-1}(z_n - z_{n-1}), \tag{6}$$

*These terms are defined in Section 15.2.

where the superscript $b$ simply denotes that the sum corresponds to choosing each $z_j$ at the beginning of its $C_j$ arc. We can just as well choose each $Q_j$ at the end of its $C_j$ interval, which choice gives

$$J_n^e = z_1(z_1 - z_0) + z_2(z_2 - z_1) + \cdots + z_{n-1}(z_{n-1} - z_{n-2}) + z_n(z_n - z_{n-1}). \tag{7}$$

Adding (6) and (7), most terms cancel, leaving

$$\begin{aligned} J_n^b + J_n^e &= z_n^2 - z_0^2 \\ &= B^2 - A^2. \end{aligned} \tag{8}$$

Finally, refining the partition over and over so that its norm tends to zero, the left-hand side of (8) tends to $2I$ so $I = (B^2 - A^2)/2$, or

$$I = \int_C z\,dz = \left.\frac{z^2}{2}\right|_A^B, \tag{9}$$

where $A$ and $B$ are the initial and final points of $C$, respectively. ∎

Why all the fuss? Doesn't $\int_C z\,dz = (z^2/2)|_A^B$ follow immediately from the fact that $d(z^2/2)/dz = z$? Yes it does, but we have not yet developed the fundamental theorem of the complex integral calculus, on which result that claim would be based; all we have, thus far, is the limit definition.

Of course, the limit definition is too unwieldy to be useful in evaluating more complicated integrals so we proceed to develop other lines of approach. In this regard it will be illuminating to distinguish two basically different strategies.

One strategy is to reduce $\int_C f(z)\,dz$ to one or more *real* integrals. To do so, one merely needs to re-express

$$\begin{aligned} \int_C f(z)\,dz &= \int_C (u + iv)(dx + i\,dy) \\ &= \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy). \end{aligned} \tag{10}$$

Thus, we can evaluate $\int_C f(z)\,dz$ by evaluating the two real line integrals

$$\int_C [u(x,y)\,dx - v(x,y)\,dy] \qquad \text{and} \qquad \int_C [v(x,y)\,dx + u(x,y)\,dy]$$

by methods developed earlier in Chapter 16. Let us illustrate with two examples.

**EXAMPLE 2.** Evaluate

$$I = \int_C z^2 dz, \tag{11}$$

where $C$ is the parabolic arc shown in Fig. 3. Parametrizing $C$ according to $y = \tau$, $x = 4 - \tau^2$, as $\tau$ goes from $+2$ to $-2$, we have

$$\begin{aligned} I &= \int_C [(x^2 - y^2)dx - 2xy\,dy] + i\int_C [2xy\,dx + (x^2 - y^2)dy] \\ &= \int_2^{-2} \left[(x^2 - y^2)\frac{dx}{d\tau} - 2xy\frac{dy}{d\tau}\right] d\tau + i\int_2^{-2} \left[2xy\frac{dx}{d\tau} + (x^2 - y^2)\frac{dy}{d\tau}\right] d\tau \\ &= \int_2^{-2} \left\{\left[(4-\tau^2)^2 - \tau^2\right](-2\tau) - 2(4-\tau^2)\tau\right\} d\tau \\ &\qquad + i\int_2^{-2} \left\{2(4-\tau^2)\tau(-2\tau) + \left[(4-\tau^2)^2 - \tau^2\right]\right\} d\tau \\ &= 0 + i\,\frac{16}{3} = \frac{16}{3}\,i. \end{aligned} \tag{12}$$

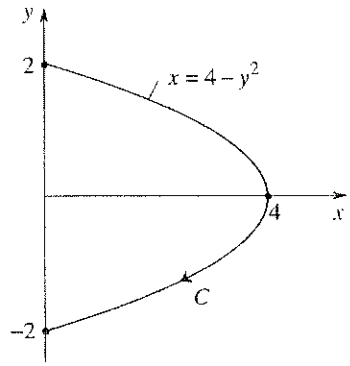



**Figure 3.** The path $C$ in (11).

COMMENT. It is natural to wonder whether we could have simplified the evaluation of $I$ by deforming $C$ into a simpler shape such as a straight line from the initial point $2i$ to the final point $-2i$. This same important question arose in Chapter 16 when we studied line integrals, and we found that path deformation (between fixed endpoints) is permissible if the given vector field is sufficiently well behaved. In view of the connection between $\int_C f(z)\,dz$ and real line integrals, displayed in (10), we expect that the same is true for the complex integral $\int_C f(z)\,dz$: the path $C$ can be deformed if $f(z)$ is sufficently well behaved. We defer discussion of this matter to Section 23.3. ∎

**EXAMPLE 3.** Evaluate

$$I = \oint_C (z - a)^n dz, \tag{13}$$

where $a$ is a given complex number, $n$ is any integer (positive, negative, or zero), and $C$ is a circle of radius $R$, centered at $z = a$ and oriented counterclockwise (Fig. 4). In this case $C$ is *closed*. Generally, we will use the $\oint_C$ notation when the contour $C$ is closed.

It is convenient to use the polar angle $\phi$ to parametrize $C$, by setting

$$z - a = Re^{i\phi}. \tag{14}$$

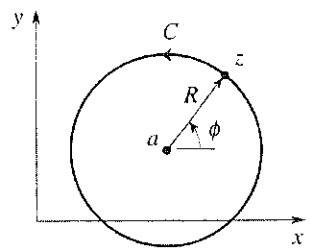



**Figure 4.** The closed contour $C$ in (13).

That is, the parametric equations of $C$ are, from the real and imaginary parts of (14), $x = \operatorname{Re} a + R\cos\phi$ and $y = \operatorname{Im} a + R\sin\phi$, as $\phi$ increases from 0 to $2\pi$. Then

$$\begin{aligned} I &= \int_0^{2\pi} (Re^{i\phi})^n (Rie^{i\phi}\,d\phi) = iR^{n+1}\int_0^{2\pi} e^{i(n+1)\phi}\,d\phi \\ &= \frac{R^{n+1}}{n+1}\,e^{i(n+1)\phi}\Big|_0^{2\pi} = 0, \end{aligned}$$

provided that $n \neq -1$ (in which case the latter yields the indeterminate form $0/0$). Treating $n = -1$ separately,

$$I = iR^0\int_0^{2\pi} e^{i0\phi}\,d\phi = i\int_0^{2\pi} d\phi = 2\pi i.$$

Thus

$$\oint_C (z-a)^n\,dz = \begin{cases} 2\pi i, & n=-1 \\ 0, & n \neq -1 \end{cases} \tag{15}$$

so $I = 0$ for all $n$'s except $n = -1$, in which case $I = 2\pi i$. ∎

The other possible strategy is to keep the complex integral intact and to develop theorems that will enable us to work with $\int_C f(z)\,dz$ directly. It will be found that this approach is by far the more fruitful and will be pursued exclusively for the remainder of this chapter and the next. Looking ahead, we shall obtain Cauchy's theorem, the fundamental theorem of the integral calculus, Cauchy's integral formula, and the residue theorem, in that order. These integral theorems will prove powerful indeed, and will also yield additional important results regarding function theory in general.

In that development we will often need to be able to "bound" complex integrals, that is, to obtain bounds on their absolute magnitude (modulus). Thus, in the remainder of this section we consider a simple upper bound that will suffice for most of our needs.

**23.2.2. Bounds.** First, observe from (2) that

$$|J_n| = \left|\sum_{j=1}^{n} f(Q_j)\Delta z_j\right| \leq \sum_{j=1}^{n} |f(Q_j)\Delta z_j| = \sum_{j=1}^{n} |f(Q_j)|\,|\Delta z_j|, \tag{16}$$

where the inequality and the final equality follow from Exercise 5(f) and 5(d), respectively, in Section 21.2. If there exists a real constant $M$ such that $|f(z)| \leq M$ everywhere on $C$, then $|f(Q_j)| \leq M$ for each $j$, and (16) gives

$$|J_n| \leq M \sum_{j=1}^{n} |\Delta z_j|. \tag{17}$$

As $n \to \infty$ and the norm of the sequence of partitions tends to zero, the sum on the right-hand side tends to $L$, the length of the (presumably rectifiable) curve $C$. Thus, the final result is the bound

$$\boxed{\left|\int_C f(z)\,dz\right| \leq ML,} \tag{18}$$

where $|f(z)| \leq M$ on $C$ and $L$ is the length of $C$. Subsequently, we shall refer to this result as the the **ML bound**.

**EXAMPLE 4.** Use (18) to bound the integral

$$I = \int_C \frac{e^z}{z^2}\,dz, \tag{19}$$

where $C$ is the straight line shown in Fig. 5. To obtain $M$, in (18), write

$$|f(z)| = \left|\frac{e^z}{z^2}\right| = \frac{|e^{x+iy}|}{|z^2|} = \frac{|e^x|\,|e^{iy}|}{|z|^2} = \frac{e^x}{x^2+y^2}. \tag{20}$$

On $C$, $e^x$ is a maximum at $Q$ and $x^2+y^2$ is a minimum at $P$ so we expect the maximum of $|f(z)|$ to lie somewhere between $P$ and $Q$. To find that point we put $y = 1 - \frac{1}{2}x$ into $e^x/(x^2+y^2)$, differentiate with respect to $x$, and set the derivative equal to zero. Those steps give $x = 4/5$. With $x = 4/5$ and $y = 3/5$, (20) gives $M = e^{4/5}/(\frac{16}{25} + \frac{9}{25}) = e^{4/5}$. And since $L = \sqrt{5}$, (18) gives

$$|I| \le e^{4/5}\sqrt{5} = 4.976. \tag{21}$$

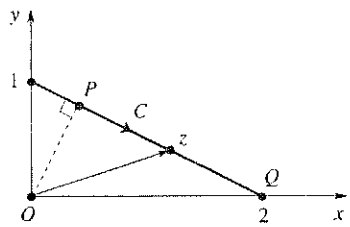



**Figure 5.** The contour $C$ in (19).

COMMENT. In this example we actually found the maximum value of $|f(z)|$ on $C$, and used that as $M$. In most applications, however, finding the maximum value of $|f(z)|$ on $C$ is so difficult that we are willing to accept a cruder value of $M$ – that is, a bound on $|f(z)|$ that is greater than the maximum value of $|f(z)|$. In the present example, for instance, the greatest value of the numerator is $e^2$ (at $Q$), and the smallest value of the denominator is the distance $OP$ squared, which, by similar triangles, is $(2/\sqrt{5})^2 = 4/5$, from which it follows that

$$|I| \le \left(\frac{e^2}{4/5}\right)\left(\sqrt{5}\right) = 20.65. \tag{22}$$

The trade-off is that (22) was more easily obtained than (21), but is somewhat cruder; we say that the inequality (22) is not as "sharp" as (21).* ∎

One more example:

**EXAMPLE 5.** Use (18) to bound the integral

$$I = \int_C \frac{\sin z}{z(z^2+9)}\,dz, \tag{23}$$

where $C$ is the circular contour shown in Fig. 6. In the spirit of the comment in Example 4, let it suffice to seek an upper bound on the numerator, $\sin z$, and a lower bound on the denominator, $z(z^2+9)$. On $C$ we have

$$\begin{aligned}
|\sin z| &= |\sin(x+iy)| \\
&= |\sin x\cosh y + i\cos x\sinh y| \\
&= \sqrt{\sin^2 x\cosh^2 y + \cos^2 x\sinh^2 y} \\
&\le \sqrt{\cosh^2 y + \sinh^2 y} \\
&\le \sqrt{\cosh^2 5 + \sinh^2 5} = \sqrt{\cosh 10},
\end{aligned} \tag{24}$$

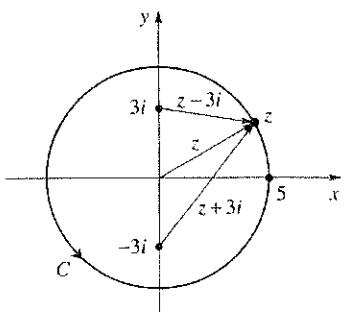



**Figure 6.** The contour $C$ in (23).

*The less sharp, the less informative. For instance, to say that the author owns less than a million neckties is less informative than saying that he owns less than seven, which is still less informative than saying that he owns exactly five.

$$|z| = 5, \tag{25}$$

$$\begin{aligned} |z^2 + 9| &= |(z - 3i)(z + 3i)| = |z - 3i|\,|z + 3i| \\ &\geq (2)(2) = 4, \end{aligned} \tag{26}$$

where the second inequality in (24) follows from the fact that $\cosh y$ and $\sinh y$ are monotonically increasing functions of $y$, and the last equality follows from the identity $\cosh(A + B) = \cosh A \cosh B + \sinh A \sinh B$ (Exercise 8 of Section 21.3). Further, the minimum values of $|z - 3i|$ and $|z + 3i|$ occur at $z = +5i$ and $-5i$, respectively, and equal 2. With $L = 2\pi(5) = 10\pi$, it follows from (24)–(26) and (18) that

$$|I| \leq \left(\frac{\sqrt{\cosh 10}}{(5)(4)}\right)(10\pi) = \frac{\pi\sqrt{\cosh 10}}{2} \tag{27}$$

is a bound on $I$. ∎

**Closure.** Beginning with the limit definition of the complex integral $I = \int_C f(z)\,dz$, we note that (as in the case of real integrals) that definition is not useful for the evaluation of such integrals, except in the simplest cases. As an alternative, we show that we can obtain the form $I = \int_C (u\,dx - v\,dy) + i\int_C (v\,dx + u\,dy)$, where the two line integrals can be evaluated by whatever methods were developed for line integrals in Chapter 16. But we promised more powerful approaches – the fundamental theorem of the complex integral calculus, Cauchy's theorem, the Cauchy integral formula, and the residue theorem, which are developed over the remainder of this chapter and the next.

We also show how to obtain a simple upper bound on $I$, the "$ML$ bound" $|I| \leq ML$, which is needed in the subsequent sections.

## EXERCISES 23.2

**1.** Evaluate the following by expressing them in terms of real line integrals and then evaluating those integrals.

(a) $\int_C |z|^2\,dz$, where $C$ is a straight line from $z = 0$ to $z = 1 + i$

(b) $\int_C \bar{z}\,dz$, where $C$ is the same as in part (a)

(c) $\int_C \bar{z}\,dz$, where $C$ is a clockwise semicircle from $z = 2$ to $z = -2$, centered at $z = 0$

(d) $\int_C dz/z$, where $C$ consists of three straight-line segments: from $z = 1$ to $z = 1 - i$, from $z = 1 - i$ to $z = -1 - i$, and then from $z = -1 - i$ to $z = -1$

(e) $\int_C e^z\,dz$, where $C$ consists of two straight-line segments: from $z = i$ to $z = 1 + i$, and then from $z = 1 + i$ to $z = 1 - 2i$

(f) $\int_C (\operatorname{Re} z)dz$, where $C$ is a clockwise quarter circle from $z = 3i$ to $z = 3$, centered at $z = 0$

(g) $\int_C (\operatorname{Im} z)dz$, where $C$ is a straight line from $z = i$ to $z = 2 + 2i$

**2.** Consider $I = \int_C \bar{z}\,dz$, where the initial and final points of $C$ are $z = 0$ and $z = 1 + i$, respectively. Show that the integral is *path dependent* by choosing two different paths and obtaining different values for $I$.

**3.** Use (18) to obtain an upper bound on $|I|$ for each given integral $I$. NOTE: Be aware that the answer is not unique, because we seek an inequality rather than an equality, just as both of the bounds (21) and (22) were correct in Example 4. Thus, it will be important for you to show your steps and reasoning.

(a) $\int_C z^5\,dz$, where $C$ is a straight line from $z = i$ to $z = 2 + 2i$

(b) $\int_C e^z\,dz$, where $C$ is a straight line from $z = -2$ to $z = 1 + 3i$

(c) $\int_C e^{-z}\,dz$, where $C$ is the same as in part (b)

(d) $\int_C dz/z$, where $C$ is a circle of radius 4 centered at $z = 2 + i$
(e) $\int_C dz/(z^2 + 1)$, where $C$ is a circle of radius 3 centered at $z = 0$
(f) $\int_C (z^2 + 1)dz/(z^2 - 1)$, where $C$ is a circle of radius 1 centered at $z = 1$
(g) $\int_C dz/[z(z + i)]$, where $C$ is a quarter circle from $z = i$ to $z = 1$, centered at $z = 0$
(h) $\int_C e^z\, dz/z$, where $C$ is a straight line from $z = i$ to $z = 2$
(i) $\int_C \cos z\, dz/z$, where $C$ is the same as in part (h)

**4.** Obtain, by any means, the maximum value of $|\sin z/[z(z^2 + 9)]|$, in Example 5, on the contour $C$ shown in Fig. 6, correct to two significant figures.

**5.** (a) If $C$ is a straight line from $z = 2$ to $z = 2 + (\pi/2)i$, show that

$$\left|\int_C \frac{dz}{e^z + 1}\right| \le \frac{\pi}{2}\,\frac{1}{\sqrt{e^4 + 1}}.$$

(b) If $C$ is a straight line from $z = 2i$ to $z = 3$, show that

$$\left|\int_C \frac{\cos z}{z}\, dz\right| \le \frac{13}{6}\cosh 2.$$

**6.** In our subsequent work we will sometimes need an upper bound on the absolute magnitude of a given integral, and (18) will generally suffice. A corresponding *lower* bound was not developed in this section because it will not be needed. Nevertheless, suppose that $|f(z)| \ge m$ on $C$. It is tempting to conjecture, by analogy with (18), that

$$\left|\int_C f(z)\, dz\right| \ge mL \tag{6.1}$$

provides a lower bound on the magnitude of the integral. Prove that (6.1) is *not* correct.

**7.** (*A sharper bound than the ML bound*) (a) Derive the upper bound

$$\boxed{\left|\int_C f(z)\, dz\right| \le \int_C |f(z)|\,|dz|,} \tag{7.1}$$

and show that

$$\left|\int_C f(z)\, dz\right| \le \int_C |f(z)|\,|dz| \le ML, \tag{7.2}$$

from which it follows that (7.1) is, in general, sharper than the $ML$ bound [i.e., closer to an equality than (18)]. NOTE: The price that we pay for this improvement is that (7.1) is generally more difficult to apply than (18) because it requires the evaluation of the (real) integral $\int_C |f(z)|\,|dz|$. For our purposes, in later sections, (18) will almost always suffice; only in one or two cases will the more refined bound (7.1) be needed. (b) To better understand these two bounds, apply each of them to the integral $I = \int_0^2 (x^3 - x)dx$, that is, $\int_C (z^3 - z)dz$ where $C$ is a straight line from $z = 0$ to $z = 2$. Compare the two results with the exact value of $I$. Further, show the graphical significance of each of the three members of (7.2), by suitable labeled sketches.

## 23.3 Cauchy's Theorem

Recall that the contour of integration may be open or closed. Cauchy's theorem, which we now derive, involves closed paths. Specifically, let $C$ be a piecewise smooth simple closed curve, and express

$$\oint_C f(z)\, dz = \oint_C (u\, dx - v\, dy) + i \oint_C (u\, dy + v\, dx). \tag{1}$$

Suppose that $f(z)$ is analytic and that $f'(z)$ is continuous in a simply connected domain $D$ containing the path $C$. If we write the first integral on the right-hand side of (1) as

$$\oint_C (u\, dx - v\, dy) = \oint_C \mathbf{w} \cdot d\mathbf{R}, \tag{2}$$

where $\mathbf{w} = u(x, y)\hat{\mathbf{i}} - v(x, y)\hat{\mathbf{j}}$ is a vector field and $d\mathbf{R} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} + dz\hat{\mathbf{k}}$ (where the $z$ in $dz$ is the real variable $z$, not $z = x + iy$), then

$$\nabla \times \mathbf{w} = 0\hat{\mathbf{i}} - 0\hat{\mathbf{j}} + \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\hat{\mathbf{k}} \tag{3}$$

vanishes everywhere in $D$ by virtue of the Cauchy–Riemann equation $\partial u/\partial y = -\partial v/\partial x$ (which holds because $f$ has been assumed analytic in $D$). Then it follows from Theorem 16.10.1 that

$$\oint_C (u\,dx - v\,dy) = 0. \tag{4}$$

Similarly, if we write the second integral on the right-hand side of (1) as

$$\oint_C (u\,dy + v\,dx) = \oint_C \mathbf{w} \cdot d\mathbf{R}, \tag{5}$$

where this time $\mathbf{w} = v(x, y)\hat{\mathbf{i}} + u(x, y)\hat{\mathbf{j}}$, then

$$\nabla \times \mathbf{w} = 0\hat{\mathbf{i}} - 0\hat{\mathbf{j}} + \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\hat{\mathbf{k}} \tag{6}$$

vanishes everywhere in $D$ by virtue of the other Cauchy – Riemann equation, $\partial u/\partial x = \partial v/\partial y$. Then it follows that

$$\oint_C (u\,dy + v\,dx) = 0. \tag{7}$$

Thus, we have shown that *if $f(z)$ is analytic and $f'(z)$ is continuous in a simply connected domain $D$, then*

$$\oint_C f(z)\,dz = 0 \tag{8}$$

*for every piecewise smooth simple closed curve $C$ in $D$.*

This result was published by Cauchy in 1825. Seventy-five years later it was shown by *Edouard Goursat* (1858–1936) that the stated result holds even if one does not assume the continuity of $f'(z)$ (i.e., of $u_x$, $u_y$, $v_x$, $v_y$).* Indeed, we will show (in Section 23.5) that if $f$ is analytic in $D$ (i.e., once differentiable), then $f$ possesses derivatives of *all* orders in $D$. In particular, since $f'$ is differentiable in $D$, it *follows* that $f'$ must be continuous there.

Thus, deleting the assumption that $f'(z)$ is continuous, we have the so-called "strong version" of Cauchy's theorem, sometimes called the Cauchy–Goursat theorem in recognition of Goursat's contribution:

*For proof, see, for example, R. V. Churchill, *Complex Variables and Applications*, 2nd ed. (New York: McGraw–Hill, 1960), Chap.5, or Konrad Knopp, *Theory of Functions*, Part I (New York: Dover, 1945), Chap. 4.

**THEOREM 23.3.1** *Cauchy's Theorem*
If $f(z)$ is analytic in a simply connected domain $D$, then

$$\boxed{\oint_C f(z)\,dz = 0} \tag{9}$$

for every piecewise smooth simple closed curve $C$ in $D$.

**EXAMPLE 1.** Consider

$$I = \oint_C \frac{dz}{z^2 - 5z + 6} = \oint_C \frac{dz}{(z-2)(z-3)}, \tag{10}$$

where $C$ is the unit circle, traversed counterclockwise (Fig. 1). The integrand $f(z) = 1/[(z-2)(z-3)]$ is analytic everywhere except at $z = 2$ and $z = 3$. Thus, it is analytic in the simply connected domain $D$ (Fig. 1) containing $C$ so $I = 0$ by Cauchy's theorem. ∎

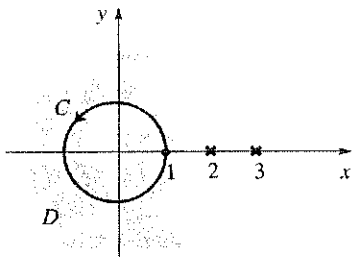



**Figure 1.** $C$ and $D$ for Example 1.

We have already mentioned the connection between the complex integral $\int_C f(z)\,dz$ of an analytic function $f$ and the real line integral $\int_C \mathbf{w}\cdot d\mathbf{R}$ of an irrotational field $\mathbf{w}$. Referring once more to Theorem 16.10.1, notice the equivalence between the vanishing of $\int_C \mathbf{w}\cdot d\mathbf{R}$ for every closed loop $C$ and the path independence of integrals $\int_C \mathbf{w}\cdot d\mathbf{R}$ between fixed endpoints. Thus, we might well anticipate such equivalence for complex integrals as well. In fact, we do have the following corollary to Theorem 23.3.2:

**THEOREM 23.3.2** *Path Independence*
If $f(z)$ is analytic in a simply connected domain $D$, then $\int_C f(z)\,dz$ is independent of path in $D$. That is, given any initial point $P$ in $D$ and any final point $Q$ in $D$, the value of $\int_C f(z)\,dz$ is the same for every piecewise smooth path $C$, lying entirely within $D$, from $P$ to $Q$.

*Partial Proof*: Let $C_1$ and $C_2$ be any two piecewise smooth paths from $P$ to $Q$. Suppose that both $C_1$ and $C_2$ are simple curves and that they intersect each other only at the endpoints $P$ and $Q$ (Fig. 2a). (The additional argument needed for the case where intersections do occur, possibly an infinite number of them, is omitted here.) Then $C_1 + (-C_2)$ is a piecewise smooth simple closed curve in $D$ (Fig. 2b) so, according to Cauchy's theorem,

$$\oint_{C_1+(-C_2)} f(z)\,dz = 0. \tag{11}$$

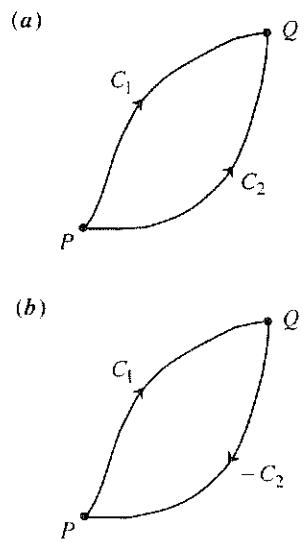



**Figure 2.** You take the high road, I'll take the low.

But

$$\oint_{C_1+(-C_2)} f(z)\,dz = \int_{C_1} f(z)\,dz + \int_{-C_2} f(z)\,dz$$
$$= \int_{C_1} f(z)\,dz - \int_{C_2} f(z)\,dz, \tag{12}$$

and it follows from (11) and (12) that

$$\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz,$$

as claimed. This step completes the proof, except for the omission noted above. ■

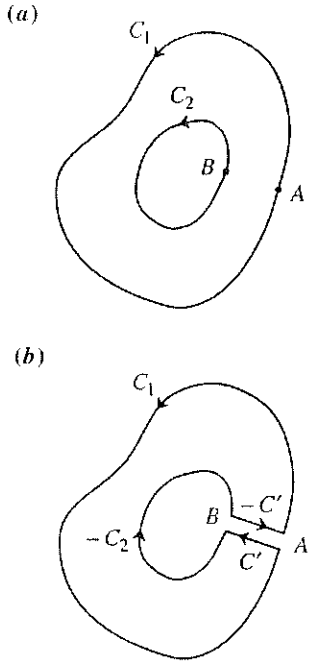



**Figure 3.** Path deformation for closed path.

It is useful to think of path independence in terms of a process of *path deformation*. That is, we can imagine deforming $C_1$, continuously, into $C_2$ (Fig. 2a), keeping the endpoints fixed. If $f$ is analytic on $C_1$ and $C_2$ and we cross no singular points in the process (thus $f$ is analytic on and between $C_1$ and $C_2$), then $\int_{C_1} f(z)\,dz$ is equal to $\int_{C_2} f(z)\,dz$. The point, of course, is that we may be able to simplify the integral by such deformation in the sense that $\int_{C_2} f(z)\,dz$ may be more easily evaluated than $\int_{C_1} f(z)\,dz$.

In fact, path deformation can be applied to closed paths as well. For example, suppose that $f$ is analytic on and between the closed paths $C_1$ and $C_2$ shown in Fig. 3a. Let us "slit" the region by introducing a piecewise smooth curve connecting a point $A$ on $C_1$ with a point $B$ on $C_2$ (with this curve intersecting $C_1$ only at $A$ and $C_2$ only at $B$). Denote this curve as $C'$ when it is oriented from $A$ to $B$ and as $-C'$ when it is oriented from $B$ to $A$ as shown in Fig. 3b. (Actually, $C'$ and $-C'$ are coincident, but we have separated them in the figure for clarity.) Since $C = C_1 + C' + (-C_2) + (-C')$ is a piecewise smooth simple closed curve and $f$ is analytic inside and on $C$, it follows from Cauchy's theorem that $\oint_C f(z)\,dz = 0$ (see Exercise 3). Thus,

$$\oint_C f(z)\,dz = \oint_{C_1+C'+(-C_2)+(-C')} f(z)\,dz$$
$$= \oint_{C_1} f(z)\,dz + \int_{C'} f(z)\,dz + \oint_{-C_2} f(z)\,dz + \int_{-C'} f(z)\,dz$$
$$= \oint_{C_1} f(z)\,dz + \int_{C'} f(z)\,dz - \oint_{C_2} f(z)\,dz - \int_{C'} f(z)\,dz$$
$$= 0, \tag{13}$$

so

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz. \tag{14}$$

That is, deforming $C_1$ into $C_2$, (14) will hold if $f$ is analytic between and on $C_1$ and $C_2$.

**EXAMPLE 2.** *An Important Little Integral.* Consider

$$I = \oint_C (z-a)^n \, dz, \tag{15}$$

where $C$ is any piecewise smooth simple closed curve, oriented counterclockwise, such as the curve $C$ shown in Fig. 4; $n$ is any integer (positive, negative, or zero), and $a$ is any given complex number lying inside $C$. According to the discussion preceding this example, we can deform the contour $C$ to a *circular* contour $C'$, lying wholly within $C$, as depicted in Fig. 4 since $(z-a)^n$ is analytic on and between $C$ and $C'$. The resulting integral, $\oint_{C'} (z-a)^n \, dz$, is evaluated in Example 3 in Section 23.2. Recalling the result of that evaluation, we have

$$\boxed{\oint_C (z-a)^n \, dz = \begin{cases} 2\pi i, & n = -1 \\ 0, & n \neq -1. \end{cases}} \tag{16}$$

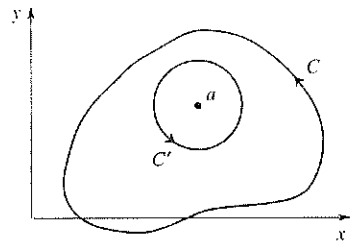



**Figure 4.** Deformation of $C$ to a circle $C'$.

That is, the result obtained in Example 3 of Section 23.2 holds even if $C$ is not a circle centered at $a$. Although rather specific, formula (16) will be used often enough so it is worth remembering. ∎

**EXAMPLE 3.** Evaluate

$$I = \oint_C \frac{dz}{z^2(z-2)(z-4)}, \tag{17}$$

where $C$ is given in Fig. 5. Expanding the integrand in partial fractions yields

$$I = \frac{3}{32}\oint_C \frac{dz}{z} + \frac{1}{8}\oint_C \frac{dz}{z^2} - \frac{1}{8}\oint_C \frac{dz}{z-2} + \frac{1}{32}\oint_C \frac{dz}{z-4}. \tag{18}$$

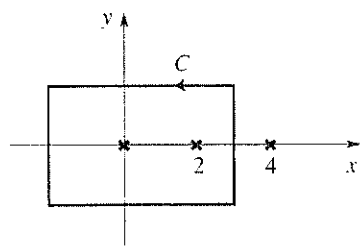



**Figure 5.** The contour $C$ in (17).

Thus

$$I = \frac{3}{32}(2\pi i) + \frac{1}{8}(0) - \frac{1}{8}(2\pi i) + \frac{1}{32}(0) = -\frac{\pi i}{16}, \tag{19}$$

because the last integral in (18) is zero by Cauchy's theorem, and the first three are covered by the "important little integral" (16). ∎

**Closure.** Roughly put, Cauchy's theorem tells us that $\oint_C f(z)\,dz = 0$ if $f(z)$ is analytic within a region containing $C$, and it follows from that result that if $C_1$ and $C_2$ are open curves with the same initial point and the same final point, then $\int_{C_1} f(z)\,dz = \int_{C_2} f(z)\,dz$ if $f(z)$ is analytic within a region containing $C_1$ and $C_2$. That is, we can deform $C_1$ into $C_2$ without changing the value of the integral. These results, Cauchy's theorem and its path independence corollary, are fundamental to the further development of the complex integral calculus in this chapter and the

next. The connection between them and the irrotational field theorem (Theorem 16.10.1) should be understood.

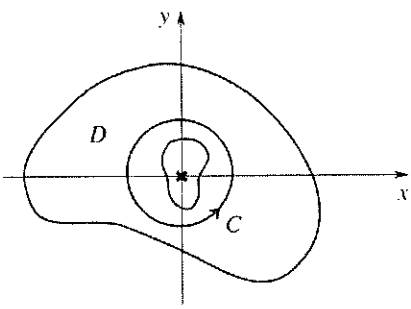
## EXERCISES 23.3

**1.** According to Example 2,

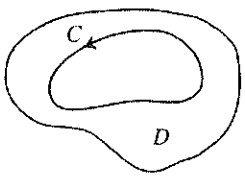
$$\oint_C \frac{dz}{z^2} = 0,$$

where $C$ is a counterclockwise circle of radius $R$, centered at the origin. Yet $f(z) = 1/z^2$ is not analytic within $C$; it is singular at $z = 0$. Explain why this result does not violate Cauchy's theorem.

**2.** Consider $I = \oint_C dz/z$, where $C$ is the counterclockwise unit circle, and the assertion that $I = 0$, from Cauchy's theorem, because $f(z) = 1/z$ is analytic in the domain $D$ containing $C$. (See the accompanying figure.) Yet we show in Example 2 that $I = 2\pi i$. Explain the apparent contradiction.




**3.** Prove that Cauchy's theorem can be stated a bit more simply, as follows: *If $C$ is a piecewise smooth simple closed curve and $f(z)$ is analytic inside and on $C$, then $\oint_C f(z)\,dz = 0$.* HINT: The idea is to show that if $f$ is analytic inside and on $C$, then there does exist a domain $D$ (see the figure) containing




$C$, such that $f$ is analytic within $D$. The desired result follows immediately from Cauchy's theorem. If needed, you may draw on the fact that $C$ is necessarily rectifiable (i.e., of finite length) because it is piecewise smooth. Also, recall from the calculus that if $F(x)$ is continuous on a closed interval $a \le x \le b$, then it has an absolute maximum and an absolute minimum on that interval.

**4.** Let $C_1, C_2, C_3$ be the following simple closed curves:

$C_1$: $|z| = 1$, counterclockwise
$C_2$: $|z| = 1$, clockwise
$C_3$: the square with vertices at $1 - i$, $1 + i$, $-1 + i$, $-1 - i$, counterclockwise.

Evaluate each of the following integrals using Cauchy's theorem if applicable, or any other method studied in Sections 23.2 or 23.3.

(a) $\oint_{C_1} \operatorname{Re} z\,dz$
(b) $\oint_{C_1} \operatorname{Im} z\,dz$
(c) $\oint_{C_3} \operatorname{Im} z\,dz$
(d) $\oint_{C_3} \dfrac{dz}{z^2 - 3}$
(e) $\oint_{C_1} \dfrac{dz}{z^4}$
(f) $\oint_{C_1} \dfrac{dz}{z(z-2)}$
(g) $\oint_{C_2} \dfrac{dz}{z(z+5)}$
(h) $\oint_{C_1} e^{\sin z}\,dz$
(i) $\oint_{C_2} \sin(\cos z)\,dz$
(j) $\oint_{C_3} \dfrac{dz}{|z|}$
(k) $\oint_{C_1} \overline{z}\,dz$
(l) $\oint_{C_3} \overline{z}\,dz$

**5.** Can we use path deformation to obtain

$$\oint_{C_3} \overline{z}\,dz = \oint_{C_1} \overline{z}\,dz, \tag{5.1}$$

where $C_1$ and $C_3$ are defined in Exercise 4? Explain.

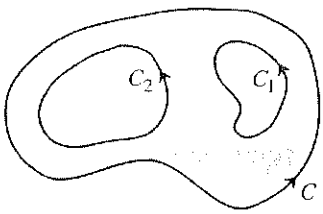
**6.** Evaluate $\int_C z^{20}\,dz$, where $C$ is the path

(a) $y = x - x^3$, from $x = 0$ to $x = 1$
(b) $y = x - x^3$, from $x = 0$ to $x = -1$
(c) $x = y - y^5$, from $y = 0$ to $y = 1$

**7.** Evaluate $\int_C \overline{z}\,dz$, where $C$ is a straight line from $z = 0$ to $z = 1 + i$, where $C$ is the parabola $y = x^2$ from $z = 0$ to $z = 1 + i$, and where $C$ is the rectilinear path from $z = 0$ to $z = 1$ to $z = 1 + i$. Are the answers the same? Is there any violation of Theorem 23.3.2? Explain.

**8.** (*Path deformation in multiply-connected domain*) Show that if $f(z)$ is analytic in the shaded region between and on the contours $C, C_1, C_2$ (see the accompanying figure), then

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz. \qquad (8.1)$$




You may use the result stated in Exercise 3. NOTE: More generally,

$$\oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \cdots + \oint_{C_n} f(z)\,dz$$

if $C_1, \ldots, C_n$ are nonintersecting counterclockwise closed contours within $C$, and $f$ is analytic between and on $C, C_1, \ldots, C_n$.

**9.** Evaluate the following integrals, where in each case $C$ is the circle $|z| = 3$, counterclockwise.

(a) $\oint_C \dfrac{dz}{z(z-1)}$ (b) $\oint_C \dfrac{dz}{z(z-5)}$

(c) $\oint_C \dfrac{z\,dz}{z^2+1}$ (d) $\oint_C \dfrac{z\,dz}{z^2-3z+2}$

(e) $\oint_C \dfrac{dz}{z^3(z^2-1)}$ (f) $\oint_C \dfrac{z\,dz}{z^3-1}$

## 23.4 Fundamental Theorem of the Complex Integral Calculus

Before obtaining the fundamental theorem of the complex integral calculus, let us, for the sake of comparison and motivation, recall the *fundamental theorem of the real integral calculus*: let $f(x)$ be continuous on the interval $x_0 \le x \le x_1$. Then the function

$$G(x) = \int_{x_0}^{x} f(\xi)\,d\xi \qquad (x_0 \le x \le x_1) \qquad (1)$$

is differentiable on $x_0 \le x \le x_1$ and is a *primitive* or *indefinite integral* of $f$; that is,

$$G'(x) = f(x) \qquad (2)$$

on that closed interval. The primitive of $f$ is unique to within an arbitrary additive constant. Finally, if $F(x)$ is any particular primitive of $f$, then

$$\int_{x_0}^{x} f(\xi)\,d\xi = F(x) - F(x_0). \qquad (3)$$

The foregoing result is found, in the elementary calculus, to be especially valuable for evaluating integrals for which a primitive can be found by inspection. For example, to evaluate $\int_2^5 x^2\,dx$ one notices that $x^3/3$ is a primitive of $x^2$ [because $d(x^3/3)/dx = x^2$] so

$$\int_2^5 x^2\,dx = \left.\frac{x^3}{3}\right|_2^5 = \frac{125-8}{3} = \frac{117}{3}.$$

Of course, the primitive $(x^3/3) + 4$, say, would have produced the same result.

Turning to the complex case, let $f(z)$ be analytic in a simply-connected domain $D$. If $z_0$ is a fixed point in $D$, then $\int_{z_0}^{z} f(\zeta)\,d\zeta$ is (Theorem 23.3.2) path independent and hence defines a single-valued function of $z$,* which we denote as $G(z)$:

$$G(z) = \int_{z_0}^{z} f(\zeta)\,d\zeta. \tag{4}$$

(Before continuing, notice that in the complex case we have asked $f$ to be analytic, for the integral to define a single-valued function, whereas in the real case it sufficed to ask $f$ to be continuous.)

Next, we show that $G'(z) = f(z)$ in $D$, analogous to equation (2) for the real case. Consider the difference quotient

$$\begin{aligned}\frac{G(z+\Delta z) - G(z)}{\Delta z} &= \frac{1}{\Delta z}\left[\int_{z_0}^{z+\Delta z} f(\zeta)\,d\zeta - \int_{z_0}^{z} f(\zeta)\,d\zeta\right] \\ &= \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(\zeta)\,d\zeta \\ &= \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(z)\,d\zeta + \frac{1}{\Delta z}\int_{z}^{z+\Delta z} [f(\zeta) - f(z)]\,d\zeta \\ &= f(z) + \frac{1}{\Delta z}\int_{z}^{z+\Delta z} [f(\zeta) - f(z)]\,d\zeta, \end{aligned} \tag{5}$$

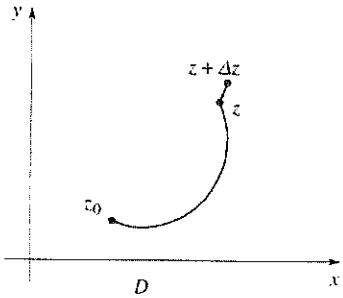



**Figure 1.** $z_0$, $z$, $z + \Delta z$, and the path.

where $z_0$, $z$, and $z + \Delta z$ are displayed in Fig. 1. By the above-noted path independence, there is no loss in taking the path from $z$ to $z + \Delta z$ to be straight (although the path from $z_0$ to $z$ may need to be curved in order to remain within $D$). Using the $ML$ bound on the last integral in (5), we have

$$\left|\frac{1}{\Delta z}\int_{z}^{z+\Delta z} [f(\zeta) - f(z)]\,d\zeta\right| \le \frac{1}{|\Delta z|} M\,|\Delta z| = M, \tag{6}$$

where $M = \max|f(\zeta) - f(z)|$ on the line from $z$ to $z + \Delta z$. Since $f$ is continuous (because it is analytic), $M \to 0$ as $\Delta z \to 0$. Thus, letting $\Delta z \to 0$ in (5) gives

$$G'(z) = f(z), \tag{7}$$

as claimed.

Any function $F(z)$ satisfying $F'(z) = f(z)$ is called an **indefinite integral** or **primitive** of $f$. It is easy to show (Exercise 1) that any two primitives corresponding to a given $f$ differ at most by an arbitrary additive constant. Thus, if $F(z)$ is any particular primitive of $f(z)$, then

$$G(z) = \int_{z_0}^{z} f(\zeta)\,d\zeta = F(z) + C, \tag{8}$$

*This is a key point. Recall that a function is to be single-valued. If $f$ is not analytic in $D$ then there is no guarantee (because Theorem 23.3.2 calls for analyticity) that it will be single-valued. See Example 2.

where $C$ is a constant. To evaluate $C$, set $z = z_0$. Then

$$0 = F(z_0) + C \tag{9}$$

gives $C = -F(z_0)$ so

$$\int_{z_0}^{z} f(\zeta)\, d\zeta = F(z) - F(z_0), \tag{10}$$

analogous to equation (3) for the real case.

Pulling these results together, we have:

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**THEOREM 23.4.1** *Fundamental Theorem of the Complex Integral Calculus*
Let $f(z)$ be analytic in a simply-connected domain $D$, and let $z_0$ be any fixed point in $D$. Then

(i) $G(z) = \int_{z_0}^{z} f(\zeta)\, d\zeta$ is analytic in $D$ and $G'(z) = f(z)$.

(ii) If $F(z)$ is any primitive of $f(z)$ [i.e., $F'(z) = f(z)$], then

$$\boxed{\int_{z_0}^{z} f(\zeta)\, d\zeta = F(z) - F(z_0).} \tag{11}$$

---

As in the analogous case of real integrals, (11) is especially useful for evaluating integrals for which a primitive can be found by inspection.

**EXAMPLE 1.** To evaluate $\int_{2i}^{3} \sin z\, dz$, observe that $F(z) = -\cos z$ (plus any constant) is a primitive of $\sin z$. Then

$$\int_{2i}^{3} \sin z\, dz = -\cos z\Big|_{z=2i}^{z=3} = -\cos 3 + \cos 2i = \cosh 2 - \cos 3. \tag{12}$$

COMMENT. Notice that the notation $\int_{2i}^{3}$ suffices. That is, we do not need to specify the path to be followed, from $2i$ to 3, because $\sin z$ is analytic in the whole plane, and hence line integrals of $\sin z$ are path independent – they depend only on the endpoints. ∎

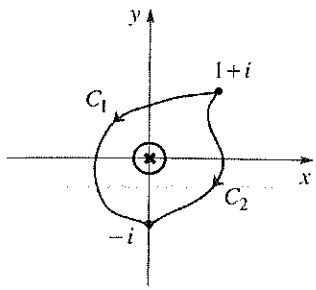


**Figure 2.** Alternative paths.

**EXAMPLE 2.** Evaluate

$$I = \int_{1+i}^{-i} \frac{dz}{z}. \tag{13}$$

With the foregoing comment in mind, we observe, first, that the problem is not clearly posed, because the integral is *not* single-valued. Specifically, the integrand $1/z$ is analytic for all $z \neq 0$, and the whole plane with $z = 0$ deleted is a multiply connected region (due to the hole at $z = 0$). Thus our path independence theorem (Theorem 23.3.2) does not guarantee that $I$ will have the same values for paths such as $C_1$ and $C_2$ in Fig. 2. In fact,

$$\int_{C_1} \frac{dz}{z} - \int_{C_2} \frac{dz}{z} = \int_{C_1+(-C_2)} \frac{dz}{z} = 2\pi i, \tag{14}$$

according to the "important little integral" in Section 23.3 so

$$\int_{C_1} \frac{dz}{z} \neq \int_{C_2} \frac{dz}{z}. \tag{15}$$

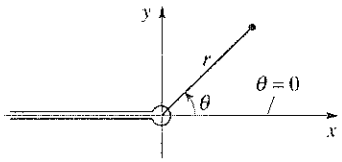


**Figure 3.** Branch cut for $\log z$.

Sure enough, this multi-valuedness of the integral shows up if we write

$$I = \int_{1+i}^{-i} \frac{dz}{z} = \log z\Big|_{1+i}^{-i}, \tag{16}$$

because $\log z$ is multi-valued! To render $I$ single-valued we need to render the $\log z$ single-valued, and we do that, as usual, by a branch cut. If, for instance, we choose to adopt the branch cut shown in Fig. 3, then (16) gives

$$I = \log(re^{i\theta})\Big|_{1+i}^{-i} = (\ln r + i\theta)\Big|_{r=\sqrt{2},\,\theta=\pi/4}^{r=1,\,\theta=-\pi/2} = -\frac{\ln 2}{2} - \frac{3\pi}{4}\, i \tag{17}$$

as the unique value of $I$. ∎

**Closure.** The purpose of this brief section is to establish the fundamental theorem of the complex integral calculus, Theorem 23.4.1. Although analogous to the familiar real variable result, we do note that whereas the real variable theorem merely asks the integrand $f(x)$ to be continuous, the complex variable theorem asks $f(z)$ to be analytic. As a simple illustration of Theorem 23.4.1, we can say that

$$\int_0^z \zeta^2\, d\zeta = \frac{\zeta^3}{3}\Big|_0^z = \frac{z^3}{3}$$

for all $z$, because $d(z^3/3)/dz = z^2$ is analytic for all $z$.

## EXERCISES 23.4

**1.** Prove the assertion, stated below equation (7), that "any two primitives corresponding to a given $f$ differ at most by an arbitrary additive constant." HINT: Let $F_1(z)$ and $F_2(z)$ be primitives of $f(z)$ so that $F_1'(z) = f(z)$ and $F_2'(z) = f(z)$. Subtract the latter two equations.

**2.** Give three different primitives for each of the following functions.

(a) $z$ (b) $z^5$ (c) $e^{2z} - z$ (d) $\cos(z-2)$

**3.** Use the fundamental theorem to evaluate each of the following.

(a) $\int_0^i z\, dz$ (b) $\int_{-i}^i z^4\, dz$
(c) $\int_4^{-1-2i} (e^{-z} - 3z^2)\, dz$ (d) $\int_i^0 \cos 3z\, dz$
(e) $\int_{1-i}^{1+i} ze^z\, dz$ (f) $\int_4^{-i} z \sin z\, dz$
(g) $\int_0^{3i} ze^{z^2}\, dz$ (h) $\int_0^{1+2i} \sin^2 z\, dz$
(i) $\int_{-2i}^3 z \cos 2z\, dz$ (j) $\int_i^{2+i} \cosh 3z\, dz$
(k) $\int_0^i \cos^3 z\, dz$ (l) $\int_i^1 z^2 e^{2z^3}\, dz$

**4.** Determine all possible values of

$$I = \int_{1-i}^{1+i} \frac{dz}{z(z-1)}.$$

**5.** At some point in the calculus one is likely to encounter the calculation

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \lim_{\substack{A\to\infty \\ B\to\infty}} \tan^{-1} x \Big|_{x=-A}^{x=B} = \pi. \qquad (5.1)$$

Surely the integral is uniquely determined, so it is interesting that the multi-valued $\tan^{-1}(\,)$ enters the picture. However, no matter which continuous branch of $\tan^{-1}(\,)$ is chosen [e.g., $-\pi/2 < \tan^{-1}(\,) < \pi/2$, $\pi/2 < \tan^{-1}(\,) < 3\pi/2$, etc.] the unique result, $\pi$, is indeed obtained. For study purposes, it may be useful to reexamine this evaluation in the light of complex variable theory. Then

$$I = \int_C \frac{dz}{z^2+1} = \lim_{\substack{A\to\infty \\ B\to\infty}} \tan^{-1} z \Big|_{z=-A}^{z=B}, \qquad (5.2)$$

where $C$ is a straight line path from $-A$ to $B$ along the real axis. To evaluate the right-hand member of (5.2), show that

$$\tan^{-1} z = \frac{1}{2i} \log \frac{i-z}{i+z}, \qquad (5.3)$$

so that

$$I = \int_C \frac{dz}{z^2+1} = \lim_{\substack{A\to\infty \\ B\to\infty}} \frac{1}{2i} \log \frac{i-z}{i+z} \Big|_{z=-A}^{z=B} \qquad (5.4)$$

[or, equivalently, derive (5.4) by applying the method of partial fractions to the integral]. Finally, evaluate the right-hand member of (5.4) by introducing any suitable branch cuts for the $\log(i-z)$ and $\log(i+z)$ functions, and show that you do obtain $\pi$, as in (5.1).

## 23.5 Cauchy Integral Formula

Recall that Cauchy's theorem is based upon analyticity of the integrand. Our purpose in the present section is to begin our consideration of the more typical case, where the integrand is singular at one or more points within the contour.

Let $f(z)$ be analytic in a simply connected domain $D$, let $C$ be any piecewise smooth simple closed curve in $D$, oriented counterclockwise, and consider the integral

$$I = \oint_C \frac{f(z)}{z-a}\, dz, \qquad (1)$$

where $a$ is any fixed point within $C$ (Fig. 1). That is, rather than a vague statement such as "Consider $\oint_C f(z)\,dz$ where $f(z)$ is singular somewhere within $C$," we consider the integrand to have a specific kind of singularity [namely, a $1/(z-a)$ behavior, which "blows up" as $z \to a$], and we make its presence explicit by writing the integrand as $f(z)/(z-a)$. Other types of singularity will be considered later.

To evaluate $I$, we begin by deforming $C$ to a circular contour $C'$ of radius $\rho$, sufficiently small so that $C'$ lies entirely inside* $C$, as shown in Fig. 1. The

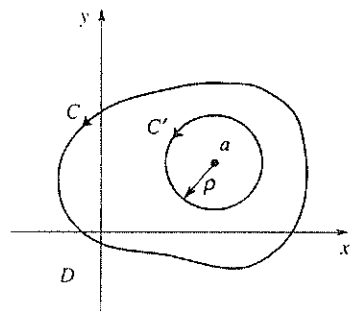


**Figure 1.** $C$ in (1). $D$ chosen as rectangular for simplicity.

*We often speak of the region *inside* or *outside* a given closed curve. Although there is little chance of misunderstanding in this regard, we wish to point out that to render the notion of inside and outside precise is not at all simple, and there exists a sophisticated result known as the **Jordan curve theorem** which clarifies the matter in a rigorous way. Discussion of this theorem lies well outside our present scope and can be found in R. N. Pederson, "The Jordan Curve Theorem for Piecewise Smooth Curves," *American Mathematical Monthly*, Vol. 76, 1969, pp. 605–610, or in G. N. Watson, *Complex Integration and Cauchy's Theorem*, Cambridge Tracts No. 15, 1914, Chap. 1.

deformation is justified since the integrand $f(z)/(z-a)$ is analytic between and on $C$ and $C'$.

Understand that $I = \oint_{C'} f(z)\,dz/(z-a)$ is independent of $\rho$, provided of course that $C'$ stays within $D$. Thus, we can let $\rho \to 0$, which is convenient because then $f(z) \sim f(a)$ on $C'$ and the integral simplifies to $I = \oint_{C'} f(a)\,dz/(z-a) = f(a)\oint_{C'} dz/(z-a) = f(a)(2\pi i)$ since $\oint_{C'} dz/(z-a) = 2\pi i$ according to our "important little integral." To prove this result rigorously, re-express (1) as

$$\begin{aligned} I &= \oint_C \frac{f(z)}{z-a}\,dz = \oint_{C'} \frac{f(z)}{z-a}\,dz \\ &= \oint_{C'} \frac{f(a)}{z-a}\,dz + \oint_{C'} \frac{f(z)-f(a)}{z-a}\,dz \\ &= f(a)(2\pi i) + \oint_{C'} \frac{f(z)-f(a)}{z-a}\,dz. \end{aligned} \tag{2}$$

The third equality is simply an identity, rigged so that the first integral on the right is the anticipated final result and the second integral is a "deviation term." By letting $\rho \to 0$ in (2), we expect to be able to show that the deviation term is zero. Specifically, the $ML$ bound gives

$$\left|\oint_{C'} \frac{f(z)-f(a)}{z-a}\,dz\right| \le \frac{M}{\rho}\,2\pi\rho = 2\pi M, \tag{3}$$

where $M = \max|f(z)-f(a)|$ on $C'$. Since $f(z)$ is continuous (because it is analytic), $M \to 0$ as $\rho \to 0$ so, letting $\rho \to 0$ in (2), we obtain $I = 2\pi i f(a)$, which result is known as the **Cauchy integral formula.***

---

**THEOREM 23.5.1** *Cauchy Integral Formula*
Let $f(z)$ be analytic in a simply-connected domain $D$, let $C$ be a piecewise smooth simple closed curve in $D$ oriented counterclockwise, and let $a$ be any point within $C$ (Fig. 1). Then

$$\boxed{\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a).} \tag{4}$$

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Let us illustrate the use of (4) in evaluating integrals.

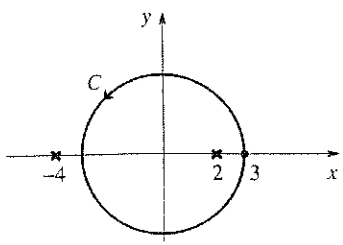



**Figure 2.** The contour $C$ in (5).

**EXAMPLE 1.** Evaluate

$$I = \oint_C \frac{e^z}{(z-2)(z+4)}\,dz, \tag{5}$$

where $C$ is a counterclockwise circle of radius 3, centered at the origin (Fig. 2).

*It would be misleading to say that the deviation term tends to zero as $\rho \to 0$, misleading because it is not a function of $\rho$. Rather, put it this way: we *find out* that the deviation is zero by letting $\rho \to 0$.

The first step is to examine the integrand to see where, if anywhere, it is singular. Evidently it has two singular points, one at $z = 2$ and one at $z = -4$. Of these, only $z = 2$ falls inside $C$ so, comparing the left-hand sides of (4) and (5), we can identify

$$f(z) = e^z/(z+4) \qquad \text{and} \qquad a = 2.$$

Thus, (4) gives

$$I = 2\pi i \left(\frac{e^z}{z+4}\right)\bigg|_{z=2} = \frac{\pi e^2}{3}\, i. \tag{6}$$

Of course, if the contour were clockwise, then the answer would be $I = -(\pi e^2/3)i$. ∎

**EXAMPLE 2.** Evaluate

$$I = \oint_C \frac{\cos z}{(z+2)(z+i)(z-2i)}\, dz, \tag{7}$$

where $C$ is as shown in Fig. 3a.

First, examine the integrand. It is singular at $z = -2$, $-i$, and $2i$. Of these three singular points, the latter two lie within $C$. Let us deform $C$ into the "dumbell" contour $C_1 + C_2 + C_3 + C_4$ shown in Fig. 3b, where $C_2$ and $C_4$ are actually coincident (lying on the $y$ axis) but are shown as slightly separated just for graphical clarity; such deformation is permissible, according to Theorem 23.3.2, because the integrand is analytic between and on $C$ and the dumbell contour. Thus, using shorthand notation,

$$I = \oint_C = \oint_{C_1+C_2+C_3+C_4} = \oint_{C_1} + \int_{C_2} + \oint_{C_3} + \int_{C_4} = \oint_{C_1} + \oint_{C_3}, \tag{8}$$

where the last step follows from the fact that the integrals on $C_2$ and $C_4$ are negatives of each other. That is,

$$\begin{aligned} I &= \oint_{C_1} \left(\frac{\cos z}{(z+2)(z-2i)}\right) \frac{dz}{z+i} + \oint_{C_3} \left(\frac{\cos z}{(z+2)(z+i)}\right) \frac{dz}{z-2i} \\ &\equiv \oint_{C_1} \frac{f_1(z)}{z-a_1}\, dz + \oint_{C_3} \frac{f_3(z)}{z-a_3}\, dz. \end{aligned} \tag{9}$$

Now, the Cauchy integral formula (4) can be used to evaluate each of these two integrals. In the first, $f_1(z) = \cos z/[(z+2)(z-2i)]$ is analytic within $C_1$ and $a_1$ is $-i$, and in the second, $f_3(z) = \cos z/[(z+2)(z+i)]$ is analytic within $C_3$ and $a_3$ is $2i$ so (4) gives

$$\begin{aligned} I &= 2\pi i f_1(-i) + 2\pi i f_3(2i) \\ &= 2\pi i \left[\frac{\cos(-i)}{(2-i)(-3i)} + \frac{\cos(2i)}{(2+2i)(3i)}\right] \\ &= \frac{\pi}{30}\left[(5\cosh 2 - 8\cosh 1) - (5\cosh 2 + 4\cosh 1)i\right]. \end{aligned} \tag{10}$$

Alternatively, let us return to (5) and use partial fractions to express

$$\frac{1}{(z+i)(z-2i)} = -\frac{1}{3i}\,\frac{1}{z+i} + \frac{1}{3i}\,\frac{1}{z-2i}. \tag{11}$$

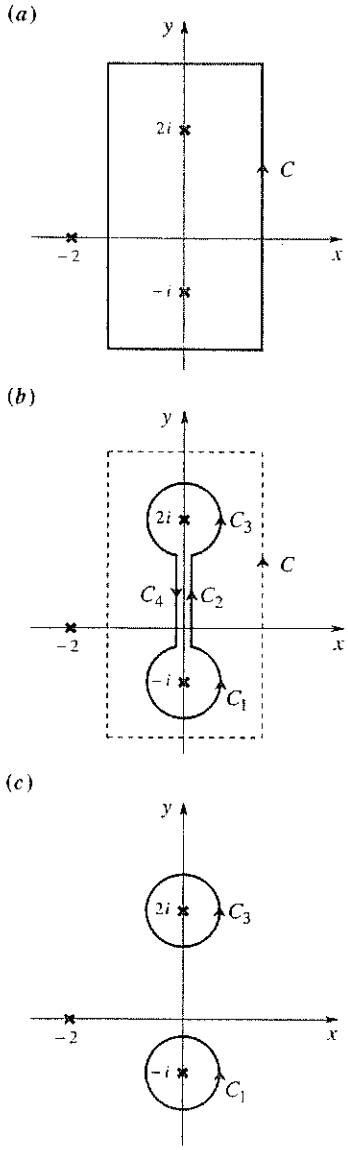



**Figure 3.** Divide and conquer.

Then (5) becomes

$$I = -\frac{1}{3i}\oint_C \left(\frac{\cos z}{z+2}\right)\frac{dz}{z+i} + \frac{1}{3i}\oint_C \left(\frac{\cos z}{z+2}\right)\frac{dz}{z-2i}. \tag{12}$$

In each integral in (12), $\cos z/(z+2)$ is analytic within $C$ so the Cauchy integral formula gives

$$I = -\frac{2\pi i}{3i}\left(\frac{\cos z}{z+2}\right)\bigg|_{z=-i} + \frac{2\pi i}{3i}\left(\frac{\cos z}{z+2}\right)\bigg|_{z=2i}, \tag{13}$$

which reduces to the same answer as was given in (10).

COMMENT. In this example there were two singular points within $C$, whereas the Cauchy integral formula (4) allows only one. However, by deforming $C$ into $C_1$ plus $C_3$ we were able to express $I$, in (9), as the sum of two integrals, each with only one singular point. Alternatively, that same objective can be met, without deforming $C$, by means of the partial fraction expansion (11) because each integrand in (12) has only one singular point within $C$. ∎

The Cauchy integral formula enables us to evaluate any integral, the integrand of which has a "first-order singularity" at some point $a$ within the contour; it does not apply if the singularity is of second order or higher – that is, if $I$ is of the form $\oint_C f(z)\,dz/(z-a)^2$, $\oint_C f(z)\,dz/(z-a)^3$, and so on. To deal with these cases, we begin by pointing out that (4) holds for *any* point $a$ within $C$. If we emphasize the allowed variability of $a$ by using the letter $z$ in its place, and adopt a dummy integration variable $\zeta$ in place of $z$, then we can re-express (4) as

$$\oint_C \frac{f(\zeta)}{\zeta - z}\,d\zeta = 2\pi i f(z). \tag{14}$$

If we differentiate (14) with respect to $z$, and can justify the step

$$\frac{d}{dz}\oint_C \frac{f(\zeta)}{\zeta - z}\,d\zeta = \oint_C \frac{\partial}{\partial z}\left(\frac{f(\zeta)}{\zeta - z}\right)d\zeta = \oint_C \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta, \tag{15}$$

then we obtain

$$\oint_C \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta = 2\pi i f'(z) \tag{16}$$

for the evaluation of an integral with a "second-order singularity" in its integrand.

The first step in (15) does indeed need justification because it amounts to an interchange in the order of the two limit processes: the integration and the differentiation. The step *looks* all right because $z$ is inside of $C$, so that $\zeta - z$ is nonzero for all $\zeta$ on $C$. Thus, $f(\zeta)/(\zeta - z)$ is an analytic function of $z$ for each $\zeta$ on $C$. Nevertheless, let us verify (15) rigorously. Not having a complex variable version of the Leibniz rule for differentiating under the integral sign, let us fall back on basics and recall that the derivative is the limit of a difference quotient. Let

$$I(z) = \oint_C \frac{f(\zeta)}{\zeta - z}\,d\zeta. \tag{17}$$

Then

$$\begin{aligned} I'(z) &= \lim_{\Delta z \to 0} \frac{I(z+\Delta z) - I(z)}{\Delta z} \\ &= \lim_{\Delta z \to 0} \frac{1}{\Delta z} \oint_C \left( \frac{1}{\zeta - z - \Delta z} - \frac{1}{\zeta - z} \right) f(\zeta)\, d\zeta \\ &= \lim_{\Delta z \to 0} \oint_C \frac{f(\zeta)\, d\zeta}{(\zeta - z)(\zeta - z - \Delta z)}. \end{aligned} \tag{18}$$

If we pass the limit across the integral sign, then the latter does give the same result as (15) but, as noted above, we cannot be sure that the order of the limit and the integration can be reversed. To proceed, we express

$$\oint_C \frac{f(\zeta)\, d\zeta}{(\zeta - z)(\zeta - z - \Delta z)} = \oint_C \frac{f(\zeta)}{(\zeta - z)^2}\, d\zeta + \Delta I \tag{19}$$

and seek to show that $\Delta I \to 0$ as $\Delta z \to 0$. From (19), $\Delta I$ is the difference

$$\begin{aligned} \Delta I &= \oint_C \left[ \frac{1}{(\zeta - z)(\zeta - z - \Delta z)} - \frac{1}{(\zeta - z)^2} \right] f(\zeta)\, d\zeta \\ &= \Delta z \oint_C \frac{f(\zeta)\, d\zeta}{(\zeta - z)^2 (\zeta - z - \Delta z)}. \end{aligned} \tag{20}$$

To bound $\Delta I$ let us deform $C$ in (20) to a circle $C'$ with its center at $z$ and radius $\rho$ small enough so that $C'$ lies entirely inside $C$. Since we are letting $\Delta z \to 0$, we can choose $|\Delta z| < \rho/5$, say. Then, from Fig. 4 we can see that $|(\zeta - z)^2|$ is exactly equal to $\rho^2$ for all points $\zeta$ on $C'$, and that $|\zeta - (z + \Delta z)| > 4\rho/5$ for all $\zeta$ on $C'$. Moreover, there must exist some finite constant $m$ such that $|f(\zeta)| \le m$ on $C'$ since $f$ is analytic on $C'$. Therefore, the $ML$ bound gives

$$|\Delta I| \le |\Delta z| \frac{m}{\rho^2 (4\rho/5)} 2\pi\rho. \tag{21}$$

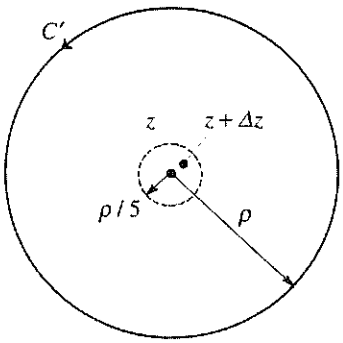



**Figure 4.** Bounding $\Delta I$.

It follows from (21) that if $\Delta z \to 0$ with $C'$ fixed (and hence with $m$ and $\rho$ fixed), then $\Delta I \to 0$, which result establishes the truth of (15) and hence (16).

Similarly, we can repeat this process of differentiation as many times as we wish, and find that

$$\oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta = \frac{2\pi i}{n!} f^{(n)}(z)$$

or, returning to the $z, a$ notation used in (4),

$$\boxed{\oint_C \frac{f(z)}{(z-a)^{n+1}}\, dz = \frac{2\pi i}{n!} f^{(n)}(a).} \tag{22}$$

For reference, let us distinguish (4) and (22) as the Cauchy integral formula and the **generalized Cauchy integral formula**, respectively.*

**EXAMPLE 3.** Evaluate

$$I = \oint_C \frac{e^z}{z^3}\, dz, \tag{23}$$

where $C$ is the counterclockwise unit circle $|z| = 1$. Rewrite (23) in the form

$$I = \oint_C \frac{e^z}{(z-0)^3}\, dz, \tag{24}$$

for comparison with (22). We see that $n = 2$, $a = 0$, and $f(z) = e^z$ so (22) gives

$$I = \frac{2\pi i}{2!}\left(\frac{d^2}{dz^2}e^z\right)\bigg|_{z=0} = \pi i. \quad \blacksquare$$

**EXAMPLE 4.** Evaluate

$$I = \oint_C \frac{z+1}{z(z-2)(z-4)^3}\, dz, \tag{25}$$

where $C$ is the counterclockwise unit circle $|z-3| = 2$ (Fig. 5a). The integrand has singularities at $z = 0, 2$, and 4, of which the latter two fall within $C$. We could apply partial fractions to the $1/[(z-2)(z-4)^3]$ part of the integrand, but it is easier to deform $C$ into the two contours shown in Fig. 5b, and to evaluate each of the two integrals using (22). Thus,

$$\begin{aligned} I &= \oint_{C_1}\left[\frac{z+1}{z(z-4)^3}\right]\frac{dz}{z-2} + \oint_{C_2}\left[\frac{z+1}{z(z-2)}\right]\frac{dz}{(z-4)^3} \\ &= 2\pi i\left[\frac{z+1}{z(z-4)^3}\right]\bigg|_{z=2} + \frac{2\pi i}{2!}\frac{d^2}{dz^2}\left[\frac{z+1}{z(z-2)}\right]\bigg|_{z=4} \\ &= -\frac{3\pi i}{8} + \frac{23\pi i}{64} = -\frac{\pi i}{64}. \quad \blacksquare \end{aligned}$$

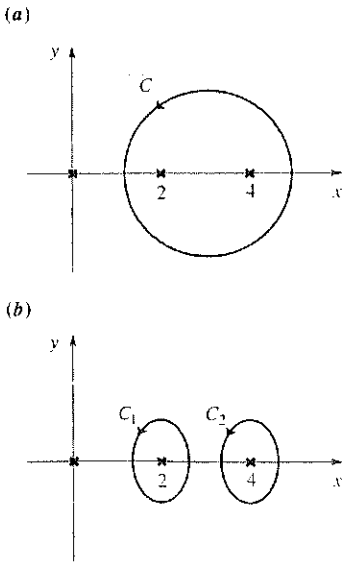



**Figure 5.** The contours in Example 4.

There are two important observations to be made. First, observe that the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\, d\zeta \tag{26}$$

gives $f(z)$ at any interior point $z$ of the region within $C$ as an integration over its boundary values on $C$. Since the real and imaginary parts of the analytic function

*This result opens up the interesting possibility of defining noninteger-order derivatives of a function, for if we interpret $n!$ as $\Gamma(n+1)$ and define the intended branch of $(\zeta - z)^{n+1}$, then the right-hand side of (22) is defined for noninteger values of $n$ as well.

$f(z) = u(x, y) + iv(x, y)$ are harmonic, we see that (26) is very close to providing the solution to the classical Dirichlet problem, namely, $\nabla^2 u = 0$ in the interior of $C$ together with a Dirichlet boundary condition on $u$ (i.e., $u$ given on $C$). Thus, as in Chapter 22, we see once again the close connection between analytic function theory and two-dimensional potential theory. We return to this idea in the exercises.

Second, observe that *having assumed only that $f(z)$ is analytic (once differentiable), one finds with no further assumption that $f(z)$ possesses derivatives of all orders*:

$$f^{(n)}(z) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - z)^{n+1}}\, d\zeta. \tag{27}$$

This remarkable result has no analog in real variable theory. For instance, observe that $f(x) = x^2 H(x)$, where $H$ is the Heaviside function, is once differentiable for all $x$ (sketch the graphs of $f$ and $f'$), but $f''$ fails to exist at $x = 0$ due to the kink in the graph of $f'$ at that point. Similarly, $f(x) = x^3 H(x)$ is differentiable twice but not three times. Thus, if $f(z)$ is "nice enough" to be once differentiable in some region, then it will be very nice indeed, *infinitely* differentiable! How can we understand this difference between real and complex function theory? The answer is that differentiability in the complex plane is much more demanding than on the real axis because $z$ can tend to $z_0$, in the formula

$$f'(z) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}, \tag{28}$$

in any manner, whereas only a horizontal approach is possible in the real variable formula

$$f'(x) = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0}. \tag{29}$$

Because more is demanded, in the complex case, the consequences of differentiability are more far reaching.

**Closure.** The chief results in this section are the Cauchy integral formula (4) and its generalized version (22). Whereas Cauchy's theorem tells us that $\oint_C f(z)\, dz = 0$ if $f$ is analytic, (22) shows us how to evaluate an integral (around a closed contour $C$) when its integrand has a singularity of any order within the region enclosed by $C$.

## EXERCISES 23.5

1. Evaluate each integral, where $C$ is the counterclockwise circle $|z| = 3$. Use Cauchy's integral formula or its extension.

(a) $\displaystyle\oint_C \frac{\cos z}{z}\, dz$

(b) $\displaystyle\oint_C \frac{\sin z}{z}\, dz$

(c) $\displaystyle\oint_C \frac{dz}{z^2 - 5z}$

(d) $\displaystyle\oint_C \frac{z^2 - 1}{z^2 + 1} e^z\, dz$

(e) $\displaystyle\oint_C \frac{z + 1}{(z - 1)(z + 2)^3}\, dz$

(f) $\displaystyle\oint_C \frac{e^{2z}}{z^5}\, dz$

(g) $\oint_C \frac{\sinh 3z}{(z^2+1)^2}\,dz$ (h) $\oint_C \frac{z+2}{z^4-1}\,dz$

(i) $\oint_C \frac{e^{z^2}}{z\cos(z/2)}\,dz$ (j) $\oint_C \frac{z\,dz}{(z+i)(z^2+1)}$

(k) $\oint_C \frac{z^3}{z^2+i}\,dz$ (l) $\oint_C \frac{dz}{z(z-2)(z-4)}$

**2.** (*Important little integral*) In Section 23.3 we show that

$$\oint_C (z-a)^n\,dz = \begin{cases} 2\pi i & (n=-1) \\ 0 & (n \neq -1), \end{cases} \tag{2.1}$$

where $n$ is any integer and $a$ is within the contour $C$. Derive (2.1) using Cauchy's theorem, the Cauchy integral formula, and the generalized Cauchy integral formula.

**3.** (a) Show from (22) that if $C$ is a circle of radius $\rho$ with center at $z$, $f(z)$ is analytic inside and on $C$, and $M$ is the maximum value of $|f(z)|$ on $C$, then

$$\left|f^{(n)}(z)\right| \leq \frac{n!\,M}{\rho^n}. \tag{3.1}$$

(b) (*Liouville's theorem*) Use (3.1) to prove **Liouville's theorem**: *If $f$ is entire (i.e., analytic for all finite $z$) and bounded for all $z$, then $f$ is a constant.*
(c) Since $f(z) = \sin z$ is entire and not a constant, it must not be bounded (according to Liouville's theorem). Demonstrate that, in fact, it is *not* bounded.
(d) (*Fundamental theorem of algebra*) Use Liouville's theorem to prove the **fundamental theorem of algebra**: *if $P(z)$ is a polynomial function of $z$, of degree 1 or greater,*

$$P(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_0 \qquad (a_n \neq 0)$$

*then $P(z) = 0$ has at least one root.* HINT: Suppose that $P(z)$ is nonzero everywhere. Then $f(z) = 1/P(z)$ is analytic everywhere and is bounded.

**4.** (*Dirichlet problems*) As mentioned in the text, just as the Cauchy integral formula

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\,d\zeta \tag{4.1}$$

expresses an analytic function $f(z) = u + iv$ in terms of its boundary values, we would expect there to exist a similar integral formula expressing a harmonic function $u(x,y)$ in terms of its boundary values. In this exercise we seek such a formula for two important cases: the case where the domain is a circular disk, and the case where the domain is the upper half plane.

(a) (*Circular disk*) Let $C$ be the counterclockwise circle $|\zeta| = R$. If we seek the desired expression for $u$ by equating real parts of the left- and right-hand sides of (4.1), we find that the right-hand side involves both $u$ *and* $v$, whereas the additional unknown $v$ is not welcome. The reason that $v$ enters is that $1/(\zeta - z)$ is not purely real. With $\zeta = Re^{i\phi}$, show that we can re-express (4.1) as

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_C \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - R^2/\overline{z}}\right) f(\zeta)\,d\zeta \\ &= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{\zeta}{\zeta - z} + \frac{\overline{z}}{\overline{\zeta} - \overline{z}}\right) f(\zeta)\,d\phi, \end{aligned} \tag{4.2}$$

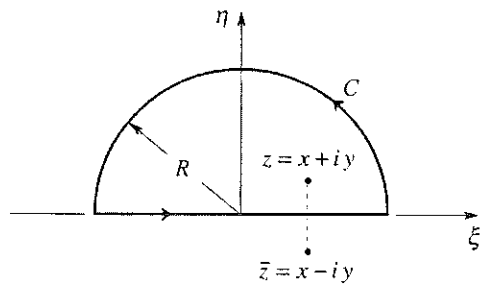
where the bracketed quantity is real. In particular, show that

$$\frac{\zeta}{\zeta - z} + \frac{\overline{z}}{\overline{\zeta} - \overline{z}} = \frac{R^2 - r^2}{|\zeta - z|^2}, \tag{4.3}$$

and hence that

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{(R^2 - r^2)\,u(R,\phi)}{R^2 - 2Rr\cos(\phi - \theta) + r^2}\,d\phi, \tag{4.4}$$

where $z = re^{i\theta}$ and $\zeta = Re^{i\phi}$. This result is also derived by separation of variables, in Section 20.3 and is known as the **Poisson integral formula for the circular disk**.
(b) (*Upper half plane*) This time let $C$ be the contour shown here. Show that (4.1) can be re-expressed as




$$f(z) = \frac{1}{2\pi i}\oint_C \left(\frac{1}{\zeta - z} - \frac{1}{\zeta - \overline{z}}\right) f(\zeta)\,d\zeta \tag{4.5}$$

for all $R > |z|$. Suppose that, as our boundary condition at infinity, $f(z) \to 0$ as $z \to \infty$. Letting $R \to \infty$ in (4.5), show that the semicircle integral tends to zero, leaving us with

$$f(z) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \left(\frac{1}{\xi - z} - \frac{1}{\xi - \overline{z}}\right) f(\xi)\,d\xi. \tag{4.6}$$

Finally, equating real parts in (4.6), show that

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{u(\xi,0)}{(\xi - x)^2 + y^2}\, d\xi \tag{4.7}$$

is the solution to the Dirichlet problem for the upper half plane, with the boundary condition $u(x,y) \to 0$ as $r = \sqrt{x^2+y^2} \to \infty$. This result is also derived by means of the Fourier transform in Section 20.4 and is the **Poisson integral formula for the upper half plane**. NOTE: The Dirichlet problems considered here in parts (a) and (b) can also be solved by an elegant technique known as the method of **Green's functions**.* For the reader who subsequently studies that method and comes back to this exercise to compare the two approaches, it may be helpful for us to note that the mysterious point $\zeta = R^2/\overline{z}$ that shows up in (4.2) and the mysterious point $\zeta = \overline{z}$ that shows up in (4.5) are (in the terminology of the method of Green's functions) the *image* points.

# Chapter 23 Review

We begin Chapter 23 by defining the complex integral $\int_C f(z)\,dz$ in essentially the same way that one defines the real integral $\int_a^b f(x)\,dx$, and note the close resemblance to line integrals in two dimensions. There are only a handful of major results, and they are as follows.

**Cauchy's Theorem.** If $f(z)$ is analytic in a simply connected domain $D$, then

$$\oint_C f(z)\,dz = 0$$

for every piecewise smooth simple closed curve $C$ in $D$.

If instead $C$ is an open curve, then Cauchy's theorem gives this corollary:

**Path Independence.** If $f(z)$ is analytic in a simply-connected domain $D$, then $\int_C f(z)\,dz$ is independent of path in $D$. That is, given any initial point $P$ in $D$ and any final point $Q$ in $D$, the value of $\int_C f(z)\,dz$ is the same for every piecewise smooth path $C$, lying entirely within $D$, from $P$ to $Q$.

Analogous to the fundamental theorem of the real integral calculus, we have:

**Fundamental Theorem of the Complex Integral Calculus.** Let $f(z)$ be analytic in a simply-connected domain $D$, and let $z_0$ be any fixed point in $D$. Then

(i) $G(z) = \displaystyle\int_{z_0}^{z} f(\zeta)\,d\zeta$ is analytic in $D$ and $G'(z) = f(z)$.

(ii) If $F(z)$ is any primitive of $f(z)$ [i.e., $F'(z) = f(z)$], then

$$\int_{z_0}^{z} f(\zeta)\,d\zeta = F(z) - F(z_0).$$

*See, for example, M. D. Greenberg, *Applications of Green's Functions in Science and Engineering* (Englewood Cliffs, NJ: Prentice Hall, 1971).

If the integrand is *not* analytic within the closed contour $C$, then we have:

**Cauchy Integral Formula.** Let $f(z)$ be analytic in a simply-connected domain $D$, let $C$ be a piecewise smooth simple closed counterclockwise curve in $D$, and let $a$ be any fixed point lying within $C$. Then

$$\oint_C \frac{f(z)}{z-a}\,dz = 2\pi i f(a).$$

In fact, for any $n = 0, 1, 2, \ldots$ we have the generalized Cauchy integral formula

$$\oint_C \frac{f(z)}{(z-a)^{n+1}}\,dz = \frac{2\pi i}{n!} f^{(n)}(a).$$

Note that we have *not* claimed that the latter formula covers all possible singular integrands, but we do claim that it suffices for most applications, and is of great importance.

Finally, we also develop the so-called $ML$ bound:

$$\left| \int_C f(z)\,dz \right| \leq ML,$$

if $|f(z)| \leq M$ on $C$ and the length of $C$ is $L$. We continue to use this bound in the next chapter, both as a theoretical tool in the derivatives, and in the applications as well.

# Chapter 24

# Taylor Series, Laurent Series, and the Residue Theorem

## 24.1 Introduction

When we first introduced complex integrals, in Section 23.2, we pointed out that we can express any complex integral $\int_C f(z)\,dz$ in the form

$$\int_C f(z)\,dz = \int_C [u(x,y)dx - v(x,y)dy] + i\int_C [v(x,y)dx + u(x,y)dy].$$

Hence, there was the temptation to stop right there and to rely on the theory of real line integrals, as is covered in Chapter 15 and 16. However, we promised that it would prove more fruitful to keep $\int_C f(z)\,dz$ intact and to develop a distinct complex integral calculus. We began that development in Chapter 23 and got as far as Cauchy's theorem, the fundamental theorem of the complex integral calculus, and the Cauchy integral formula. Those results now enable us to derive the Taylor and Laurent series expansions of functions of a complex variable which, in turn, permits us to clarify the notion of singularities and, finally, to complete our development of the complex integral calculus by deriving the powerful residue theorem.

## 24.2 Complex Series and Taylor Series

**24.2.1. Complex series.** By a **complex series** we mean any sum of the form

$$\sum_{n=1}^{\infty} c_n = c_1 + c_2 + c_3 + \cdots, \tag{1}$$

where the $c_n$'s are complex numbers. As in the real case, we say that the series converges if the limit of the sequence of partial sums, $s_n = c_1 + c_2 + \cdots + c_n$, exists as $n \to \infty$. That is, the series **converges** to $s$ if to each (real) number $\epsilon > 0$,

no matter how small, there exists an integer $N(\epsilon)$ such that $|s_n - s| < \epsilon$ for all $n > N$; otherwise, it **diverges**.

The sum $\sum c_n$ can be expressed in terms of real series, for if $c_n = a_n + ib_n$ where $a_n$ and $b_n$ are real, then $\sum c_n$ converges if and only if $\sum a_n$ and $\sum b_n$ do, in which case $\sum c_n = \sum a_n + i \sum b_n$ (Exercise 1). However, just as it is best to develop a complex integral calculus that does not rely on expressing $\int_C f(z)\,dz$ in terms of real integrals, likewise it is best to develop a theory of complex series.

As a start, a necessary and sufficient condition for convergence of any series, real or complex, is given by the Cauchy convergence theorem:

---

**THEOREM 24.2.1** *Cauchy Convergence Theorem*
An infinite series is convergent if and only if its sequence of partial sums $s_n$ is a Cauchy sequence – that is, if to each $\epsilon > 0$ (no matter how small) there corresponds an integer $N(\epsilon)$ such that $|s_m - s_n| < \epsilon$ for all $m$ and $n$ greater than $N$.

---

Unfortunately, this theorem is difficult to apply so one develops (in the calculus) an array of theorems (i.e., tests for convergence/divergence), that are more specialized than the Cauchy convergence theorem, but easier to apply. For instance, if in Theorem 24.2.1 we set $m = n - 1$, then the stated condition becomes: to each $\epsilon > 0$ (no matter how small) there corresponds an integer $N(\epsilon)$ such that $|s_m - s_n| = |c_n| < \epsilon$ for all $n > N$, which is equivalent to saying that $c_n \to 0$ as $n \to \infty$. Thus, we have the following specialized, but readily applied, result:

---

**THEOREM 24.2.2** *A Necessary Condition for Convergence*
For the series $\sum_{n=1}^{\infty} c_n$ to converge it is necessary, but not sufficient, that $c_n \to 0$ as $n \to \infty$.

---

Two more such theorems follow.

---

**THEOREM 24.2.3** *Comparison Test*
If the series $\sum_{n=1}^{\infty} M_n$ converges, where the $M_n$'s are positive constants and $|c_n| \leq M_n$ for each $n$ greater than some integer $N$, then the series $\sum_{n=1}^{\infty} c_n$ converges too.

---

*Proof*: According to the Cauchy criterion, the assumed convergence of $\sum_{n=1}^{\infty} M_n$ implies that to each $\epsilon > 0$ there corresponds an $N_0$ such that $M_p + M_{p+1} + \cdots + M_{p+q} < \epsilon$ for all $p > N_0$ and all $q \geq 0$. Then (Exercise 2),

$$\begin{aligned} |c_p + c_{p+1} + \cdots + c_{p+q}| &\leq |c_p| + |c_{p+1}| + \cdots + |c_{p+q}| \\ &\leq M_p + M_{p+1} + \cdots + M_{p+q} < \epsilon \end{aligned} \tag{2}$$

for all $p > N_0$ (if $N_0 < N$, then use $N$ instead) and for all $q \geq 0$, so $\sum_{n=1}^{\infty} c_n$ converges too. ■

**EXAMPLE 1.** Determine the convergence or divergence of the series

$$\sum_{n=0}^{\infty} \frac{i^n}{n!} = 1 + i + \frac{i^2}{2!} + \frac{i^3}{3!} + \cdots. \tag{3}$$

Since

$$\left|\frac{i^n}{n!}\right| = \frac{1}{n!} < \frac{1}{2^n} \tag{4}$$

for all $n \geq 4$, and $\sum_{n=0}^{\infty} 1/2^n = \sum_{n=0}^{\infty} (\frac{1}{2})^n$ is a convergent geometric series, it follows from the comparison test that (3) is convergent. ■

Observe that sometimes our series start at $n = 0$ and sometimes at $n = 1$ or some other integer. These differences are inconsequential insofar as convergence/divergence are concerned (assuming that each of the initial terms in question is finite) though they do of course affect the sum if the series is convergent.

---

**THEOREM 24.2.4** *Ratio Test*
If $\lim_{n\to\infty} |c_{n+1}/c_n| = L$, then $\sum_{n=0}^{\infty} c_n$ converges if $L < 1$ and diverges if $L > 1$. No information is obtained if $L = 1$ or if the limit does not exist.

---

*Proof*: Suppose that $L < 1$. Choose any number $p$ such that $L < p < 1$. Then it must be true that $|c_{n+1}/c_n| < p$ for all sufficiently large $n$'s, say for $n \geq N$. Thus,

$$\begin{aligned} |c_{N+1}| &< p\,|c_N|\,, \\ |c_{N+2}| &< p\,|c_{N+1}| < p^2\,|c_N|\,, \\ |c_{N+3}| &< p\,|c_{N+2}| < p^3\,|c_N|\,, \end{aligned}$$

and so on. According to the comparison test, $\sum_{n=0}^{\infty} c_n$ converges by comparison with the series $\sum_{n=0}^{\infty} p^n\,|c_N|$, which is $|c_N|$ times the convergent geometric series $\sum_{n=0}^{\infty} p^n$, convergent because $p < 1$. The case $L > 1$ is left for the exercises. ■

**EXAMPLE 2.** Determine the convergence or divergence of the series

$$\sum_{n=0}^{\infty} \frac{(1+i)^n}{n!}. \tag{5}$$

In this case,

$$\lim_{n\to\infty} \left|\frac{c_{n+1}}{c_n}\right| = \lim_{n\to\infty} \left|\frac{1+i}{n+1}\right| = \sqrt{2} \lim_{n\to\infty} \frac{1}{n+1} = 0 \tag{6}$$

so $L = 0$ and, according to the ratio test, the series (5) is convergent. ∎

**EXAMPLE 3.** Determine the convergence or divergence of the series

$$\sum_{n=2}^{\infty} e^{-(2+3i)n}. \tag{7}$$

Again applying the ratio test,

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| &= \lim_{n\to\infty}\left|e^{-(2+3i)}\right| = \left|e^{-(2+3i)}\right| \\ &= \left|e^{-2}e^{-3i}\right| = e^{-2}\left|e^{-3i}\right| = e^{-2}\end{aligned} \tag{8}$$

so $L = e^{-2} < 1$ and the series (7) is convergent. ∎

More generally, the terms may be functions of $z$ rather than constants:

$$\sum_{n=0}^{\infty} f_n(z) = f_0(z) + f_1(z) + \cdots. \tag{9}$$

Then the set of all points in the $z$ plane for which the series converges is called the *region of convergence* of the series.

Since we are leading up to complex Taylor series we shall not consider the general case (9). Rather, we suppose that $f_n(z)$ is of the form $c_n(z-a)^n$, where the $c_n$'s and $a$ are, in general, complex numbers. The resulting form,

$$\boxed{\sum_{n=0}^{\infty} a_n(z-a)^n = a_0 + a_1(z-a) + a_2(z-a)^2 + \cdots,} \tag{10}$$

is called a **power series**. If, to determine the region of convergence of (10), we apply the ratio test, we obtain the following result, proof of which is left for the exercises.

---

**THEOREM 24.2.5** *Power Series Convergence by Ratio Test*
If

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = L, \tag{11}$$

then the power series (10) converges in the disk $|z-a| < 1/L$ and diverges in $|z-a| > 1/L$. If $L = \infty$ the series converges only at the point $z = a$, and if $L = 0$ it converges for all $z$, that is, in $|z-a| < \infty$.

---

**EXAMPLE 4.** *Geometric Series.* For the **geometric series**

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots, \tag{12}$$

$a_n = 1$ for each $n$ so (11) gives $L = 1$. Hence, the geometric series (12) converges in the disk $|z| < 1$ and diverges in $|z| > 1$. Theorem 24.2.5 gives no information regarding convergence or divergence for points on the circele $|z| = 1$. However, $|z^n| = |z|^n = 1^n = 1$ fails to tend to zero as $n \to \infty$ so, by Theorem 24.2.2, the series diverges at all points on the circle $|z| = 1$. ∎

It is tempting to conclude from Theorem 24.2.5 that a power series inevitably converges in a *circular disk* centered at $a$ (that disk being the single point $z = a$ if $L = \infty$ and the whole $z$ plane if $L = 0$). However, notice that the theorem yields no information if $\lim_{n\to\infty} |a_{n+1}/a_n|$ does not exist (as occurs if, for example, $a_n = \sin n$), and in that event it is conceivable that a noncircular region of convergence may exist. Toward clarifying this point, we first present the following lemma.*

---

**LEMMA 24.2.1** *Power Series Convergence*
If the power series (10) converges at $z_0$, then it converges everywhere in the open disk $|z - a| < |z_0 - a|$ (Fig. 1).

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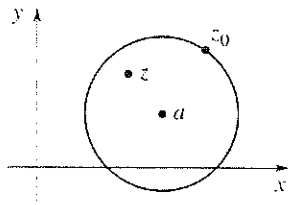



**Figure 1.** The disk $|z - a| < |z_0 - a|$.

*Proof*: Since the series converges for $z = z_0$, it follows that $a_n(z_0 - a)^n \to 0$ as $n \to \infty$. Thus, there must certainly exist a constant $K$ such that $|a_n(z_0 - a)^n| < K$ for all $n = 0, 1, 2, \ldots$ and from this result we infer that $|a_n| < K/|z_0 - a|^n$ $(n = 0, 1, 2, \ldots)$. Then, for any $z$ such that $|z - a| < |z_0 - a|$, we have

$$|a_n(z_0 - a)^n| < K \left| \frac{z - a}{z_0 - a} \right|^n. \tag{13}$$

Observing that $|(z - a)/(z_0 - a)| < 1$, we see that $K\,|(z - a)/(z_0 - a)|^n$ is the $n$th term of a convergent geometric series (scaled by $K$). Thus, it follows from (13) and the comparison test that (10) is convergent, as stated. ∎

With the help of this lemma we can now show that the region of convergence of a power series is necessarily a circular disk.

---

**THEOREM 24.2.6** *Power Series Convergence in a Disk*
For the power series (10) there exist three possibilities regarding its region of convergence:

*Not an end in itself, a lemma is developed to help prove a subsequent theorem.

(i) the series converges only at the point $a$;
(ii) the series converges in the whole plane;
(iii) there exists a constant $R > 0$, called the **radius of convergence** of (10), such that (10) converges in $|z - a| < R$ and diverges in $|z - a| > R$.

*Proof*: To prove that (i) and (ii) are possibilities it suffices to put forward an example of each case. We leave this part for the exercises, and turn to (iii). If (i) does *not* apply, there must be a point $z_1 \neq a$ at which (10) converges, and if (ii) does *not* apply, there must be a point $z_2$ at which (10) diverges. From the lemma it is evident that $|z_2 - a| \geq |z_1 - a|$. If $|z_2 - a| = |z_1 - a|$, it follows from the lemma that (10) converges everywhere in $|z - a| < |z_1 - a|$. Furthermore, the lemma also implies that (10) diverges everywhere in $|z - a| > |z_1 - a|$ for if it converged anywhere in that region it could not diverge at $z_2$ as assumed. Thus, if $|z_2 - a| = |z_1 - a|$, then $R = |z_2 - a| = |z_1 - a|$ is the radius of convergence referred to in (iii), and it remains to consider the case where $|z_2 - a| > |z_1 - a|$ as depicted in Fig. 2. Let $S$ be the set of values $\rho$ such that (10) converges for all $|z - a| < \rho$. From the lemma it follows that if some value $\rho_0$ belongs to $S$ so do all $\rho$'s satisfying $0 \leq \rho < \rho_0$. Thus, $S$ is necessarily an interval (on a $\rho$ axis), either $0 \leq \rho < R$ or $0 \leq \rho \leq R$ for some $R$ such that $|z_1 - a| \leq R \leq |z_2 - a|$. Furthermore, (10) must diverge everywhere in $|z - a| > R$ because if it converged anywhere in that region, the right endpoint of the $S$ interval would be greater than $R$. Then $R$ is the radius of convergence referred to in (iii), and the proof is complete. ■

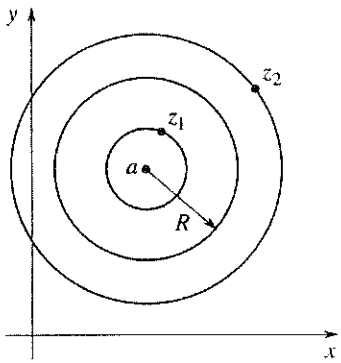



**Figure 2.** Disk of convergence.

We are now ready to develop the concept of the Taylor series of a function of a complex variable.

**24.2.2. Taylor series.** In Section 23.5 we found that if a function $f(z)$ is analytic at $z = a$, then it admits derivatives of all orders there. Since $f(a), f'(a), f''(a), \ldots$ all exist, it is at least formally possible to write down the **Taylor series**

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z - a)^n \tag{14}$$

of $f(z)$ about the point $a$. However, whether or not the Taylor series is *useful* depends upon the answers to these questions: Does the series converge and, if so, in what domain? If it does converge in some domain $D$, does it converge to $f(z)$ there; that is, does it *represent* $f$ in $D$? Finally, if (14) does represent $f$ in $D$, might there exist *other* power series representations of $f$ as well? That is, might there exist more than one representation of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - a)^n,$$

where $a_n \neq f^{(n)}(a)/n!$ ? These questions are answered by the following theorem.

**THEOREM 24.2.7** *Taylor Series*
Let $f(z)$ be analytic in a domain $D$. If $a$ lies within $D$ and $|z-a| < R$ is any disk centered at $a$ and lying entirely within $D$, then $f(z)$ admits precisely one power series representation in $|z-a| < R$, its Taylor series (14):

$$\boxed{f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n = f(a) + f'(a)(z-a) + \cdots.} \tag{15}$$

*Proof*: Let $C$ be the circle $|z-a| = R$ (Fig. 3), counterclockwise. For any point $z$ inside $C$ we have, according to the Cauchy integral formula,

$$f(z) = \frac{1}{2\pi i}\oint_C \frac{f(\zeta)}{\zeta - z}\,d\zeta, \tag{16}$$

which formula serves as our starting point. Let us re-express

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a}\,\frac{1}{1 - \dfrac{z-a}{\zeta - a}} \tag{17}$$

in the integrand, and then use the formula

$$\frac{1}{1-t} = 1 + t + t^2 + \cdots + t^{n-1} + \frac{t^n}{1-t}, \tag{18}$$

which is simply an algebraic identity for all $t \neq 1$ [as can be verified by multiplying (18) through by $1-t$]. Specifically, let $(z-a)/(\zeta - a)$ be $t$ in (17), and then use (18), These steps give

$$f(z) = \frac{1}{2\pi i}\left[\oint_C \frac{f(\zeta)}{\zeta - a}\,d\zeta + (z-a)\oint_C \frac{f(\zeta)}{(\zeta-a)^2}\,d\zeta + \cdots \right.$$
$$\left. + (z-a)^{n-1}\oint_C \frac{f(\zeta)}{(\zeta-a)^n}\,d\zeta\right] + R_n(z), \tag{19}$$

where the *remainder term* is*

$$R_n(z) = \frac{(z-a)^n}{2\pi i}\oint_C \frac{f(\zeta)}{(\zeta-a)^n(\zeta-z)}\,d\zeta. \tag{20}$$

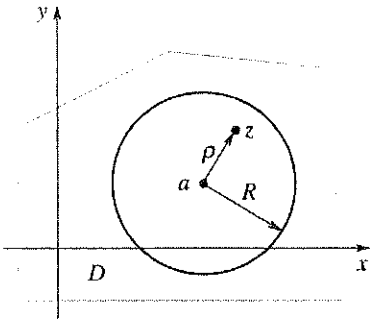



**Figure 3.** The contour $C$ in (16).

*At first glance, it may appear that in deriving (19) we have been guilty of interchanging two limit processes without justification, namely, expressing the integral of a sum as the sum of the integrals. However, there is no need for concern because $1 + t + t^2 + \cdots + t^{n-1}$ is not an infinite series; it is only a finite sum.

Then, recalling the generalized Cauchy integral formula

$$f^{(n)}(a) = \frac{n!}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta, \tag{21}$$

we see that (19) becomes

$$f(z) = f(a) + f'(a)(z-a) + \cdots + \frac{f^{(n-1)}(a)}{(n-1)!}(z-a)^{n-1} + R_n, \tag{22}$$

which is the complex version of Taylor's formula with remainder.

Finally, we wish to show that $R_n \to 0$ as $n \to \infty$. We expect that to be true since $R_n$ comes from the $t^n/(1-t)$ term in (18), and $t^n$ does tend to zero as $n \to \infty$ because $|t| = |(z-a)/(\zeta - a)| = \rho/R < 1$ for all points $\zeta$ on $C$ (Fig. 3).

Since $f(z)$ is analytic on $C$ it must be bounded on $C$. Thus, suppose that $|f(z)| < m$ on $C$. Noting further, from Fig. 3, that $|z-a| = \rho$, that $|\zeta - a| = R$, and that $|\zeta - z| \geq R - \rho$ for all $\zeta$ on $C$, we have

$$\begin{aligned} |R_n(z)| &= \frac{\rho^n}{2\pi} \left| \oint_C \frac{f(\zeta)}{(\zeta-a)^n(\zeta - z)}\, d\zeta \right| \\ &\leq \frac{\rho^n}{2\pi} \frac{m}{R^n(R-\rho)} 2\pi R \qquad \text{(by the } ML \text{ bound)} \\ &= \frac{mR}{R-\rho}\left(\frac{\rho}{R}\right)^n, \end{aligned} \tag{23}$$

which expression tends to zero as $n \to \infty$ for all $z$ in $|z-a| < R$. Then, by the definition of series convergence, letting $n \to \infty$ in (23) gives the resulting Taylor series representation

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots \tag{24}$$

of $f(z)$ in $|z-a| < R$.

We will omit the proof of uniqueness, that (24) is the only representation of $f(z)$ of the form $\sum_{n=0}^{\infty} a_n(z-a)^n$. ■

The upshot is that if $f(z)$ is analytic in $|z-a| < R$, then it can be represented in that disk by a convergent power series, namely, its Taylor series about $a$. It can also be shown that the converse is true:

---

**THEOREM 24.2.8** *Convergent Power Series*
If a power series $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges in $|z-a| < R$, then its sum function $f(z)$ is analytic there and the power series is the Taylor series of $f$; that is, $a_n = f^{(n)}(a)/n!$.

---

**EXAMPLE 5.** To obtain the Taylor series expansion of $f(z) = e^z$ about $z = 0$, say, we could work out the coefficients $f(0), f'(0), f''(0)/2!$, and so on, but it is easier to note that (15) is of exactly the same form as in real variable theory so we need merely change the $x$'s to $z$'s in the familiar formula

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots. \tag{25}$$

Observing also that $e^z$ is analytic in the whole plane, we see from Theorem 24.2.7 that the Taylor series representation

$$e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots \tag{26}$$

holds in $|z| < \infty$. As in real variable theory, we call the Taylor expansion of $f(z)$ about $z = 0$ the Maclaurin expansion of $f(z)$.

If, however, we wish to expand $e^z$ about some point other than the origin, say $z = i$, then the remembered formula (25) is of no help, and we need to work out the $f^{(n)}(a)/n!$ coefficients in (15). Doing so, we obtain

$$e^z = e^i + e^i(z - i) + \frac{e^i}{2!}(z - i)^2 + \cdots, \tag{27}$$

which holds in the whole plane, $|z - i| < \infty$. Of course, $e^i = \cos 1 + i \sin 1$. ∎

**EXAMPLE 6.** *Geometric Series Revisited.* In Example 4 we used Theorem 24.2.5 to determine that the geometric series $1 + z + z^2 + \cdots$ converges in $|z| < 1$ and diverges in $|z| > 1$. (We also showed that it diverges on the circle $|z| = 1$ because there the general term $z^n$ fails to go to zero as $n \to \infty$.) In fact, the series sums to $1/(1 - z)$ in $|z| < 1$ and is the Taylor series of that function:

$$\frac{1}{1 - z} = 1 + z + z^2 + \cdots. \qquad (|z| < 1) \tag{28}$$

It is important to understand that the preceding three theorems enable us to determine the region of convergence directly from the function being expanded, without having to test the series itself. To illustrate, consider $f(z) = 1/(1 - z)$. Theorem 24.2.6 tells us that the region of convergence of the Taylor series of $f$ about $z = 0$ will be a disk centered at $z = 0$, having zero, finite, or infinite radius. To determine that radius, examine $f$ to see where it is analytic and where it is not. In this example $f(z) = 1/(1 - z)$ is readily seen to be analytic everywhere except at $z = 1$, which is a "singular point" of $f$. Theorem 24.2.7 tells us that $R$ is at least 1, and Theorem 24.2.8 tells us that $R$ is no greater than that, because the sum function $f$ is necessarily analytic in $|z| < R$ whereas $f$ is singular at $z = 1$.

In physical terms we can visualize the disk of convergence spreading from the point of expansion, like ripples on a pond, until it encounters a singular point; it can spread no further.

COMMENT 1. Whereas $1/(1 - z)$ is meaningful everywhere except at the point $z = 1$, the series $1 + z + z^2 + \cdots$ is meaningful only inside the disk $|z| < 1$ and is therefore only a *partial representation* of the original function. It is interesting to imagine being handed

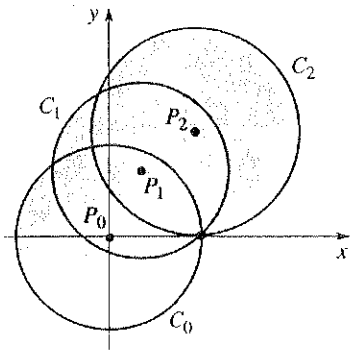



**Figure 4.** Analytic continuation.

the series $1 + z + z^2 + \cdots$ and not knowing that the sum function is $1/(1-z)$. Applying the ratio test to the series, we ascertain that it converges only in $|z| < 1$. Nevertheless, if we wish to compute values of the mysterious function (of which the series is a partial representation) *outside* of $|z| < 1$, we can (at least in principle) proceed as follows. Denote the function, of which $1 + z + z^2 + \cdots$ is a partial representation, as $f(z)$. Then we can compute $f, f', f'', \ldots$ at some point $P_1$, of our choosing, within $|z| < 1$ (Fig. 4). With these values in hand, we can expand $f$ in Taylor series about $P_1$, obtaining

$$f(z) = f(P_1) + f'(P_1)(z - P_1) + \frac{f''(P_1)}{2!}(z - P_1)^2 + \cdots. \tag{29}$$

Then, applying the ratio test (or some other convergence test) to this new Taylor series, we would find that it converges inside the circle $C_1$. [Since we are pretending not to know that $f$ happens to be $1/(1-z)$, we do not know about the singular point at $z = 1$.] Similarly, we can use the representation (29) to compute $f, f', f'', \ldots$ at some other point $P_2$ (Fig. 4) and thus derive still another Taylor series representation of $f$, this time valid inside $C_2$, and so on. This "stepping-stone" process is an example of **analytic continuation.*** Incidentally, the best we could possibly do in this example, given the series $1+z+\cdots$, is to obtain $f(z) = 1/(1-z)$, which is the most complete possible analytic continuation of this particular series.

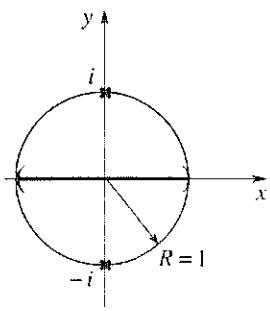



**Figure 5.** Disk of convergence.

COMMENT 2. Consider, instead, the function

$$g(z) = \frac{1}{1 + z^2}. \tag{30}$$

We can see that $g$ is analytic everywhere except at $z = \pm i$. Thus, we know in advance, even before generating its Taylor series about $z = 0$, that the series will converge in $|z| < 1$ and diverge in $|z| \geq 1$ (Fig. 5). To generate the Taylor series we can, of course, use (15). However, it is easier to recall the geometric series

$$\frac{1}{1-t} = 1 + t + t^2 + \cdots \tag{31}$$

and to notice that (30) is of that form, with $t = -z^2$. Then, without further ado, (31) gives

$$\frac{1}{1 + z^2} = 1 - z^2 + z^4 - z^6 + \cdots. \tag{32}$$

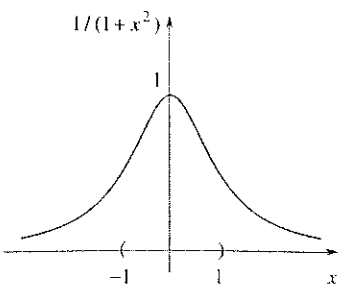



**Figure 6.** Graph of $1/(1 + x^2)$.

Since (31) converges in $|t| < 1$, and $t$ is related to $z$ by $t = -z^2$, it follows from $|z| = \sqrt{|t|}$ that (32) converges in $|z| < 1$, as we predicted from the fact that $g$ has singularities at $\pm i$.

If we restrict $z$ to lie on the real axis, then (32) becomes the real Taylor series

$$\frac{1}{1 + x^2} = 1 - x^2 + x^4 - x^6 + \cdots \qquad (|x| < 1). \tag{33}$$

If one encounters (33) in studying the real variable calculus, the limitation $|x| < 1$ must surely seem paradoxical. For why should the series diverge outside $|x| < 1$ if the function $1/(1 + x^2)$ is so beautifully behaved (e.g., infinitely differentiable) for *all* $x$ (Fig. 6)?

*For discussion of analytic continuation, see for example E. B. Saff and A. D. Snider, *Fundamentals of Complex Analysis* (Englewood Cliffs, NJ: Prentice Hall, 1976), or Konrad Knopp, *Theory of Functions*, Part I (New York: Dover, 1945).

However, our study of complex function theory now renders the situation transparent: the restriction $|x| < 1$ in (33) occurs because the function $1/(1+z^2)$ has singularities *off the real axis* at $z = +i$ and $z = -i$. ∎

**EXAMPLE 7.** As a second example in which the results of real variable theory appear paradoxical, consider the function

$$f(x) = \begin{cases} e^{-1/x^2}, & x \neq 0 \\ 0, & x = 0 \end{cases} \tag{34}$$

the graph of which is shown in Fig. 7. Besides $f(0) = 0$, one can show (which is left for the exercises) that $f'(0) = f''(0) = f'''(0) = \cdots = 0$ so the Taylor series of $f$ about $x = 0$ is simply $0 + 0 + 0 + \cdots$. Surely, the latter converges for *all* $x$ (namely, to zero), but it does not *represent* the given $f$ anywhere except at $x = 0$, the only point at which $f(x)$ is exactly zero. Why this striking failure of the Taylor series? After all, $f$ is infinitely differentiable for all $x$ and hence seems quite well behaved.

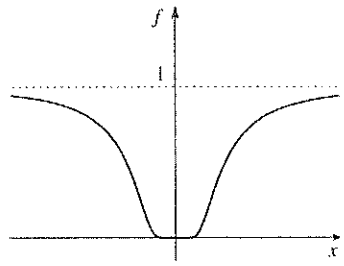



**Figure 7.** Graph of $f$ given by (34)

Again the "paradox" is resolved by examining the behavior of

$$f(z) = \begin{cases} e^{-1/z^2}, & z \neq 0 \\ 0, & z = 0 \end{cases} \tag{35}$$

in the complex plane. Specifically, observe that if $z$ is on the real axis, but $z \neq 0$, then $f(z) = e^{-1/x^2} \to 0$ as $x \to 0$; but if $z \to 0$ along the *imaginary axis*, then $f(z) = e^{-1/(iy)^2} = e^{1/y^2} \to \infty$ as $y \to 0$. Thus, $f(z)$ is not even *continuous* at $z = 0$, let alone analytic. Hence, its Taylor series expansion about $z = 0$ is doomed to converge to $f$ only at the expansion point $z = 0$. ∎

**EXAMPLE 8.** *Undetermined Coefficients.* In this final example we emphasize the practical matter of how to generate the Taylor series of a given function in a convenient manner. To illustrate, consider the expansion of

$$f(z) = \frac{e^z}{\cos z} \tag{36}$$

about $z = 0$. First, observe that $f$ is singular only at the zeros of $\cos z$, namely, at $z = \pm\pi/2, \pm 3\pi/2, \ldots$. Of these, the closest to the origin are $\pm\pi/2$, and these will limit the radius of convergence of the Maclaurin expansion to $R = \pi/2$ (Fig. 8).

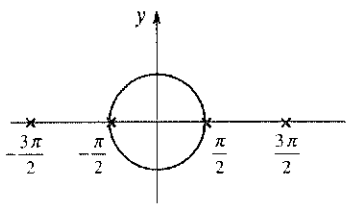



**Figure 8.** Disk of convergence.

Naturally, we can work out $f(0), f'(0), \ldots$ and use the Taylor series formula. However, the repeated differentiations grow increasingly tedious and we wonder whether we might, instead, benefit from the fact that the expansions of $e^z$ and $\cos z$ are known (i.e., remembered or easily derived). Thus, let us proceed as follows. First, express

$$\frac{e^z}{\cos z} = a_0 + a_1 z + a_2 z^2 + \cdots, \tag{37}$$

where the $a_j$'s are to be determined. Multiplying both sides of (37) by $\cos z$ and then recalling the expansions of $e^z$ and $\cos z$, write

$$1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots = (a_0 + a_1 z + a_2 z^2 + \cdots)\left(1 - \frac{z^2}{2} + \frac{z^4}{24} + \cdots\right). \tag{38}$$

The first series on the right converges in $|z| < \pi/2$, and the second (the cosine series) converges in $|z| < \infty$ so it is justified (Theorem 24.2.10, below) to multiply them term by term, in $|z| < \pi/2$, and to rearrange the result in ascending powers of $z$. That step gives

$$1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \cdots = a_0 + a_1 z + \left(a_2 - \frac{1}{2}a_0\right) z^2 + \left(a_3 - \frac{1}{2}a_1\right) z^3 + \cdots. \quad (39)$$

Finally, equating coefficients of like powers of $z$ (Theorem 24.2.9, below), we obtain $a_0 = 1$, $a_1 = 1$, $a_2 = 1$, and $a_3 = \frac{2}{3}$ so

$$\frac{e^z}{\cos z} = 1 + z + z^2 + \frac{2}{3}z^3 + \cdots \quad (40)$$

in $|z| < \pi/2$. ∎

In the preceding example we discovered a need for additional theorems covering the manipulation of power series. For instance, we solved for $a_0, a_1, \ldots$ by equating coefficients of like powers of $z$ on the left- and right-hand sides of (39). Surely

$$\sum_{n=0}^{\infty} a_n (z-a)^n = \sum_{n=0}^{\infty} b_n (z-a)^n \quad (41)$$

holds if $a_n = b_n$ for each $n$, but is $a_n$ *necessarily* equal to $b_n$, for each $n$, for (41) to hold? The answer is yes:

---

**THEOREM 24.2.9** *Uniqueness of Power Series*
If $\sum_{n=0}^{\infty} a_n (z-a)^n$ and $\sum_{n=0}^{\infty} b_n (z-a)^n$ both converge in $|z| < R$, then (41) holds in $|z| < R$ if and only if $a_n = b_n$ for each $n = 0, 1, 2, \ldots$.

---

*Proof*: Since the sum functions of the two series are identical, say $f(z)$, it follows from Theorem 24.2.8 that $a_n = f^{(n)}(a)/n!$ and $b_n = f^{(n)}(a)/n!$. Hence, $a_n = b_n$ for each $n$. ∎

Also, in proceeding from (38) to (39) we multiplied two power series term by term and rearranged the resulting terms in ascending powers. The following theorem justifies such manipulation.

---

**THEOREM 24.2.10** *Termwise Product of Power Series*
If $\sum_{n=0}^{\infty} a_n (z-a)^n$ converges to $f(z)$ in $|z-a| < R_1$ and $\sum_{n=0}^{\infty} b_n (z-a)^n$ converges to $g(z)$ in $|z-a| < R_2$, then the termwise product

$$\left(\sum_{n=0}^{\infty} a_n (z-a)^n\right)\left(\sum_{n=0}^{\infty} b_n (z-a)^n\right)$$

$$= a_0b_0 + (a_0b_1 + a_1b_0)(z-a) + (a_0b_2 + a_1b_1 + a_2b_0)(z-a)^2 + \cdots$$
$$= \sum_{n=0}^{\infty}(a_0b_n + a_1b_{n-1} + \cdots + a_nb_0)(z-a)^n \tag{42}$$

converges to $f(z)g(z)$ in $|z-a| < \min(R_1, R_2)$.

---

The resulting series, on the right-hand side of (42), is known as the **Cauchy product** of the two original series.

**EXAMPLE 9.** Obtain the Cauchy product of the Maclaurin expansions of $1/(1-z)$ and $1/(1+2z)$.

$$\begin{aligned}\frac{1}{1-z}\,\frac{1}{1+2z} &= (1+z+z^2+\cdots)(1-2z+4z^2-\cdots)\\ &= (1)(1) + [(1)(-2)+(1)(1)]z + [(1)(4)+(1)(-2)+(1)(1)]z^2 + \cdots\\ &= 1 - z + 3z^2 + \cdots,\end{aligned} \tag{43}$$

which result does agree with the Maclaurin expansion of the product function $1/(1+z-2z^2)$. Since the original series converge in $|z| < 1$ and $|z| < 1/2$, respectively, their Cauchy product converges to $1/(1+z-2z^2)$ in $|z| < 1/2$. ∎

We give three more theorems about the manipulation of power series. Incidentally, we now know that when we speak of convergent power series we might as well call them Taylor series since Theorem 24.2.8 revealed that a convergent power series *is* a Taylor series, the Taylor series of the sum function.

---

**THEOREM 24.2.11** *Termwise Linear Combination of Power Series*
If $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges to $f(z)$ in $|z-a| < R_1$ and $\sum_{n=0}^{\infty} b_n(z-a)^n$ converges to $g(z)$ in $|z-a| < R_2$, then

$$\alpha\sum_{n=0}^{\infty} a_n(z-a)^n + \beta\sum_{n=0}^{\infty} b_n(z-a)^n = \sum_{n=0}^{\infty}(\alpha a_n + \beta b_n)(z-a)^n \tag{44}$$

converges to $\alpha f(z) + \beta g(z)$ in $|z-a| < \min(R_1, R_2)$.

---

*Proof*:

$$\begin{aligned}\lim_{N\to\infty}\sum_{n=0}^{N}(\alpha a_n + \beta b_n)(z-a)^n &= \alpha\lim_{N\to\infty}\sum_{n=0}^{N} a_n(z-a)^n + \beta\lim_{N\to\infty}\sum_{n=0}^{N} b_n(z-a)^n\\ &= \alpha f(z) + \beta g(z),\end{aligned}$$

where each of the series on the right-hand side converge. ■

In particular, the cases $\alpha = 1$ and $\beta = \pm 1$ show that the power series can be added and subtracted termwise within the smaller disk of convergence.

---

**THEOREM 24.2.12** *Termwise Differentiation and Integration of Power Series*
If $\sum_{n=0}^{\infty} a_n(z-a)^n$ converges to $f(z)$ in $|z-a| < R$, then

(a) the series $\sum_{n=1}^{\infty} na_n(z-a)^{n-1}$, obtained by termwise differentiation, converges to $f'(z)$ in $|z-a| < R$, and

(b) the series $\sum_{n=0}^{\infty} \dfrac{a_n}{n+1}(z-a)^{n+1}$, obtained by termwise integration, converges in $|z-a| < R$.

---

*Proof of (a)*: Since the given series converges to $f(z)$, it must be the Taylor series of $f$ (Theorem 24.2.8); hence, $a_n = f^{(n)}(a)/n!$. We learned, from our discussion of the Cauchy integral formula, that if $f$ is analytic in a given region then so is $f', f'', f'''$, and so on. Thus, $f'(z)$ can be represented by a Taylor series in $|z-a| < R$:

$$\begin{aligned} f'(z) &= \sum_{n=0}^{\infty} \frac{f'^{(n)}(a)}{n!}(z-a)^n = \sum_{n=0}^{\infty}(n+1)\frac{f^{(n+1)}(a)}{(n+1)!}(z-a)^n \\ &= \sum_{n=0}^{\infty}(n+1)a_{n+1}(z-a)^n = \sum_{n=1}^{\infty} na_n(z-a)^{n-1}, \end{aligned}$$

as was to be shown. ■

**Closure.** In Section 24.2.1 we introduce the concept of complex series, and cover several tests for the convergence of complex series of constants and complex power series. Whereas a real power series $\sum a_n(x-a)^n$ converges in an interval $|x-a| < R$ on the $x$ axis, a complex power series $\sum a_n(z-a)^n$ converges in a *disk* $|z-a| < R$ in the $z$ plane.

In Section 24.2.2 we focus on the Taylor series of a function of a complex variable and use the Cauchy integral formula to derive the Taylor series of $f(z)$ about $z = a$ and to show that the series converges to $f(z)$, and hence "represents $f$," in the largest possible disk, centered at $a$, throughout which $f$ is analytic. Since the theory is really an extension of the real variable theory, the expansions of the various elementary functions carry over intact, with $x$'s changed to $z$'s and intervals of convergence changed to disks of convergence. For instance,

$$\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n \qquad \text{in } |z| < 1, \tag{45}$$

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \qquad \text{in } |z| < \infty, \tag{46}$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \qquad \text{in } |z| < \infty, \tag{47}$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \qquad \text{in } |z| < \infty, \tag{48}$$

$$\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} \qquad \text{in } |z| < \infty, \tag{49}$$

$$\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} \qquad \text{in } |z| < \infty. \tag{50}$$

With Theorem 24.2.8 the distinction between power series and Taylor series virtually disappears since a convergent power series is indeed the Taylor series of its sum function.

Examples 6 and 7 reveal that the Taylor series of real variable theory affords only a partial view of the whole, and can therefore sometimes appear paradoxical. For instance, the Taylor series of $1/(1+x^2)$ about $x = 0$ converges only in $-1 < x < 1$, even though the function is well behaved at $x = \pm 1$ and, indeed, is infinitely differentiable for all $x$. The source of the paradox is revealed by considering instead the Taylor series of $1/(1+z^2)$ about $z = 0$ in the complex plane; namely, it is the presence of singularities at $z = \pm i$, *off* of the real axis, that limits the disk of convergence in the $z$ plane and hence the interval of convergence on the $x$ axis.

Example 8, on a method of undetermined coefficients, reveals a general need for more theorems on the manipulation of power series, and we close this section with several such theorems, the gist of which is that convergent power series can be added, subtracted, multiplied, differentiated, and integrated termwise within their common disk of convergence.

## EXERCISES 24.2

**1.** In the second paragraph of Section 24.2.1 we state that $\sum c_n$ converges if and only if $\sum a_n$ and $\sum b_n$ do, where $c_n = a_n + ib_n$. Prove that claim, using the "$\epsilon$ definition" of convergence given in the first paragraph.

**2.** Cite the justification for the first inequality in (2).

**3.** Prove Theorem 24.2.4 for the case where $L > 1$.

**4.** Use any of Theorems 24.2.1–4 to prove Theorem 24.2.5.

**5.** Use any of the theorems of Section 24.2.1 to determine the convergence or divergence of the following complex series.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{(2+i)^n}$

(b) $\displaystyle\sum_{n=5}^{\infty} \frac{n^{50}}{3^n}$

(c) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{(2i)^n \ln n}$

(d) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1+n}{2+n}\right)^3$

(e) $\displaystyle\sum_{n=1}^{\infty} \frac{(1+3i)^n}{n^{100}}$

(f) $\displaystyle\sum_{n=1}^{\infty} n^4 e^{-(5-i)n}$

(g) $\displaystyle\sum_{n=1}^{\infty} e^{-in}$

(h) $\displaystyle\sum_{n=1}^{\infty} (\sin n) \left(\frac{1+i}{2-i}\right)^n$

**6.** Use any of the theorems of Section 24.2.1 to determine, insofar as possible, the regions of convergence and divergence of the following power series.

(a) $\sum_{n=0}^{\infty} z^{2n}$ (b) $\sum_{n=2}^{\infty} n^2(z-3)^n$

(c) $\sum_{n=0}^{\infty} n!(z+5)^n$ (d) $\sum_{n=4}^{\infty} e^n(z+i)^n$

(e) $\sum_{n=1}^{\infty} e^{-n}z^n$ (f) $\sum_{n=0}^{\infty} (n^{100}/n!)z^n$

(g) $\sum_{n=0}^{\infty} e^{in}z^n$ (h) $\sum_{n=0}^{\infty} \frac{\cos n}{n^2+1} z^n$

(i) $\sum_{n=1}^{\infty} [(2-i)z]^n$ (j) $\sum_{n=3}^{\infty} (e^{n^2}/n!)z^n$

**7.** For the function $f(x)$ defined by (34), it is claimed that derivatives of all orders exist and are zero at $x = 0$. Verify that claim for the derivatives of first and second order.

**8.** Is the following power series the Taylor series of some function? If so, in what region does the series represent the function; if not, why not?

(a) $\sum_{n=1}^{\infty} (-1)^{n+1} n(z-1)^n$ (b) $\sum_{n=0}^{\infty} \frac{z^{n-1}}{4^{n+1}}$

(c) $\sum_{n=4}^{\infty} \left(\frac{z+i}{1+i}\right)^{2n}$ (d) $1 + z^3$

**9.** What is the radius of convergence of the Taylor series of $1/(z^2 - 3z + 2)$ about

(a) $z = 0$ (b) $z = 3i$ (c) $z = 1 - 5i$ (d) $z = 5 - i$

**10.** What is the radius of convergence of the Taylor series of $(z^2 - 3z + 2)/(z^2 - 2z + 3i + 1)$ about

(a) $z = 0$ (b) $z = 10i$ (c) $z = 2 - 5i$ (d) $z = 20$

**11.** Obtain the Taylor series of the given function, about $z = a$, and give its radius of convergence $R$.

(a) $\sin z,\ a = 0$ (b) $\sin z,\ a = 2 - i$

(c) $\cos 2z,\ a = 3i$ (d) $e^{z^6},\ a = 0$

(e) $\frac{1}{i+z},\ a = 0$ (f) $\frac{z^3}{2-iz},\ a = 0$

(g) $\sin z^8,\ a = 0$ (h) $z^3,\ a = -2i$

(i) $\frac{1}{1+2z^{35}},\ a = 0$ (j) $z^2 - iz,\ a = 2i$

**12.** (*Binomial series*) (a) Derive the **binomial series**

$$\frac{1}{(1-z)^m} = 1 + mz + \frac{m(m+1)}{2!} z^2 + \frac{m(m+1)(m+2)}{3!} z^3 + \cdots = \sum_{n=0}^{\infty} \frac{(m+n-1)!}{(m-1)!\,n!} z^n \qquad (|z| < 1) \tag{12.1}$$

for any positive integer $m$.

(b) Use (12.1) to obtain the Taylor series of $1/(3-z)^2$ about $z = i$. Identify its disk of convergence. HINT: Write

$$\frac{1}{(3-z)^2} = \frac{1}{[3-(z-i)-i]^2} = \frac{1}{(3-i)^2\left[1-\left(\frac{z-i}{3-i}\right)\right]^2} \tag{12.2}$$

and use $(z-i)/(3-i)$ in place of $z$ in (12.1).

**13.** Use (12.1) in Exercise 12 to obtain the Taylor expansions

(a) $\frac{1}{(2z+1)^3} = \frac{1}{2}\sum_{n=0}^{\infty} (-1)^n(n+1)(n+2)2^n z^n$ in $|z| < \frac{1}{2}$

(b) $\frac{1}{(2z+1)^3} = \frac{1}{250}\sum_{n=0}^{\infty} (-1)^n(n+1)(n+2)\left(\frac{2}{5}\right)^n (z-2)^n$ in $|z-2| < \frac{5}{2}$

(c) $\frac{1}{z^2-z-6} = \frac{1}{20}\sum_{n=0}^{\infty} \frac{(-4)^{n+1}-1}{4^n}(z+1)^n$ in $|z+1| < 1$

**14.** Let $\sqrt{z}$ be defined by a principal-value branch cut (i.e., with the origin and negative $x$ axis deleted and $-\pi < \theta < \pi$). Work out the first several terms of the Taylor series, about the given point, and give the region of validity of the Taylor series representation—that is, the region in which the series converges to the given function.

(a) $a = 1$ (b) $a = -i$ (c) $a = i$

(d) $a = -1 - i$ (e) $a = -2 + i$ (f) $a = 2 + i$

**15.** Determine the coefficients of the next two terms in (40) (i.e., the $z^4$ and $z^5$ terms).

**16.** Use the method of undetermined coefficients to determine the first several terms of the Maclaurin series of the given function, and give its region of convergence.

(a) $\tan z = \dfrac{\sin z}{\cos z}$ (b) $\sec z = \dfrac{1}{\cos z}$ (g) $\dfrac{1}{2 - \sin z}$ (h) $\dfrac{1}{3 + \cos z}$

(c) $\operatorname{cosec} z = \dfrac{1}{\sin z}$ (d) $\dfrac{1 + z}{1 + 2z + 3z^2}$

(e) $\dfrac{3 - z}{2 + 3z^2 + z^4}$ (f) $\dfrac{e^z}{\sin 2z}$

**17.** Show that $f'(0) = f''(0) = \cdots = 0$, as claimed in Example 7. HINT: Use the difference quotient definition of the derivative.

## 24.3 Laurent Series

Consider the function

$$f(z) = \frac{1}{(z-1)(z-2i)}, \tag{1}$$

which is analytic for all $z$ except at the two (singular) points $z = 1$ and $z = 2i$. If we develop a Taylor expansion about $z = 0$, that expansion will be valid [that is, will converge to $f(z)$] in the disk $|z| < 1$. Thus, expanding about $z = 0$, the region $|z| \geq 1$ is inaccessible to us. However, there exists a more general representation known as a *Laurent series*, which includes the Taylor series as a special case and which permits expansions in any *annulus* throughout which $f$ is analytic. Thus, for the $f$ given above, there would be three possible Laurent expansions about $z = 0$: one in $2 < |z| < \infty$, one in $1 < |z| < 2$, and one in $|z| < 1$ (Fig. 1a). (The latter would simply be the *Taylor series*.) In fact, with the Laurent series one is even able to expand about a singular point! For the function $f$ given above, for example, we can expand about $z = 1$ in $0 < |z - 1| < \sqrt{5}$ or in $\sqrt{5} < |z - 1| < \infty$ (Fig. 1b); similarly, we could expand about the other singular point $z = 2i$.

The relevant theorem is as follows.

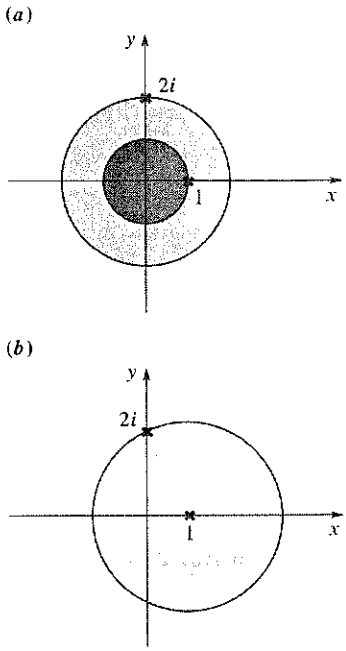



**Figure 1.** Laurent series regions for $f$ given by (1): (a) expansions about $z = 0$, (b) expansions about $z = 1$.

**THEOREM 24.3.1** *Laurent Series*
Let $D$ be the closed region between, and including, concentric circles $C_i$, $C_o$, with their centers at $z = a$. If $f(z)$ is analytic in $D$, then it admits the **Laurent series** representation

$$\boxed{f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n} \tag{2}$$

in $D$, with the $c_n$'s given uniquely by

$$c_n = \frac{1}{2\pi i} \oint_C \frac{f(\zeta)}{(\zeta - a)^{n+1}}\, d\zeta, \tag{3}$$

where $C$ is any piecewise smooth simple closed counterclockwise path in $D$.

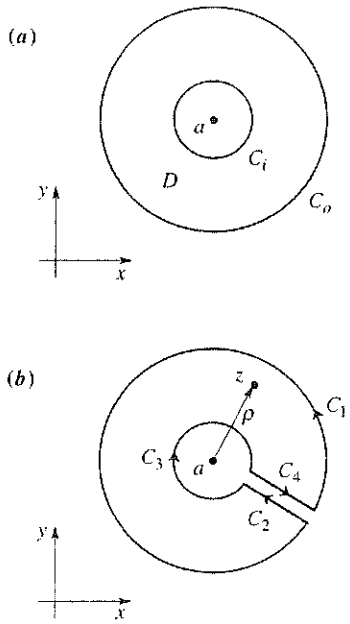



**Figure 2.** Deriving (2).

*Partial Proof*: The circles $C_i$ and $C_o$ are shown in Fig. 2a. Our derivation of (2) will be similar to our derivation of the Taylor series formula (15) in Section 24.2, which derivation we urge you to review before proceeding. Once again our starting point is the Cauchy integral formula, but since the region of interest (the shaded annulus in Fig. 2a) is not bounded by a single piecewise smooth simple closed curve we cannot yet apply that formula. Thus, we first introduce a slit through the annulus, so as to create the piecewise smooth simple curve $C_1 + C_2 + C_3 + C_4$ shown in Fig. 2b, where $C_1$ is the circle $C_o$ taken counterclockwise, $C_3$ is the circle $C_i$ taken clockwise, and $C_2, C_4$ are abutting and oppositely oriented, where $C_2$ and $C_4$ are shown as slightly separated for clarity. Further, $f(z)$ is analytic inside and on that contour so Cauchy's integral formula applies and gives

$$\begin{aligned} f(z) &= \frac{1}{2\pi i}\oint_{C_1+C_2+C_3+C_4} \frac{f(\zeta)}{\zeta - z}\,d\zeta \\ &= \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z}\,d\zeta + \frac{1}{2\pi i}\oint_{C_3} \frac{f(\zeta)}{\zeta - z}\,d\zeta, \end{aligned} \tag{4}$$

where the second equality in (4) holds because the contributions from $C_2$ and $C_4$ are equal and opposite and hence cancel.

Considering the $C_1$ integral first, express

$$\frac{1}{\zeta - z} = \frac{1}{\zeta - a}\,\frac{1}{1 - \dfrac{z-a}{\zeta - a}} = \frac{1}{\zeta - a}\sum_{n=0}^{\infty}\left(\frac{z-a}{\zeta - a}\right)^n, \tag{5}$$

where the first step in (5) is arranged so that the resulting series, on the right-hand side, converges for all $\zeta$ on $C_1$ because $\left|\frac{z-a}{\zeta-a}\right| = |z-a|\,/\,|\zeta - a| < 1$ for all $\zeta$ on $C_1$. Thus,

$$\begin{aligned} \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - z}\,d\zeta &= \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{\zeta - a}\sum_{n=0}^{\infty}\left(\frac{z-a}{\zeta - a}\right)^n d\zeta \\ &= \sum_{n=0}^{\infty}\left[\frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{(\zeta - a)^{n+1}}\,d\zeta\right](z-a)^n \\ &= \sum_{n=0}^{\infty} c_n (z-a)^n, \end{aligned} \tag{6}$$

where

$$c_n = \frac{1}{2\pi i}\oint_{C_1} \frac{f(\zeta)}{(\zeta - a)^{n+1}}\,d\zeta.$$

The path $C_1$ can be deformed to any piecewise smooth simple closed counterclockwise path $C$ lying entirely in $D$ because the integrand is analytic between and on $C_1$ and $C$ so let us re-express $c_n$ as

$$c_n = \frac{1}{2\pi i}\oint_{C} \frac{f(\zeta)}{(\zeta - a)^{n+1}}\,d\zeta. \tag{7}$$

Next, consider the $C_3$ integral in (4). This time, express

$$\frac{1}{\zeta - z} = -\frac{1}{z-a}\,\frac{1}{1-\dfrac{\zeta - a}{z-a}} = -\frac{1}{z-a}\sum_{n=0}^{\infty}\left(\frac{\zeta - a}{z-a}\right)^n, \tag{8}$$

where the series, on the right-hand side converges for all $\zeta$ on $C_3$ because $\left|\frac{\zeta-a}{z-a}\right| = |\zeta - a| / |z-a| < 1$ for all $\zeta$ on $C_3$. Thus,

$$\begin{aligned}
\frac{1}{2\pi i}\oint_{C_3}\frac{f(\zeta)}{\zeta - z}\,d\zeta &= -\frac{1}{2\pi i}\oint_{C_3}\frac{f(\zeta)}{z-a}\sum_{n=0}^{\infty}\left(\frac{\zeta-a}{z-a}\right)^n d\zeta \\
&= \sum_{n=0}^{\infty}\left[-\frac{1}{2\pi i}\oint_{C_3} f(\zeta)(\zeta-a)^n\,d\zeta\right](z-a)^{-(n+1)} \\
&= \sum_{n=-1}^{-\infty}\left[\frac{1}{2\pi i}\oint_{-C_3}\frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta\right](z-a)^n \\
&= \sum_{n=-1}^{-\infty}\left[\frac{1}{2\pi i}\oint_{C}\frac{f(\zeta)}{(\zeta-a)^{n+1}}\,d\zeta\right](z-a)^n \\
&= \sum_{n=-1}^{-\infty} c_n(z-a)^n.
\end{aligned} \tag{9}$$

In the third equality in (9) we shifted the summation index so as to obtain powers of $z - a$, as are present in (6). In the fourth equality we deformed $-C_3$ (i.e., the contour $C_3$ with its direction reversed so as to be counterclockwise) to the same counterclockwise contour $C$ as in (7). Observe that the $c_n$'s in (9) are defined by the same integral expression as those in (7). Thus, putting (6), (7), and (9) into (4) does give the series (2), with the $c_n$'s given by (3).

Observe that we have not justified the interchanges in the order of the infinite summation and the integration, which occurred in the second equality in (6) and in the second equality in (9). We could do that in either of two ways. First, instead of using infinite series, in (5) and (8), we could have used finite series with remainder terms [as we did in equations (18)–(20) in Section 24.2], and then shown, by *ML* bounds, that the remainder term integrals tend to zero as $n \to \infty$. We have omitted those steps for the sake of brevity. Second, we could have justified the limit interchanges by showing that the infinite series in (5) and (8) converge *uniformly*, but–again for brevity–we have not included the discussion of *uniform convergence* that would be needed. ■

Observe that the sum in (2) is unlike the power series and Taylor series of Section 24.2 in that it contains both positive and negative powers: $n$ runs from $-\infty$ to $+\infty$. Since such a sum is new, for us, we need to define it, just as we defined

$\sum_{n=1}^{\infty} c_n$ as $\lim_{N\to\infty} \sum_{n=1}^{N} c_n$ in Section 24.2.1. Reviewing the foregoing proof, we see that (2) came into being as the sum of two "ordinary" series,

$$\sum_{n=-\infty}^{\infty} c_n(z-a)^n = \sum_{n=0}^{\infty} c_n(z-a)^n + \sum_{n=-1}^{-\infty} c_n(z-a)^n \tag{10}$$

so it follows that we are to understand the Laurent series in (2) as

$$\sum_{n=-\infty}^{\infty} c_n(z-a)^n \equiv \lim_{M\to\infty} \sum_{n=0}^{M} c_n(z-a)^n + \lim_{N\to\infty} \sum_{n=1}^{N} c_{-n}(z-a)^{-n}, \tag{11}$$

where $M$ and $N$ tend to infinity independently.

Observe further that $f(z)$ has been assumed analytic only in the annulus $D$. Thus, Theorem 24.3.1 allows for singularities inside $C_i$ and outside $C_o$.

If, in fact, there are no singular points inside $C_i$, then (3) gives

$$c_n = \begin{cases} \dfrac{f^{(n)}(a)}{n!}, & n = 0, 1, 2, \ldots \\ 0, & n = -1, -2, \ldots \end{cases} \tag{12}$$

by the generalized Cauchy integral formula and Cauchy's theorem, respectively, and the Laurent series (2) reduces to the Taylor series

$$f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(z-a)^n, \tag{13}$$

as indeed it should.

However, if $f(z)$ *does* have singular points inside $C_i$ (so that one does not merely have a Taylor series), then (12) does not hold, and we are "stuck" with the integral expression of the $c_n$'s that is given in (3). Thus, whereas Taylor series coefficients are the easily computed quantities $f^{(n)}(a)/n!$, Laurent series coefficients are given by the unwieldy integral expression (3). Do we really need to evaluate that integral to obtain the $c_n$'s? No, *practically speaking we are always able to avoid (3) in developing Laurent series, as illustrated in the examples to follow.*

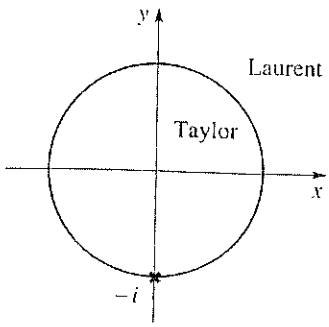



**Figure 3.** Two possible expansions about $z = 0$.

**EXAMPLE 1.** Obtain all possible expansions of

$$f(z) = \frac{1}{z+i} \tag{14}$$

about $z = 0$. Since $f$ is singular only at $z = -i$, it admits exactly two possible expansions about $z = 0$, a Taylor series in $|z| < 1$ and a Laurent series in the annulus $1 < |z| < \infty$ (Fig. 3). The Taylor series is

$$\begin{aligned} \frac{1}{z+i} &= \frac{1}{i}\,\frac{1}{1+\dfrac{z}{i}} = -i\frac{1}{1-iz} \\ &= -i[1 + (iz) + (iz)^2 + \cdots] \\ &= -i + z + iz^2 - \cdots, \qquad (|z| < 1) \end{aligned} \tag{15}$$

where we factored out the $i$ in the denominator simply so that we could use the remembered geometric series formula $1/(1-t) = 1+t+t^2+\cdots$ ($|t| < 1$). Alternatively, we could have applied the Taylor series formula $f(z) = f(a) + f'(a)(z-a) + \cdots$ to $f(z) = 1/(z+i)$.

Next, determine the Laurent expansion in $1 < |z| < \infty$. This time, write

$$\frac{1}{z+i} = \frac{1}{z}\frac{1}{1+\dfrac{i}{z}}. \tag{16}$$

Since we are expanding about $z = 0$, the desired Laurent series will proceed in powers of $z$. The first factor in (16), $1/z$, is already in powers of $z$, so leave it intact. In the second factor, set $t = i/z$ and observe that $|t| = |i/z| = 1/|z| < 1$ because $|z| > 1$, so we can write the *Taylor* expansion

$$\frac{1}{1+i/z} = \frac{1}{1+t} = 1 - t + t^2 - \cdots \qquad (|t| < 1) \tag{17}$$

in the variable $t$. With the expansion accomplished, we revert to the $z$ variable by putting $t = i/z$. Thus,

$$\begin{aligned}\frac{1}{1+i/z} &= 1 - \frac{i}{z} + \left(\frac{i}{z}\right)^2 - \cdots \\ &= 1 - \frac{i}{z} - \frac{1}{z^2} + \cdots, \qquad (1 < |z| < \infty)\end{aligned} \tag{18}$$

so (16) becomes

$$f(z) = \frac{1}{z+i} = \left(\frac{1}{z}\right)\left(1 - \frac{i}{z} - \frac{1}{z^2} + \cdots\right). \tag{19}$$

Since the first factor is valid in the annulus $0 < |z| < \infty$ (i.e., the plane with the origin removed because $1/z$ is undefined at $z = 0$), and the second is valid in the annulus $1 < |z| < \infty$, the product

$$f(z) = \frac{1}{z} - \frac{i}{z^2} - \frac{1}{z^3} + \cdots \tag{20}$$

is valid in the overlap annulus $1 < |z| < \infty$ and is the desired Laurent expansion there. More generally, Laurent series contain both positive and negative powers; (20) happens to contain only negative powers.

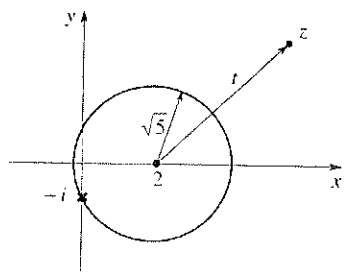



**Figure 4.** Expanding about $z = 2$.

COMMENT 1. The expansions (15) and (20) are about the same point, $z = 0$, yet they are different from each other. That difference does not violate uniqueness for they are valid in different annuli: (15) is the unique expansion in $|z| < 1$, and (20) is the unique expansion in $1 < |z| < \infty$.

COMMENT 2. Suppose we wish to expand about a point other than $z = 0$, say $z = 2$. We can obtain a Taylor series in $|z-2| < \sqrt{5}$, or a Laurent series in $\sqrt{5} < |z-2| < \infty$ (Fig. 4). Let us derive the latter. It is convenient, though not essential, to shift the origin to the point of expansion by settting $t = z - 2$; thus, $t$ is the "vector" from 2 to any point $z$ in the annulus (Fig. 4). Then

$$\frac{1}{z+i} = \frac{1}{t+2+i} = \frac{1}{t}\frac{1}{1+\dfrac{2+i}{t}}$$

$$= \frac{1}{t}\left[1 - \frac{2+i}{t} + \frac{(2+i)^2}{t^2} - \cdots\right]$$
$$= \frac{1}{z-2} - (2+i)\frac{1}{(z-2)^2} + (3+4i)\frac{1}{(z-2)^3} - \cdots \tag{21}$$

is the desired expansion, valid in $\sqrt{5} < |z| < \infty$.

COMMENT 3. As noted above, we can even expand about a singular point. In the present example, the Laurent expansion about the singular point $-i$ is simply

$$f(z) = \frac{1}{z+i}; \tag{22}$$

that is, $1/(z+i)$ is already a Laurent expansion about $-i$ because it proceeds in powers of $z+i$, a one-term expansion, just as $5x^3$ is a one-term Taylor expansion about $x = 0$. ∎

Again, we did *not* use (3) to determine the $c_n$ coefficients in the Laurent series (20). Rather, the rearrangement accomplished in (17) enabled us to recast the problem in terms of a *Taylor* series expansion in the new variable $t$. That idea, recasting the problem in terms of Taylor series, will be relied on exclusively in our derivations of Laurent series.

**EXAMPLE 2.** The function

$$f(z) = \frac{1}{\sin z} \tag{23}$$

is singular at the zeros of $\sin z$, namely, at $z = 0, \pm\pi, \pm 2\pi, \ldots$. Derive the Laurent expansion of $f$ about $z = \pi$, in the annulus $0 < |z - \pi| < \pi$. (See Fig. 5, where we have made the outer radius a bit less than $\pi$, and the inner radius a bit greater than zero, to emphasize that $0 < |z - \pi| < \pi$.) With $t = z - \pi$, for convenience,

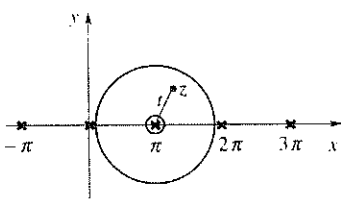



**Figure 5.** Expansion of $1/\sin z$ in $0 < |z - \pi| < \pi$.

$$\frac{1}{\sin z} = \frac{1}{\sin(t+\pi)} = -\frac{1}{\sin t} = -\frac{1}{t}\frac{t}{\sin t}. \tag{24}$$

To motivate the last equality, observe that $\sin t = t - t^3/3! + \cdots = (t)(1 - t^2/3! + t^4/5! - \cdots)$. The first factor (i.e., $t$) vanishes at $t = 0$ and causes $1/\sin t$ to be singular there; the second factor is *not* zero at $t = 0$, so $t/\sin t = (1 - t^2/3! + \cdots)^{-1}$ is analytic at $t = 0$. The idea, then, is that the $1/t$ on the right-hand side of (24) is singular at $t = 0$ and is already a one-term Laurent series about $t = 0$. The $t/\sin t$ has been "desingularized" at $t = 0$, by the $t$ in the numerator, but is still singular at $t = \pm\pi, \pm 2\pi, \ldots$ (i.e., at $z = 2\pi, 0, 3\pi, -\pi, \ldots$) so it admits a Taylor series in $0 \le |t| < \pi$, which Taylor series

$$\frac{t}{\sin t} = a_0 + a_1 t + a_2 t^2 + \cdots \tag{25}$$

can be determined, by the method of undetermined coefficients, as

$$\frac{t}{\sin t} = 1 + \frac{1}{6}t^2 + \frac{7}{360}t^4 + \cdots. \tag{26}$$

Thus,

$$\begin{aligned}\frac{1}{\sin z} &= -\left(\frac{1}{t}\right)\left(1+\frac{1}{6}t^2+\frac{7}{360}t^4+\cdots\right)\\ &= -\frac{1}{z-\pi}-\frac{1}{6}(z-\pi)-\frac{7}{360}(z-\pi)^3-\cdots \qquad (27)\end{aligned}$$

is the desired Laurent series. Since the $1/t$ factor converges in the annulus $0<|t|<\infty$, and the $1+t^2/6+\cdots$ factor converges in $0\le|t|<\pi$, their product converges in the overlap annulus $0<|t|<\pi$, that is, in $0<|z-\pi|<\pi$. ∎

**EXAMPLE 3.** Expand the function

$$f(z)=\frac{1}{z(z-2)}, \qquad (28)$$

which is singular at $z=0$ and $z=2$, about $z=i$, in the annulus $\sqrt{5}<|z-i|<\infty$ (Fig. 6). We could expand each of the factors, $1/z$ and $1/(z-2)$, and multiply their expansions term by term, or we could use partial fractions to express

$$f(z)=-\frac{1}{2}\frac{1}{z}+\frac{1}{2}\frac{1}{z-2}, \qquad (29)$$

and then add the expansions of $-1/(2z)$ and $1/[2(z-2)]$. Since addition is easier than multiplication, let us do the latter. With $t=z-i$, we have

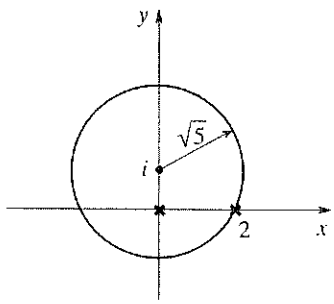



**Figure 6.** Expansion in $\sqrt{5}<|z-i|<\infty$.

$$\begin{aligned}\frac{1}{z} &= \frac{1}{t+i}=\frac{1}{t}\frac{1}{1+\frac{i}{t}}=\frac{1}{t}\left(1-\frac{i}{t}+\frac{i^2}{t^2}-\cdots\right)\\ &= \frac{1}{t}-i\frac{1}{t^2}-\frac{1}{t^3}-\cdots, \qquad (1<|t|<\infty) \qquad (30)\end{aligned}$$

and

$$\begin{aligned}\frac{1}{z-2} &= \frac{1}{t+i-2}=\frac{1}{t}\frac{1}{1+\frac{i-2}{t}}\\ &= \frac{1}{t}\left[1-\frac{i-2}{t}+\frac{(i-2)^2}{t^2}-\cdots\right]\\ &= \frac{1}{t}-(i-2)\frac{1}{t^2}+(3-4i)\frac{1}{t^3}-\cdots, \qquad (\sqrt{5}<|t|<\infty) \qquad (31)\end{aligned}$$

so (29) becomes

$$f(z)=\frac{1}{(z-i)^2}+(2-2i)\frac{1}{(z-i)^3}+\cdots. \qquad (\sqrt{5}<|z-i|<\infty) \qquad (32)$$

COMMENT 1. Working term by term is useful pedagogically because the approach is so concrete, but if possible it is preferable to work with summation notation so as to obtain

the full expansion rather than just the first few terms. In the present case we have, from (29)–(31),

$$f(z) = -\frac{1}{2}\frac{1}{t}\sum_{n=0}^{\infty}\left(-\frac{i}{t}\right)^n + \frac{1}{2}\frac{1}{t}\sum_{n=0}^{\infty}\left(-\frac{i-2}{t}\right)^n$$
$$= \frac{1}{2}\sum_{n=1}^{\infty}\left[(2-i)^n - (-i)^n\right]\frac{1}{(z-i)^{n+1}} \tag{33}$$

in $\sqrt{5} < |z-i| < \infty$, where we have changed the lower summation limit to 1 simply because $(2-i)^0 - (-i)^0 = 1 - 1 = 0$.

COMMENT 2. The expansion of $f$ about $z = i$ in the other annuli, $0 \le |z-i| < 1$ and $1 < |z-i| < \sqrt{5}$, is left for the exercises. ∎

**EXAMPLE 4.** Expand

$$f(z) = e^{1/z} \tag{34}$$

about $z = 0$.

Evidently, $f$ is analytic for all $z \neq 0$ and singular for $z = 0$, so the expansion of $f$ about $z = 0$ will be a Laurent expansion valid in the annulus $0 < |z| < \infty$. To obtain that expansion, let $1/z = t$ and observe that $e^{1/z} = e^t$ admits the *Taylor* expansion

$$e^{1/z} = e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots \tag{35}$$

in $|t| < \infty$. Since $|t| < \infty$ corresponds to $|z| > 0$, it follows [by setting $t = 1/z$ in (35)] that the desired Laurent expansion is

$$e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots \tag{36}$$

in the annulus $0 < |z| < \infty$. ∎

**Closure.** The focus of this section is that we can expand a function $f(z)$, about any point $z = a$, within any annulus centered at $a$ and throughout which $f(z)$ is analytic. In fact, $f(z)$ need not even be analytic at $a$, but if it is, then the innermost such annulus is a disk, and the Laurent series in that disk is simply the Taylor series of $f(z)$ about $a$. Besides that theoretical base, provided by Theorem 24.3.1, we emphasize the technique involved in actually generating the Laurent series of a given function in a particular annulus – namely, that the unwieldy integral formula (3) can be avoided, and that the coefficients $c_n$ can be evaluated by recasting the expansion(s) in terms of Taylor series.

## EXERCISES 24.3

**1.** Derive the right-hand side of (26), up to and including the $t^6$ term, by the method of undetermined coefficients.

**2.** Expand the function $f(z) = 1/[z(z-2)]$, in Example 3, about $z = i$, in the annulus between the two singular points, and show that

$$f(z) = -\frac{1}{2}\sum_{n=0}^{\infty}(-i)^n(z-i)^{-n-1} - \frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{2+i}{5}\right)^{n+1}(z-i)^n$$

in that annulus.

**3.** In Example 3 we use partial fractions to re-express (28) as (29), then we expand each of the two terms in (29) and add their series. The result is given in (33). Here, we ask you to work with the product form (28) rather than the sum form (29), and to show that the same final result is obtained. Specifically, show that (28) gives

$$\begin{aligned} f(z) &= \frac{1}{t^2}\sum_{n=0}^{\infty}\left(-\frac{i}{t}\right)^n \sum_{m=0}^{\infty}\left(\frac{2-i}{t}\right)^m \\ &= \frac{1}{t^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-i)^n(2-i)^m t^{-(m+n)}, \end{aligned} \tag{3.1}$$

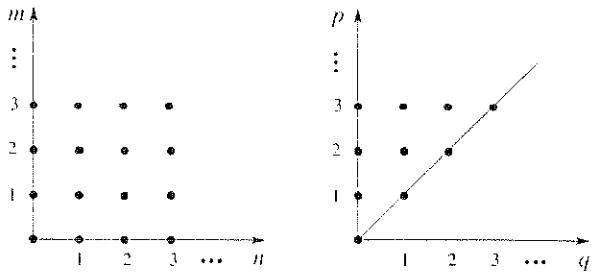
where $t = z - i$. The latter is an iterated sum, and we can handle it in somewhat the same manner as an iterated integral. Since the $m+n$ exponent is "begging" to be a new variable, let us change variables from $m, n$ to $p, q$ according to

$$p = m + n, \qquad q = m \tag{3.2}$$

(or we could use $q = n$: it doesn't matter). Next, show that the regions of summation in the $m, n$ and $p, q$ planes are as shown below, and that

$$f(z) = \sum_{p=0}^{\infty}\left[\sum_{q=0}^{p}(-i)^{p-q}(2-i)^q\right]t^{-(p+2)}. \tag{3.3}$$

Finally, write out the first four terms of (33), and of (3.3), and verify that they agree. In fact, show (33) and (3.3) are identical for *all* terms. HINT: Recall the identity (5.1) in Exercise 5 of Section 4.2. NOTE: It should be clear from this exercise that the partial fraction approach pursued in Example 3 is simpler than the product approach pursued in this exercise.




**4.** Obtain the first three nonvanishing terms of the Laurent expansion of each of the following.

(a) $\dfrac{1}{z}$ in $1 < |z - i| < \infty$

(b) $\dfrac{1}{z^2+1}$ in $1 < |z| < \infty$

(c) $\dfrac{z^2+3}{z}$ in $0 < |z| < \infty$

(d) $\dfrac{1}{e^z - 1}$ in $0 < |z| < 2\pi$

(e) $\dfrac{1}{z(z^3+2)}$ in $0 < |z| < \sqrt[3]{2}$

(f) $\dfrac{1}{z} + \dfrac{z}{z+i}$ in $1 < |z| < \infty$

(g) $\dfrac{1}{z^3}$ in $2 < |z - 2| < \infty$

(h) $\dfrac{1}{z^2}$ in $1 < |z + i| < \infty$

(i) $\dfrac{1}{\cos z}$ in $0 < \left|z - \dfrac{\pi}{2}\right| < \pi$

(j) $\tan z$ in $0 < \left|z + \dfrac{\pi}{2}\right| < \pi$

**5.** Determine all possible Taylor and Laurent expansions about the given point $z = a$, and state their regions of validity.

(a) $\sin\dfrac{1}{z}$, $a = 0$

(b) $\dfrac{1}{z}$, $a = -2$

(c) $e^{-1/z^3}$, $a = 0$

(d) $\dfrac{z^2+5}{z^2+4}$, $a = -1$

(e) $\dfrac{1+z}{2+z}$, $a = -i$

(f) $\dfrac{\sin z}{z^4}$, $a = 0$

(g) $\dfrac{\cos 2z}{(z+i)^2}$, $a = -i$

(h) $e^{-z^2}$, $a = 0$

(i) $e^{-1/z}$, $a = 0$ (j) $\dfrac{1}{z(z^2+1)}$, $a = 1$

(k) $\dfrac{1}{z^2+iz+2}$, $a = i$ (l) $\dfrac{1}{z^2}$, $a = 1+i$

6. A certain function $f(z)$ is represented by the expansion

$$\frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \cdots$$

in $1 < |z| < \infty$. Determine the value of $f(z)$ at $z = 2i$ and at $z = i/3$.

7. A certain function $f(z)$ is represented by the expansion

$$\frac{1}{z} - \frac{1}{z^2} + \frac{1}{z^3} - \cdots$$

in $1 < |z| < \infty$. Determine the value of $f(z)$ at $z = 2$ and at $z = 1/3$.

8. Let $f(z) = \log z$ be defined by a principal value branch cut. Can $f$ be expanded in a Laurent series about $z = 0$? Explain.

9. (*Bessel functions*) In an exercise in Section 4.6 we noted that $\exp\left[\frac{x}{2}(z - \frac{1}{z})\right]$ is the *generating function* for the Bessel functions $J_n(x)$ inasmuch as

$$e^{\frac{x}{2}(z-\frac{1}{z})} = \sum_{n=-\infty}^{\infty} J_n(x) z^n. \tag{9.1}$$

(Here, $x$ is not the real part of $z$, it is an independent real variable.)

(a) Considering the analytic nature of the generating function on the left-hand side, show that (9.1) is valid in $0 < |z| < \infty$.
(b) Use (3), with $C$ taken to be the unit circle, to derive the integral representation of $J_n(x)$,

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(n\theta - x\sin\theta)\, d\theta. \tag{9.2}$$

## 24.4 Classification of Singularities

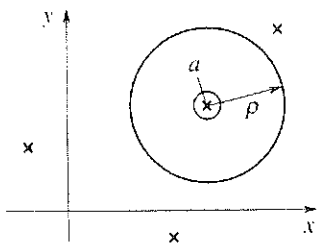



**Figure 1.** Isolated singular point at $z = a$.

Recall that if $f(z)$ is not analytic at a given point then it is singular there. Now that we have studied Laurent series we are in a position to examine the nature of singularities and to distinguish and classify them into different types.

Let $f(z)$ be singular at $z = a$. If it is analytic in an annulus $0 < |z - a| < \rho$ (for some $\rho > 0$), then $z = a$ is said to be an **isolated singular point** of $f$ (Fig. 1); otherwise it is a **nonisolated singular point**. For example, $1/[z(z+2)]$ is singular only at $z = 0$ and $z = -2$, each of which singular point is isolated.

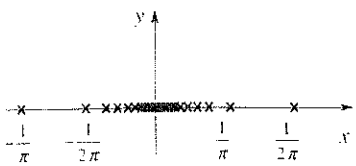



**Figure 2.** The singular points of $1/\sin(1/z)$.

**EXAMPLE 1.** The function

$$f(z) = \frac{1}{\sin(1/z)} \tag{1}$$

is singular at $z = 1/k\pi$ ($k = \pm 1, \pm 2, \ldots$) because $\sin(1/z) = 0$ at those points, and at $z = 0$ as well, because $\sin(1/z)$ is not even defined there (Fig. 2). Each of the former is isolated, but the singular point $z = 0$ is *not* because every annulus $0 < |z| < \rho$ inevitably contains at least one singular point (in fact, an infinite number of them) no matter how small we choose $\rho$. ∎

**EXAMPLE 2.** The function $f(z) = \log z$, made single-valued by a branch cut, is singular at $z = 0$, its singularity being a branch point. The latter is *not* an isolated singular

point because there is no annulus $0 < |z| < \rho$ throughout which $f$ is analytic. Indeed, $f$ is not even *defined* throughout such an annulus because of the intrusion of the branch cut. Similarly for *any* branch point singularity, such as the one at the origin for the function $z^{5/3}$, and the one at $4i$ for the function $\sqrt{z-4i}$. ∎

**EXAMPLE 3.** The function $f(z) = |z|^2 = x^2 + y^2$ is analytic *no*where. Thus, every point in the plane is a nonisolated singular point. ∎

In the remainder of this section we consider only isolated singular points. If $f(z)$ has an isolated singularity at $z = a$, there necessarily exists an annulus $0 < |z-a| < \rho$, for some positive number $\rho$, in which $f$ admits the Laurent series representation

$$\begin{aligned} f(z) &= \sum_{n=-\infty}^{\infty} c_n (z-a)^n \\ &= \cdots + c_{-2}\frac{1}{(z-a)^2} + c_{-1}\frac{1}{z-a} + c_0 + c_1(z-a) + \cdots. \end{aligned} \tag{2}$$

If the expansion terminates on the left so that it is of the form

$$f(z) = c_{-N}\frac{1}{(z-a)^N} + c_{-N+1}\frac{1}{(z-a)^{N-1}} + \cdots, \tag{3}$$

then we categorize the singularity of $f$ at $z = a$ as an **$N$th-order pole**. (A *first*-order pole is sometimes called a *simple* pole.) If not (i.e., if there are an infinite number of negative powers of $z - a$ present), then we categorize the singularity as an **essential singularity**.

**EXAMPLE 4.** The function $f(z) = 1/[z^2(1-z)]$ admits two possible expansions about the singular point $z = 0$:

$$\frac{1}{z^2(1-z)} = \frac{1}{z^2} + \frac{1}{z} + 1 + z + z^2 + \cdots \tag{4}$$

in $0 < |z| < 1$, and

$$\frac{1}{z^2(1-z)} = -\frac{1}{z^3} - \frac{1}{z^4} - \frac{1}{z^5} - \cdots \tag{5}$$

in $1 < |z| < \infty$. The former seems to indicate that $f$ has a second-order pole at $z = 0$, and the latter seems to indicate that it has an essential singularity there. The key is to remember that the classification is based on the Laurent series (2) which is valid in an annulus in $0 < |z-a| < \rho$, that is, in an annulus with an infinitely tight inner circle! Since (5) is valid in $1 < |z| < \infty$ it is irrelevant, gives no information, insofar as the classification of the singularity at $z = 0$. Indeed, that fact is perfectly reasonable in that the region of validity is at a nonzero distance from the point in question. On the other hand, (4) is valid in $0 < |z| < 1$, hence right up to the point in question, and reveals that $f$ has a second-order pole at $z = 0$. ∎

**EXAMPLE 5.** We saw, in Example 4 of Section 24.3, that

$$f(z) = e^{1/z} = 1 + \frac{1}{z} + \frac{1}{2!}\frac{1}{z^2} + \frac{1}{3!}\frac{1}{z^3} + \cdots \tag{6}$$

in $0 < |z| < \infty$. Since (6) contains an infinite number of negative powers, $f$ has an essential singularity at $z = 0$. ∎

To proceed further it is useful to consider the vanishing of a function at a given point. We say that $f(z)$ has a **zero** at $z = a$ if $f(a) = 0$. Suppose, further, that $f(z)$ is analytic at $a$ so that it admits a Taylor series represantation

$$f(z) = f(a) + f'(a)(z-a) + \frac{f''(a)}{2!}(z-a)^2 + \cdots \tag{7}$$

in some neighborhood of $a$. If $f(a) = f'(a) = \cdots = f^{(k-1)}(a) = 0$, and $f^{(k)}(a) \neq 0$, then we say that the zero is of order $k$. Thus, if $f$ has a ***k*th-order zero** at $a$ then

$$\begin{aligned} f(z) &= \frac{f^{(k)}(a)}{k!}(z-a)^k + \frac{f^{(k+1)}(a)}{(k+1)!}(z-a)^{k+1} + \cdots \\ &\sim \frac{f^{(k)}(a)}{k!}(z-a)^k \end{aligned} \tag{8}$$

as $z \to a$, where $f^{(k)}(a) \neq 0$. For example, $\sin z$ has a first-order zero at $z = 0$, $1 - \cos z$ has a second-order zero there, $\sin^3 z$ and $\sin(z^3)$ have third-order zeros there, and so on. If $f(a) \neq 0$ so that $f$ does not have a zero at $a$, it will nevertheless be convenient to say that $f$ has a **zeroth-order zero** there.

---

**THEOREM 24.4.1** *Nth-Order Pole*
If $p(z)$ and $q(z)$ have zeros of order $P$ and $Q$, respectively, at $z = a$, then $f(z) = p(z)/q(z)$ has a pole of order $N = Q - P$ there if $Q > P$, and is analytic there if $Q \leq P$.

---

**EXAMPLE 6.** Locate and classify all singularities of

$$f(z) = \frac{(\pi - z)(z^4 - 3z^2)}{\sin^2 z}. \tag{9}$$

Candidates for singular points are the zeros of the denominator, $z = n\pi$ $(n = 0, \pm 1, \pm 2, \ldots)$. Of these, $z = 0$ and $z = \pi$ also happen to be zeros of the numerator so they need to be considered separately. At $z = 0$, $p(z) = (\pi - z)(z^4 - 3z^2) = -3\pi z^2 + \cdots$ has a second-order zero, and $q = \sin^2 z = z^2 + \cdots$ has a second-order zero, so $P = Q = 2$ and hence (by Theorem 24.4.1) $f$ is analytic there.

At $z = \pi$, $p(z) = (3\pi^2 - \pi^4)(z - \pi) + \cdots$ has a first order zero. Since

$$\sin z = \sin \pi + (\cos \pi)(z - \pi) + \cdots = -(z - \pi) + \cdots, \tag{10}$$

it follows that $q(z) = \sin^2 z = (z - \pi)^2 + \cdots$ has a second-order zero. Thus, $N = Q - P = 2 - 1 = 1$ so $f$ has a first-order pole at $\pi$.

At $z = n\pi$ for $n \neq 0$ and $n \neq 1$, $p(z)$ has a zeroth-order zero so $P = 0$. And since

$$\sin z = \sin n\pi + (\cos n\pi)(z - n\pi) + \cdots = (-1)^n(z - n\pi) + \cdots, \tag{11}$$

we see that $q(z) = \sin^2 z = (z - n\pi)^2 + \cdots$ has a second-order zero so $N = Q - P = 2 - 0 = 2$, and $f$ has a second-order pole at each of those points. ∎

Having distinguished isolated singularities as poles or essential singularities, it is of interest to see how $f(z)$ behaves near such points.

Suppose that $f(z)$ has a pole at $z = a$, say a second-order pole for definiteness. Then

$$f(z) = \frac{c_{-2}}{(z-a)^2} + \frac{c_{-1}}{z-a} + c_0 + c_1(z-a) + \cdots \qquad (c_{-2} \neq 0) \tag{12}$$

holds in $0 < |z - a| < \rho$ for some $\rho$. Since $c_0 + c_1(z - a) + \cdots$ is a convergent power series in $|z - a| < \rho$, its sum, say $g(z)$, is analytic in $|z - a| < \rho$, and we can express (12) more compactly as

$$f(z) = \frac{c_{-2}}{(z-a)^2} + \frac{c_{-1}}{z-a} + g(z). \tag{13}$$

Since $g(z)$ is analytic at $z = a$ it is surely continuous there so

$$(z-a)^2 f(z) = c_{-2} + c_{-1}(z-a) + g(z)(z-a)^2 \to c_{-2}$$

as $z \to a$. Hence, $\left|(z-a)^2 f(z)\right| = |z-a|^2 |f(z)| \to |c_{-2}| \neq 0$, and since $|z-a|^2 \to 0$ as $z \to a$, it follows that $|f(z)| \to \infty$ as $z \to a$ (Fig. 3). Similarly for a pole of *any* order so we can state the following:

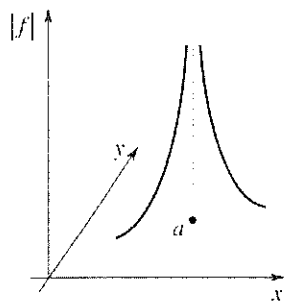



**Figure 3.** Growth of $|f|$ as $z \to a$.

---

**THEOREM 24.4.2** *Behavior Near Pole*
If $f(z)$ has a pole at $z = a$, then $|f(z)| \to \infty$ as $z \to a$.

---

Thus, one says that $f(z)$ "blows up" as $z$ approaches $a$ (from any direction).

The behavior near an essential singularity is more subtle. To illustrate, consider $f(z) = \exp(-1/z^2)$, which has an essential singularity at $z = 0$, as is evident from the Laurent expansion

$$e^{-1/z^2} = 1 - \frac{1}{z^2} + \frac{1}{2!}\frac{1}{z^4} + \frac{1}{3!}\frac{1}{z^6} + \cdots, \tag{14}$$

valid in $0 < |z| < \infty$. Now, if $z \to 0$ along the real axis, then $\exp(-1/z^2) = \exp(-1/x^2) \to 0$ as $x \to 0$ from the left or right, but if $z \to 0$ along the imaginary axis, then $\exp(-1/z^2) = \exp[-1/(iy)^2] = \exp(1/y^2) \to \infty$ as $y \to 0$ from above or below.

In fact, *Emile Picard* (1856–1941) proved the following:

---

**THEOREM 24.4.3** *Behavior Near Essential Singularity; Picard's Theorem*
Let $f(z)$ have an essential singularity at $z = a$. Then $f(z)$ takes on *every* complex number as a value, with at most one exception, and it does so an infinite number of times within any given neighborhood of $a$.

---

Let us illustrate Picard's theorem with an example.

**EXAMPLE 7.** Recall from Example 5 that $e^{1/z}$ has an essential singularity at $z = 0$. Now, $e^{1/z}$ does not equal 0 for any $z$ so 0 must be the "exceptional value" referred to in the theorem, in this case. Let $c$ be any given complex number other than this exceptional value; $c \neq 0$. Then $e^{1/z} = c$ gives $1/z = \log c = \ln|c| + i(\theta_0 + 2n\pi)$, where $\theta_0$ is the principal argument of $c$. Thus, $e^{1/z}$ takes on the value $c$ at the points

$$z_n = \frac{1}{\ln|c| + i(\theta_0 + 2n\pi)} \tag{15}$$

for $n = 0, \pm 1, \pm 2, \ldots$, and (because we can choose $n$ as large as we like) an infinite number of these points are to be found in every neighborhood $|z| < \epsilon$, no matter how small we choose $\epsilon$. ∎

One final idea. Recall from Section 22.3 that when we wish to include the point at infinity in the $z$ plane we speak of the *extended* $z$ plane. Thus, when we say that $f(z) = e^z$, say, is analytic everywhere in the $z$ plane, it is to be understood that we are not including the point at infinity. If we do wish to examine $e^z$ in the extended $z$ plane, we also need to examine it at infinity. To do so, we make the change of variables $z = 1/t$ so $z = \infty$ corresponds to $t = 0$, and then examine $e^z = e^{1/t}$ at $t = 0$. In this case $e^{1/t}$ has an essential singularity at $t = 0$ so we say that $e^z$ has an essential singularity at $z = \infty$. Thus, $e^z$ is analytic everywhere in the extended $z$ plane except at $z = \infty$, where it has an essential singularity. Similarly,

$$f(z) = \frac{1}{z^2} + 3 + z - 4z^3 \tag{16}$$

has a second-order pole at $z = 0$ and a third-order pole at infinity because

$$f\left(\frac{1}{t}\right) = t^2 + 3 + \frac{1}{t} - \frac{4}{t^3} \tag{17}$$

has a third-order pole at $t = 0$.

**Closure.** Distinguishing isolated singularities from nonisolated ones, we limit our attention to the former. If $f(z)$ has an isolated singular point at $z = a$, then there is necessarily an annulus $0 < |z - a| < \rho$, for some $\rho$, within which $f$ admits a Laurent series representation. If that series has an infinite number of negative powers of $z - a$, then the singularity of $f$ at $a$ is called an essential singularity. If instead there are only a finite number of negative powers, the degree of the most negative power being $-N$, then the singularity is called an $N$th-order pole. In making that assessment, it is critical to remember that the Laurent series under consideration must hold in an infinitely tight annulus. Remember that branch point singularities are always nonisolated and therefore fall outside our discussion of poles and essential singularities.

We find that the behavior of $f$ near a pole is quite different from its behavior near an essential singularity. Specifically, $|f(z)| \to \infty$ as $z$ approaches a pole from any direction, whereas if $f$ has an essential singularity at $a$, then the limit of $f$ as $z \to a$ depends upon the direction of approach.

## EXERCISES 24.4

**1.** Prove Theorem 24.4.1.

**2.** Determine the location of the zeros (if any) of order 1 or higher, and their order, for each given function.

(a) $z^2 - z$ (b) $e^z - 1$
(c) $z \sin z$ (d) $z \cos^2 z$
(e) $(z^2 + 1)^3$ (f) $2 + e^z$
(g) $z^2 + z + 1$ (h) $1 - z^4$

**3.** Determine all singular points, in the finite $z$ plane, of the following functions. If isolated, classify them further as $N$th-order poles or essential singularities.

(a) $\dfrac{e^z - 1}{z^3}$ (b) $\dfrac{z^2}{e^z - 1}$

(c) $\dfrac{1}{z^3 - 1}$ (d) $\dfrac{1}{1 + 1/(1 + z)}$

(e) $\dfrac{1}{\sinh z}$ (f) $\dfrac{1}{\cosh z}$

(g) $\dfrac{z}{\sin^3 z}$ (h) $\sin \dfrac{1}{z}$

(i) $\cos \dfrac{1}{z}$ (j) $\sinh \dfrac{1}{z}$

(k) $\cosh \dfrac{1}{z}$ (l) $\sin \dfrac{1}{z^3}$

(m) $e^{1/z^3}$ (n) $4 - \dfrac{z}{(z-1)^2}$

(o) $\dfrac{1}{e^z}$ (p) $\tan(z^2)$

(q) $\tan \dfrac{1}{z^2}$ (r) $\dfrac{1}{e^{z^2}}$

(s) $\dfrac{1}{\sin(z - 2)}$ (t) $\dfrac{1}{\sin \dfrac{1}{z}}$

**4.** (a)–(t) Is the function given in the corresponding part of Exercise 3 analytic at infinity? If not, classify its singularity there.

**5.** A function $f(z)$ is represented in a certain annulus by the given Laurent series. Classify the singularity (if any) of $f$ at $z = 0$.

(a) $\dfrac{1}{z^2} + \dfrac{1}{z^3} + \dfrac{1}{z^4} + \cdots$ (b) $\dfrac{1}{2z} - \dfrac{1}{(2z)^2} + \dfrac{1}{(2z)^3} - \cdots$

(c) $\dfrac{2}{z^4} + 3z^4$ (d) $\dfrac{1}{2!\,z^2} + \dfrac{1}{3!\,z^3} + \dfrac{1}{4!\,z^4} + \cdots$

(e) $\dfrac{1}{z^5}\left[1 - \left(\dfrac{2}{z^3}\right) + \left(\dfrac{2}{z^3}\right)^2 - \left(\dfrac{2}{z^3}\right)^3 + \cdots\right]$

**6.** Give a critical assessment of the following claim: $f(z) = 1/e^z$ has an essential singularity at the origin because

$$\frac{1}{e^z} = \frac{1}{1 + z + \dfrac{z^2}{2!} + \cdots}$$

has an infinite number of negative powers of $z$.

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## 24.5 Residue Theorem

In this final section we derive the powerful residue theorem for the evaluation of complex integrals (and real integrals as well) and give numerous applications.

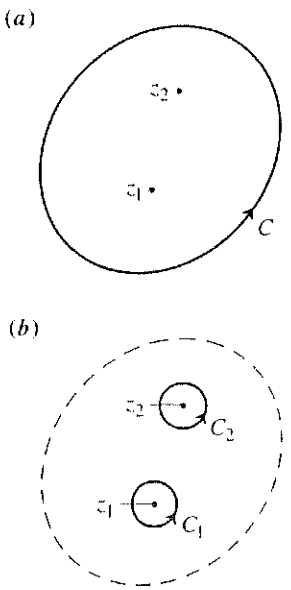



**Figure 1.** Deformation of contour.

**24.5.1. Residue theorem.** Consider the contour integral

$$I = \oint_C f(z)\,dz, \tag{1}$$

where $f$ has a finite number of singular points within $C$, and each of them is isolated. We do not permit singularities on the contour $C$, and whether or not $f$ has singular points outside $C$ will not be relevant.

For definiteness, suppose that $f$ has two isolated singular points within $C$, $z_1$ and $z_2$, as shown in Fig. 1a. Using Cauchy's theorem to deform the contour into $C_1$ and $C_2$ (Fig. 1b), we have

$$I = \oint_C f(z)\,dz = \oint_{C_1} f(z)\,dz + \oint_{C_2} f(z)\,dz. \tag{2}$$

To evaluate the latter two integrals, which we denote as $I_1$ and $I_2$, respectively, expand $f$ in Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n^{(1)}(z - z_1)^n \qquad \text{in } 0 < |z - z_1| < \rho_1 \tag{3a}$$

and

$$f(z) = \sum_{n=-\infty}^{\infty} c_n^{(2)}(z - z_2)^n \qquad \text{in } 0 < |z - z_2| < \rho_2, \tag{3b}$$

and be sure that $C_1$ is small enough to fit entirely within the $0 < |z - z_1| < \rho_1$ annulus, and that $C_2$ is small enough to fit entirely within the $0 < |z - z_2| < \rho_2$ annulus. Then (3a) holds on $C_1$ and (3b) on $C_2$, so we can re-express (2) as

$$I = \oint_{C_1} \sum_{n=-\infty}^{\infty} c_n^{(1)}(z - z_1)^n\,dz + \oint_{C_2} \sum_{n=-\infty}^{\infty} c_n^{(2)}(z - z_2)^n\,dz$$

$$= \sum_{n=-\infty}^{\infty} c_n^{(1)} \oint_{C_1} (z - z_1)^n \, dz + \sum_{n=-\infty}^{\infty} c_n^{(2)} \oint_{C_2} (z - z_2)^n \, dz$$
$$= 2\pi i \, c_{-1}^{(1)} + 2\pi i \, c_{-1}^{(2)}, \tag{4}$$

where the last step follows our "important little integral" (Example 2 in Section 24.3). Understand that (3a) and (3b) are two different Laurent series – of the same function, $f(z)$, but about $z_1$ and $z_2$, respectively. Hence, their coefficients are in general different, and we denote them as $c_n^{(1)}$ and $c_n^{(2)}$, respectively.

Observe that of the infinite number of terms in the $C_1$ integral, only the $n = -1$ term survives and contributes to the answer, its contribution being $2\pi i \, c_{-1}^{(1)}$; similarly for the $C_2$ integral. Thus, the surviving coeficients in (4), $c_{-1}^{(1)}$ and $c_{-1}^{(2)}$, are called the **residues** of $f(z)$ at $z_1$ and $z_2$, and we have found in (4) that $I = \oint_C f(z)\, dz$ is equal to $2\pi i$ times the sum of the residues.

The foregoing derivation is attractive in its directness, but it does beg justification of the termwise integration of the two Laurent series, expressed by the second equality in (4). Though that step can indeed be justified, let us sidestep the issue altogether and simply use equation (3) in Theorem 24.3.1. With $n = -1$, that equation gives, immediately,

$$\oint_{C_1} f(z)\, dz = 2\pi i \, c_{-1}^{(1)} \qquad \text{and} \qquad \oint_{C_2} f(z)\, dz = 2\pi i \, c_{-1}^{(2)}, \tag{5}$$

in agreement with the result obtained above.

Surely the same method applies if $C$ contains any finite number of isolated singular points so we can state the residue theorem:

---

**THEOREM 24.5.1** *Residue Theorem*
Let $C$ be a piecewise smooth simple closed curve oriented counterclockwise, and let $f(z)$ be analytic inside and on $C$ except at finitely many isolated points $z_1, \ldots, z_k$ within $C$. If $c_{-1}^{(j)}$ denotes the residue of $f$ at $z_j$, then

$$\boxed{\oint_C f(z)\, dz = 2\pi i \sum_{j=1}^{k} c_{-1}^{(j)}.} \tag{6}$$

That is, the integral equals $2\pi i$ times the sum of the residues of $f$ within $C$.

---

Again, by the residue of $f$ at $z_j$ we mean the $c_{-1}^{(j)}$ coefficient in the Laurent expansion

$$f(z) = \sum_{n=-\infty}^{\infty} c_n^{(j)} (z - z_j)^n \tag{7}$$

of $f$ about $z_j$, in some annulus $0 < |z - z_j| < \rho_j$.

**EXAMPLE 1.** Evaluate

$$I = \oint_C z^4 \sin \frac{1}{z}\, dz, \tag{8}$$

where $C$ is the circle $|z| = 1$, counterclockwise. There is only one singular point, the essential singular point at $z = 0$. The relevant Laurent series is

$$\begin{aligned} z^4 \sin \frac{1}{z} &= z^4 \left( \frac{1}{z} - \frac{1}{3!\, z^3} + \frac{1}{5!\, z^5} - \frac{1}{7!\, z^7} + \cdots \right) \\ &= z^3 - \frac{z}{3!} + \frac{1}{5!\, z} - \frac{1}{7!\, z^3} + \cdots, \qquad (0 < |z| < \infty) \end{aligned} \tag{9}$$

so the residue is $1/5! = 1/120$, and $I = 2\pi i(1/120) = \pi i/60$.

COMMENT 1. If $C$ were clockwise instead, then we would have $I = -\pi i/60$.

COMMENT 2. If the integrand were $z^3 \sin \frac{1}{z}$ instead, then in place of (9) we would have

$$z^3 \sin \frac{1}{z} = z^2 - \frac{1}{3!} + \frac{1}{5!\, z^2} - \frac{1}{7!\, z^4} + \cdots. \tag{10}$$

It would be incorrect to say "there is no residue" because there is no $1/z$ term. Rather, the coefficient of the $1/z$ term happens to be 0 so the residue is 0 and $I = (2\pi i)(0) = 0$. ∎

In Example 1 it was easy to write out the desired Laurent series, and hence to pick out the residue (as the coefficient of $1/z$). However, the beauty of the residue theorem is that we don't need the entire Laurent series, all we need is one coefficient in it, the $c_{-1}$ residue. Thus, let us develop a method for evaluating the residue without having to generate the entire Laurent series.

**24.5.2. Calculating residues.** To begin, suppose that $f(z)$ has a first-order pole at $a$ so that

$$f(z) = c_{-1} \frac{1}{z - a} + c_0 + c_1(z - a) + \cdots \tag{11}$$

in $0 < |z - a| < \rho$ for some $\rho$. Then

$$(z - a)f(z) = c_{-1} + c_0(z - a) + c_1(z - a)^2 + \cdots, \tag{12}$$

and letting $z \to a$ in both sides gives

$$c_{-1} = \lim_{z \to a} [(z - a)f(z)]. \tag{13}$$

Next, suppose that $f(z)$ has an $N$th-order pole at $z = a$ so

$$f(z) = c_{-N} \frac{1}{(z - a)^N} + c_{-N+1} \frac{1}{(z - a)^{N-1}} + \cdots. \tag{14}$$

Then

$$(z-a)^N f(z) = c_{-N} + c_{-N+1}(z-a) + c_{-N+2}(z-a)^2 + \cdots. \tag{15}$$

Unfortunately, letting $z \to a$ gives $c_{-N}$ rather than the desired coefficient $c_{-1}$. However, the right side of (15) is the Taylor series of $(z-a)^N f(z)$ so it follows that the coefficient $c_{-N+j}$ of $(z-a)^j$ is the $j$th derivative of $(z-a)^N f(z)$ evaluated at $a$ and divided by $j!$. Since $c_{-N+j}$ becomes $c_{-1}$ when $j = N-1$, it follows that

$$\boxed{c_{-1} = \frac{1}{(N-1)!} \lim_{z \to a} \left\{ \frac{d^{N-1}}{dz^{N-1}} \left[ (z-a)^N f(z) \right] \right\}.} \tag{16}$$

To use (16) we need to know the order $N$ of the pole; *if the singularity is an essential singularity, then (16) does not apply.*

**EXAMPLE 2.** Evaluate all residues of

$$f(z) = \frac{1}{(z+4)(z-1)^3}. \tag{17}$$

The denominator has first- and third-order zeros at $-4$ and 1, respectively, and the numerator has zeroth-order zeros at those points so $f$ has a first-order pole ($N = 1$) at $z = -4$ and a third-order pole ($N = 3$) at $z = 1$. Thus, (16) gives

$$\begin{aligned} \operatorname*{Res}_{z=-4} f &= \frac{1}{0!} \lim_{z \to -4} \left[ (z+4) \frac{1}{(z+4)(z-1)^3} \right] \\ &= \lim_{z \to -4} \frac{1}{(z-1)^3} = -\frac{1}{125}, \end{aligned} \tag{18}$$

and

$$\begin{aligned} \operatorname*{Res}_{z=1} f &= \frac{1}{2!} \lim_{z \to 1} \left\{ \frac{d^2}{dz^2} \left[ (z-1)^3 \frac{1}{(z+4)(z-1)^3} \right] \right\} \\ &= \frac{1}{125} \end{aligned} \tag{19}$$

as the desired residues. ∎

**24.5.3. Applications of the residue theorem.** Let us consider several applications.

**EXAMPLE 3.** Evaluate $I = \oint_C f(z)\,dz$, where $f(z)$ is given by (17) and $C$ is counterclockwise. If $C$ encloses both singular points, then the residue theorem gives

$I = 2\pi i(-\frac{1}{125} + \frac{1}{125}) = 0$; if $C$ encloses only one of them, say the one at $z = 1$, then $I = 2\pi i(\frac{1}{125}) = 2\pi i/125$; and if $C$ encloses neither of them, then $I = 0$. ∎

Actually, most practical applications of the residue theorem are to *real* integrals. How can that be? We shall see, in the following examples.

**EXAMPLE 4.** Evaluate the real integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^2+1}. \tag{20}$$

That (20) is a "real integral" is no problem because we can just as well write

$$I = \int_C \frac{dz}{z^2+1},$$

where $C$ is the path from $-\infty$ to $+\infty$ along the $x$ axis. What *is* a problem, however, is that to apply the residue theorem we need $C$ to be a *closed* path. Thus, we will consider, in place of $I$, the contour integral

$$J = \int_C \frac{dz}{z^2+1}, \tag{21}$$

where $C$ is the closed path shown in Fig. 2, not from $-\infty$ to $\infty$ but from $-R$ to $+R$ and then closed with a semicircle.

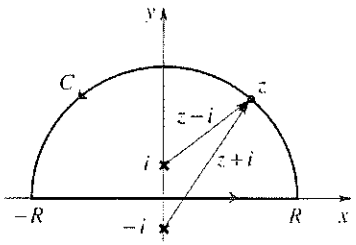



**Figure 2.** The closed contour $C$ in (21).

The integrand has first-order poles at $z = +i$ and $-i$, and is analytic elsewhere. Of these, only the pole at $+i$ is within $C$. Thus, on the one hand, the residue theorem gives

$$J = 2\pi i \operatorname*{Res}_{z=i} f = 2\pi i \lim_{z\to i}\left[(z-i)\frac{1}{(z+i)(z-i)}\right] = \pi \tag{22}$$

(provided that $R > 1$, so that $z = i$ is within $C$). On the other hand,

$$J = \int_{-R}^{R} \frac{dx}{x^2+1} + \int_{C_R} \frac{dz}{z^2+1}, \tag{23}$$

where $C_R$ denotes the semicircular part of $C$. Equating these two results gives

$$\pi = \int_{-R}^{R} \frac{dx}{x^2+1} + \int_{C_R} \frac{dz}{z^2+1}. \tag{24}$$

Now, (24) holds for all $R$'s (greater than 1) so it must hold as $R \to \infty$. Taking that limit, (24) becomes*

$$\pi = \int_{-\infty}^{\infty} \frac{dx}{x^2+1} + \lim_{R\to\infty}\int_{C_R} \frac{dz}{z^2+1}. \tag{25}$$

*Recall from Section 4.5 that the singular integral $\int_a^\infty f(x)\,dx$ is defined as the limit of $\int_a^B f(x)\,dx$ as $B \to \infty$. The case $\int_{-\infty}^{\infty} f(x)\,dx$ was not covered there. The latter integral is defined as the limit of $\int_{-A}^{B} f(x)\,dx$ as $A$ and $B$ tend to infinity *independently*. *If* that limit does indeed exist, then there is no harm in letting $A = B$ and writing $\int_{-\infty}^{\infty} f(x)\,dx$ as the limit of $\int_{-A}^{A} f(x)\,dx$ as $A \to \infty$, as occurs in our passage from (24) to (25).

To evaluate the last term in (25) we use the $ML$ bound. From Fig. 2 we see that as $z$ traverses $C_R$ the $z - i$ "vector" is smallest when $z = Ri$, so its smallest magnitude is $R - 1$. Further, the $z + i$ "vector" is smallest when $z = \pm R$ so its smallest magnitude is $\sqrt{R^2 + 1}$. Thus,

$$\left|\frac{1}{z^2+1}\right| = \frac{1}{|z-i|\,|z+i|} \le \frac{1}{(R-1)\sqrt{R^2+1}} \tag{26}$$

for all $z$ on $C_R$, so the latter can be used as $M$. And since the length of $C_R$ is $\pi R$, we have from the $ML$ bound,

$$\left|\int_{C_R} \frac{dz}{z^2+1}\right| \le \frac{1}{(R-1)\sqrt{R^2+1}}\,\pi R \sim \frac{\pi}{R}, \tag{27}$$

which tends to zero as $R \to \infty$. Thus, (25) becomes $\pi = I + 0$, so $I = \pi$.

COMMENT 1. The method employed in this example is fairly general so let us review it. Given an integral $I$ from $-\infty$ to $\infty$ on a real axis, we considered instead a contour integral $J$ on a closed contour in a $z$ plane. On the one hand, we could evaluate $J$ by the residue theorem and, on the other hand, $J$ could be expressed as the desired integral $I$ plus a computable integral: specifically, the upper semicircle contribution was shown to tend to zero as $R \to \infty$.

If we were given the semi-infinite integral

$$I = \int_0^\infty \frac{dx}{x^2+1}$$

instead, we could use the fact that the integrand $1/(x^2+1)$ is an even function to re-express $I$ as

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{x^2+1},$$

and then proceed as before.

COMMENT 2. Observe that, alternatively, we could have closed the contour below, as in Fig. 3. In that case

$$J = -2\pi i \operatorname*{Res}_{z=-i} f \qquad \text{(the minus sign because } C \text{ is clockwise)}$$

$$= -2\pi i \lim_{z\to -i}\left[(z+i)\frac{1}{(z+i)(z-i)}\right] = \frac{-2\pi i}{-2i} = \pi,$$

so we obtain (25) once again, where this time $C_R$ is the lower semicircle. In the same manner as before, we can show that the $C_R$ integral tends to zero as $R \to \infty$ so we obtain the same final result, $I = \pi$. ∎

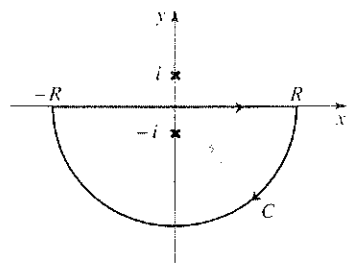



**Figure 3.** Closing $C$ below.

It is true that the integral (20) was simple, and could have been evaluated by a trigonometric change of variables, giving $I = \tan^{-1} x|_{-\infty}^{\infty} = \pi$, but our chief purpose was to develop the contour integration solution method. That method applies equally well to any convergent integral of the form

$$I = \int_{-\infty}^{\infty} \frac{p(x)}{q(x)}\,dx, \tag{28}$$

where $p(x)$ and $q(x)$ are finite-degree polynomials.

**EXAMPLE 5.** Evaluate

$$I = \int_0^\infty \frac{\cos ax}{x^2+1}\,dx. \qquad (a > 0) \tag{29}$$

First, extend the lower integration limit back to $-\infty$ by noting that the integrand is an even function of $x$ so that

$$I = \frac{1}{2}\int_{-\infty}^\infty \frac{\cos ax}{x^2+1}\,dx. \tag{30}$$

Next, consider the contour integral

$$J = \oint_C \frac{\cos az}{z^2+1}\,dz, \tag{31}$$

where $C$ is shown in Fig. 2. Looking ahead, we will wish to show that the $C_R$ integral tends to zero as $R \to \infty$. Does that step appear to bo feasible? Recall that

$$\cos az = \frac{e^{iaz} + e^{-iaz}}{2}.$$

If we consider the single point $z = Ri$ on $C_R$, we see that the $e^{-iaz}$ term in $\cos az$ is $e^{-ia(Ri)} = e^{aR}$ there. This exponential growth (as $R \to \infty$) makes it appear highly unlikely that the $C_R$ integral tends to zero as $R \to \infty$. Furthermore, if we try closing the contour below instead, as in Fig. 3, then at the point $z = -Ri$ on $C_R$ the $e^{+iaz}$ term in $\cos az$ is $e^{ia(-Ri)} = e^{aR}$. Thus, if the $e^{-iaz}$ term does not cause trouble, the $e^{+iaz}$ term does!

To overcome this difficulty, let us consider a *single* exponential in place of $\cos az$. Specifically, consider

$$J = \oint_C \frac{e^{iaz}}{z^2+1}\,dz \tag{32}$$

in place of (31), with $C$ as shown in Fig. 2. The integrand $f(z)$ is analytic everywhere except at $z = \pm i$, where it has first-order poles. Thus, on the one hand,

$$J = 2\pi i \operatorname*{Res}_{z=i} f = 2\pi i\,\frac{e^{ia(i)}}{2i} = \pi e^{-a}, \tag{33}$$

and on the other hand,

$$J = \int_{-R}^R \frac{e^{iax}}{x^2+1}\,dx + \int_{C_R} \frac{e^{iaz}}{z^2+1}\,dz \tag{34}$$

or, with $R \to \infty$,

$$J = \int_{-\infty}^\infty \frac{\cos ax + i\sin ax}{x^2+1}\,dx + \lim_{R\to\infty}\int_{C_R} \frac{e^{iaz}}{z^2+1}\,dz. \tag{35}$$

Since $|e^{iaz}| = |e^{ia(x+iy)}| = |e^{iax}|\,|e^{-ay}| = e^{-ay} \le 1$ on $C_R$, it follows that

$$\left|\frac{e^{iaz}}{z^2+1}\right| \le \left|\frac{1}{z^2+1}\right| \le \frac{1}{(R-1)\sqrt{R^2+1}} \tag{36}$$

everywhere on $C_R$, where the last inequality follows from (26). Furthermore, the length of $C_R$ is $\pi R$, so the $ML$ bound gives

$$\left|\int_{C_R} \frac{e^{iaz}}{z^2+1}\,dz\right| \le \frac{\pi R}{(R-1)\sqrt{R^2+1}} \sim \frac{\pi}{R}, \tag{37}$$

which does tend to zero as $R \to \infty$. With this result, comparison of (33) and (35) gives

$$\pi e^{-a} = \int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx + i\int_{-\infty}^{\infty} \frac{\sin ax}{x^2+1}\,dx. \tag{38}$$

Finally, equating real parts in (38) gives

$$\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx = \pi e^{-a}, \tag{39}$$

and hence the answer

$$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{\cos ax}{x^2+1}\,dx = \frac{\pi}{2}e^{-a}. \tag{40}$$

Of course, we can also equate imaginary parts in (38), and that step gives the "bonus" result

$$\int_{-\infty}^{\infty} \frac{\sin ax}{x^2+1}\,dx = 0, \tag{41}$$

but (41) is not very interesting since it follows from the fact that the integrand is an odd function. ∎

**EXAMPLE 6.** Evaluate

$$I = \int_0^{\infty} \frac{x^{1/3}}{(x+1)^2}\,dx. \tag{42}$$

To do so, consider the contour integral

$$J = \oint_C \frac{z^{1/3}}{(z+1)^2}\,dz, \tag{43}$$

where $C$ is to be a suitably chosen contour. Before selecting $C$ it is important to notice that although the $x^{1/3}$ appearing in (42) is single-valued (e.g., $8^{1/3} = 2$), the $z^{1/3}$ in (43) is *multi*-valued (e.g., $8^{1/3} = 2,\ 2e^{i2\pi/3},\ 2e^{i4\pi/3}$). Having written down the $z^{1/3}$, we need to assume responsibility for defining it. As we will see, a branch cut to the right, as depicted in Fig. 4a, will be convenient.

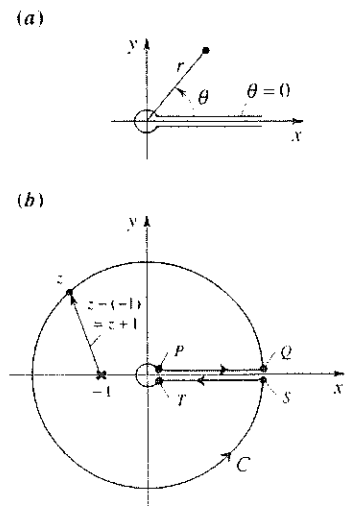



**Figure 4.** Branch cut for $z^{1/3}$, and contour $C$.

Next, let us select the contour $C$ shown in Fig. 4b, where the outer circle is of radius $R$ and the inner circle is of radius $\epsilon$. (We plan to let $R \to \infty$ and $\epsilon \to 0$.) It consists of four parts: $PQ$, $QS$, $ST$, and $TP$. Denote the counterclockwise circle $QS$ as $C_R$, and the clockwise circle $TP$ as $C_\epsilon$.

Observe carefully that on $PQ$ we have $z = xe^{i0}$ so

$$z^{1/3} = x^{1/3}e^{i0/3} = x^{1/3}, \tag{44a}$$

whereas on $ST$ we have $z = xe^{2\pi i}$ so

$$z^{1/3} = x^{1/3}e^{2\pi i/3}. \tag{44b}$$

And since the integrand $f(z)$ is analytic inside and on $C$, except for a second-order pole at $z = -1$, we have

$$\begin{aligned} 2\pi i \operatorname*{Res}_{z=-1} f(z) &= \int_{\epsilon}^{R} \frac{x^{1/3}}{(x+1)^2}\,dx + \int_{C_R} \frac{z^{1/3}}{(z+1)^2}\,dz \\ &\quad + \int_{R}^{\epsilon} \frac{x^{1/3}e^{2\pi i/3}}{(x+1)^2}\,dx + \int_{C_\epsilon} \frac{z^{1/3}}{(z+1)^2}\,dz. \end{aligned} \tag{45}$$

Applying the $ML$ bound to the $C_R$ integral, observe that

$$\left|\frac{z^{1/3}}{(z+1)^2}\right| = \frac{|R^{1/3}e^{i\theta/3}|}{|z+1|^2} = \frac{R^{1/3}}{|z-(-1)|^2} \le \frac{R^{1/3}}{(R-1)^2} \tag{46}$$

on $C_R$ and that the length of $C_R$ is $2\pi R$ so that

$$\left|\int_{C_R} \frac{z^{1/3}}{(z+1)^2}\,dz\right| \le \frac{R^{1/3}}{(R-1)^2}\,2\pi R \sim \frac{2\pi}{R^{2/3}}, \tag{47}$$

which tends to zero as $R \to \infty$. Similarly,

$$\left|\int_{C_\epsilon} \frac{z^{1/3}}{(z+1)^2}\,dz\right| \le \frac{\epsilon^{1/3}}{(1-\epsilon)^2}\,2\pi\epsilon \sim 2\pi\epsilon^{1/3}, \tag{48}$$

which tends to zero as $\epsilon \to 0$.

Furthermore,

$$\begin{aligned} \operatorname*{Res}_{z=-1} f(z) &= \frac{1}{1!}\lim_{z\to -1}\frac{d}{dz}\left[(z+1)^2\frac{z^{1/3}}{(z+1)^2}\right] = \frac{1}{3}(-1)^{-2/3} \\ &= \frac{1}{3}(1e^{\pi i})^{-2/3} = \frac{e^{-2\pi i/3}}{3}, \end{aligned} \tag{49}$$

where, in evaluating $(-1)^{-2/3}$, we have expressed $-1 = 1e^{\pi i}$, in accordance with the branch cut shown in Fig. 4a.

Thus, letting $R \to \infty$ and $\epsilon \to 0$ in (45) gives

$$\begin{aligned} 2\pi i\,\frac{e^{-2\pi i/3}}{3} &= \int_0^\infty \frac{x^{1/3}}{(x+1)^2}\,dx + 0 + \int_\infty^0 \frac{x^{1/3}e^{2\pi i/3}}{(x+1)^2}\,dx + 0 \\ &= (1 - e^{2\pi i/3})\int_0^\infty \frac{x^{1/3}}{(x+1)^2}\,dx, \end{aligned} \tag{50}$$

or,

$$I = \int_0^\infty \frac{x^{1/3}}{(x+1)^2}\,dx = \frac{2\pi i}{3}\,\frac{e^{-2\pi i/3}}{1-e^{2\pi i/3}} = \frac{2\pi}{3\sqrt{3}}. \tag{51}$$

COMMENT 1. Observe that as $R$ tended to infinity and $\epsilon$ tended to zero the contributions from $PQ$, $QS$, $ST$, and $TP$ tended either to zero or to a scalar multiple of $I$, so the final equation (50) was one equation in the one unknown $I$.

COMMENT 2. Although it was important to choose the branch cut to the right, the choice $\theta = 0$ on the top of the cut was not critical; any integer multiple of $2\pi$ would have worked just as well. ∎

From Examples 4, 5, and 6 we can see the general pattern involved in using the residue theorem to evaluate real integrals. The key is in suitably choosing the "$J$ integral," namely, its integrand and its closed contour. We suggest choosing the same integrand as in the $I$ integrand (with the $x$'s changed to $z$'s), and then modifying it only if we find that it doesn't work. For instance, in Example 5 we found that $(\cos az)/(z^2+1)$ did not work since the contribution from the semicircular part of the contour $C_R$ did not tend to zero as $R \to \infty$, so we modified the integrand, slightly, to $e^{iaz}/(z^2+1)$. As for the contour $C$, the idea is to choose $C$ so that one segment of it gives the desired $I$ integral, and each other segment is either known or is some multiple of $I$. In Example 6, for instance, $PQ$ gave $I$, $QS$ went to zero, $ST$ gave $-e^{2\pi i/3}$ times $I$, and $TP$ went to zero as $\epsilon$ tended to zero and $R$ tended to infinity.

It should be appreciated that each example given in this section is representative of an entire class of applications. Example 4, for instance, is representative of the class of integrals of the form $I = \int_{-\infty}^{\infty} p(x)\,dx/q(x)$, where $p$ and $q$ are polynomials.

Next, consider the class of real integrals of the form

$$I = \int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta, \tag{52}$$

where $F$ is a rational function of each of its two arguments. A **rational function** is the ratio of two finite-degree polynomials so $F(x, y)$ is a rational function of each of its two arguments if it is a finite linear combination of terms of the form $x^m y^n$ divided by another such finite linear combination. For example,

$$F_1(x, y) = \frac{5 - x^4 + xy^3}{x + 4y^2}$$

is a rational function of both $x$ and $y$, and

$$F_2(\cos\theta, \sin\theta) = \frac{2 - \cos^2\theta \sin\theta}{1 + \cos\theta}$$

and

$$F_3(\cos\theta, \sin\theta) = \frac{\sin 4\theta}{(1 + \cos\theta)^2}$$

are rational functions of $\cos\theta$ and $\sin\theta$. [In the case of $F_3$, we need to recall the trigonometric identity

$$\sin 4\theta = \sin\theta(4\sin\theta - 8\sin^3\theta).$$

Similar identities exist for $\cos k\theta$ and $\sin k\theta$ for $k = 2, 3, 4, \ldots$.]

Integrals of the form shown in (52) arise, for example, in evaluating the coefficients in Fourier series. To evaluate $I$, enter the change of variables

$$z = e^{i\theta}. \tag{53}$$

This change of variables converts $I$ from an integral on a real $\theta$ axis to a contour integral in a complex $z$ plane for, as $\theta$ varies from 0 to $2\pi$, $z$ undergoes one complete counterclockwise trip around the unit circle (the unit circle because $z = e^{i\theta}$ is of the form $z = re^{i\theta}$ with $r = 1$).

From (53), $dz = ie^{i\theta}\,d\theta = iz\,d\theta$ so $d\theta = dz/iz$, and

$$\begin{aligned} \cos\theta &= \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + z^{-1}}{2} = \frac{z^2 + 1}{2z}, \\ \sin\theta &= \frac{e^{i\theta} - e^{-i\theta}}{2i} = \frac{z - z^{-1}}{2i} = \frac{z^2 - 1}{2iz}, \end{aligned} \tag{54}$$

so

$$I = \int_C F\left(\frac{z + z^{-1}}{2}, \frac{z - z^{-1}}{2i}\right)\frac{dz}{iz}. \tag{55}$$

Now, we see from (54) that $\cos\theta$ and $\sin\theta$ are rational functions of $z$. In turn, $F$ is a rational function of $\cos\theta$ and $\sin\theta$ so $F$ (and hence the new integrand $F/iz$) is a rational function of $z$. Thus, the integrand in (55) is analytic everywhere except at those zeros of the denominator that lie within the contour $C$, that is, within the unit circle. Let us illustrate.

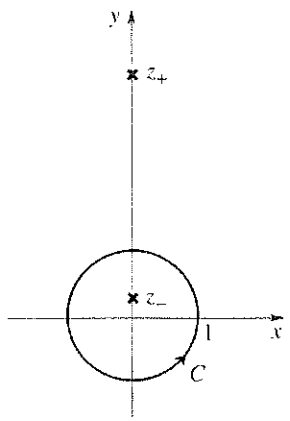



**Figure 5.** Contour and singular points.

**EXAMPLE 7.** Evaluate

$$I = \int_0^{2\pi} \frac{d\theta}{2 - \sin\theta}. \tag{56}$$

With the change of variables $z = e^{i\theta}$, we obtain

$$\begin{aligned} I &= \oint_C \frac{1}{2 - \dfrac{z^2 - 1}{2iz}}\,\frac{dz}{iz} = -2\oint_C \frac{dz}{z^2 - 4iz - 1} \\ &= -2\oint_C \frac{dz}{(z - z_+)(z - z_-)} \equiv \oint_C f(z)\,dz, \end{aligned} \tag{57}$$

where $z_+ = (2 + \sqrt{3})i$ and $z_- = (2 - \sqrt{3})i$. Since $z_+$ lies outside $C$ and $z_-$ lies inside (Fig. 5), the residue theorem gives

$$I = 2\pi i \operatorname*{Res}_{z=z_-} f = 2\pi i \lim_{z\to z_-}\left[(z-z_-)\frac{-2}{(z-z_+)(z-z_-)}\right]$$
$$= 2\pi i\left(\frac{-2}{z_- - z_+}\right) = \frac{2\pi}{\sqrt{3}}. \tag{58}$$

COMMENT 1. Remember that *the residue theorem does not apply if there are singular points on* $C$ for the theorem requires $f(z)$ to be analytic inside and on $C$. For example, if we generalize (15) to the form

$$I = \int_0^{2\pi} \frac{d\theta}{a - \sin\theta}, \tag{59}$$

where $a$ is real, we find that the residue theorem does not apply if $-1 \le a \le 1$ because in that case the singular points $\pm\sqrt{1-a^2} + ai$ fall on the contour $C$.

COMMENT 2. It is also to be noted that if the integral limits in (59) were 0 and $\pi$, say, the residue theorem could not have been applied because the corresponding contour $C$ in the $z$ plane would not have been closed; it would have been only a semicircle. ∎

As our final applications, we indicate how the residue theorem can be used to evaluate inverse Laplace and Fourier transforms. Recall that we introduced the Laplace transform of a function $f(t)$ as

$$L\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}\,dt, \tag{60}$$

and regarded the transform variable $s$ as a real number. For the transform $F(s)$ to *exist*—that is, for the integral in (60) to converge, we asked $f(t)$ to be of exponential order, whereby there exist real constants $K, c$, and $T$ such that $|f(t)| \le Ke^{ct}$ for all $t > T$, for then we can ensure the existence of the transform $F(s)$ by asking $s$ to exceed $c$.

Later, in Section 17.11, we show how to derive both the transform formula (60) and also the *Laplace inversion formula*

$$L^{-1}\{F(s)\} = f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}\,ds. \tag{61}$$

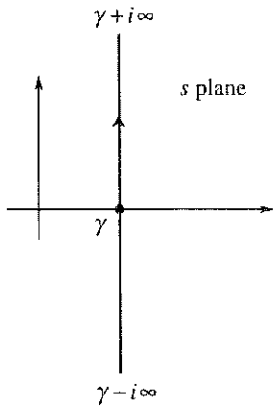



**Figure 6.** The contour in (61).

Thus, even though it is permissible, in working with the Laplace transform in Chapter 5, to consider $s$ as real, the inversion integral is actually along the path shown in Fig. 6 in a complex $s$ plane. Just as we needed $s > c$ in (60) when we regarded $s$ as real, we need $\operatorname{Re} s = \gamma > c$ now that we are considering $s$ to be complex.

**EXAMPLE 8.** *Inverse Laplace Transform.* To illustrate the use of (61), let us determine the inverse of the Laplace transform

$$F(s) = \frac{1}{(s-2)^3}, \tag{62}$$

namely,

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{st}}{(s-2)^3}\,ds. \tag{63}$$

As noted above, we need $\gamma$ to be sufficiently positive, that is, we need $\gamma > c$, but we don't know $c$ yet because we don't know $f(t)$ yet! We assert, without proof, that $\gamma$ *will be sufficiently positive if the line of integration (from $\gamma - i\infty$ to $\gamma + i\infty$) is placed to the right of all singularities of $F(s)$.* In the present example $F(s)$ is singular only at $s = 2$, where it has a third-order pole, so we choose $\gamma > 2$. To obtain a closed contour, shall we close on the left or on the right? Let us close on the left as shown in Fig. 7, with a semicircle of radius $R$ centered at $s = \gamma$. Thus, consider the contour integral

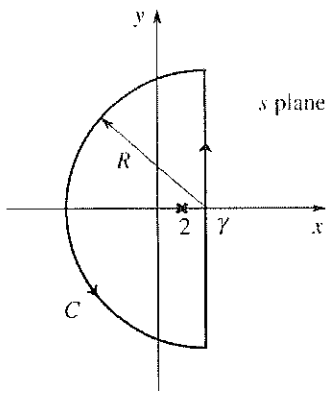



**Figure 7.** Laplace inversion contour.

$$I = \frac{1}{2\pi i}\oint_C \frac{e^{st}}{(s-2)^3}\,ds, \tag{64}$$

where $C$ is shown in the figure.

On the one hand,

$$J = 2\pi i \operatorname*{Res}_{s=2} \frac{e^{st}}{2\pi i(s-2)^3} = \frac{1}{2}t^2e^{2t}, \tag{65}$$

and on the other hand,

$$J = \frac{1}{2\pi i}\int_{\gamma-iR}^{\gamma+iR} \frac{e^{st}}{(s-2)^3}\,ds + \frac{1}{2\pi i}\int_{C_R} \frac{e^{st}}{(s-2)^3}\,ds, \tag{66}$$

where $C_R$ is the semicircular part of $C$. Equating these results, letting $R \to \infty$ and recalling (63) gives

$$\frac{1}{2}t^2e^{2t} = f(t) + \frac{1}{2\pi i}\lim_{R\to\infty}\int_{C_R} \frac{e^{st}}{(s-2)^3}\,ds. \tag{67}$$

Now,

$$\left|e^{st}\right| = \left|e^{(x+iy)t}\right| = \left|e^{xt}\right|\left|e^{iyt}\right| = e^{xt} \le e^{\gamma t}, \tag{68}$$

and

$$|s-2| \ge R-2 \tag{69}$$

for all points $s$ on $C_R$ so

$$\left|\frac{e^{st}}{(s-2)^3}\right| \le \frac{e^{\gamma t}}{(R-2)^3} \tag{70}$$

on $C_R$. Hence the $ML$ bound gives

$$\left|\int_{C_R} \frac{e^{st}}{(s-2)^3}\,ds\right| \le \frac{e^{\gamma t}}{(R-2)^3}\,\pi R \sim \frac{\pi e^{\gamma t}}{R^2}, \tag{71}$$

which tends to zero as $R \to \infty$ (with $t$ held constant). Thus, (67) gives

$$f(t) = \frac{1}{2}t^2e^{2t}, \tag{72}$$

which result agrees with the corresponding entry in Appendix C.

COMMENT 1. With hindsight, we can now see from (72) that $|f(t)| \leq Ke^{ct}$ for some constant $K$ and $t$ sufficiently large, if $c$ is any number greater than 2. Thus, we need $\gamma > 2$, and that is precisely in accord with the italicized assertion below equation (63).

COMMENT 2. You may recall, from Section 17.11, that the inversion integral in (61) gives $f(t)$ for $t > 0$; for $t < 0$ it gives zero:

$$L^{-1}\{F(s)\} = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{st}\,ds = \begin{cases} f(t), & t > 0 \\ 0, & t < 0. \end{cases} \tag{73}$$

Thus far, in this example, we have understood $t$ to be positive, which is of course the interval of interest in applications. However, it will be instructive to show that we do indeed obtain zero, in this example, for $t < 0$ in accordance with (73). Specifically, observe that the $e^{xt} \leq e^{\gamma t}$ inequality in (68) holds on $C_R$ only if $t > 0$. Thus, the foregoing analysis breaks down for $t < 0$, and we need to close the contour on the right instead as shown in Fig. 8. We leave it for the exercises to show that with this contour used in (64) we do obtain $L^{-1}\{F(s)\} = 0$.

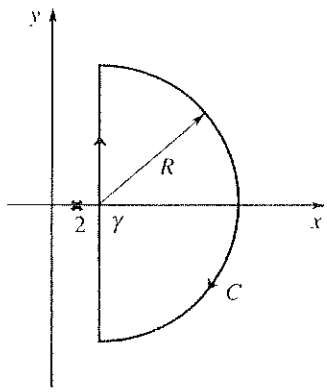



**Figure 8.** Inversion contour for $t < 0$.

COMMENT 3. In summary, the idea is to keep the line from $\gamma - i\infty$ to $\gamma + i\infty$ to the right of all singularities of $F(s)$ and to close the contour on the left for $t > 0$ and on the right for $t < 0$. Of course, the case $t < 0$ need not be carried out because the result will always be zero. ∎

Similarly, to find the inverse of a Fourier transform $\hat{f}(\omega)$ we use the *Fourier inversion formula*

$$F^{-1}\{\hat{f}(\omega)\} = f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}(\omega)e^{i\omega x}\,d\omega. \tag{74}$$

**EXAMPLE 9.** *Inverse Fourier Transform.* Determine the inverse of the Fourier transform

$$\hat{f}(\omega) = \frac{1}{\omega^2+1}, \tag{75}$$

namely,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{\omega^2+1}\,e^{i\omega x}\,d\omega. \tag{76}$$

To do so, consider the contour integral

$$J = \frac{1}{2\pi}\oint_C \frac{e^{ix\omega}}{\omega^2+1}\,d\omega \tag{77}$$

in the complex $\omega$ plane, where $C$ is the closed contour shown in Fig. 9a for $x > 0$ and the closed contour shown in Fig. 9b for $x < 0$. (We close the contour above for $x > 0$ and below for $x < 0$ so that in each case the integral on the semicircular part of $C$ will tend to zero as $R \to \infty$.)

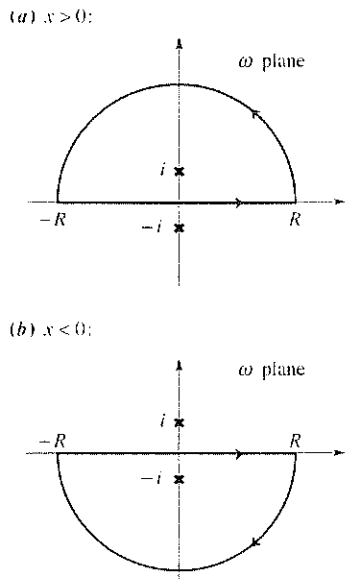



**Figure 9.** Contour $C$ for (77).

First, consider the case $x > 0$. On the one hand,

$$J = 2\pi i \operatorname*{Res}_{\omega=i} \left( \frac{1}{2\pi} \frac{e^{ix\omega}}{\omega^2+1} \right) = 2\pi i \frac{1}{2\pi} \frac{e^{ix(i)}}{2i} = \frac{e^{-x}}{2}, \tag{78}$$

and on the other hand,

$$J = \frac{1}{2\pi} \int_{-R}^{R} \frac{e^{ix\omega}}{\omega^2+1}\, d\omega + \frac{1}{2\pi} \int_{C_R} \frac{e^{ix\omega}}{\omega^2+1}\, d\omega, \tag{79}$$

where $C_R$ denotes the semicircular part of $C$ in Fig. 9a. As $R \to \infty$ the first integral in (79) tends to $f(x)$ [recall (76)] and the second integral tends to zero. [The bounding process is the same as in (36) and (37), with $z$ changed to $\omega$ and $a$ changed to $x$, so we will not repeat it here.] Thus, letting $R \to \infty$ in (79) gives

$$J = f(x) + 0, \tag{80}$$

and comparison of (78) and (80) shows that

$$f(x) = \frac{e^{-x}}{2}. \qquad (x > 0) \tag{81}$$

Next, consider $x < 0$, remembering that this time we close the contour below (Fig. 9b). On the one hand,

$$J = -2\pi i \operatorname*{Res}_{\omega=-i} \left( \frac{1}{2\pi} \frac{e^{ix\omega}}{\omega^2+1} \right) = -2\pi i \frac{1}{2\pi} \frac{e^{ix(-i)}}{-2i} = \frac{e^{x}}{2}, \tag{82}$$

the first minus sign being necessary because this time $C$ is clockwise, and on the other hand

$$J = \frac{1}{2\pi} \int_{-R}^{R} \frac{e^{ix\omega}}{\omega^2+1}\, d\omega + \frac{1}{2\pi} \int_{C_R} \frac{e^{ix\omega}}{\omega^2+1}\, d\omega, \tag{83}$$

where $C_R$ is the semicircular part of $C$ in Fig. 9b. Letting $R \to \infty$ once again, (83) gives

$$J = f(x) + 0, \tag{84}$$

and comparison of (82) and (84) shows that

$$f(x) = \frac{e^{x}}{2}. \qquad (x < 0) \tag{85}$$

Finally, (81) and (85) give

$$f(x) = \frac{e^{-|x|}}{2} \tag{86}$$

for all $x$, which result agrees with entry 4 of Appendix D for the choice $a = 1$. ∎

**Closure.** This section underscores the power of analytic function theory by revealing that the integral around a closed path is simply $2\pi i$ times the sum of the residues

contained within the path. (For a precise statement, see Theorem 24.5.1.) And it is not difficult to calculate residues, even if the integrand is quite unwieldy.

Besides closed loop integrals in the complex plane, we find that the residue theorem can be used to evaluate a wide variety of integrals on the real axis, and inverse Laplace transforms as well. A key step is in setting up the "$J$ integral" – that is, in choosing its integrand and the closed contour $C$. The integrand is often, though not always, the same as the original integrand but with the original real integration variable replaced by the complex variable $z$. As an example of this rule, compare (21) with (20). As an example of an exception to the rule, compare (32) with (30). If the contour is not closed we need to close it, so we can apply the residue theorem.

A basic question arises, which we have not yet addressed: Is the residue theorem more powerful than the generalized Cauchy integral formula? If so, how? In fact, if $f(z)$ has an $N$th-order pole inside a piecewise smooth simple closed counterclockwise curve $C$, at $z = a$, then both the residue theorem and the generalized Cauchy integral formula give

$$I = \oint_C f(z)\,dz = 2\pi i\,\frac{1}{(N-1)!}\lim_{z\to a}\left\{\frac{d^{N-1}}{dz^{N-1}}\left[(z-a)^N f(z)\right]\right\}. \tag{87}$$

However, it is traditional to rely on the residue theorem, for two reasons. First, the residue theorem applies to essential singularities as well as poles, whereas the generalized Cauchy integral formula applies only to poles. Second, and admittedly more superficial, is that the residue theorem is easily remembered: the integral is equal to $2\pi i$ times the sum of the residues.

## EXERCISES 24.5

**1.** Let $C_1$ be a closed rectangular contour, traversed counterclockwise, with vertices at $-1-i$, $3-i$, $3+3i$, $-1+3i$. Let $C_2$ be a closed triangular contour, traversed clockwise, with vertices at $-2$, $2$, and $-2+3i$. Evaluate the given integral by means of the residue theorem.

(a) $\displaystyle\oint_{C_1} \frac{dz}{\sin 2z}$ (b) $\displaystyle\oint_{C_1} \frac{dz}{z^2 e^z}$

(c) $\displaystyle\oint_{C_1} \frac{z^2 dz}{\sinh 2z}$ (d) $\displaystyle\oint_{C_1} \left(\frac{z+1}{z-1}\right)^3 dz$

(e) $\displaystyle\oint_{C_2} \frac{dz}{z^2-2iz-2}$ (f) $\displaystyle\oint_{C_2} \frac{dz}{\cosh^2(\pi z/2)}$

**2.** Evaluate by means of the residue theorem.

(a) $\displaystyle\int_0^\infty \frac{dx}{x^4+a^4}$ HINT: The zeros of $z^4+a^4=0$ are $ae^{\pi i/4}$, $ae^{3\pi i/4}$, $ae^{5\pi i/4}$, and $ae^{7\pi i/4}$. Denote them as $z_1$, $z_2$, $z_3$, $z_4$, respectively. To work out the residue of $1/(z^4+a^4)$ at $z_1$, for instance, it will be easier to evaluate

$$\lim_{z\to z_1}\left[(z-z_1)\frac{1}{z^4+a^4}\right]$$

by *l'Hôpital's rule* than to cancel the $(z-z_1)$'s and evaluate $1/[(z_1-z_2)(z_1-z_3)(z_1-z_4)]$. That step immediately gives the residue as $1/(4z_1^3)$.

(b) $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)}$ $(a>0,\ b>0)$

(c) $\displaystyle\int_0^\infty \frac{x^2}{x^4+1}\,dx$

(d) $\displaystyle\int_0^\infty \frac{dx}{(x^2+1)^2}$

(e) $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{4x^2+2x+1}$

(f) $\displaystyle\int_{0}^{\infty} \frac{x^2}{x^6+1}\,dx$

(g) $\displaystyle\int_{0}^{\infty} \frac{\cos 2x}{(x^2+1)^2}\,dx$

(h) $\displaystyle\int_{0}^{\infty} \frac{x\sin x}{x^4+16}\,dx$

(i) $\displaystyle\int_{-\infty}^{\infty} \frac{\cos x}{8x^2+12x+5}\,dx$

(j) $\displaystyle\int_{-\infty}^{\infty} \frac{\sin 3x}{2x^2+2x+1}\,dx$

**3.** Evaluate the given integral by means of the residue theorem. HINT: The contour in the $z$ plane must be a closed loop, one time around. Thus, the limits on the polar angle must be any $2\pi$ interval, such as 0 to $2\pi$ or $-\pi$ to $\pi$. Thus, in (a) for instance, use the evenness of $\sin^2 x$ to first rewrite the integral as $\frac{1}{2}\int_{-\pi}^{\pi} \sin^2 x\,dx$.

(a) $\displaystyle\int_{0}^{\pi} \sin^2 x\,dx$ (b) $\displaystyle\int_{0}^{\pi} \cos^2 x\,dx$

(c) $\displaystyle\int_{0}^{\pi/2} \sin^2 x\,dx$ (d) $\displaystyle\int_{\pi/2}^{\pi} \cos^2 x\,dx$

(e) $\displaystyle\int_{0}^{\pi} \sin^4 x\,dx$ (f) $\displaystyle\int_{0}^{\pi} \cos^4 x\,dx$

(g) $\displaystyle\int_{0}^{\pi} \sin^6 x\,dx$ (h) $\displaystyle\int_{0}^{4\pi} \cos^6 x\,dx$

(i) $\displaystyle\int_{-\pi}^{\pi} \frac{dt}{7+\cos t}$ (j) $\displaystyle\int_{0}^{\pi} \frac{dt}{1+\cos^2 t}$

(k) $\displaystyle\int_{0}^{2\pi} \frac{d\theta}{1+\sin^2\theta}$ (l) $\displaystyle\int_{0}^{2\pi} \frac{d\theta}{2+\sin 2\theta}$

**4.** Using the residue theorem, show that

$$\int_{0}^{\pi} \frac{\cos t\,dt}{1-2a\cos t+a^2} = \frac{\pi a}{1-a^2}. \qquad (-1<a<1)$$

**5.** Use the inversion formula and the residue theorem to evaluate the inverse of each of these Laplace transforms, where $a>0$ and $b>0$.

(a) $\dfrac{1}{s^2}$ (b) $\dfrac{1}{s^6}$ (c) $\dfrac{1}{s^2+a^2}$

(d) $\dfrac{1}{s^2-a^2}$ (e) $\dfrac{1}{(s-a)^4}$ (f) $\dfrac{s}{(s^2+a^2)^2}$

(g) $\dfrac{1}{(s-a)^2+b^2}$ (h) $\dfrac{e^{-as}}{s^3}$ (i) $\dfrac{e^{-as}}{(s-b)^2}$

**6.** Evaluate each by the residue theorem.

(a) $\displaystyle\int_{0}^{\infty} \frac{x^{a-1}}{x+1}\,dx \quad (0<a<1)$

(b) $\displaystyle\int_{0}^{\infty} \frac{\sqrt{x}}{x^3+1}\,dx$ (c) $\displaystyle\int_{0}^{\infty} \frac{dx}{\sqrt{x}(x^2+1)^2}$

(d) $\displaystyle\int_{0}^{\infty} \frac{\ln x}{x^2+1}\,dx$ (e) $\displaystyle\int_{0}^{\infty} \frac{x^{1/3}}{x^2+4}\,dx$

**7.** Evaluate

$$\int_{0}^{\infty} e^{-x^2}\cos 2ax\,dx = \frac{\sqrt{\pi}}{2}e^{-a^2} \qquad (a>0)$$

by integrating $e^{-z^2}$ around a rectangle with vertices at 0, $R$, $R+ia$ and $ia$, and using the known integral

$$\int_{0}^{\infty} e^{-x^2}\,dx = \frac{\sqrt{\pi}}{2}.$$

**8.** ($\int_0^\infty f(x)\,dx$ *where* $f(x)$ *is not even*) We saw in the examples that if $f(x)$ is even then $\int_0^\infty f(x)\,dx$ can be evaluated by first re-expressing it as $\frac{1}{2}\int_{-\infty}^{\infty} f(x)\,dx$. In this exercise we show what to do if $f(x)$ is not even. To illustrate, we will consider

$$I = \int_{0}^{\infty} \frac{dx}{x^2+x+1}.$$

(a) Show that considering

$$J = \oint_C \frac{dz}{z^2+z+1},$$

where $C$ is the closed contour shown in Fig. 2 is of no help because using the residue theorem and letting $R\to\infty$ gives one equation in the *two* unknown integrals

$$\int_{0}^{\infty} \frac{dx}{x^2+x+1} \quad \text{and} \quad \int_{0}^{\infty} \frac{dx}{x^2-x+1}.$$

(b) Show that $I$ can, however, be evaluated by inserting $\log z$ in the integrand and considering

$$J = \oint_C \frac{\log z}{z^2+z+1}\,dz,$$

where $\log z$ is defined by the branch cut in Fig. 4a and $C$ is the contour shown in Fig. 4b; let $\epsilon\to 0$ and $R\to\infty$. Thus, show that $I = 2\pi/(3\sqrt{3})$.

**9.** Use the idea put forward in Exercise 8(b) to evaluate these integrals:

(a) $\displaystyle\int_0^\infty \frac{dx}{x^3+1}$ (b) $\displaystyle\int_0^\infty \frac{dx}{(x^2+2x+10)^2}$

(c) $\displaystyle\int_0^\infty \frac{x\,dx}{x^3+1}$ (d) $\displaystyle\int_0^1 \frac{dx}{x^2+1}$

HINT: For (d), set $t=(1-x)/x$.

**10.** (*Inversion of Fourier transforms*) Use the inversion formula and the residue theorem to evaluate the inverse of the given transform.

(a) $\dfrac{1}{\omega^2+i\omega+2}$ (b) $\dfrac{1}{\omega^2-3i\omega-2}$

(c) $\dfrac{1}{\omega^2+3i\omega-2}$ (d) $\dfrac{1}{(2-i\omega)^2}$

(e) $\dfrac{1}{(1+i\omega)^2}$ (f) $\dfrac{1}{(1+i\omega)^3}$

(g) $\dfrac{1}{(\omega^2+1)^2}$ (h) $\dfrac{1}{(\omega^2+4)^3}$

**11.** (*Cauchy principal value*) In Section 4.5 we defined the singular integral $\int_a^b f(x)\,dx$, where $f$ is unbounded as $x\to a$, as the limit of the sequence of regular integrals $\int_{a+\epsilon}^b f(x)\,dx$ as $\epsilon\to 0$. The case where $f$ is unbounded at an interior point was not considered there. In this exercise we consider the integral $\int_a^c f(x)\,dx$, where $|f(x)|\to\infty$ as $x\to b$, with $a<b<c$. In the same spirit as the limit definition given above, one defines

$$\int_a^c f(x)\,dx = \lim_{\substack{\epsilon_1\to 0\\ \epsilon_1\to 0}}\left[\int_a^{b-\epsilon_1} f(x)\,dx + \int_{b+\epsilon_2}^c f(x)\,dx\right], \tag{11.1}$$

where $\epsilon_1$ and $\epsilon_2$ tend to zero independently. If the limit exists, we say that $\int_a^c f(x)\,dx$ *exists*, or *converges*; otherwise, it *does not exist*, or *diverges*. Now, it is possible that the stated limit fails to exist, but does exist if we restrict the $\epsilon$'s to tend to zero together: that is, if $\epsilon_1=\epsilon_2$ ($=\epsilon$, say). In that case we write

$$\boxed{\oint_a^c f(x)\,dx = \lim_{\epsilon\to 0}\left[\int_a^{b-\epsilon} f(x)\,dx + \int_{b+\epsilon}^c f(x)\,dx\right],} \tag{11.2}$$

and call $\oint_a^c f(x)\,dx$ the **Cauchy principal value** of the integral. In place of the $\oint$ notation, some authors use $PV\int_a^c f(x)\,dx$.

(a) Show that

$$\int_{-1}^3 \frac{dx}{x}$$

is divergent [i.e., in the sense of (11.1)], but convergent in the Cauchy principal-value sense. Determine its Cauchy principal value.

(b) Repeat part (a) for $\int_1^4 dx/[x(x-2)]$.

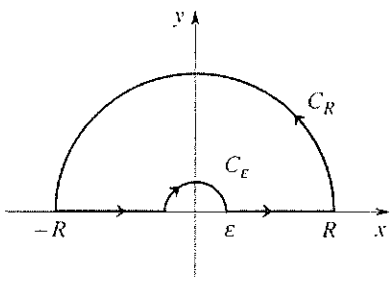
**12.** The integral

$$I=\int_0^\infty \frac{\sin x}{x}\,dx = \frac{\pi}{2} \tag{12.1}$$

is well known, and there are several ways of evaluating it. Here, we ask you to evaluate it using the residue theorem. HINT: Consider

$$J=\oint_C \frac{e^{iz}}{z}\,dz, \tag{12.2}$$

where $C$ is the contour shown in the accompanying figure.




The circular indentation of radius $\epsilon$ is needed to avoid having the first-order pole of $e^{iz}/z$, at $z=0$, lie on the path of integration. (The function $\sin z/z$ is analytic everywhere but, as in Example 5, we consider $e^{iz}/z$ instead so that the $C_R$ integral will tend to zero as $R\to\infty$.) Then, show that

$$J=0=\oint_{-\infty}^{\infty}\frac{\cos x+i\sin x}{x}\,dx + \lim_{R\to\infty}\int_{C_R}\frac{e^{iz}}{z}\,dz + \lim_{\epsilon\to 0}\int_{C_\epsilon}\frac{e^{iz}}{z}\,dz, \tag{12.3}$$

where $C_R$ and $C_\epsilon$ are the semicircular contours of radius $R$ and $\epsilon$, respectively. Show that the $ML$ bound gives

$$\left|\int_{C_R}\frac{e^{iz}}{z}\,dz\right| \le \frac{1}{R}\pi R=\pi, \tag{12.4}$$

which is simply not sharp enough to show that $\int_{C_R}\to 0$ as $R\to\infty$. Thus, and this is the first time the $ML$ bound has not sufficed, use the sharper bound given in Exercise 7 of

Section 24.2 to show that

$$\begin{aligned}\left|\int_{C_R} \frac{e^{iz}}{z}\,dz\right| &\le \int_0^{\pi} \left|\frac{e^{iz}}{z}\right| R\,d\theta \\ &= \int_0^{\pi} e^{-R\sin\theta}\,d\theta = 2\int_0^{\pi/2} e^{-R\sin\theta}\,d\theta \\ &< 2\int_0^{\pi/2} e^{-2R\theta/\pi}\,d\theta \\ &= \frac{\pi}{R}(1-e^{-R}) \to 0\end{aligned} \tag{12.5}$$

as $R \to \infty$, explaining each step. Turning to the $C_\epsilon$ integral, observe that

$$\begin{aligned}\frac{e^{iz}}{z} &= \frac{1}{z} + i - \frac{z}{2} + \cdots \qquad (0 < |z| < \infty) \\ &= \frac{1}{z} + g(z),\end{aligned} \tag{12.6}$$

and show that $g(z)$ is analytic for all $z$. Thus,

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{C_\epsilon} \frac{e^{iz}}{z}\,dz &= \lim_{\epsilon\to 0}\left[\int_{C_\epsilon} \frac{1}{z}\,dz + \int_{C_\epsilon} g(z)\,dz\right] \\ &= -\pi i + \lim_{\epsilon\to 0}\int_{C_\epsilon} g(z)\,dz.\end{aligned} \tag{12.7}$$

Show that the latter limit is zero so that the limit of $\int_{C_\epsilon} e^{iz}\,dz/z$, as $\epsilon \to 0$, is $-\pi i$. Finally, note that

$$\oint_{-\infty}^{\infty} \frac{\cos x + i\sin x}{x}\,dx = \oint_{-\infty}^{\infty} \frac{\cos x}{x}\,dx + i\int_{-\infty}^{\infty} \frac{\sin x}{x}\,dx, \tag{12.8}$$

where the latter integral does not need the Cauchy principal-value sign because the integral is not singular; that is, whereas $(\cos x)/x \sim 1/x$ as $x \to 0$, $(\sin x)/x \sim 1$ there.

# Chapter 24 Review

After introducing the concept of complex series in Section 24.2.1, we use the Cauchy integral formula to derive Taylor series in Section 24.2.2. Whereas the Taylor series of $f(x)$ about $x = a$ converges in an *interval* $|x - a| < R$ (or perhaps only at the point $x = a$), the Taylor series of $f(z)$ about $z = a$ converges in a *disk* $|z - a| < R$ (or perhaps only at the point $z = a$), the radius of which is the distance from $a$ to the nearest singularity of $f$. Theorem 24.2.8 reveal an intimate connection between Taylor series and power series: if a power series converges in $|z - a| < R$, then its sum function $f(z)$ is analytic there and, indeed, the power series is the Taylor series of $f(z)$.

Besides the Taylor series representation of $f$ in the disk $|z - a| < R$, one can develop power series, called Laurent series, that represent $f$ in *annuli* of analyticity. Laurent series differ from Taylor series in that they necessarily include one or more negative powers of $z - a$, and in that their coefficients are given by the unwieldy integral expression $c_n = (1/2\pi i)\oint_C f(\zeta)\,d\zeta/(\zeta - a)^{n+1}$. However, we showed that we can bypass that formula and determine the $c_n$'s using only the methods of Taylor series.

With Laurent series in hand, we are then able to make precise the notion of singularities, by categorizing them into different types. Distinguishing, first, isolated singularities from nonisolated ones, we limit our attention to the former, which are of chief interest to us. Among isolated singularities, we distinguish $N$th-order

poles and essential singularities.

Finally, we are able to return to contour integration and use Laurent series to derive the powerful and elegant residue theorem which states, essentially, that if $f(z)$ has only isolated singularities within $C$, then $\oint_C f(z)\,dz$ is simply $2\pi i$ times the sum of the residues within $C$, where the residue of $f$ at a singular point $z_j$ is the $c_{-1}$ coefficient in the Laurent series of $f$ about $z_j$, within an annulus $0 < |z - z_j| < \rho$. Besides its obvious applicability to integrals on closed paths in the $z$ plane, the residue theorem can, as we see in the examples and exercises, be applied to real integrals and Laplace and Fourier inversions.

# References

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27 explains the method of characteristics for the numerical solution of hyperbolic PDE's, and Chapter 28 explains the method of Green's functions for solving ODE's and PDE's.

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22. E. Zauderer, *Partial Differential Equations of Applied Mathematics*. NY: John Wiley, 1983. This graduate-level text includes nonlinear problems, and perturbation and asymptotic methods, as well as discussion of the method of characteristics for the numerical solution of hyperbolic PDE's.

# Appendix A

# Review of Partial Fraction Expansions

Generally, one meets the method of partial fraction expansions in the integral calculus, where the method is used to express a difficult integral as a linear combination of simpler ones. For example,

$$\begin{aligned}\int \frac{dx}{x^2+3x+2} &= \int \left(\frac{1}{2}\frac{1}{x+1} - \frac{1}{2}\frac{1}{x+3}\right) dx \\ &= \frac{1}{2}\int \frac{dx}{x+1} - \frac{1}{2}\int \frac{dx}{x+3} \\ &= \frac{1}{2}\ln|x+1| - \frac{1}{2}\ln|x+3| + \text{constant}.\end{aligned}$$

In this text we use the method primarily to help us to invert Laplace and Fourier transforms such as the Laplace transform $F(s) = 1/(s^2+3s+2)$. For convenient reference, this appendix contains a review of the method.

Let $p(x)$ and $q(x)$ be finite-degree polynomials in $x$, of degree $P$ and $Q$, respectively. Then

$$f(x) = \frac{p(x)}{q(x)} \tag{A1}$$

is called a **rational function** of $x$. Let $P$ be less than $Q$. [If $P \geq Q$, then we can, by the long division of $q$ into $p$, express $f$ as a polynomial of degree $P - Q$ plus a rational function $r(x)/q(x)$, where the degree $R$ of $r$ *is* less than $Q$. For instance, long division gives

$$\frac{x^5+6x^2-5x+6}{x^3-x^2+x+1} = x^2 - x + 2 + \frac{x^2-4x+1}{x^3+x^2-x+3}.$$

Whereas the method of partial fractions cannot be applied to the rational function on the left (because $P = 5$ is not less than $Q = 3$), it can be applied to the one on the right (because $P = 2$ *is* less than $Q = 3$).]

**Distinct roots.** Let $q(x)$ in (A1) have the distinct roots $x_1, \ldots, x_Q$. Then $f$ admits the **partial fraction expansion**

$$\boxed{f(x) = \frac{p(x)}{q(x)} = \frac{a_1}{x - x_1} + \frac{a_2}{x - x_2} + \cdots + \frac{a_Q}{x - x_Q},} \tag{A2}$$

where the $a_j$'s are constants. One way to determine the $a_j$'s is to recombine the terms on the right-hand side over a common denominator [namely, $q(x)$] and require its numerator to be identical to $p(x)$.

**EXAMPLE 1.** Expand $f(x) = (x - 1)/(x^2 + 5x + 6)$ in partial fractions. Since $x^2 + 5x + 6 = (x + 2)(x + 3)$, we can expand $f$ as

$$f(x) = \frac{x - 1}{x^2 + 5x + 6} = \frac{a_1}{x + 2} + \frac{a_2}{x + 3}.$$

To determine $a_1$ and $a_2$, write

$$\frac{x - 1}{x^2 + 5x + 6} = \frac{(a_1 + a_2)x + (3a_1 + 2a_2)}{x^2 + 5x + 6}.$$

For the numerators to be identical we need

$$\begin{aligned} x^0 &: \quad -1 = 3a_1 + 2a_2, \\ x^1 &: \quad 1 = a_1 + a_2. \end{aligned}$$

Solving these equations gives $a_1 = -3$ and $a_2 = 4$ so

$$\frac{x - 1}{x^2 + 5x + 6} = -\frac{3}{x + 2} + \frac{4}{x + 3}. \quad \blacksquare$$

However, it is simpler to proceed as follows. To calculate $a_1$, multiply (A2) by $x - x_1$, then let $x \to x_1$ in the result. That step gives

$$\lim_{x \to x_1} \left[ (x - x_1) \frac{p(x)}{q(x)} \right] = a_1 + 0 + \cdots + 0.$$

To find $a_2$ multiply by $x - x_2$ and let $x \to x_2$, and so on. Thus,

$$a_j = \lim_{x \to x_j} \left[ (x - x_j) \frac{p(x)}{q(x)} \right] \tag{A3}$$

or, applying l'Hôpital's rule to the indeterminate part, $(x - x_j)/q(x)$,

$$\boxed{a_j = \frac{p(x_j)}{q'(x_j)}.} \tag{A4}$$

In Example 1, $x_1 = -2$ and $x_2 = -3$ so (A4) gives $a_1 = [(x-1)/(2x+5)]|_{x=-2} = -3$ and $a_2 = [(x-1)/(2x+5)]|_{x=-3} = 4$ as obtained above. In summary, if $q(x)$ has distinct roots, then $f(x) = p(x)/q(x)$ can be expanded in partial fractions according to (A2) and the $a_j$'s can be found readily according to (A4).

**Repeated roots.** If any root $x_j$ of $q(x)$ is of multiplicity $k$ [i.e., $(x - x_j)^k$ is a factor of $q(x)$], then the $j$th term on the right-hand side of (A2) must be modified to the form

$$\boxed{\frac{a_{j1}}{x - x_j} + \frac{a_{j2}}{(x - x_j)^2} + \cdots + \frac{a_{jk}}{(x - x_j)^k}} \tag{A5}$$

or, equivalently,

$$\frac{b_{j0} + b_{j1}x + \cdots + b_{j,k-1}x^{k-1}}{(x - x_j)^{k-1}}.$$

To solve for $a_{j1}, \ldots, a_{jk}$ in (A5), we can recombine terms over a common denominator [namely, $q(x)$] and equate coefficients of powers of $x$ in the numerator (since powers of $x$ are linearly independent functions of $x$) as we did in Example 1.

**EXAMPLE 2.** To expand $(4x^2 + 5)/[(x-2)^3(x+3)]$ in partial fractions, write

$$\begin{aligned}
\frac{4x^2+5}{(x-2)^3(x+3)} &= \left[\frac{a}{x-2} + \frac{b}{(x-2)^2} + \frac{c}{(x-2)^3}\right] + \frac{d}{x+3} \\
&= \frac{\left[a(x-2)^2 + b(x-2) + c\right](x+3) + d(x-2)^3}{(x-2)^3(x+3)} \\
&= [(a+d)x^3 + (-a+b-6d)x^2 + (-8a+b+c+12d)x \\
&\qquad + (12a - 6b + 3c - 8d)]/[(x-2)^3(x+3)],
\end{aligned}$$

where the notation $a, b, c, d$ will be simpler than using subscripted $a_{jk}$'s. Thus,

$$\begin{aligned}
x^0 &: \quad 5 = 12a - 6b + 3c - 8d, \\
x^1 &: \quad 0 = -8a + b + c + 12d, \\
x^2 &: \quad 4 = -a + b - 6d, \\
x^3 &: \quad 0 = a + d,
\end{aligned}$$

with solution $a = 41/125$, $b = 59/25$, $c = 21/5$, $d = -41/125$ so

$$\frac{4x^2+5}{(x-2)^3(x+3)} = \frac{41}{125}\frac{1}{x-2} + \frac{59}{25}\frac{1}{(x-2)^2} + \frac{21}{5}\frac{1}{(x-2)^3} - \frac{41}{125}\frac{1}{x+3}. \quad \blacksquare$$

In Example 2 we could have used (A4) to compute $d$ but we could not have used it to compute $a, b, c$ because $x = 2$ is a repeated root. To compute $a_{j1}, \ldots, a_{jk}$ in (A5), we need a modified version of (A4), namely,

$$\boxed{a_{jm} = \frac{1}{(k-m)!}\frac{d^{(k-m)}}{dx^{(k-m)}}\left[(x - x_j)^k \frac{p(x)}{q(x)}\right]\Bigg|_{x \to x_j}} \tag{A6}$$

for $m = 1, \ldots, k$. For instance, if we apply (A6) to the calculation of $a, b, c$ in Example 2 we obtain

$$a = a_{11} = \frac{1}{2!}\frac{d^2}{dx^2}\left[(x-2)^3\frac{4x^2+5}{(x-2)^3(x+3)}\right]\Bigg|_{x\to 2} = \frac{41}{125},$$

$$b = a_{12} = \frac{1}{1!}\frac{d}{dx}\left[(x-2)^3\frac{4x^2+5}{(x-2)^3(x+3)}\right]\Bigg|_{x\to 2} = \frac{59}{25},$$

and

$$c = a_{13} = \frac{1}{0!}\left[(x-2)^3\frac{4x^2+5}{(x-2)^3(x+3)}\right]\Bigg|_{x\to 2} = \frac{21}{5},$$

which results agree with those obtained above.

Partial fraction expansions can also be carried out using computer software. For example, the relevant *Maple* command is **convert**, and the commands

$$\text{convert}((x-1)/(x\hat{}2 + 5 * x + 6), \text{parfrac}, x);$$

and

$$\text{convert}((4 * x\hat{}2 + 5)/((x-2)\hat{}3 * (x+3)), \text{parfrac}, x);$$

give the results that we obtained in Examples 1 and 2, respectively.

We leave the derivation of (A6) for you as an exercise.

# Appendix B

# Existence and Uniqueness of Solutions of Systems of Linear Algebraic Equations

This appendix is intended as a minimal prerequisite for Chapter 3. Alternatively, to integrate linear algebra more heavily with the ODE chapters, we suggest following Chapter 2 with Sections 8.1–10.6 before beginning Chapter 3.

**Definitions.** We call the $m$ equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= c_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= c_2, \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= c_m, \end{aligned} \tag{B1}$$

where the $a_{jk}$ coefficients and the $c_j$'s are known, a **system** of $m$ linear algebraic equations on the $n$ unknowns $x_1, \ldots, x_n$. Any set of numbers $x_1, \ldots, x_n$ that renders each of the $m$ equations a numerical equality is called a **solution** of the system. A system is said to be **consistent** if it admits at least one solution, and **inconsistent** if it admits none. It is shown in Chapter 8 that if (B1) is consistent, then it admits either a **unique** solution (one solution) or an infinity of them, never three solutions or 27 solutions, for instance.

We call the array of coefficients

$$\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \tag{B2}$$

the **coefficient matrix**, and enclose the $a_{jk}$ elements between brackets simply to show that the entire array is being regarded as a single entity. We say that A has

$m$ rows and $n$ columns. For instance, the second row consists of the elements $a_{21}, a_{22}, \ldots, a_{2n}$.

**The case where $m$=$n$.** In applications, the number of equations ($m$) is usually, but not always, equal to the number of unknowns ($n$); in general, $m$ can be less than, equal to, or greater than $n$. Consider first the case where $m = n$. As the simplest case, let $m = n = 1$ so (B1) becomes $a_{11}x_1 = c_1$, or simply

$$ax = c. \tag{B3}$$

If $a \neq 0$, then (B3) admits a unique solution, namely, $x = c/a$. If $a = 0$, however, then there are two possibilities, as can be seen from (B3): if $c \neq 0$, then there is no solution and (B3) is inconsistent, and if $c = 0$, then (B3) is consistent and there is the infinity of solutions $x = \alpha$, where $\alpha$ is arbitrary.

The upshot, for this simple case, is that whether or not the coefficient $a$ is zero is crucial: if $a \neq 0$ (the generic case), then there is a unique solution, and if $a = 0$ (the nongeneric or *singular* case) then there is either no solution or an infinity of them, depending upon the value of $c$. This idea generalizes to the case where $m = n > 1$ as indicated in (i) and (ii) below. First, we define a so-called **determinant** of the **A** matrix, and denote it as

$$\det\mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}, \tag{B4}$$

that is, with straight line braces instead of square brackets. The determinant of **A** is a number (which can be positive, negative, or zero), defined as

$$\begin{vmatrix} a_{11} \end{vmatrix} \equiv a_{11}, \tag{B5a}$$

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} \equiv a_{11}a_{22} - a_{21}a_{12}, \tag{B5b}$$

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \equiv \begin{aligned} & a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) \\ & + a_{13}(a_{21}a_{32} - a_{31}a_{22}) \end{aligned} \tag{B5c}$$

for $n = 1, 2$, and 3, respectively. The general definition for any $n$ is given in Section 10.4, but the cases (B5a,b,c) should suffice for Chapter 3. Do not confuse determinant with absolute value, especially in (B5a) where we cannot tell, from the left-hand side, whether we are signifying the absolute value of $a_{11}$ or the determinant of the tiny matrix $\mathbf{A} = [a_{11}]$, without being told or from the context. For instance, $|-6| = -6$, $\begin{vmatrix} 2 & 5 \\ -3 & 4 \end{vmatrix} = 8 - (-15) = 23$, and

$$\begin{vmatrix} 3 & -2 & 5 \\ 0 & 4 & 6 \\ 1 & 1 & 8 \end{vmatrix} = (3)(32-6) - (-2)(0-6) + (5)(0-4) = 46.$$

Now, whether or not $\det\mathbf{A}$ is zero is crucial insofar as the existence and uniqueness of solutions of the system (B1):

**(i)** If $\det\mathbf{A} \neq 0$, then (B1) is consistent and has a unique solution.

**(ii)** If $\det\mathbf{A} = 0$, then (B1) has either no solution (is inconsistent) or an infinity of solutions, depending upon the $c_j$'s.

For instance, consider the three systems

$$\begin{aligned} 2x_1 - x_2 &= 4 \\ 3x_1 + x_2 &= 11, \end{aligned} \qquad \begin{aligned} 2x_1 - x_2 &= 5 \\ 4x_1 - 2x_2 &= 3, \end{aligned} \qquad \text{and} \qquad \begin{aligned} 2x_1 - x_2 &= 5 \\ 4x_1 - 2x_2 &= 10. \end{aligned}$$

In the first case, $\det\mathbf{A} = 5 \neq 0$ so there is a unique solution (namely, $x_1 = 3$ and $x_2 = 2$); in the second case, $\det\mathbf{A} = 0$ and there is no solution (as is not surprising since the second left-hand side is twice the first left-hand side whereas 3 is not twice 5); and in the third case, $\det\mathbf{A} = 0$ and there is an infinity of solutions [namely, $x_2 = \alpha$ and $x_1 = (5+\alpha)/2$, where $\alpha$ is arbitrary].

In fact, if $\det\mathbf{A} \neq 0$, then we can give a formula for the solution of (B1). Namely, for each $j$ from 1 to $n$, $x_j$ is given as the ratio of two determinants: the one in the denominator is $\det\mathbf{A}$, and the one in the numerator is the same except that its $j$th column is replaced by the column of $c$'s on the right-hand side of (B1). This result is known as **Cramer's rule**. For instance, if

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 1 \\ 2x_1 - 2x_2 + x_3 &= 0 \\ 4x_1 + 3x_2 - x_3 &= -5, \end{aligned} \tag{B6}$$

then Cramer's rule gives

$$x_1 = \frac{\begin{vmatrix} 1 & -1 & 1 \\ 0 & -2 & 1 \\ -5 & 3 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & -1 & 1 \\ 2 & -2 & 1 \\ 4 & 3 & -1 \end{vmatrix}} = \frac{-6}{5} = -\frac{6}{5}, \qquad x_2 = \frac{\begin{vmatrix} 3 & 1 & 1 \\ 2 & 0 & 1 \\ 4 & -5 & -1 \end{vmatrix}}{\begin{vmatrix} 3 & -1 & 1 \\ 2 & -2 & 1 \\ 4 & 3 & -1 \end{vmatrix}} = \frac{11}{5}, \tag{B7}$$

and, in similar fashion, $x_3 = 34/5$. Notice from Cramer's rule that since each $x_j$ is given as the ratio of two determinants, with the determinant in each denominator being the determinant of $\mathbf{A}$, it follows that if $\det\mathbf{A} \neq 0$, then there exists a unique solution for each $x_j$. That result is identical to (i) stated above.

Remember that (i) and (ii), and Cramer's rule, hold only in the case where $m = n$. If $m \neq n$, then the "determinant of $\mathbf{A}$" is not even defined. More about

this in Section 10.4.

**The homogeneous case.** If all of the $c_j$'s in (B1) are zero, then we say that the system is homogeneous. It should be evident that *homogeneous systems are always consistent* since they necessarily admit the solution $x_1 = x_2 = \cdots = x_n = 0$, which is known as the **trivial solution**. (When we give it that name, we do not mean to imply that it is in some way beneath our dignity; it is a perfectly legitimate solution.) Further, *if (B1) is a homogeneous system with $m < n$, then it necessarily admits not only the trivial solution, but also an infinity of nontrivial solutions.* For instance, it can be verified by substitution that the system

$$\begin{aligned} x_1 + 2x_2 + x_3 &= 0 \\ 2x_1 + x_2 - x_3 &= 0 \end{aligned} \tag{B8}$$

admits the solution $x_1 = x_3 = \alpha$ and $x_2 = -\alpha$ for any $\alpha$ (which solutions include the trivial solution for the choice $\alpha = 0$).

# Appendix C

# Table of Laplace Transforms

| $f(t)$ | $\overline{f}(s) = \int_0^\infty f(t)e^{-st}\,dt$ |
|---|---|

NOTE: $s$ is regarded as real here.

| | | |
|---|---|---|
| 1. | $1$ | $\frac{1}{s} \quad (s > 0)$ |
| 2. | $e^{at}$ | $\frac{1}{s-a} \quad (s > a)$ |
| 3. | $\sin at$ | $\frac{a}{s^2+a^2} \quad (s > 0)$ |
| 4. | $\cos at$ | $\frac{s}{s^2+a^2} \quad (s > 0)$ |
| 5. | $\sinh at$ | $\frac{a}{s^2-a^2} \quad (s > \lvert a\rvert)$ |
| 6. | $\cosh at$ | $\frac{s}{s^2-a^2} \quad (s > \lvert a\rvert)$ |
| 7. | $t^n \quad (n = \text{positive integer})$ | $\frac{n!}{s^{n+1}} \quad (s > 0)$ |
| 8. | $t^p \quad (p > -1)$ | $\frac{\Gamma(p+1)}{s^{p+1}} \quad (s > 0)$ |
| 9. | $e^{at}\sin bt$ | $\frac{b}{(s-a)^2+b^2} \quad (s > a)$ |
| 10. | $e^{at}\cos bt$ | $\frac{s-a}{(s-a)^2+b^2} \quad (s > a)$ |
| 11. | $t\sin at$ | $\frac{2as}{(s^2+a^2)^2} \quad (s > 0)$ |
| 12. | $t\cos at$ | $\frac{s^2-a^2}{(s^2+a^2)^2} \quad (s > 0)$ |
| 13. | $t\sinh at$ | $\frac{2as}{(s^2-a^2)^2} \quad (s > a)$ |

| $f(t)$ | $\overline{f}(s) = \int_0^\infty f(t)e^{-st}\,dt$ |
|---|---|
| 14. $t\cosh at$ | $\dfrac{s^2+a^2}{(s^2-a^2)^2} \quad (s>a)$ |
| 15. $t^n e^{at} \quad (n = \text{positive integer})$ | $\dfrac{n!}{(s-a)^{n+1}} \quad (s>a)$ |
| 16. $t^p e^{at} \quad (p>-1)$ | $\dfrac{\Gamma(p+1)}{(s-a)^{p+1}} \quad (s>a)$ |
| 17. $\ln t$ | $-\dfrac{\gamma+\ln s}{s} \quad (s>0)$ |
| | $(\gamma = \text{Euler's constant} \approx 0.577215665)$ |
| 18. $H(t-a) \quad (a \geq 0)$ | $\dfrac{e^{-as}}{s} \quad (s>0)$ |
| 19. $\delta(t-a) \quad (a>0)$ | $e^{-as}$ |
| 20. $\dfrac{e^{-a^2/t}}{\sqrt{t}} \quad (a \geq 0)$ | $\sqrt{\dfrac{\pi}{s}}\,e^{-2a\sqrt{s}} \quad (s>0)$ |
| 21. $\dfrac{e^{-a^2/t}}{t^{3/2}} \quad (a>0)$ | $\dfrac{\sqrt{\pi}}{a}\,e^{-2a\sqrt{s}} \quad (s>0)$ |
| 22. $J_0(t)$ | $\dfrac{1}{\sqrt{s^2+1}} \quad (s>0)$ |

**Linearity of transform and inverse (Theorems 5.3.1, 5.3.2):**

| | |
|---|---|
| 23. $\alpha u(t) + \beta v(t)$ | $\alpha\overline{u}(s) + \beta\overline{v}(s)$ |

**Transform of derivative (Theorem 5.3.3):**

| | |
|---|---|
| 24. $f'(t)$ | $s\overline{f}(s) - f(0)$ |
| 25. $f''(t)$ | $s^2\overline{f}(s) - sf(0) - f'(0)$ |
| 26. $f^{(n)}(t)$ | $s^n\overline{f}(s) - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots$ $-sf^{(n-2)}(0) - f^{(n-1)}(0)$ |

**Transform of integral (Theorem 5.7.3):**

| | |
|---|---|
| 27. $\int_0^t f(\tau)\,d\tau$ | $\dfrac{\overline{f}(s)}{s}$ |

**Laplace convolution theorem (Theorem 5.3.4):**

| | |
|---|---|
| 28. $(f*g)(t) = \int_0^t f(\tau)g(t-\tau)d\tau$ | $\overline{f}(s)\overline{g}(s)$ |

| $f(t)$ | $\overline{f}(s) = \int_0^\infty f(t)e^{-st}\,dt$ |
|---|---|
| **$s$-Shift (Theorem 5.7.1):** | |
| 29. $e^{-at}f(t)$ | $\overline{f}(s+a)$ |
| **$t$-Shift (Theorem 5.7.2):** | |
| 30. $H(t-a)f(t-a)$ | $e^{-as}\overline{f}(s)$ |
| **Multiplication by $t$ and $1/t$ (Theorems 5.7.4 and 5.7.5):** | |
| 31. $tf(t)$ | $-\dfrac{d\overline{f}(s)}{ds}$ |
| 32. $\dfrac{f(t)}{t}$ | $\int_s^\infty \overline{f}(s')\,ds'$ |
| **Transform of periodic function (Theorem 5.7.8):** | |
| 33. $f(t)$, of period $T$ | $\dfrac{1}{1-e^{-sT}}\int_0^T f(t)e^{-st}\,dt$ |

# Appendix D

# Table of Fourier Transforms

| | $f(x)$ | $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$ |
|---|---|---|
| 1. | $\dfrac{1}{x^2+a^2} \quad (a>0)$ | $\dfrac{\pi}{a}e^{-a\lvert\omega\rvert}$ |
| 2. | $H(x)e^{-ax} \quad (\mathrm{Re}\,a>0)$ | $\dfrac{1}{a+i\omega}$ |
| 3. | $H(-x)e^{ax} \quad (\mathrm{Re}\,a>0)$ | $\dfrac{1}{a-i\omega}$ |
| 4. | $e^{-a\lvert x\rvert} \quad (a>0)$ | $\dfrac{2a}{\omega^2+a^2}$ |
| 5. | $e^{-x^2}$ | $\sqrt{\pi}e^{-\omega^2/4}$ |
| 6. | $\dfrac{1}{2a\sqrt{\pi}}e^{-x^2/(2a)^2} \quad (a>0)$ | $e^{-a^2\omega^2}$ |
| 7. | $\dfrac{1}{\sqrt{\lvert x\rvert}}$ | $\sqrt{\dfrac{2\pi}{\lvert\omega\rvert}}$ |
| 8. | $e^{-a\lvert x\rvert/\sqrt{2}}\sin\left(\dfrac{a}{\sqrt{2}}\lvert x\rvert+\dfrac{\pi}{4}\right) \quad (a>0)$ | $\dfrac{2a^3}{\omega^4+a^4}$ |
| 9. | $H(x+a)-H(x-a)$ | $\dfrac{2\sin\omega a}{\omega}$ |
| 10. | $\delta(x-a)$ | $e^{-i\omega a}$ |
| 11. | $f(ax+b) \quad (a>0)$ | $\dfrac{1}{a}e^{ib\omega/a}\hat{f}\left(\dfrac{\omega}{a}\right)$ |
| 12. | $\dfrac{1}{a}e^{-ibx/a}f\left(\dfrac{x}{a}\right) \quad (a>0,\ b\,\text{real})$ | $\hat{f}(a\omega+b)$ |
| 13. | $f(ax)\cos cx \quad (a>0,\ c\,\text{real})$ | $\dfrac{1}{2a}\left[\hat{f}\left(\dfrac{\omega-c}{a}\right)+\hat{f}\left(\dfrac{\omega+c}{a}\right)\right]$ |
| 14. | $f(ax)\sin cx \quad (a>0,\ c\,\text{real})$ | $\dfrac{1}{2ai}\left[\hat{f}\left(\dfrac{\omega-c}{a}\right)-\hat{f}\left(\dfrac{\omega+c}{a}\right)\right]$ |
| 15. | $f(x+c)+f(x-c) \quad (c\,\text{real})$ | $2\hat{f}(\omega)\cos\omega c$ |

| $f(x)$ | $\hat{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$ |
|---|---|
| 16. $f(x+c) - f(x-c)$ ($c$ real) | $2i\hat{f}(\omega)\sin\omega c$ |
| 17. $x^n f(x)$ $(n = 1, 2, \ldots)$ | $i^n \dfrac{d^n}{d\omega^n}\hat{f}(\omega)$ |
| **Linearity of transform and inverse:** | |
| 18. $\alpha f(x) + \beta g(x)$ | $\alpha\hat{f}(\omega) + \beta\hat{g}(\omega)$ |
| **Transform of derivative:** | |
| 19. $f^{(n)}(x)$ | $(i\omega)^n\hat{f}(\omega)$ |
| **Transform of integral:** | |
| 20. $f(x) = \int_{-\infty}^{x} g(\xi)\,d\xi$, where $f(x) \to 0$ as $x \to \infty$ | $\hat{f}(\omega) = \dfrac{1}{i\omega}\hat{g}(\omega)$ |
| **Fourier convolution theorem:** | |
| 21. $(f * g)(x) = \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi$ | $\hat{f}(\omega)\hat{g}(\omega)$ |

# Appendix E

# Table of Fourier Cosine and Sine Transforms

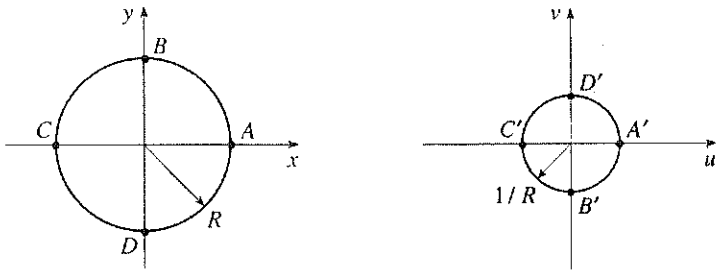
| | $f(x)$ | $\hat{f}_C(\omega) = \int_0^\infty f(x)\cos\omega x\,dx$ |
|---|---|---|
| 1C. | $e^{-ax} \quad (a > 0)$ | $\dfrac{a}{\omega^2 + a^2}$ |
| 2C. | $x^n e^{-ax} \quad (a > 0)$ | $\dfrac{n!\,\mathrm{Re}\,(a + i\omega)^{n+1}}{(\omega^2 + a^2)^{n+1}}$ (Re = real part) |
| 3C. | $\dfrac{1}{x^2 + a^2} \quad (a > 0)$ | $\dfrac{\pi}{2a}e^{-a\omega}$ |
| **Linearity of transform and inverse:** | | |
| 4C. | $\alpha f(x) + \beta g(x)$ | $\alpha\hat{f}_C(\omega) + \beta\hat{g}_C(\omega)$ |
| **Transform of derivative:** | | |
| 5C. | $f'(x)$ | $\omega\hat{f}_S(\omega) - f(0)$ |
| 6C. | $f''(x)$ | $-\omega^2\hat{f}_C(\omega) - f'(0)$ |
| **Convolution theorem:** | | |
| 7C. | $\dfrac{1}{2}\int_0^\infty [f(\lvert x - \xi\rvert) + f(x + \xi)]g(\xi)\,d\xi$ | $\hat{f}_C(\omega)\hat{g}_C(\omega)$ |

| $f(x)$ | $\hat{f}_S(\omega) = \int_0^\infty f(x)\sin\omega x\,dx$ |
|---|---|
| 1S. $e^{-ax}$ $(a > 0)$ | $\dfrac{\omega}{\omega^2 + a^2}$ |
| 2S. $x^n e^{-ax}$ $(a > 0)$ | $\dfrac{n!\,\mathrm{Im}\,(a + i\omega)^{n+1}}{(\omega^2 + a^2)^{n+1}}$ (Im = imaginary part) |
| 3S. $\dfrac{x}{x^2 + a^2}$ $(a > 0)$ | $\dfrac{\pi}{2}e^{-a\omega}$ |
| **Linearity of transform and inverse:** | |
| 4S. $\alpha f(x) + \beta g(x)$ | $\alpha\hat{f}_S(\omega) + \beta\hat{g}_S(\omega)$ |
| **Transform of derivative:** | |
| 5S. $f'(x)$ | $-\omega\hat{f}_C(\omega)$ |
| 6S. $f''(x)$ | $-\omega^2\hat{f}_S(\omega) + \omega f(0)$ |
| **Convolution theorem:** | |
| 7S. $\dfrac{1}{2}\int_0^\infty [f(\lvert x - \xi\rvert) - f(x + \xi)]g(\xi)\,d\xi$ | $\hat{f}_C(\omega)\hat{g}_S(\omega)$ |

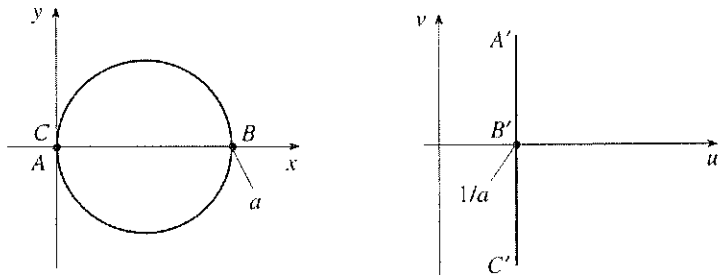
# Appendix F

# Table of Conformal Maps

1. $w = 1/z$

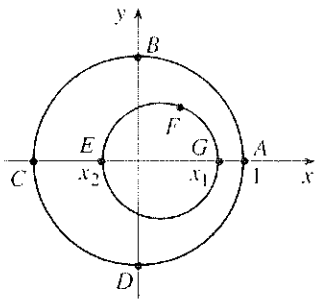


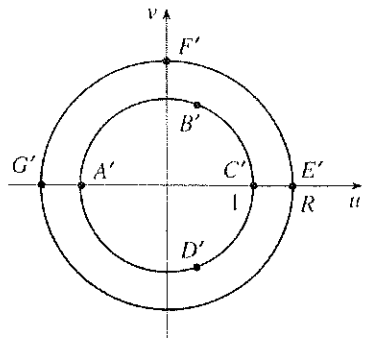

2. $w = 1/z$



3. $w = \dfrac{z-a}{az-1}$; $\quad a = \dfrac{1 + x_1x_2 + \sqrt{(1-x_1^2)(1-x_2^2)}}{x_1 + x_2}$,

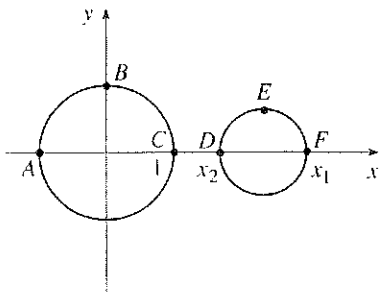
$$R = \frac{1 - x_1x_2 + \sqrt{(1-x_1^2)(1-x_2^2)}}{x_1 - x_2}, \quad -1 < x_2 < x_1 < 1$$

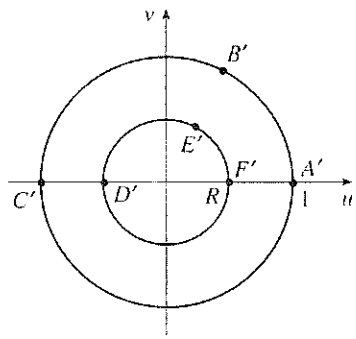


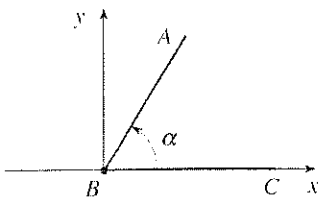

4. $w = \dfrac{z-a}{az-1}$; $\quad a = \dfrac{x_1x_2 + 1 + \sqrt{(x_1^2-1)(x_2^2-1)}}{x_1 + x_2}$,

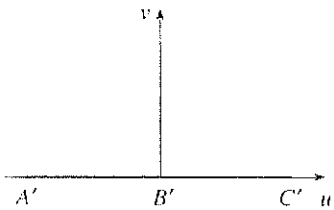
$$R = \frac{x_1x_2 - 1 - \sqrt{(x_1^2-1)(x_2^2-1)}}{x_1 - x_2}, \quad 1 < x_2 < x_1$$



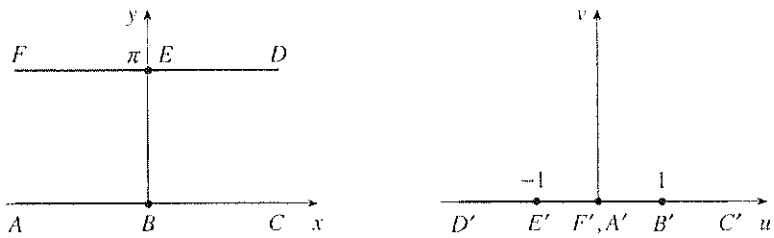




5. $w = z^{\pi/\alpha}$

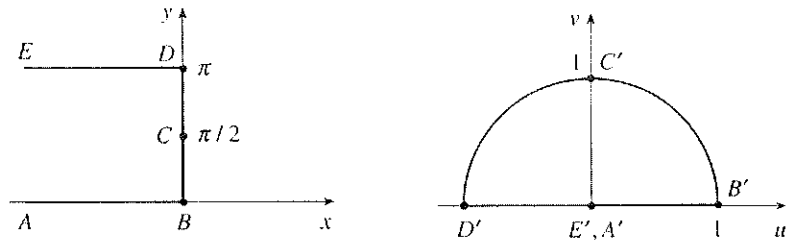


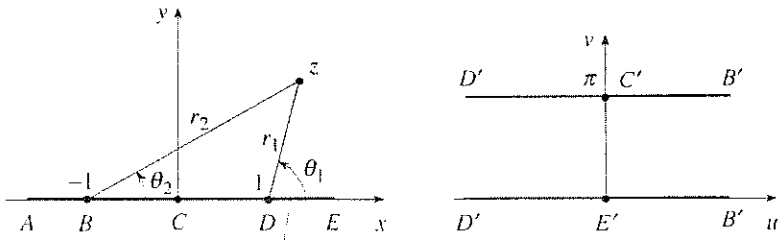
6. $w = e^z$




7. $w = e^z$

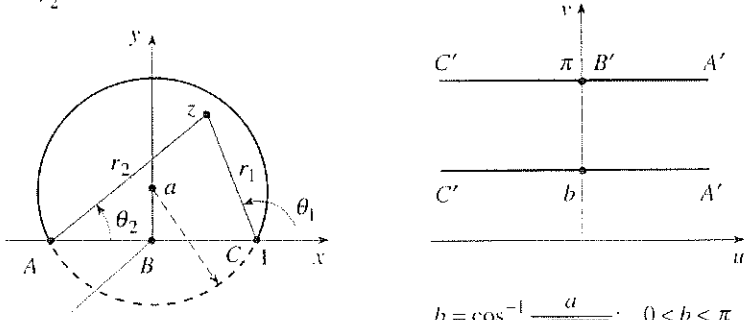



8. $w = \log \dfrac{z-1}{z+1} = \ln \dfrac{r_1}{r_2} + i(\theta_1 - \theta_2)$




$\theta_1 = \theta_2 = 0$

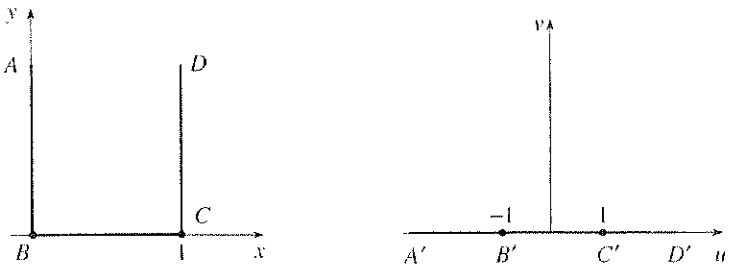
9. $w = \log \dfrac{z-1}{z+1} = \ln \dfrac{r_1}{r_2} + i(\theta_1 - \theta_2)$



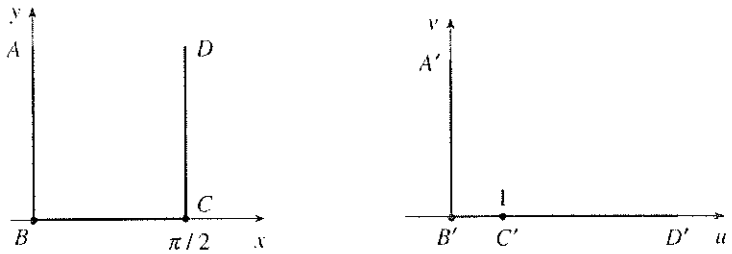

$\theta_1 = \pi,\ \theta_2 = 0$

$b = \cos^{-1} \dfrac{a}{\sqrt{a^2+1}}; \quad 0 < b < \pi$

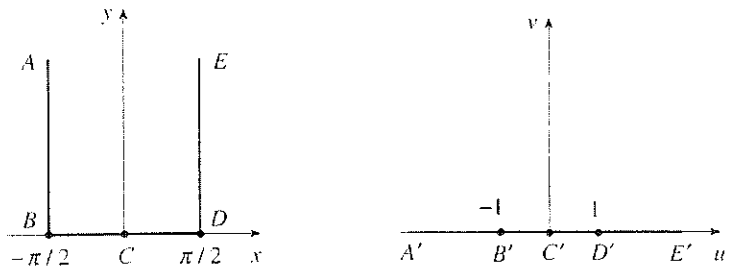
10. $w = -\cos \pi z$



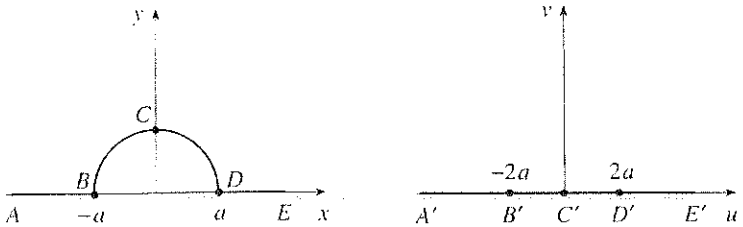

11. $w = \sin z$




12. $w = \sin z$




13. $w = z + \dfrac{a^2}{z}$



# Answers to Selected Exercises

## Chapter 1

### Section 1.2

**1.** (a) First order, $y_1$ yes, $y_2$ no, $y_3$ no (d) First order, $y_1$ no, $y_2$ yes (g) Third order, $y_1$ yes **5.** (a) $-3$ (d) 1 (g) $\pm 1$, $\pm\sqrt{5}$ **6.** (a) $A = 0$, $B = 2$ **7.** (a) Linear (d) Nonlinear (because of the $\exp y$ term) (g) Nonlinear (because of the $yy'''$ term)

### Section 1.3

**2.** When $H \to \infty$, with $L$ fixed, the cable hangs straight down at $x = L/2$, and the tension $T$ there is simply half the total weight of the cable, $wL/2$. [In fact, (16) also gives $T(0) = 0$ as $H \to \infty$. Do you see, from the physics, why that result is correct as well?]

## Chapter 2

### Section 2.2

**2.** (a) $y = (3x + C)e^x$, $\sigma = e^{-x}$ (d) $\frac{1}{5}(\sin 2x + 2\cos 2x) + Ce^x$, $\sigma = e^{-x}$ (g) $y = x^3 + Cx^2$, $\sigma = x^{-2}$ (for the equation $y' - \frac{2}{x}y = x^2$) (m) $x = t^5 + Ct$, $\sigma = 1/t$ (for the equation $x' - \frac{1}{t}x = 4t^4$). Whereas $p(t) = -1/t$ and $q(t) = 4t^4$ are continuous on $0 < t < \infty$ and $-\infty < t < 0$, the solution $t^5 + Ct$ holds on $-\infty < t < \infty$.
**5.** (a) $y(x) = 2x^2 - 2/x$ on $0 < x < \infty$ (d) $y(x) = 2x^2$ on $-\infty < x < \infty$
**6.** (a) $y(x) = (x^3 + 3x^2 - 20)/(3x^2)$ on $-\infty < x < 0$ or on $0 < x < \infty$ (d) $y(x) = (x^3 + 3x^2 - 1)/(3x^2)$ on $-\infty < x < 0$ or on $0 < x < \infty$ **7.** (a) Solution for $y(x)$ given in implicit form by $x = e^y(3y + C)$ **8.** (a) Direction field reveals the straight-line integral curve $y = 2x$ (b) Direction field reveals the straight-line integral curves $y = \pm 2$
**9.** (a) For $n = 0$, $y = e^{-\int p\,dx}\left(e^{\int p\,dx} q\,dx + C\right)$; for $n = 1$, $y = Ce^{-\int (p-q)\,dx}$ **10.** (a) $y = e^{4x}/(C - e^{4x})$ (d) $y = (x - 2 + Ce^{-x/2})^{2/3}$ (g) $y = -\ln(x - C_1) + C_2$, where both $C_1$ and $C_2$ are arbitrary constants. Do you see that this result is equivalent to the solution $y = -\ln(C_1 x + C_2)$, where both $C_1$ and $C_2$ are arbitrary constants?
**12.** (a) $y = 4e^{4x}/(C - e^{4x})$, which does give $y = Y(x) = -4$ for the choice $C = 0$ (d) $y = 2/(Ce^x + e^{-x})$ (g) $y = -2(e^{4x} + C)/(e^{4x} - C)$ **13.** (b) $x(p) = -\frac{3}{2} + \frac{C}{p^2}$, $y(p) = \frac{2C}{p}$ (In this example we can eliminate the parameter $p$ between these equations and obtain $y(x) = \sqrt{2C}\sqrt{2x+3} = A\sqrt{2x+3}$, where $\sqrt{2C} \equiv A$. (f) $P_0 = 1$ gives $y_1(x) = x + e$, $P_0 = 2$ gives $y_2(x) = 2x + e^2$ (i) $P_0 = 0$ gives $y_1(x) = 0$, $P_0 = +2$ gives $y_2(x) = 2x - 2\sin 2$, $P_0 = -2$ gives $y_3(x) = -2x + 2\sin 2$

### Section 2.3

**1.** (a) $t = 4.605L/R$ **2.** (a) For $0 \le t < t_1$, (11) gives $i(t) = i_0 e^{-Rt/L} + \frac{1}{L}\int_0^t e^{R(\tau - t)/L} E_0\,d\tau = \frac{E_0}{R} + \left(i_0 - \frac{E_0}{R}\right)e^{-Rt/L}$. For $t_1 < t < \infty$, (11) gives $i(t) = i_0 e^{-Rt/L} + \frac{1}{L}\int_0^{t_1} e^{R(\tau - t)/L} E_0\,d\tau$

$= \left[i_0 + \frac{E_0}{R}\left(r^{Rt_1/L} - 1\right)\right] e^{-Rt/L}$. Steady state: $i(t) \to 0$ as $t \to \infty$. **3.** (a) If $R^2C \neq L$, $i(t) = [(E_0RC)/(R^2C - L)]\left(e^{-Rt/L} - e^{-t/(RC)}\right)$; if $R^2C = L$, $i(t) = -\frac{E_0}{L}te^{-t/(RC)}$. **5.** 847 years **9.** $N(t) = aN_0/[bN_0 + (a - N_0b)e^{-at}]$ **10.** (a) $c(t) = c_1 + (c_0 - c_1)e^{-Q_0t/v}$ (b) For $0 < t < 1$, $c(t) = e^{-\int_0^t 4\,d\tau}\left(\int_0^t e^{\int_0^\tau 4\,d\xi} 4\,d\tau + 0\right) = 1 - e^{-4t}$. Thus, $c(1) = 0.9817$. Then, for $t > 1$, $c(t) = e^{-\int_1^t 2\,d\tau}\left(\int_1^t e^{\int_1^\tau 2\,d\xi} 2\,d\tau + 0.9817\right) = 1 - 0.0183e^{-2(t-1)}$.
**11.** $x(t) = (mg\sin\alpha/c^2)(ct - m + me^{-ct/m})$ **13.** 0.2299 inches **14.** (a) It is simplest to treat $x < 0$ and $x > 0$ separately. For $x < 0$ we have $Q(x) = 0$ so $c' + (\beta/U)c = 0$ with $c(-\infty) = 0$. Applying $c(-\infty) = 0$ to the general solution $c(x) = Be^{-\beta x/U}$ gives $B = 0$ so $c(x) = 0$ for $x < 0$. For $x > 0$ we have $Q(x) = \text{constant} = Q$ and $c' + (\beta/U)c = Q/(AU)$ with $c(0) = 0$ [from the solution $c(x) = 0$ on $x < 0$]. This problem gives $c(x) = \frac{Q}{\beta A}(1 - e^{-\beta x/U})$. Note that $c(x) \to \frac{Q}{\beta A}$ as $x \to \infty$. (b) As in (a), $c(x) = 0$ for $x < 0$ and $c(x) = \frac{Q}{\beta A}(1 - e^{-\beta x/U})$ in $x > 0$, but only up to $x = L$, where it gives $c(L) = \frac{Q}{\beta A}(1 - e^{-\beta L/U})$ as the initial condition for the problem $c' + (\beta/U)c = 0$ on $L < x < \infty$. Applying that initial condition to the general solution $c(x) = Be^{-\beta x/U}$ gives $B = \frac{Q}{\beta A}(e^{\beta L/U} - 1)$ and hence $c(x) = \frac{Q}{\beta A}(e^{\beta(L-x)/U} - e^{-\beta x/U})$ for $x > L$. This time $c(x) \to 0$ as $x \to \infty$.
**15.** (b) 2.20 hrs

## Section 2.4

**1.** (a) $y = \ln(x^3 + 1)$. We do not claim that these solutions hold for all $x$. For example, this one holds only on $-1 < x < \infty$ because $\ln(x^3 + 1) \to -\infty$ as $x \to -1$. (d) $y = \tan(x - 0.6266)$ (g) $y = 12e^{3x}/(1 - 4e^{3x})$ (j) $y = -\sqrt{x/3}$ **4.** $N(t) = aN_0/[bN_0 + (a - N_0b)e^{-at}]$ **6.** (a) $y = (1 - \sqrt{8x^3 + 4x + 25})/2$ (d) $y = (1 + \sqrt{8x^3 + 4x + 13})/2$ **7.** (a) $y = -\sqrt{x^3 - x + 9}$ (d) $y = +\sqrt{x^3 - x}$ and $-\sqrt{x^3 - x}$, nonunique solution **8.** (a) No (b) Yes, degree zero. **10.** (a) $y = x(\frac{9}{2}\ln x + C)^{2/3}$ (d) $y = -x + C/x$ **11.** (b) Implicit solution: $y^2 + (6 - 2x)y + 2x^2 - 12x + C = 0$. Can solve for $y$ by quadratic formula.

## Section 2.5

**1.** (a) $y = 3x + C$, $y = 3x + 6$ (d) $10\sin 2u + e^{-5v} = C$, $10\sin 2u + e^{-5v} = e^{30}$ (g) $x^2 + z^2 - 4xz = C$, $x^2 + z^2 - 4xz = 94$ (j) $x^3\sin 2y - x^2y = C$, $x^3\sin 2y - x^2y = -0.7854$ **4.** (a) Exact if and only if $b = A$. **5.** (a) $\sigma(x) = e^{3x}$, $e^{3x}y = C$ (d) $\sigma(y) = e^y$, $xe^y - y = C$ (g) $\sigma(y) = \cos y$, $(x - y)\cos^2 y = C$ (j) $\sigma = 1$, $u^3\sinh 3v - u^2 = C$ **7.** (a) $\sigma = xy$, $x^3y^2 - x^2y^3 = C$ **8.** (a) $\sigma = e^{-(x+y)}$, $e^{-x} + e^{-y} = C$
**9.** (a) $x^2 - 2xy - y^2 = C$

# Chapter 3

## Section 3.2

**1.** (a) No, from Definition 3.2.1. **2.** (a) For instance, $(3x - 5) = 3(x + 2) - 11(1)$. (d) For instance, $\cosh x = -1(\sinh x) + 1(e^x) + 0(e^{2x})$. **3.** (a) The Wronskian determinant $W[1, x, \ldots, x^n](x)$ is upper triangular, with $0!, 1!, 2!, \ldots, n!$ as its diagonal elements. Thus, by property D3 in Section 10.4, $W(x) = (0!)(1!)\cdots(n!)$ which is nonzero. Then, by Theorem 3.2.2 the set is LI (on $-\infty < x < \infty$). An alternative approach that avoids determinants is as follows. Let $n = 2$, say. The question is whether $a + bx + cx^2 = 0$ can be satisfied by constants $a, b, c$ other than $a = b = c = 0$. Repeated differentiation gives $b + 2cx = 0$ and then $2c = 0$, and these three equations give $c = 0$, $b = 0$, $a = 0$ so, by Theorem 3.2.1, $\{1, x, x^2\}$ is LI. The argument is easily generalized to $\{1, x, \ldots, x^n\}$. (d) Surely, $e^{2x}$ is not a scalar multiple of $e^x$ (nor vice versa). For if $e^{2x} = ae^x$ then $x = 0$ gives $a = 1$ and $x = 1$ gives $a = e$, which cannot both be true. Then, by Theorem 3.2.4, the set is LI. **4.** (a) $W = \begin{vmatrix} e^x & e^{2x} & e^{3x} \\ e^x & 2e^{2x} & 3e^{3x} \\ e^x & 4e^{2x} & 9e^{3x} \end{vmatrix} = 2e^{6x} \neq 0$, so LI. Theorem 3.2.4 does not apply. (d) $W = 0$ so, by Theorem 3.2.3, LD.

## Section 3.3

**1.** (a) $e^x$ and $e^{2x}$ satisfy the ODE. Further, $W[e^x, e^{2x}](x) = e^{3x} \neq 0$, so $C_1e^x + C_2e^{2x}$ is a general solution. (d) $e^{-x}$ and $e^{2x}$ satisfy the ODE. Further, $W[e^{-x}, e^{2x}](x) = 3e^x \neq 0$, so $C_1e^{-x} + C_2e^{2x}$ is a general solution. **2.** (a) $e^{3x}, \cosh 3x, \sinh 3x$ are indeed solutions, but they are not LI because $\cosh 3x = 1(\sinh 3x) + 1(e^{-3x})$, for instance. Thus, the set is not a basis. (d) Each is a solution, and $W[e^x, xe^x, x^2e^x](x) = 2e^{3x} \neq 0$ so the set is LI. Thus, the set is a basis. **3.** (a) No; for second-order we need *two* LI solutions. (b) Yes **4.** (a) No; $x^2$ is not a solution. (c) Yes on $0 < x < \infty$, on $-\infty < x < 0$, and on $6 < x < 10$, but no on $-\infty < x < \infty$ because $x \ln|x|$ is not differentiable at $x = 0$. **5.** (a) No. A general solution of a seventh-order ODE must include seven LI solutions, whereas the given expression contains only six. **7.** No. Note that if $y_1$ and $y_2$ are solutions of a homogeneous linear ODE, then $C_1y_1 + C_2y_2$ is a solution too, for arbitrary constants $C_1, C_2$, but the ODE given here is *not* homogeneous. **8.** (a) Given $y(0) = 4, y'(0) = 3$. Then ODE gives $y'' = -y, y''' = -y', y^{(iv)} = -y'', \ldots$ so $y''(0) = -y(0) = -4$, $y'''(0) = -y'(0) = -3$, $y^{(iv)}(0) = -y''(0) = 4$, $y^{(v)}(0) = -y'''(0) = 3$. Hence, $y(x) = 4 + 3x - 4x^2/2! - 3x^3/3! + 4x^4/4! + 3x^5/5! - \cdots$ (d) $y(0) = 1, y'(0) = 0$. ODE gives $y'' = -xy, y''' = -y - xy', y^{(iv)} = -2y' - xy'', y^{(v)} = -3y'' - xy'''$, so $y''(0) = 0, y'''(0) = -1, y^{(iv)}(0) = 0, y^{(v)}(0) = 0$. Hence, $y(x) = 1 + 0x + 0x^2 + (-1)x^3/3! + 0x^4 + 0x^5 + \cdots$ **9.** (a) $p_1(x) = 2$ and $p_2(x) = 3$ are continuous on $-\infty < x < \infty$ so it follows from Theorem 3.3.1 that there exists a unique solution for $y(x)$ on every interval (no matter how broad) containing the initial point $x = 0$. (e) $p_1(x) = 0, p_2(x) = 1, p_3(x) = -1/x$ are continuous on the interval $-\infty < x < 0$ containing the initial point $x = -1$, so there does exist a unique solution on that interval. **11.** (a) Unique solution $y(x) = 0$. (b) Boundary conditions give $C_1 = 0$ and $C_1 = -3$. Hence, no solution. **12.** (a) Unique solution $y(x) = -(2/\pi)x \sin x$.

## Section 3.4

**3.** Trying to avoid the messy Wronskian determinant, fall back on Theorem 3.2.1. Accordingly, the question is whether $\alpha_1e^{\lambda_1x} + \alpha_2xe^{\lambda_1x} + \cdots + \alpha_kx^{k-1}e^{\lambda_1x} = 0$ can be satisfied by $\alpha_j$'s not all zero. Since $e^{\lambda_1x} \neq 0$ for all $x$, the exponential $e^{\lambda_1x}$ factors can be cancelled, leaving $\alpha_1 + \alpha_2x + \cdots + \alpha_kx^{k-1} = 0$. Thus, the given set is LI if and only if $\{1, x, \ldots, x^{k-1}\}$ is, and proof of the linear independence of that set is given in the answer to Exercise 3(a) of Section 3.2. **4.** (a) $y(x) = A + Be^{-5x}$ (d) $y(x) = Ae^x + Be^{2x}$ is general solution, and initial conditions give $y(x) = 2e^{x-1} - e^{2(x-1)}$ (g) $y(x) = e^{2x}(2\cos x + \sin x)$ (m) $y(x) = Ae^x + e^{-x}(B\cos x + C\sin x)$
**6.** (a) $y(x) = 2 - x$ (d) $y(x) = 1$ (g) $y(x) = Ae^{-x} + (B + Cx)e^x$ **8.** (a) $(\lambda - 2)(\lambda - 6) = \lambda^2 - 8\lambda + 12$, so $y'' - 8y' + 12y = 0$, $y(x) = Ae^{2x} + Be^{6x}$ (d) $y''' - 6y'' - y' + 30y = 0$, $y(x) = Ae^{-2x} + Be^{3x} + Ce^{5x}$
(g) $y^{(v)} - 12y^{(iv)} + 49y''' - 76y'' + 48y' - 64y = 0$, $y(x) = (A + Bx + Cx^2)e^{4x} + D\cos x + E\sin x$
**9.** (a) $y(x) = Ae^{i(1+\sqrt{2})x} + Be^{i(1-\sqrt{2})x} = e^{ix}(C\cos\sqrt{2}\,x + D\sin\sqrt{2}\,x)$ (d) $y(x) = (A + Bx)e^{ix}$
**12.** (a) $\lambda = 0.0776, 1.4612 + 4.8619i, 1.4612 - 4.8619i$, hence unstable (d) $\lambda = +i, -i, -0.5 + 1.9369i, -0.5 - 1.9369i$: in this case there are no roots to the right of the imaginary axis and the two roots *on* the imaginary axis are nonrepeated, hence stable

## Section 3.5

**1.** (a) $E = \sqrt{6^2 + 1^2} = \sqrt{37}$, $\phi = \tan^{-1}(6/1) = 1.406$, $\omega = 1$, so $\sqrt{37}\sin(t + 1.406)$ (d) $\sqrt{8}\sin(3t - 0.785)$ or $\sqrt{8}\sin(3t - \pi/4)$ **3.** $A = x_0$, $B = (cx_0 + 2mx_0')/\sqrt{4m^2\omega^2 - c^2}$ **10.** (d) Yes

## Section 3.6

**1.** (a) $y(x) = C/x$ on $x < 0$ or $x > 0$ (or $-\infty < x < \infty$ if $C = 0$) (d) $y(x) = 3(x^5 - 1)/5$ (g) $y(x) = (2/x)\sin(\ln x)$ on $x > 0$ (p) $y(x) = [A + B\ln|x| + C(\ln|x|)^2]x$ since $\lambda = 1, 1, 1$ **7.** (b) $Y'' - 4Y = 0$, $Y(t) = Ae^{2t} + Be^{-2t}$, $y(x) = Ax^2 + B/x^2$ (f) $Y'' - Y' + Y = 0$, $Y(t) = Ae^{(\frac{1}{2}+i\frac{\sqrt{3}}{2})t} + Be^{(\frac{1}{2}-i\frac{\sqrt{3}}{2})t} = e^{t/2}(C\cos\frac{\sqrt{3}}{2}t + D\sin\frac{\sqrt{3}}{2}t)$, $y(x) = \sqrt{x}\,[C\cos(\frac{\sqrt{3}}{2}\ln|x|) + D\sin(\frac{\sqrt{3}}{2}\ln|x|)]$ on $x < 0$ or $x > 0$ **9.** (a) $\phi(r) = [(\phi_2\ln r_1 - \phi_1\ln r_2) + (\phi_1 - \phi_2)\ln r]/(\ln r_1 - \ln r_2)$ **10.** (a) $u(r) = (u_1r_1 - u_2r_2)/(r_1 - r_2) + [(r_2r_1(u_2 - u_1)/(r_1 - r_2)](1/r)$ **11.** (a) $y(x) = Ax + B\left(e^{-x^2/2} + x\int_0^x e^{-t^2/2}\,dt\right)$ or, in terms of the tabulated function $\operatorname{erf}(x)$

defined by (60), $y(x) = Ax + B[e^{-x^2/2} + \sqrt{\frac{\pi}{2}}x\text{erf}(\frac{x}{\sqrt{2}})]$. **16.** In that case (50a,b) become $a' = a^2 + (a_1)a + (a_2)$ and $b' = -b^2 - (a_1)b - (a_2)$. Because $a_1$ and $a_2$ are constants these two equations admit solutions $a(x) =$ constant and $b(x) =$ constant, which constants are roots of the characteristic equation $\lambda^2 + a_1\lambda + a_2 = 0$. Thus, $a, b$ are the roots $\lambda_1, \lambda_2$ of the characteristic equation, so the factored equation is $(D-\lambda_2)(D-\lambda_1)y = 0$. The latter can be solved by setting $(D-\lambda_1)y \equiv u(x)$. Then $(D-\lambda_2)u = u' - \lambda_2 u = 0$ gives $u(x) = Ae^{\lambda_2 x}$. Finally, solve $(D-\lambda_1)y = u$, namely, $y' - \lambda_1 y = Ae^{\lambda_2 x}$. If $\lambda_1 \neq \lambda_2$, then that step gives $y = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$; if $\lambda_1 = \lambda_2$, then that step gives $y = (C_1 + C_2x)e^{\lambda_1 x}$. **17.** (a) $a_1(x) = -2/x$ and $a_2(x) = 2/x^2$ so (50a) gives the equation $a' = a^2 - \frac{2}{x}a$ on $a(x)$. By inspection, the latter has a particular solution $a(x) = 0$. Rather than try to solve (50b) for $b(x)$, we obtain $b(x)$ readily from (49a) as $b(x) = -a_1(x) - a(x) = 2/x$. Thus, the factored form (47) is $D(D - \frac{2}{x})y = 0$. Then $(D - \frac{2}{x})y = A$ or $y' - \frac{2}{x}y = A$ and (using the general solution of the first-order linear equation) $y(x) = -Ax + Bx^2$ or $C_1x + C_2x^2$. NOTE: Rather than solve (50a) for $a(x)$ and (50b) for $b(x)$, solve (50a) for $a(x)$ and then (49a) for $b(x)$. Or, solve (50b) for $b(x)$ and then (49a) for $a(x)$. In the present example (50a) gives, by inspection, a solution $a(x) = 0$; (50b) gives, by inspection, solutions $b(x) = 1/x$ *and* $b(x) = 2/x$. Of these, the pair $a(x) = 0$ and $b(x) = 2/x$ does satisfy (49a); the pair $a(x) = 0$ and $b(x) = 1/x$ does *not*.

## Section 3.7

**1.** (a) Yes, $\{x^2\cos x, x^2\sin x, x\cos x, x\sin x, \cos x, \sin x,\}$ (b) No, $x^2\ln x \to \{x^2\ln x, x\ln x, x, 1, 1/x, 1/x^2, \ldots,\}$ without end. **2.** (a) $y_h(x) = C_1e^{3x}, xe^{2x} \to \{xe^{2x}, e^{2x}\}, 6 \to \{1\}$. Seek $y_p(x) = Axe^{2x} + Be^{2x}$ for the $xe^{2x}$ term and $y_p(x) = C$ for the 6 term. Obtain $A = -1, B = -1, C = -2$, so $y(x) = C_1e^{3x} - xe^{2x} - e^{2x} - 2$.
(d) $y_h(x) = C_1e^{3x}, xe^{3x} \to \{xe^{3x}, e^{3x}\}, 4 \to \{1\}$. Seek $y_p(x) = x(Axe^{3x} + Be^{3x})$ for the $xe^{3x}$ term and $y_p(x) = C$ for the 4 term. Obtain $y(x) = C_1e^{3x} + \frac{1}{2}x^2e^{3x} - \frac{4}{3}$. (g) $y(x) = C_1 + C_2e^x + \frac{1}{2}\cos 2x - \sin 2x$.
(k) $y(x) = C_1\cos x + C_2\sin x + 3x\sin x + 2$ (o) $y(x) = C_1 + C_2e^x + (x^2 - 2x)e^x$ **4.** (a) $y(x) = C_1e^{-2x} + e^{2x}$
(d) $y(x) = C_1\frac{1}{x} + \frac{\ln x}{x}$ (g) $y(x) = C_1e^x + C_2e^{-x} + (4x-2)e^x$ (m) $y(x) = C_1x^2 + C_2/x^2 - 1/(3x)$
(o) $y(x) = C_1e^x + (C_2 + C_3x)e^{-x} + 1 - x$

## Section 3.8

**5.** (a) As $c \to \infty$ the graph of $E$ becomes discontinuous at $\Omega = 0$: $E = F_0/k$ at $\Omega = 0$, $E = 0$ for all $\Omega > 0$.
(b) For $c = 0$, $\Phi$ is 0 for $\Omega < \omega$ and $\pi$ for $\Omega > \omega$; for $c \to \infty$, $\Phi$ is 0 at $\Omega = 0$ and $\pi/2$ for $\Omega > 0$.
**8.** Of course we can see from (19b) that $E \to 0$ as $\Omega \to \infty$. One way of interpreting this result physically is to let $\Omega t = \tau$ in (15), in which case (15) becomes $m\Omega^2 d^2x/d\tau^2 + c\Omega dx/d\tau + kx = F_0\cos\tau$ in which the "effective mass" $m\Omega^2$ tends to $\infty$ as $\Omega \to \infty$ (as does the effective damping $c\Omega$). **12.** (b) (12.2) is $mw'' + cw' + kw = F_0e^{i\Omega t}$ and seeking $w_p(t) = Ae^{i\Omega t}$ gives $A = F_0/(-m\Omega^2 + ic\Omega + k)$ so $x_p(t) = \text{Re}\,[F_0e^{i\Omega t}/(-m\Omega^2 + ic\Omega + k)] = F_0[(k - m\Omega^2)\cos\Omega t + c\Omega\sin\Omega t]/[(k - m\Omega^2)^2 + c^2\Omega^2]$ (e) $x_p(t) = \text{Im}\,[e^{i3t}/(3i-1)] = -\frac{6}{5}\cos 3t - \frac{2}{5}\sin 3t$
**13.** (a) $Q(t) = 5 - (5/3)e^{-t}(3\cos 3t + \sin 3t)$. Steady state: $Q(t) \to 5$.

## Section 3.9

**4.** (c) $R(i_1 - i_2) = E(t)$, $L(i_2' - i_3') = E(t)$, $(1/C)i_3 = E'(t)$, of which the first and third happen to be algebraic rather than differential equations. **5.** (a) $x(t) = -2Ae^{2t} - Be^{-t}$, $y(t) = Ae^{2t} + 2Be^{-t}$ (d) $x(t) = Ae^{t/2} - 3Be^{-t/2} - 1$, $y(t) = Ae^{t/2} + Be^{-t/2} - t - 1$ (g) $x(t) = -\frac{13}{16} + \frac{1}{8}t + 3Ae^{4t} + Be^{-4t}$, $y(t) = -\frac{7}{8} + \frac{1}{8}t + Ae^{4t} - Be^{-4t}$
(k) $x(t) = -2Ae^{at} - 2Be^{-at} + C\sin at + D\cos at$, $y(t) = Ae^{at} + Be^{-at} + C\sin at + D\cos at$ $(a = \sqrt{3})$... **8.** Let us give only $z(t)$:
(a) $z(t) = \gamma + \gamma(\beta e^{-\alpha t} - \alpha e^{-\beta t})/(\alpha - \beta)$ (b) $z(t) = \gamma(1 - e^{-\alpha t} - \alpha te^{-\alpha t})$ **10.** (a) Given $(D-1)x + y = t$, $(D^2 - 1)x + (D+1)y = t^2$, operate on the first with $D + 1$ and obtain $(D^2 - 1)x + (D+1)y = 1 + t$. Since $t^2 \neq 1 + t$ there is no solution. (b) Operating on the first equation with $D + 1$ gives an equation that is identical to the second equation. Thus, we can discard the second equation. In the first equation we can set $x(t) = f(t)$, an arbitrary (twice-differentiable) function and solve, by algebra, for $y(t)$. Doing so, we obtain $y(t) = t - f'(t) + f(t)$.

# Chapter 4

## Section 4.2

**1.** (a) $a_n = n$, $\lim |a_{n+1}/a_n| = \lim |(n+1)/n| = 1$ so $R = 1$ (d) $a_n = 0$ for all $n > 1000$, so the "series" converges for all $x$; i.e., it consists of only a finite number of terms so convergence is not an issue. $R = \infty$ (g) $\lim |a_{n+1}/a_n| = \lim |(\frac{n+1}{n})^{50}\frac{1}{n}| = 0$ so $R = \infty$ **2.** (a) Denominator has zeros at $\pm i$ in $z$ plane so $R = 1$. (d) Denominator vanishes at $x = -2$ so $R = 24$. (g) Factoring gives $(x+2)(x-1)/[(x^2+4)(x-1)] = (x+2)/[(x+2i)(x-2i)]$. Distance, in $z$ plane, from 2 to $\pm 2i$ is $\sqrt{8}$, so $R = \sqrt{8}$. **3.** (a) $e^x = e\sum_0^\infty (x-1)^n/n!$, $R = \infty$ (d) $\sin x = 1 - \frac{1}{2!}(x-\frac{\pi}{2})^2 + \frac{1}{4!}(x-\frac{\pi}{2})^4 - \cdots = \sum_0^\infty \frac{(-1)^n}{(2n)!}(x-\frac{\pi}{2})^{2n}$, $R = \infty$ (g) $\cos x = \cos 5 - \frac{\sin 5}{1!}(x-5) - \frac{\cos 5}{2!}(x-5)^2 + \frac{\sin 5}{3!}(x-5)^3 + \frac{\cos 5}{4!}(x-5)^4 - \cdots$, $\lim |a_{n+1}/a_n| = 0$, $R = \infty$ (j) $2x^3 - 4 = -4 + 0x + 0x^2 + \frac{12}{3!}x^3 + 0x^4 + \cdots = -4 + 2x^3$, $R = 0$ **5.** (c) At $x = 1$ the geometric series is $1 + 1 + 1 + \cdots$, which diverges because the $n$th term does not tend to zero as $n \to \infty$ (see the italics below Theorem 4.2.1). Alternatively, $s_n = n \to \infty$ as $n \to \infty$, so the series diverges. At $x = -1$ the series is $1 - 1 + 1 - 1 + \cdots$, which diverges because the $n$th term does not tend to zero. **7.** (a) $p(x) = 2$, $q(x) = 1$, $R = \infty$. $y(x) = \sum_0^\infty a_n x^n$ gives the recursion formula $(n+2)(n+1)a_{n+2} + 2(n+1)a_{n+1} + a_n = 0$ for $n = 0, 1, 2, \ldots$. Obtain $y(x) = a_0 + a_1 x - (\frac{a_0}{2} + a_1)x^2 + (\frac{a_0}{3} + \frac{a_1}{2})x^3 - (\frac{a_0}{8} + \frac{a_1}{6})x^4 + \cdots$ so $y_1(x) = 1 - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{8}x^4 + \cdots$, $y_2(x) = x - x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^4 + \cdots$. NOTE: Since the analytical solution is $y(x) = (A + Bx)e^{-x}$ we might expect $y_1$ and $y_2$ to be $e^{-x}$ and $xe^{-x}$. Actually, $y_2$ *is* $xe^{-x}$, but $y_1$ is not $e^{-x}$; it is $(1+x)e^{-x}$. That's fine; $y_1 = (1+x)e^{-x}$ and $y_2 = xe^{-x}$ are, indeed, two LI solutions of the ODE, LI because neither is a scalar multiple of the other. (d) $p(x) = 1/x$, $q(x) = 1/x$, $R = 5$. Don't forget to expand the $x$ in the ODE as $-5 + (x+5)$. $y(x) = \sum_0^\infty a_n(x+5)^n$ gives the recursion formula $-5(n+2)(n+1)a_{n+2} + (n+1)^2 a_{n+1} + a_n = 0$ for $n = 0, 1, 2, \ldots$. Obtain $y(x) = a_0 + a_1(x+5) + (\frac{a_0}{10} + \frac{a_1}{10})(x+5)^2 + (\frac{a_0}{75} + \frac{7a_1}{150})(x+5)^3 + (\frac{11a_0}{3000} + \frac{13a_1}{1500})(x+5)^4 + \cdots$ so $y_1(x) = 1 + \frac{1}{10}(x+5)^2 + \frac{1}{75}(x+5)^3 + \frac{11}{3000}(x+5)^4 + \cdots$, $y_2(x) = (x+5) + \frac{1}{10}(x+5)^2 + \frac{7}{150}(x+5)^3 + \frac{13}{1500}(x+5)^4 + \cdots$. (g) $p(x) = (3+x)/x$, $q(x) = 1$, $R = 3$. Recursion formula is $-3(n+2)(n+1)a_{n+2} + n(n+1)a_{n+1} + (n-3)a_n + a_{n-1} = 0$ for $n = 0, 1, 2, \ldots$, with $a_{-1} = 0$. Obtain $y_1(x) = 1 - \frac{1}{2}(x+3)^2 + \frac{1}{72}(x+3)^4 - \frac{1}{180}(x+3)^5 + \cdots$, $y_2(x) = (x+3) - \frac{1}{9}(x+3)^3 + \frac{1}{108}(x+3)^4 + \frac{1}{540}(x+3)^5 + \cdots$. **11.** (a) $a_{n+2} = \frac{(n+1)^2}{(n+3)(n+2)}a_{n+1} - \frac{n}{(n+3)(n+2)}a_n \sim a_{n+1} - \frac{1}{n}a_n$ so $\lim |a_{n+2}/a_{n+1}| = 1$, $R = 1$ (d) Because of the subscripts $n+2$ and $n$, successive terms in $y_1(x)$ and $y_2(x)$ differ by a factor of $(x - x_0)^2$ rather than $(x - x_0)$. Thus, the ratio test requires $\lim_{n\to\infty} |a_{n+2}(x-x_0)^{n+2}/[a_n(x-x_0)^n]| = \lim_{n\to\infty} \left|\frac{a_{n+2}}{a_n}\right|(x-x_0)^2 < 1$ for convergence. If $\lim |a_{n+2}/a_n| \equiv L$, then we have convergence in $|x - x_0| < 1/\sqrt{L}$. In the present case $a_{n+2}/a_n = 3(n+2)/(n+1) \to 3 = L$, so $R = 1/\sqrt{3}$.

## Section 4.3

**1.** (a) $p = -x^3$ and $q(x) = x$ are infinitely differentiable and hence analytic (recall our italicized rule of thumb in the sentence preceding Section 4.2.2) for all $x$ so there are no singular points; every point is an ordinary point. (d) $p(x) = 0$ and $q(x) = 1/[x(x^2+3)]$ are analytic for all $x$ except $x = 0$, which is a singular point, a regular singular point because $xp(x) = 0$ and $x^2q(x) = x/(x^2+3)$ are analytic there. (g) $p(x) = q(x) = (x-1)^{-1}(x+3)^{-2}$. Singular points at $x = 1$, $x = -3$. $(x-1)p(x) = (x+3)^{-2}$ and $(x-1)^2q(x) = (x-1)(x+3)^{-2}$ are analytic at $x = 1$, so $x = 1$ is a regular singular point. $(x+3)p(x) = (x-1)/(x+3)$ and $(x+3)^2q(x) = 1/(x-1)$. The latter is analytic at $x = -3$ but the former is not, so $x = -3$ is an irregular singular point. **3.** (a) $tY'' + Y' - Y = 0$ **5.** (a) The indicial equation is (38): $r^2 + (p_0 - 1)r + q_0 = 0$. If $r = 1$ and 4 then $(r-1)(r-4) = r^2 - 5r + 4 = 0$ so $p_0 - 1 = 5$ and $q_0 = 4$; hence $p_0 = -4$, $q_0 = 4$. Hence, $xp(x) = p_0 + p_1x + \cdots = -4$ and $x^2q(x) = q_0 + q_1x + \cdots = 4$, say, so $p(x) = -4/x$, $q(x) = 4/x^2$, and the ODE is $y'' - (4/x)y' + (4/x^2)y = 0$ or $x^2y'' - 4xy' + 4y = 0$. Or we could take $xp(x) = -4 + 3x - 5x^3$ and $x^2q(x) = 4 + x^2$, or $xp(x) = -4\cos x$ and $x^2q(x) = 4e^x$, and so on. (d) $r^2 + (p_0 - 1)r + q_0 = 0$ and $(r + 1/2)(r - 1/2) = r^2 - 1/4 = 0$ give $p_0 = 1$, $q_0 = -1/4$. Can choose $xp(x) = 1$ and $x^2q(x) = -1/4$, for instance. Thus, $x^2y'' + xy' - \frac{1}{4}y = 0$, say. **6.** (a) $p(x) = 1/(2x)$ is singular

at $x = 0$, but $xp(x) = 1/2$ and $x^2q(x) = x^4/2$ are analytic there (with radii of convergence $R_1 = R_2 = \infty$), so $x = 0$ is a regular singular point; $p_0 = 1/2$ and $q_0 = 0$ so indicial equation is $r^2 - \frac{1}{2}r = 0$; $r = 0, 1/2$; case (i). $y_1(x) = 1 - \frac{1}{28}x^4 + \frac{1}{3360}x^8 - \cdots$, $y_2(x) = \sqrt{x}\left(1 - \frac{1}{36}x^4 + \frac{1}{4896}x^8 - \cdots\right)$, both valid on $0 < x < \infty$.
(d) $p(x) = 1/x$ is singular at $x = 0$, but $xp(x) = 1$ and $x^2q(x) = x^2$ are analytic there (with $R_1 = R_2 = \infty$), so $x = 0$ is a regular singular point; $p_0 = 1$ and $q_0 = 0$ so indicial equation is $r^2 = 0$; $r = 0, 0$; case (ii). $y_1(x) = 1 - \frac{1}{2^2}x^2 + \frac{1}{2^24^2}x^4 - \frac{1}{2^24^26^2}x^6 + \cdots = \sum_0^\infty (-1)^n x^{2n}/[1^2 \cdot 2^2 \cdot 4^2 \cdots (2n)^2]$ and (41b), with $r = 0$, gives $y_2(x) = y_1(x)\ln x + (\frac{1}{4}x^2 - \frac{3}{128}x^4 + \frac{11}{13824}x^6 - \cdots)$, both valid on $0 < x < \infty$. (g) $p(x) = 1/x$ and $q(x) = -(1 + 2x)/x^2$ are singular at $x = 0$, but $xp(x) = 1$ and $x^2q(x) = -1 - 2x$ are analytic there (with $R_1 = R_2 = \infty$), so $x = 0$ is a regular singular point; $p_0 = 1$ and $q_0 = -1$ so indicial equation is $r^2 - 1 = 0$; $r = -1, 1$; case (iii). $y_1(x) = x + \frac{2}{3}x^2 + \frac{1}{6}x^3 + \cdots$, $y_2(x) = 4y_1(x)\ln x + \frac{1}{x}(-2 + 4x - \frac{32}{9}x^3 + \cdots)$, both valid on $0 < x < \infty$.

## Section 4.4

**10.** (a) It is convenient to first shift the origin to the point of expansion, $x = 1$, by setting $x - 1 \equiv t$. Then the ODE becomes $t(2 + t)Y'' + 2(1 + t)Y' - 2Y = 0$ on $Y(t)$. The indicial equation is $r^2 = 0$ so $r = 0, 0$; case (ii) of Theorem 4.3.1. Obtain one solution $Y_1(t) = 1 + t = x$ [i.e., the bounded solution $P_1(x)$] and the second solution $Y_2(t) = (\ln t)(1 + t) + \left(-\frac{5}{2}t - \frac{3}{8}t^2 + \frac{1}{12}t^3 + \cdots\right)$. Observe that the $\ln t$ term reveals the singular nature of that solution as $t \to 0$ [i.e., as $x \to 1$ since $\ln t = \ln(1 - x)$]. Similarly, at $x = -1$ we obtain one solution $x$ and a second solution with a $\ln(1 + x)$ term, which "blows up" as $x \to -1$.

## Section 4.5

**3.** (a) $\int_0^\infty dx/(x^4 + 2) = \int_0^1 dx/(x^4 + 2) + \int_1^\infty dx/(x^4 + 2)$. $\int_0^1$ converges because $1/(x^4 + 2)$ is continuous on $0 \le x \le 1$, and $\int_1^\infty$ converges by Theorem 4.5.2 (b) because $1/(x^4 + 2) \sim 1/x^4$ as $x \to \infty$ and $\int_1^\infty dx/x^4$ is a convergent $p$-integral (Theorem 4.5.1). Observe that it was *necessary* to break up the integral, for example as $\int_0^1 + \int_1^\infty$, because otherwise our $I_2$ integral (in Theorem 4.5.2) would be $\int_0^\infty dx/x^4$ and $1/x^4$ is not bounded on $0 \le x < \infty$, as assumed in the theorem. (d) Break up $\int_0^\infty = \int_0^1 + \int_1^\infty$. $\int_0^1$ converges because the integrand is continuous on $0 \le x \le 1$, but $\int_1^\infty$ diverges [Theorem 4.5.2 (b)] because $x^{3.2}/(x^4 + 100) \sim 1/x^{0.8}$ and $\int_1^\infty dx/x^{0.8}$ is a divergent horizontal $p$-integral (Theorem 4.5.1). Thus, the given integral diverges. (g) The integral is positive on $4 \le x < \infty$ and $\frac{\sin^2 x}{\sqrt{x}(x-1)} \le \frac{1}{\sqrt{x}(x-1)} \sim \frac{1}{x^{3/2}}$. Since $\int_4^\infty dx/x^{3/2}$ converges (Theorem 4.5.1 with $p = 3/2 > 1$), the given integral converges as well [Theorem 4.5.2 (b) and (a)]. (j) $1/(x^2 \cos x) \sim 1/x^2$ as $x \to 0$ so, by Theorem 4.5.5, the integral diverges. **6.** (a) Obtain $I = \int_0^{1/2} \xi^2\, d\xi/(1 + 2\xi^4)$, which integral is not singular; the limits are finite and the integrand is continuous on $0 \le \xi \le 1/2$, so $I$ converges. Thus, we may be able to "desingularize" a convergent singular integral by a suitable change of variables. This idea is especially important regarding the *numerical* evaluation of integrals. For instance, it would be much easier to evaluate the regular $\xi$ integral (above) than the original singular integral, because of the *finite* extent of the $\xi$ integration interval. **7.** (a) $1 < \alpha < \infty$ (c) As $x \to \infty$, $x^\alpha/(x + 1) \sim 1/x^{1-\alpha}$ so we need $1 - \alpha > 1$ or $\alpha < 0$; as $x \to 0$, $x^\alpha/(x + 1) \sim x^\alpha = 1/x^{-\alpha}$ so we need $-\alpha < 1$ or $\alpha > -1$. Thus, for convergence we need $-1 < \alpha < 0$. (e) $I = \int_1^2 (x - 1)^\alpha (x + 1)^\alpha\, dx$ so the integrand blows up at $x = 1$ if $\alpha < 0$. By vertical $p$-integral test, we need $\alpha > -1$ for convergence. If this is not clear, let $x - 1 = t$, to move the singularity to the origin. Then $I = \int_0^1 t^\alpha (t + 2)^\alpha\, dt$ and $t^\alpha(t + 2)^\alpha \sim 2^\alpha/t^{-\alpha}$ as $t \to 0$, so we need $-\alpha < 1$ or $\alpha > -1$.
**8.** (a) (16) gives $\Gamma(3.5) = 2.5\Gamma(2.5) = (2.5)(1.5)\Gamma(1.5) = (2.5)(1.5)(0.5)\Gamma(0.5) = (2.5)(1.5)(0.5)\sqrt{\pi} = 3.323$, which appears to be consistent with Fig. 3. (b) (22) gives $\Gamma(-3.5) = \Gamma(-2.5)/(-3.5) = \Gamma(-1.5)/(-3.5)(-2.5) = \Gamma(-0.5)/(-3.5)(-2.5)(-1.5) = \Gamma(0.5)/(-3.5)(-2.5)(-1.5)(-0.5) = \sqrt{\pi}/(3.5)(2.5)(1.5)(0.5) = 0.270$, which appears to be consistent with Fig. 3. **10.** (a) With $x^p = t$, $I = \frac{1}{p}\int_0^\infty e^{-t}t^{(1/p)-1}\, dt = \frac{1}{p}\Gamma(\frac{1}{p})$. (c) Let $x^2 = t$.
**13.** HINT: Sketch the graph of $\exp(-x^p)$, on $0 \le x < \infty$, as $p \to \infty$. **19.** (a) $F(x) \sim 4 = O(1)$ as $x \to 0$

(d) $G(x) \sim -3/4 = O(1)$ as $x \to \infty$ (h) $I(x) \sim 2x/x^2 = 2/x = O(1/x)$ as $x \to \infty$

## Section 4.6

**12.** (a) Comparing $y'' + 4x^2y = 0$ with $y'' + (a/x)y' + bx^{c-a}y = 0$ gives $a = 0, b = 4, c = 2$ so $\alpha = 1/2, \nu = 1/4$ and $y(x) = x^{1/2}Z_{1/4}(\frac{1}{2}\sqrt{4}x^2)$, $y(x) = A\sqrt{x}J_{1/4}(x^2) + B\sqrt{x}Y_{1/4}(x^2)$. (d) Comparing $y'' + (9/4)xy = 0$ with $y'' + (a/x)y' + bx^{c-a}y = 0$ gives $a = 0, b = 9/4, c = 1$ so $\alpha = 2/3, \nu = 1/3$ and $y(x) = x^{1/2}Z_{1/3}(\frac{2}{3}\sqrt{\frac{9}{4}}x^{3/2})$, $y(x) = A\sqrt{x}J_{1/3}(x^{3/2}) + B\sqrt{x}Y_{1/3}(x^{3/2})$. (g) $a = 3, b = -1, c = 3$ so $\alpha = 1, \nu = -1$ and $y(x) = x^{-1}Z_{-1}(\sqrt{|-1|}x) = x^{-1}Z_1(x)$, $y(x) = Ax^{-1}I_1(x) + Bx^{-1}K_1(x)$. NOTE: $b = -1 < 0$ so $Z_1$ denotes $I$ and $K$, not $J$ and $Y$. [See sentence following (50).] (j) $a = 3, b = -4, c = 3$ so $\alpha = 1, \nu = -1$ and $y(x) = x^{-1}Z_{-1}(\sqrt{|-4|}x) = x^{-1}Z_1(2x)$, $y(x) = Ax^{-1}I_1(2x) + Bx^{-1}K_1(2x)$. **13.** (a) $a = 3, b = 9, c = 3$ so $\alpha = 1$, $\nu = -1$ and $y(x) = Ax^{-1}J_1(3x) + Bx^{-1}Y_1(3x)$. (b) Because $x^{-1} \to \infty$, $J_1(3x) \to 0$, and $Y_1(3x) \to -\infty$ as $x \to 0$, we need to be careful in applying the initial conditions. Using (16b) we have $y(x) = \frac{A}{x}\left(\frac{3x}{2} - \frac{27x^3}{16} + \cdots\right) + \frac{B}{x}Y_1(3x) = A\left(\frac{3}{2} - \frac{27x^2}{16} + \cdots\right) + \frac{B}{x}Y_1(3x)$. Since both $1/x \to \infty$ and $Y_1(3x) \to -\infty$ as $x \to 0$, we need to set $B = 0$ if we are to satisfy the condition $y(0) = 6$. Then $y(0) = 6 = 3A/2$ gives $A = 4$. Further, $y'(x) = A(-\frac{27}{8}x + \cdots)$ does happen to satisfy the other initial condition $y'(0) = 0$, so $y(x) = 4x^{-1}J_1(3x)$. (c) We saw that the condition $y(0) = 6$ implied both that $B = 0$ and that $A = 4$, giving $y(x) = 4x^{-1}J_1(3x)$. The latter gives $y'(0) = 0$ so there is no solution satisfying the initial conditions $y(0) = 6$, $y'(0) = 2$. This result does not contradict Theorem 3.3.1 because that theorem supposes that $p_1(x), \ldots, p_n(x)$ are continuous on the closed interval, whereas in this example $p_1(x) = 3/x$ is not continuous at the left endpoint $x = 0$.

# Chapter 5

## Section 5.2

**1.** (a) Yes; $K = 5$ (or greater), $c = 4$ (or greater), $T = 0$ (or greater). (d) Yes; $|\cosh 3t| = (e^{3t} + e^{-3t})/2 \le e^{3t}$ for all $t \ge 0$, so we can take $K = 1, c = 3, T = 0$. (g) Yes; $|\cos t^3| \le 1 = 1e^{0t}$ so we can take $K = 1, c = 0, T = 0$. (j) No; $e^{t^4}/e^{ct} = e^{t^4 - ct} \sim e^{t^4} \to \infty$ as $t \to \infty$, so there do not exist constants $K, c, T$ such that $\exp(t^4) \le Ke^{ct}$ for all $t \ge T$.

## Section 5.3

**1.** (a) $3/[s(s+8)] = \frac{3}{8}\frac{1}{s} - \frac{3}{8}\frac{1}{s+8}$ so linearity and entries 1 and 2 give $f(t) = (3/8)(1 - e^{-8t})$. Alternatively, $1/s \to 1$ and $1/(s+8) \to e^{-8t}$ so the convolution theorem (and linearity) give $3/[s(s+8)] = 3 * e^{-8t} = 3\int_0^t e^{-8\tau}\,d\tau = (3/8)(1 - e^{-8t})$. (d) $5(e^{-2t/3} - e^{-t})$ **3.** (a) Choose $a = -4, b = 4i$ (or $-4i$) in entry 9 and obtain $e^{-4t}\sin(4it)/4i = e^{-4t}(e^{i(4it)} - e^{-i(4it)})/[(2i)(4i)] = (1 - e^{-8t})/8$. (d) Choose $a = 1/2, b = 3i/2$ in entry 9 and obtain $(e^{2t} - e^{-t})/3$. **10.** (a) $f(t) = \int_0^t e^{t-\tau}\sin 2\tau\,d\tau = e^t * \sin 2t$ so $L\{f(t)\} = L\{e^t\}L\{\sin 2t\} = \frac{1}{s-1}\frac{2}{s^2+4}$. (d) $f(t) = 1 * \cosh t$ so $L\{f(t)\} = \frac{1}{s}\frac{s}{s^2-1} = \frac{1}{s^2-1}$. **11.** (a) $F(s)G(s) = \frac{1}{s^2}\frac{1}{s-1}$ and $L^{-1}\{\frac{1}{s^2(s-1)}\} = e^t - t - 1 \ne te^t$.

## Section 5.4

**1.** (a) $x(t) = 2t^2 - 2t + 1 + x(0)e^{-2t}$ (d) $x(t) = 2 - t + t^3$ (g) $x(t) = 5e^t - 2e^{2t}$ (j) $x(t) = [1 - (1 + 21t)e^{-3t}]/9$ (m) $x(t) = \frac{1}{6}t^3 + \frac{1}{4}t^2 - \frac{1}{4}t + \frac{5}{8} + \frac{1}{56}e^{t/2}[21\cos(\frac{\sqrt{7}t}{2}) + \sqrt{7}\sin(\frac{\sqrt{7}t}{2})]$ (s) $x(t) = 2t^2 - \frac{1}{3}t + \frac{1}{9}(1 - e^{-3t})$

## Section 5.5

**1.** (a) $f(t) = t[H(t) - H(t-2)] + (4-t)[H(t-2) - H(t-4)] = tH(t) + (4-2t)H(t-2) - (4-t)H(t-4)$, $F(s) = \frac{1}{s} + (e^{-4s} - 2e^{-2s})/s^2$ (d) $f(t) = (t^3 - t)[H(t) - H(t-1)] - 6H(t-1)$, $F(s) = \frac{6}{s^4} - \frac{1}{s^2} - 2\frac{e^{-s}}{s^4}(3 + 3s + s^2 + 3s^3)$

**3.** (a) $-t-10$ NOTE: Recall that $H(t)=1$ for $t>0$. (d) $(t-a)H(t-a)-(t-b)H(t-b)$ (g) $(5-t)H(5-t)$ (j) $\int_0^t e^{-(t-\tau)}H(\tau-5)\,d\tau=0$ for $t<5$ and $e^{-t}\int_5^t e^{\tau}\,d\tau=1-e^{5-t}$ for $t>5$. Thus, we obtain $(1-e^{5-t})H(t-5)$. **5.** (a) $X(s)=e^{-s}/[s(s-1)]$, $x(t)=(e^{t-1}-1)H(t-1)$ (d) $f(t)=10[H(t-5)-H(t-7)]$, $x(t)=10(e^{t-5}-1)H(t-5)-10(e^{t-7}-1)H(t-7)$ (h) $f(t)=t[1-H(t-1)]+(2-t)[H(t-1)-H(t-2)]$, $X(s)=1/[s^2(s-1)]-2e^{-s}/[s^2(s-1)]+e^{-2s}/[s^2(s-1)]$, $x(t)=-t-1+e^t+2(t-e^{t-1})H(t-1)+(1-t+e^{t-2})H(t-2)$

## Section 5.6

**1.** (a) $X(s)=e^{-2s}/(s^2-1)$, $x(t)=H(t-2)\sinh(t-2)$ by entries 30 and 5. (d) $x(t)=t+2(1-e^{-t})+H(t-2)(1-e^{-(t-2)})$ (g) $x(t)=8e^t-4e^{2t}+100H(t-3)(e^{2(t-3)}-e^{t-3})$ **2.** (b) Letting $\kappa t=\tau$, $\int_{-\infty}^{\infty}g(t)\delta(\kappa t)\,dt=\int_{-\infty}^{\infty}g(\tau/\kappa)\delta(\tau)\,d\tau/\kappa=g(0)/\kappa$ if $\kappa>0$ and $\int_{\infty}^{-\infty}g(\tau/\kappa)\delta(\tau)\,d\tau/\kappa=-g(0)/\kappa$ if $\kappa<0$. Thus, $\int_{-\infty}^{\infty}g(t)\delta(\kappa t)\,dt=-g(0)/|\kappa|$. Similarly, $\int_{-\infty}^{\infty}g(t)\frac{\delta(t)}{|\kappa|}\,dt=g(0)/|\kappa|$, so $\delta(\kappa t)=\delta(t)/|\kappa|$. (c) $\int_{-\infty}^{\infty}g(t)[f(t)\delta(t)]\,dt=\int_{-\infty}^{\infty}[g(t)f(t)]\delta(t)]\,dt=g(0)f(0)$. Also, $\int_{-\infty}^{\infty}g(t)[f(0)\delta(t)]\,dt=f(0)\int_{-\infty}^{\infty}g(t)\delta(t)\,dt=f(0)g(0)$, so $f(t)\delta(t)=f(0)\delta(t)$, for $f(0)\neq 0$. If $f(0)=0$, then $\int_{-\infty}^{\infty}g(t)[f(t)\delta(t)]\,dt=\int_{-\infty}^{\infty}[g(t)f(t)]\delta(t)]\,dt=g(0)f(0)=0$. Also, $\int_{-\infty}^{\infty}g(t)[0]\,dt=0$, so $0\delta(t)=0$. (d) $\int_{-\infty}^{t}\delta(\tau)\,d\tau=0$ if $t<0$ and 1 if $t>0$, which is $H(t)$.

## Section 5.7

**1.** (a) $L^{-1}\{\frac{1}{s}\frac{s}{(s^2+a^2)^2}\}=f(t)$ so $f(t)=1*\frac{t\sin at}{2a}$ (entry 11) so $f(t)=\frac{1}{2a}\int_0^t\tau\sin a\tau\,d\tau=(\sin at-at\cos at)/(2a^3)$. $sF(s)\to 0$ as $s\to\infty$, and $f(0)$ does equal 0. (d) Taylor expanding $s^2$ about $s=-1$ gives $1-2(s+1)+(s+1)^2$, so $s^2/(s+1)^3=1/(s+1)^3-2/(s+1)^2+1/(s+1)$ and, using entries 1, 7, 29, $f(t)=e^{-t}\frac{t^2}{2}-2e^{-t}t+e^{-t}$. $sF(s)\to 1$ as $s\to\infty$ and $f(0)$ does equal 1. (g) $1/s^5\to t^4/4!=t^4/24$ by entry 7 and $e^{-s}/s^5\to H(t-1)(t-1)^4/24$ by entry 30. $sF(s)\to 0$ as $s\to\infty$ and $f(0)$ does equal 0. (j) $F(s)=\ln(s^2+a^2)-\ln s^2$, $dF/ds=\frac{2s}{s^2+a^2}-\frac{2}{s}\to 2\cos at-2$ (entries 4 and 1) so, by entry 31, $-tf(t)=2\cos at-2$, $f(t)=2(1-\cos at)/t$. $sF(s)\to s\ln(1+a^2/s^2)\to(\infty)(0)$ as $s\to 0$. For l'Hôpital we need $0/0$ or $\infty/\infty$, so write $s\ln(1+a^2/s^2)=\frac{\ln(1+a^2/s^2)}{1/s}\to 0$ as $s\to\infty$ by l'Hôpital and, also by l'Hôpital, $f(t)\to 0$ as $t\to 0$. (m) $1/(s^2+s+1)=1/[(s+\frac{1}{2})^2+\frac{3}{4}]\to\frac{e^{-t/2}\sin\sqrt{3/4}\,t}{\sqrt{3/4}}$ (entries 3, 29), so $e^{-s}/(s^2+s+1)\to\frac{2}{\sqrt{3}}H(t-1)e^{-(t-1)/2}\sin 2\sqrt{3}(t-1)$ (entry 30). $sF(s)\to 0$ as $s\to\infty$ and $f(0)$ does equal 0. (p) Proceed as in Example 5. $\frac{d}{ds}\ln\left(\frac{s^2+1}{s^2+s}\right)=\frac{2s}{s^2+1}-\frac{2s+1}{s^2+s}$ so $\ln\left(\frac{s^2+1}{s^2+s}\right)=-\int_s^{\infty}\left(\frac{2\sigma}{\sigma^2+1}-\frac{2\sigma+1}{\sigma^2+\sigma}\right)d\sigma$, "$f(t)$" $=L^{-1}\left\{-\frac{2s}{s^2+1}+\frac{2s+1}{s^2+s}\right\}=-2\cos t+1+e^{-t}$, so entry 32 gives $f(t)=$ "$f(t)$"$/t=(-2\cos t+1+e^{-t})/t$. $sF(s)\to 0$ as $s\to\infty$ and l'Hôpital does give $f(t)\to 0$ as $t\to 0$. (s) $\frac{1}{s}\tanh s=\frac{1}{s}\frac{e^s-e^{-s}}{e^s+e^{-s}}=\frac{1}{s}\frac{1-e^{-2s}}{1+e^{-2s}}=\frac{1}{s}(1-e^{-2s})(1-e^{-2s}+e^{-4s}-e^{-6s}+\cdots)=\frac{1}{s}(1-2e^{-2s}+2e^{-4s}-2e^{-6s}+\cdots)\to H(t)-2H(t-2)+2H(t-4)-H(t-6)+\cdots$, the graph of which is seen to be a square wave $f(t)=1$ on $0<t<2$, $-1$ on $2<t<4$, and of period 4. As a check, we can verify that the right-hand side of (37) does give us back $\frac{1}{s}\tanh s$. $sF(s)\to 1$ as $s\to\infty$ and we do have $f(t)\to 1$ as $t\to 0$. **5.** (a) $F(s)=\frac{1}{1-e^{-\pi s}}\int_0^{\pi}(\sin t)e^{-st}\,dt=\frac{1+e^{-\pi s}}{1-e^{-\pi s}}\frac{1}{s^2+1}=\coth(\pi s/2)/(s^2+1)$ **8.** $x(t)=x_0\cos t+x_0'\sin t+f(t)*\sin t=x_0\cos t+x_0'\sin t+\int_0^t[2-4H(\tau-2)+4H(\tau-4)-4H(\tau-6)-\cdots]\sin(t-\tau)\,d\tau=x_0\cos t+x_0'\sin t+2t-4H(t-2)[1-\cos(t-2)]+4H(t-4)[1-\cos(t-4)]-4H(t-6)[1-\cos(t-6)]+\cdots$. With $x_0=x_0'=1$, $x(5)=3.2036$.

# Chapter 6

## Section 6.2

**2.** (a) $y_1=0.8, y_2=0.64, y_3=0.512$ (d) $y_1=-0.2, y_2=0.0192, y_3=0.2194$ (g) $y_1=y_2=y_3=0$. Indeed, the solution is $y(x)=0$. **3.** (g) Exact solution is $y(x)=4e^x-x^2-2x-3$. Sample results: $y_5=2.253091, y_{10}=$

4.534344; $y(x_5) = y(0.5) = 2.344885$, $y(x_{10}) = y(1) = 4.873127$, so $E_{10} = 0.338783$.
**4.** (a) For $h = 0.1, 0.05, 0.01, 0.005, 0.001$ Euler gives the values $1.2, 1.225, 1.245, 1.2475, 1.2495$, respectively. Since the exact value is $y(0.5) = 1.25$, the accumulated truncation error is $0.05, 0.025, 0.005, 0.0025, 0.0005$, respectively. These results are consistent with the method being a first-order method, i.e., with the accumulated truncation error being proportional to the step size $h$, because each time we reduce $h$ the error reduces proportionately. (d) For $h = 0.1, 0.05, 0.01, 0.005, 0.001$ Euler gives the values $1.101786, 1.115792, 1.127303, 1.128761, 1.129932$. The solution is $y(x) = \exp(1 - \cos x)$, so the exact value is $y(0.5) = 1.130226$, hence the accumulated truncation error is $0.02844, 0.014434, 0.002923, 0.001465, 0.000294$, which does indeed diminish proportional to $h$ (approximately).

## Section 6.3

**1.** (a) Second-order: $y_1 = 2.15, y_2 = 2.486476$. Fourth-order: $y_1 = 2.140149, y_2 = 2.477369$. Exact: $y(x) = (4500x^2 + 8)^{1/3}$, so $y(x_1) = y(0.02) = 2.13997$, $y(x_2) = y(0.04) = 2.47712$. (d) Second-order: $y_1 = -7.749214, y_2 = -7.495329$. Fourth-order: $y_1 = -7.749231, y_2 = -7.495363$. Exact: $y(x) = -8\cos x/\cos 1$, so $y(x_1) = y(1.02) = -7.749231$, $y(x_2) = y(1.04) = -7.495363$. **2.** (a) Second-order: $y_{10} = 10.68520$. Fourth-order: $y_{10} = 10.44818$. Exact: $y(x) = (4500x^2 + 1)^{1/3}$, so $y(x_{10}) = y(0.5) = 10.40350$. (d) Second-order: $y_{10} = 0.877527$. Fourth-order: $y_{10} = 0.877583$. Exact: $y(x) = \cos x$, so $y(x_{10}) = y(0.5) = 0.877583$.
**3.** (a) $x = 1$: Euler gives $y_{10} = 3.434368$ for $h = 0.1$ and $y_{20} = 3.570633$ for $h = 0.05$. Exact $y(1) = 3.718282$ so (28) gives $p \approx 0.943$. $x = 2$: Euler gives $y_{20} = 10.154750$ for $h = 0.1$ and $y_{40} = 10.737989$ for $h = 0.05$. Exact $y(2) = 11.389056$ so (28) gives $p \approx 0.923$. (b) $x = 1$: Second-order R-K gives $y_{10} = 3.705918$ for $h = 0.1$ and $y_{20} = 3.715097$ for $h = 0.05$. Exact $y(1) = 3.718282$ so (28) gives $p \approx 1.957$. $x = 2$: Second-order R-K gives $y_{20} = 11.335919$ for $h = 0.1$ and $y_{40} = 11.3753429$ for $h = 0.05$. Exact $y(2) = 11.389056$ so (28) gives $p \approx 1.954$
**4.** (a) $x(0) = 0$, $x(600) = 3.314$, $x(1200) = 3.852$, $x(1800) = 3.967$, $x(2400) = 3.993$, $x(3000) = 3.998$, $x(3600) = 4.000$ (d) $x(0) = 6$, $x(600) = 4.487$, $x(1200) = 4.111$, $x(1800) = 4.025$, $x(2400) = 4.006$, $x(3000) = 4.001$, $x(3600) = 4.000$ NOTE: Setting $x'(t) = 0$ in the differential equation gives $0 = 0.02 - 0.01\sqrt{x}$ so the steady-state solution is $x \to 4$. **7.** (b) $C = 0.09824$, $x(600) = 3.313860967, \ldots, x(3600) = 3.999636784$
**10.** (a) $h = 0.05$ gives $3.718281474$, $h = 0.02$ gives $3.718281819$, exact $y(1) = 3.718281828$ so (28) gives $p \approx 4.008$. (b) $h = 0.05$ gives $3.701259429$, $h = 0.02$ gives $3.711558039$ so (28) gives $p \approx 1.01$. Evidently, the error has reduced the method to a first-order method. **13.** (a) Exact $y(x) = 1/(1 - x^2)$ so $y_1 = y(0.1) = \underline{1.01010}$, $y_2 = y(0.2) = \underline{1.04167}$, $y_3 = y(0.3) = \underline{1.09890}$. Then, A-B predictor (31a) gives $y_4 = y_3 + [55(2)(0.3)y_3^2 - 59(2)(0.2)y_2^2 + 37(2)(0.1)y_1^2 - 9(2)(0)y_0^2](0.1/24) = 1.18970$ and taking this as $y_4^{(0)}$, A-M corrector (33) gives $y_4^{(1)} = y_3 + [9(2)(0.4)y_4^{(0)\,2} + 19(2)(0.3)y_3^2 - 5(2)(0.2)y_2^2 + (2)(0.1)y_1^2](0.1/24) = 1.09890 + (7.2y_4^{(0)\,2} + 11.80033)(0.1/24) = 1.19053$, $y_4^{(2)} = 1.09890 + (7.2y_4^{(1)\,2} + 11.80033)(0.1/24) = 1.19059$, $y_4^{(3)} = 1.09890 + (7.2y_4^{(2)\,2} + 11.80033)(0.1/24) = \underline{1.19059}$. (b) Using the four underlined values in (a), the A-B predictor gives $y_5 = y_4 + [55(2)(0.4)y_4^2 - 59(2)(0.3)y_3^2 + 37(2)(0.2)y_2^2 - 9(2)(0.1)y_1^2](0.1/24) = 1.33161$, which we use as $y_5^{(0)}$ in the A-M corrector formula $y_5^{(1)} = y_4 + [9(2)(0.5)y_5^{(0)\,2} + 19(2)(0.4)y_4^2 - 5(2)(0.3)y_3^2 + (2)(0.2)y_2^2](0.1/24) = 1.19059 + (9y_5^{(0)\,2} + 18.35736)(0.1/24) = 1.33357$, $y_5^{(2)} = 1.19059 + (9y_5^{(1)\,2} + 18.35736)(0.1/24) = 1.33377$, $y_5^{(3)} = 1.19059 + (9y_5^{(2)\,2} + 18.35736)(0.1/24) = \underline{1.33379}$.

## Section 6.4

**2.** (a) Exact: $y(x_1) = 0.9800666$, $z(x_1) = -0.1986693$. Euler: $y_1 = 1$, $z_1 = -0.2$. Second-order RK: $y_{n+1} = y_n + \frac{1}{2}(k_1 + k_2)$, $z_{n+1} = z_n + \frac{1}{2}(l_1 + l_2)$. With $n = 0$, $k_1 = hf(x_n, y_n, z_n) = hz_n = hz_0 = 0$, $l_1 = hg(x_n, y_n, z_n) = -hy_n = -hy_0 = -0.2$, $k_2 = hf(x_{n+1}, y_n + k_1, z_n + l_1) = h(z_0 + l_1) = -0.04$, $l_2 = hg(x_{n+1}, y_n + k_1, z_n + l_1) = -h(y_0 + k_1) = -0.2$, so $y_1 = 1 + \frac{1}{2}(0 - 0.04) = 0.98$, $z_1 = 0 + \frac{1}{2}(-0.2 - 0.2) = -0.2$. Fourth-order RK: $y_0 = 1$, $z_0 = 0$, $k_1 = 0$, $l_1 = -0.2$, $k_2 = -0.02$, $l_2 = -0.2$, $k_3 = -0.02$, $l_3 = -0.198$, $k_4 = -0.0396$, $l_4 = -0.196$, so $y_1 = 0.9800667$, $z_1 = -0.1986667$ (d) Exact: $y(x_1) = 1.44$, $z(x_1) = 1.2$. Euler: $y_1 = 1.4$, $z_1 = 1.2$. Second-order RK: $k_1 = 0.4$, $l_1 = 0.2$, $k_2 = 0.49371$, $l_2 = 0.19444$, $y_1 = 1.4468$, $z_1 = 1.1972$. Fourth-order RK: $k_1 = 0.4$, $l_1 = 0.2$, $k_2 = 0.4436667$, $l_2 = 0.1983471$, $k_3 = 0.4350841$, $l_3 = 0.2022597$, $k_4 = 0.4834599$,

$l_4 = 0.1985687$, so $y_1 = 1.4401603$, $z_1 = 1.1999637$ **5.** (a) $x' = u$, $x(0) = x_0$; $u' = [f(t) - kx - cu]/m$, $u(0) = x_0'$ (d) $y' = u$, $y(0) = 2$; $u' = 3x+4y-u$, $u(0) = 1$ **6.** (a) $y(1) = 1.0418532996$, $z(1) = -1.3328381583$, $y(2) = 4.4563094187$, $z(2) = -0.9372059061$ **8.** (a) $L[Y_1] = 0$, $Y_1(0) = 1$, $Y_1'(0) = 0$; $L[Y_2] = 0$, $Y_2(0) = 0$, $Y_2'(0) = 1$; $L[Y_p] = 3\sin x$, $Y_p(0) = Y_p'(0) = 0$. Then $y(x) = C_1Y_1(x) + C_2Y_2(x) + Y_p(x)$. $y(0) = 1$ gives $C_1 = 1$ and $y(2) = 3$ gives $C_2 = [3 - Y_1(2) - Y_p(2)]/Y_2(2)$. Using *Maple* dsolve command with abserr = Float(1, −6) (to obtain the desired accuracy) gives $Y_1(1) = 0.338991$, $Y_2(1) = 1.219163$, $Y_p(1) = 0.626700$, $Y_1(2) = -8.366992$, $Y_2(2) = 7.639154$, $Y_p(2) = 14.523026$. Thus, $C_2 = -0.413139$. Hence, $y(1) = 0.46201$.

## Section 6.5

**1.** (a) No. $y(x) = Ce^{2x} + 4x$ and $y(0) = 0$ gives $C = 0$ so $y(x) = 4x$. In effect, numerical error causes $C$ to differ slightly from 0, in which case the $e^{2x}$ term will cause the solution to diverge dramatically from the exact solution $4x$. **6.** (a) $\partial f/\partial y = e^{x+y} > 0$ on $0 \le x \le 4$; stable (b) $\partial f/\partial y = -e^{x-y} < 0$ on $0 \le x \le 4$; unstable **11.** (a) General solution $y_n = A(4^n)$; particular solution $y_n = 5(4^n)$. (d) General solution $y_n = A + B(3^n)$; particular solution $y_n = 4 - 3^n$. **15.** (a) Particular solution $Y_n = -\frac{1}{2}n - \frac{1}{4}$; general solution $y_n = -\frac{1}{2}n - \frac{1}{4} + A(3^n)$. (d) Particular solution $Y_n = n^2$; general solution $y_n = n^2 + A(2^n) + B(3^n)$. (g) Particular solution $Y_n = e^n/(e^2 - 1)$; general solution $y_n = e^n/(e^2 - 1) + A + B(-1)^n$.

# Chapter 7

## Section 7.2

**4.** (a) $x^2 + y^2 = C^2$; concentric circles; flow clockwise. **5.** (a) Trajectories are the straight lines $y = -x + C$; rightward flow for $y > 0$, leftward for $y < 0$; singular points all along the $x$ axis. (d) $x^2 - (y/3)^2 = C^2$; equilibrium point at origin.
**6.** Use the *Maple* command

```
phaseportrait ([y, -x + x^3], [t, x, y], t = 0..20, {[0, .05, 0], [0, .3, 0], [0, .6, 0], [0, .9, 0], [0, .95, 0], [0, .99, 0]},
stepsize = .05, scene = [t, x]);
```

The periods are (approximately, from the plot) 6.3, 6.7, 7.5, 10.5, 12, 17, respectively. They do approach $2\pi$ as $A \to 0$ and show signs of tending to infinity as $A \to 1$.

**9.** (a) $y^2 + x^4 = C$. **11.** Equilibrium points at $(0, 0)$ and $(1, 1)$. The axes are trajectories, with the flow downward on the $y$ axis and rightward on the $x$ axis. Within the first quadrant the trajectories are closed orbits containing the equilibrium point $(1, 1)$, although the lineal element field does not really permit us to distinguish between periodic orbits and weak spirals, by eye.

## Section 7.3

**1.** (a) $0 = 0$ and $2x - y = 0$ give the *line* of singular points $y = 2x$; hence, not isolated. (d) Isolated singular points at $(-n\pi, n\pi)$ for $n = 0, \pm1, \pm2, \ldots$. (g) Isolated singular points at $(2\sqrt{2}, \sqrt{2})$ and $(-2\sqrt{2}, -\sqrt{2})$. **4.** No; for a given $\epsilon$ the corresponding $\delta$ could be enormous **9.** (a) Saddle; $y = 2x$ (unstable manifold), $y = -2x$ (stable manifold) (d) Saddle; $y = x/\sqrt{3}$ (unstable), $y = -x/\sqrt{3}$ (stable) (g) Unstable node; $y = x$ (unstable), $y = -x$ (unstable) **11.** (a) Unstable focus (d) Center (g) Unstable node

## Section 7.4

**1.**(b) If we write $x'' + \epsilon x'^3 + x = 0$ as $x'' + (\epsilon x'^2)x' + x = 0$ we can think of $(\epsilon x'^2)$ as the (variable) damping coefficient. For large motions $x'^2$ is large over much of the motion so the effective damping is greater than for the linear case $x'' + \epsilon x' + x = 0$. But as the motion diminishes $x'^2$ becomes small so the effective damping tends to zero. This result should be observable from the phaseportrait in part (a). **2.**(a) Saddle point at $(-1, 0)$, center at (1,0) (d) Singular points at $x = y = n\pi/2$ ($n = 0, \pm1, \pm2, \ldots$), saddles for $n$ odd and unstable foci for $n$

even. (g) Borderline case: stable focus, stable proper node, or stable improper node at (0,0); a phaseportrait plot reveals it to be a stable improper node. (j) Stable focus at (0,0), saddle at (4,8). **4.** (a) No, because of the $x'$ term. (c) Yes **6.** (a) If $r = 0.3$, (32) gives $S_+ = (x_+, y_+) = (3, 0.9)$ and $S_- = (x_-, y_-) = (1/3, 0.1)$. At $S_+$ obtain $X' = -0.3X + Y, Y' = 0.06X - Y$, a stable improper node with straight-line trajectories $Y = -0.7772X$ and $Y = 0.0772X$. At $S_-$ obtain $X' = -0.3X + Y, Y' = 0.54X - Y$, a saddle with straight-line trajectories $Y = -1.16394X$ (stable manifold) and $Y = 0.46394X$ (unstable manifold). At the origin obtain $x' = -0.3x + y$, $y' = 0x - y$, a stable node with straight- line trajectories $y = 0$ and $y = -0.7x$. **8.** (a) Singular points at $x_\pm = (1 \pm \sqrt{(1-4r)})/2$, $y = 0$, where $0 < x_- < x_+ < 1$. $(x_+, 0)$ is a saddle and $(x_-, 0)$ is a center or a focus, but we can rule out a focus because the system is conservative. Trajectories given by $y^2 + x^2 + 2r \ln|x-1| =$ constant$= C$. Of these, the separatrix is $y^2 + x^2 + 2r \ln|x-1| = x_+^2 + 2r \ln(1 - x_+)$. At the right, the vertical line $x = 1$ is a trajectory, with upward flow direction. [As $r \to 1/4$, $x_\pm$ merge at $x = 1/2$ and the merging center and saddle produce a higher-order singularity with no periodic motions, and for $r > 1/4$ the singularity disappears altogether. Physically, for $r > 1/4$ the force of attraction is so great as to rule out the possibility of oscillation and the vertical line $x = 1$ is an symptote for every trajectory. These points are covered in parts (c)–(g).]

## Section 7.5

**2.** (a) $f(x) = x^2 - 1, g(x) = x^3$. $f$ is even; $g$ is odd; $g(x) > 0$ for all $x > 0$; $g'(x) = 3x^2$ is continuous for all $x$. $F(x) = \int_0^x f(\zeta)d\zeta = x^3/3 - x$ so, with $x_0 = \sqrt{3}$, $F(x) < 0$ for all $0 < x < x_0$; $F(x) > 0$ for $x > x_0$; $F(x)$ is monotone increasing for $x > x_0$; $F(x) \to \infty$ as $x \to \infty$. Thus, there exists a single limit cycle enclosing the origin.

## Section 7.6

**4.** Saddle at $(0,0)$; stable foci at $(1,0)$ and $(-1,0)$ if $r < \sqrt{8}$, stable nodes if $r > \sqrt{8}$.

# Chapter 8

## Section 8.2

**1.** (a) No; e.g., $6x^2 - x + 3 = 0$ (d) Yes (g) A linear equation cannot be transcendental

## Section 8.3

**1.** Final results:
(a) $x = 7/17, y = -1/17$ (unique solution)
(d) $z = \alpha, y = 6 + 3\alpha, x = 7 + 2\alpha$ (nonunique solution: a 1– parameter family of solutions)
(g) $z = \alpha, y = 3 - 2\alpha, x = -2 + \alpha$ (nonunique solution: a 1– parameter family of solutions)
(j) $x_4 = \alpha, x_3 = \beta, x_2 = 2 + \alpha + \beta, x_1 = 1 - \beta$ (nonunique solution: a 2–parameter family of solutions)
**4.** Yes; yes; yes; yes; no. At most we can have a 14- parameter family of solutions, and that will occur if and only if *all* of the $a_{ij}$'s and $c_j$'s in (1) are zero. **7.** (a) The equations are homogeneous because we can write them as $(2-\lambda)x + y = 0$, $x + (2-\lambda)y = 0$. There is only the unique trivial solution $x = y = 0$ unless $\lambda = 1$ or $\lambda = 3$. If $\lambda = 1$ there is the 1– parameter family of solutions $y = \alpha$, $x = -\alpha$ (which contains, but is not limited to the trivial solution). If $\lambda = 3$ there is the 1–parameter family of solutions $y = \alpha$, $x = \alpha$.
(d) By inspection, if $\lambda = 0$ there is the 2–parameter family of solutions $x = \alpha$, $y = \beta$, $z = 0$, and if $\lambda \neq 0$, there is the 1– parameter family of solutions $x = y = \alpha$, $z = \lambda\alpha$. **8.** (a) Slow down: if we add $-2$ times the first equation to the second, as a replacement for the second we obtain $x_1 - 2x_2 = 0$ and $0 = 0$. Now if we add $-1/2$ times the second to the first we still have $x_1 - 2x_2 = 0$ and $0 = 0$. **10.** Don't physical systems such as this always have unique solutions? For any choice of values of $R_1, R_2, R_3$ the system (10.1) will indeed have a *unique* solution for $i_1, i_2, i_3$, except for these cases: Case 1. If $R_2 = R_3 = 0$ there is a *nonunique* solution: $i_1 = E/R_1$ and $i_2$ and $i_3$ sum to $i_1$, but are not uniquely determined ($i_3 = \alpha$, $i_2 = E/R_1 - \alpha$). Case 2. If $R_1$ and either $R_2$ or $R_3$ are zero, then there is *no solution*: the system is *inconsistent*. If $R_1 = R_3 = 0$, for example, then the last equation is $0 = E$, which has no

solutions. More physically, if $R_1 = R_3 = 0$, then we have a "short circuit" and the currents will be infinite. That is what the last equation ($0i_1 + 0i_2 + 0i_3 = E$) is trying to tell us for, crudely speaking, zero times something can equal a nonzero number only if that something is infinite.

# Chapter 9

## Section 9.3

**1.** (a) $-25$. **4.** (a) $\mathbf{OC} \cdot \mathbf{AB} = (\mathbf{OA} + \mathbf{AD} + \mathbf{DC}) \cdot \mathbf{AB} = \mathbf{OA} \cdot \mathbf{AB} + \mathbf{AD} \cdot \mathbf{AB} + \mathbf{DC} \cdot \mathbf{AB} = (1)(1)\cos 90^o + (1)(1)\cos 90^o + (1)(1)\cos 0^o = 1$. By expressing $\mathbf{OC}$ as $\mathbf{OA} + \mathbf{AD} + \mathbf{DC}$ and using (3.1) we obtain several simple dot products, simple because $\theta$ is $0^o$ or $90^o$. (d) $\mathbf{OC} \cdot \mathbf{CP} = (\mathbf{OA} + \mathbf{AD} + \mathbf{DC}) \cdot (\frac{1}{2}\mathbf{CD} + \frac{1}{2}\mathbf{AO}) = -1$. NOTE: There is no name for the midpoint of $CD$ so for the vector from that point to $P$ we have used $\frac{1}{2}\mathbf{AO}$.
**5.** (a) $APO = \cos^{-1}[(\mathbf{PA} \cdot \mathbf{PO})/(\|\mathbf{PA}\|\|\mathbf{PO}\|)]$, where $\mathbf{PA} \cdot \mathbf{PO} = (\frac{1}{2}\mathbf{OA} + \frac{1}{2}\mathbf{CD} + \mathbf{DA}) \cdot (\frac{1}{2}\mathbf{AO} + \frac{1}{2}\mathbf{CD} + \mathbf{DA}) =$ etc. $= 1$; $\|\mathbf{PA}\| = \sqrt{\mathbf{PA} \cdot \mathbf{PA}} = \sqrt{(\frac{1}{2}\mathbf{OA} + \frac{1}{2}\mathbf{CD} + \mathbf{DA}) \cdot (\frac{1}{2}\mathbf{OA} + \frac{1}{2}\mathbf{CD} + \mathbf{DA})} = \sqrt{6}/2$, $\|\mathbf{PO}\| =$ etc. $= \sqrt{6}/2$, so $APO = \cos^{-1}(2/3) = 48.19^o$.

## Section 9.4

**1.** (a) $(24, -7, 23, 32)$ (d) Not defined because of the $\mathbf{tu}$ product. NOTE: Actually, one can extend our mathematical system to include products of vectors, which are called **dyads** or **second order tensors**, but even if we do that we cannot add the dyad $4\mathbf{tu}$ to the vector $\mathbf{w}$. **2.** (a) $\mathbf{x} = [-5/9, 1, 20/9, -2/3]$ **4.** Only the trivial solution exists: $\alpha_1 = \alpha_2 = \alpha_3 = 0$.

## Section 9.5

**1.** (a) $\|\mathbf{u}\| = 5$, $\|\mathbf{v}\| = \sqrt{5}$, $\theta = 63.4^o$ (d) $\|\mathbf{u}\| = 2\sqrt{3}$, $\|\mathbf{v}\| = \sqrt{77}$ **2.** (a) Yes, it is a scalar times a vector. (b) No, it is a vector dotted with a scalar. **3.** (a) $ABC = \cos^{-1}(-1/\sqrt{10}) = 108.4^o$, $BCA = \cos^{-1}(2/\sqrt{5}) = 26.6^o$, $CAB = \cos^{-1}(1/\sqrt{2}) = 45^o$. **4.** (a) $\hat{\mathbf{u}} = [4/5, 3/5]$, $\hat{\mathbf{v}} = [2/\sqrt{5}, -1/\sqrt{5}]$ **6.** (a) $\mathbf{u}_1 = \mathbf{u} = [1, 3, 0]$, $\mathbf{u}_2 = \mathbf{u} - \frac{10}{11}\mathbf{v} = [-9/11, 3/11, 0]$, $\mathbf{u}_3 = \mathbf{u} - \frac{5}{4}\mathbf{v} + \frac{3}{4}\mathbf{w} = [0, 0, -9/4]$ **7.** (a) $3\sqrt{3}$ **9.** (a) $\mathbf{u} \cdot [3, 0, -1] = 3u_1 - u_3 = 0$, so $u_3 = \alpha$, $u_2 = \beta$, $u_1 = \alpha/3$ and $\mathbf{u} = [\alpha/3, \beta, \alpha]$ ($\alpha, \beta$ arbitrary) (d) $\mathbf{u} = [-15\alpha - 4\beta, 5\alpha - 8\beta, 7\beta, 7\alpha]$ ($\alpha, \beta$ arbitrary) **10.** (c) $\mathbf{u}_1 = [0, 18/13, 27/13]$, $\mathbf{u}_2 = [2, 21/13, -14/13]$ **12.** (b) $l_1 = 2/\sqrt{30}$, $l_2 = -1/\sqrt{30}$, $l_3 = 5/\sqrt{30}$ **14.** (a) Yes **15.** (a) Gauss elimination gives $x_3 = 3\alpha$, say, $x_2 = 8/3 + 2\alpha$, $x_1 = 8/3 - \alpha$, so $\mathbf{x} = [8/3 - \alpha, 8/3 + 2\alpha, 3\alpha] = [8/3, 8/3, 0] + \alpha[-1, 2, 3]$ so the desired vector is $\pm\frac{1}{\sqrt{14}}[-1, 2, 3]$.

## Section 9.6

**1.** (a) Yes (f) No; (6) is not satisfied

## Section 9.7

**1.** (a) Yes (d) No (g) No **3.** (a) No: there is no negative inverse $-\mathbf{u}$ for each $\mathbf{u}$ in the set. Also, the set is not closed under scalar multiplication because $\alpha\mathbf{u}$ is not in the set if $\alpha < 0$. **4.** (a) span$\{[-4, 0, 1], [1, 1, 0]\}$ (d) span$\{[-1, 0, 0, 1], [-2, 1, 1, 0]\}$ **5.** (a) $[-4, 0, 1]$, $[2, 1, 0]$ **6.** (a) Yes (b) No **7.** $[1, 2]$, $[2, -1]$

## Section 9.8

**1.** (a) No; a set is one or the other. **2.** (a) $[3, 4] = 2[1, 1] + [1, 2]$ **3.** (a) LD: e.g., $[7, 3] = [1, 3] + 3[2, 0] + 0[-1, 3]$ (d) LD: e.g., $[2, 3, 0] = -\frac{1}{3}[1, -2, 4] + [1, 1, 0] + \frac{1}{3}[1, 1, 1]$

## Section 9.9

**1.** (a) No (d) No (g) Yes (o) Yes **2.** (a) $[9,-2,4] = 2\mathbf{e}_1 + \frac{13}{5}\mathbf{e}_2 + \frac{2}{5}\mathbf{e}_3$ **3.** (a) $[9,-2,4] = 2\sqrt{14}\,\hat{\mathbf{e}}_1 + \frac{13}{\sqrt{5}}\hat{\mathbf{e}}_2 + \frac{2\sqrt{70}}{5}\hat{\mathbf{e}}_3$ **4.** (a) $[1,0,0,0] = \frac{1}{15}\mathbf{e}_1 + \frac{1}{3}\mathbf{e}_2 + 0\mathbf{e}_3 + \frac{1}{5}\mathbf{e}_4$ **8.** (a) 1 (d) 3 **9.** (a) 3 (d) 1 **10.** (a) 2 **12.** (a) [1,0], [0,1] (d) $\frac{1}{\sqrt{2}}[1,1,0]$, $\frac{1}{\sqrt{22}}[3,-3,2]$, $\frac{1}{\sqrt{11}}[-1,1,3]$ **13.** (b) $\mathbf{e}_1^* = [1,-1]$, $\mathbf{e}_2^* = [0,1]$, $\mathbf{u} = (\mathbf{u}\cdot\mathbf{e}_1^*)\mathbf{e}_1 + (\mathbf{u}\cdot\mathbf{e}_2^*)\mathbf{e}_2 = 2\mathbf{e}_1 + \mathbf{e}_2$ (e) $\mathbf{e}_1^* = [1,-1,0]$, $\mathbf{e}_2^* = [0,1,-1]$, $\mathbf{e}_3^* = [0,0,1]$, $\mathbf{u} = 5\mathbf{e}_1 - 6\mathbf{e}_2 + 5\mathbf{e}_3$, $\mathbf{v} = 0\mathbf{e}_1 - 2\mathbf{e}_2 + 2\mathbf{e}_3$, $\mathbf{w} = 7\mathbf{e}_1 - 5\mathbf{e}_2 + 3\mathbf{e}_3$

## Section 9.10

**2.** $\mathbf{u} \approx [64/15, 0, -19/30, 0, 11/6]$, $\|\mathbf{E}\| = 4.0042$ **3.** In span$\{\hat{\mathbf{e}}_1\}$, $\mathbf{u} \approx -\frac{4}{3}[1,1,0,-1]$ with $\|\mathbf{E}\| = 7.188$; in span$\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2\}$, $\mathbf{u} \approx \frac{1}{3}[1,-9,-5,4]$ with $\|\mathbf{E}\| = 6.236$; in span$\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3\}$, $\mathbf{u} \approx [4,-3,2,5]$ with $\|\mathbf{E}\| = 1.732$; in span$\{\hat{\mathbf{e}}_1, \hat{\mathbf{e}}_2, \hat{\mathbf{e}}_3, \hat{\mathbf{e}}_4\}$, $\mathbf{u} = [4,-2,1,6]$ with $\|\mathbf{E}\| = 0$

# Chapter 10

## Section 10.2

**2.** (a) undefined (d) $6 \times 1$ (g) $4 \times 1$ **4.** No; $\mathbf{A} - c$ is undefined because $\mathbf{A}$ is $n \times n$ and $c$ is a scalar.
**5.** (a) $(\mathbf{A}+\mathbf{B})(\mathbf{A}+\mathbf{B}) = \mathbf{A}^2 + \mathbf{AB} + \mathbf{BA} + \mathbf{B}^2 = \mathbf{A}^2 + 2\mathbf{AB} + \mathbf{B}^2$ only if $\mathbf{AB} = \mathbf{BA}$, that is, if $\mathbf{A}$ and $\mathbf{B}$ commute; in *general* $\mathbf{AB} \neq \mathbf{BA}$ so the given formula is incorrect. **7.** (a) $2\mathbf{A}^2 + 4\mathbf{AB} + \mathbf{BA} + 2\mathbf{B}^2$
**8.** (a) $\mathbf{A}^{100}$ is a $2 \times 2$ matrix with each element equal to $2^{99}$. (d) $\mathbf{D}^{100}$ is a $3 \times 3$ zero matrix.
**10.** (a) Dimensionally, $[m \times n][4 \times 1] = 1 \times 1$ so $n = 4$ and $m = 1$. Thus, $\mathbf{A}$ is a $1 \times 4$ matrix, i.e., a single row $[a_{11}, a_{12}, a_{13}, a_{14}]$. Multiplying the latter into $\mathbf{x} = [x_1, x_2, x_3, x_4,]^{\mathrm{T}}$ gives $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4$ and comparing the latter with $x_1 - 3x_4$ shows that $a_{11} = 1$, $a_{12} = 0$, $a_{13} = 0$, $a_{14} = -3$, so $\mathbf{A} = [1,0,0,-3]$.
**11.** (a) For example, $\mathbf{AB} = \begin{bmatrix} 3 & 0 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 6 & 5 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \mathbf{0}$ **16.** (c) $a_{11} = \alpha$ (arbitrary), $a_{21} = \beta$ (arbitrary), $a_{12} = (1-\alpha^2)/\beta$, $a_{22} = -\alpha$ or $a_{11} = \pm 1$, $a_{12} = a_{21} = 0$, $a_{22} = \pm 1$ with the signs of $a_{11}$ and $a_{22}$ being independent of each other.

## Section 10.3

**1.** (a) $\mathbf{x}^{\mathrm{T}}\mathbf{y} = [-3]$, $\mathbf{xy}^{\mathrm{T}} = \begin{bmatrix} 3 & 6 \\ -3 & -6 \end{bmatrix}$ **7.** $\mathbf{A} = \begin{bmatrix} 3 & 1.5 \\ 1.5 & -5 \end{bmatrix} + \begin{bmatrix} 0 & 0.5 \\ -0.5 & 0 \end{bmatrix}$ = symmetric + skew symmetric **8.** (a) $\mathbf{A} = \begin{bmatrix} 6 & -4 \\ -4 & 1 \end{bmatrix}$

## Section 10.4

**2.** (a) 0 (g) 132 **14.** (a) $a_j$'s $> 0$, $\Delta_1 = 6$, $\Delta_2 = \begin{vmatrix} 6 & 1 \\ 4 & 5 \end{vmatrix} = 26 > 0$, $\Delta_3 = \begin{vmatrix} 6 & 1 & 0 \\ 4 & 5 & 6 \\ 0 & 1 & 4 \end{vmatrix} = 68 > 0$,

$\Delta_4 = \begin{vmatrix} 6 & 1 & 0 & 0 \\ 4 & 5 & 6 & 1 \\ 0 & 1 & 4 & 5 \\ 0 & 0 & 0 & 1 \end{vmatrix} = 68 > 0$; stable. As a check, the *Maple* solution, using fsolve, gives $\lambda = -5.18, -0.34, -0.24 \pm 0.72i$, each of which has a negative real part.

## Section 10.5

**1.** (a) $r = 1$, nullity $= 3$, number of LI rows $= 1$, number of LI columns $= 1$ (e) $r = 2$, nullity $= 1$, number of LI rows $= 2$, number of LI columns $= 2$ (i) $r = 3$, nullity $= 1$, number of LI rows $= 3$, number of

LI columns = 3 **3.** (a) $r(\mathbf{A}) = 1$, $r(\mathbf{A}|\mathbf{c}) = 1$ so consistent; $p$-parameter family of solutions with $p = n - r = 4 - 1 = 3$
(e) $r(\mathbf{A}) = 2$, $r(\mathbf{A}|\mathbf{c}) = 3$ so inconsistent (i) $r(\mathbf{A}) = 3$, $r(\mathbf{A}|\mathbf{c}) = 3$ so consistent; $p$-parameter family of solutions with $p = n - r = 4 - 3 = 1$. **8.** (a) $r = 2$ so (Theorem 10.5.2) LD **16.** (a) 4; e.g., $H_2 + O_2 \rightleftharpoons 2\,OH$, $O_2 + 2H_2O \rightleftharpoons 4OH$, $H + 3OH \rightleftharpoons 2H_2O + O$, $OH \rightleftharpoons O + H$

## Section 10.6

**1.** (a) $\dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ (g) $\begin{bmatrix} 0 & 1/2 & -5/6 \\ 1 & 0 & 0 \\ 0 & 0 & 1/3 \end{bmatrix}$ (q) $\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ **5.** (a) $x_1 = 24/13$, $x_2 = -6/13$ (d) $x_1 = 72/11$, $x_2 = -3/22$ **7.** (a) $\mathbf{A} = \begin{bmatrix} 2/3 & -1/3 \\ 1 & -1 \end{bmatrix}$ **11.** (a) $\begin{bmatrix} 1 & -2 & 13 \\ 0 & 1 & -8 \\ 0 & 0 & 1 \end{bmatrix}$

## Section 10.7

**1.** $\mathbf{Q} = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$. Not orthogonal because the column vectors are not ON. $[\mathbf{x}]_{B'} = \begin{bmatrix} 3 \\ 6 \end{bmatrix}$, $[\mathbf{x}]_B = \begin{bmatrix} 8/3 \\ -1/3 \end{bmatrix}$
**3.** (a) $\mathbf{Q} = \begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{bmatrix}$, which is orthogonal. **4.** (b) $[\mathbf{x}]_{B'} = [\sqrt{2}, 2, 5/\sqrt{3}, -10/\sqrt{6}]^T$ **5.** (a) No
(e) Yes **9.** $\mathbf{Q}^n = \begin{bmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{bmatrix}$

## Section 10.8

**2.** (a) Nonlinear (d) Linear **3.** (b) The $n$th-order identity matrix $\mathbf{I}$ **5.** (a) dim $R = 2$, dim $K = 1$, dim $V = 3$. $\mathbf{F}$ is onto, is not one-to-one and is not invertible. A basis for $K$ is $[0, -1, 1]^{\mathrm{T}}$, and a basis for $R$ is $\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.
(d) dim $R = 2$, dim $K = 0$, dim $V = 2$. $\mathbf{F}$ is not onto, is one-to-one, and is not invertible. A basis for $R$ is $[4, 3, 0]^{\mathrm{T}}$, $[0, 5, -4]^{\mathrm{T}}$, and $K$ has no basis. **6.** (a) To be one-to-one we need $r(\mathbf{A}) = 2$. Also, to be onto we need $r(\mathbf{A}) = 2$. Thus, if $r(\mathbf{A}) = 2$ (i.e., $\mathbf{A}$ is any $2 \times 2$ matrix with nonzero determinant) then $\mathbf{F}$ is one-to-one and onto; if $r(\mathbf{A}) < 2$ (e.g., a $2 \times 2$ matrix with only one nonzero element) then it is neither. Can't be one and not the other. **7.** $\mathbf{A} = \begin{bmatrix} v_1^2 & v_1v_2 & v_1v_3 \\ v_2v_1 & v_2^2 & v_2v_3 \\ v_3v_1 & v_3v_2 & v_3^2 \end{bmatrix}$ **10.** (b) $\mathbf{A} = \begin{bmatrix} 2L_1^2 - 1 & 2L_2L_1 \\ 2L_1L_2 & 2L_2^2 - 1 \end{bmatrix}$ **11.** (a) $(\mathbf{GF})(\mathbf{x}) = \begin{bmatrix} -20 \\ -40 \end{bmatrix}$, $\mathbf{GF}$ transformation matrix $= \begin{bmatrix} 2 & 10 & 6 & -4 \\ 4 & 20 & 12 & -8 \end{bmatrix}$ **13.** (c) $\mathbf{F}(\mathbf{X}) = [2, 1, \sqrt{2} - 3, 1]^{\mathrm{T}}$
(d) $\mathbf{F}(\mathbf{X_p}) = [1.2439, 3.5859, 2.2637, 1]^{\mathrm{T}}$, $\mathbf{F}(\mathbf{X}_e) = [1.5394, 3.7757, 1.3275, 1]^{\mathrm{T}}$

# Chapter 11

## Section 11.2

**3.** (a) $\lambda_1 = 0$ (multiplicity 2), eigenspace is $\mathbf{e}_1 = \alpha[1, 0]^{\mathrm{T}} + \beta[0, 1]^{\mathrm{T}}$ where $\alpha$ and $\beta$ are arbitrary or, equivalently, span $\{[1, 0]^{\mathrm{T}}, [0, 1]^{\mathrm{T}}\}$. Basis for eigenspace is $\{[1, 0]^{\mathrm{T}}, [0, 1]^{\mathrm{T}}\}$ (i) $\lambda_1 = 0$, $\mathbf{e}_1 = \alpha[0, -1, 1]^{\mathrm{T}}$ with basis $[0, -1, 1]^{\mathrm{T}}$; $\lambda_2 = 2$ (multiplicity 2), $\mathbf{e}_2 = \beta[0, 1, 1]^{\mathrm{T}} + \gamma[1, 0, 0]^{\mathrm{T}}$ with basis $\{[0, 1, 1]^{\mathrm{T}}, [1, 0, 0]^{\mathrm{T}}\}$. (n) $\lambda_1 = 0$, $\mathbf{e}_1 = \alpha[13, 1, 0]^{\mathrm{T}}$ with basis $\{[13, 1, 0]^{\mathrm{T}}\}$, $\lambda_2 = 2$, $\mathbf{e}_2 = \beta[3, 1, 2]^{\mathrm{T}}$ with basis $\{[3, 1, 2]^{\mathrm{T}}\}$, $\lambda_3 = 5$, $\mathbf{e}_3 = \gamma[3, 1, 5]^{\mathrm{T}}$ with basis $\{[3, 1, 5]^{\mathrm{T}}\}$
**5.** (a) No (b) Yes, with $\lambda = -6$. **6.** (a) $\mathbf{e} = \alpha[1, 1, -1, -1]^{\mathrm{T}}$ **16.** (a) $\mathbf{A}$ has $\lambda_1 = 1$, $\mathbf{e}_1 = \alpha[1, -1, 0]^{\mathrm{T}}$; $\lambda_2 = 2$;

$\mathbf{e}_2 = \beta[0,1,-1]^{\mathrm{T}}$; $\lambda_3 = 3$, $\mathbf{e}_3 = \gamma[0,0,1]^{\mathrm{T}}$. Hence, $\mathbf{A}^{10}$ has $\lambda_1 = 1$, $\lambda_2 = 2^{10}$, $\lambda_3 = 3^{10}$, with the same eigenspaces as $\mathbf{A}$. **22.** (a) $\lambda = 1, (-1+\sqrt{3})/4, (-1-\sqrt{3})/4$ **23.** (a) $\lambda_1 = 0$, $\mathbf{e}_1 = \alpha[2,-1]^{\mathrm{T}}$; $\lambda_2 = 7$, $\mathbf{e}_2 = \beta[1,3]^{\mathrm{T}}$ so $[x,y]^{\mathrm{T}} = \alpha[2,-1]^{\mathrm{T}}e^{0t} + \beta[1,3]Te^{7t}$ or $x(t) = 2\alpha + \beta e^{7t}$, $y(t) = -\alpha + 3\beta e^{7t}$. (c) $\lambda_1 = r^2 = 5$, $\mathbf{e}_1 = \alpha\begin{bmatrix}1\\3\end{bmatrix}$ gives $\begin{bmatrix}x\\y\end{bmatrix} = C_1\begin{bmatrix}1\\3\end{bmatrix}e^{+\sqrt{5}t}, C_2\begin{bmatrix}1\\3\end{bmatrix}e^{-\sqrt{5}t}$
$\lambda_2 = r^2 = -1$, $\mathbf{e}_2 = \beta\begin{bmatrix}1\\-3\end{bmatrix}$ gives $\begin{bmatrix}x\\y\end{bmatrix} = C_3\begin{bmatrix}1\\-3\end{bmatrix}e^{it}, C_4\begin{bmatrix}1\\-3\end{bmatrix}e^{-it}$ so, by superposition, $\begin{bmatrix}x(t)\\y(t)\end{bmatrix} = C_1\begin{bmatrix}1\\3\end{bmatrix}e^{\sqrt{5}t} + C_2\begin{bmatrix}1\\3\end{bmatrix}e^{-\sqrt{5}t} + C_3\begin{bmatrix}1\\-3\end{bmatrix}e^{it} + C_4\begin{bmatrix}1\\-3\end{bmatrix}e^{-it}$ or $x(t) = C_1e^{\sqrt{5}t} + C_2e^{-\sqrt{5}t} + C_3e^{it} + C_4e^{-it}$; $y(t) = 3C_1e^{\sqrt{5}t} + 3C_2e^{-\sqrt{5}t} - 3C_3e^{it} - 3C_4e^{-it}$ or $x(t) = C_5\cosh\sqrt{5}t + C_6\sinh\sqrt{5}t + C_7\cos t + C_8\sin t$; $y(t) = 3C_5\cosh\sqrt{5}t + 3C_6\sinh\sqrt{5}t - 3C_7\cos t - 3C_8\sin t$ (e) $\lambda_1 = 0$, $\mathbf{e}_1 = \alpha[1,-4,5]^{\mathrm{T}}$; $\lambda_2 = 1$, $\mathbf{e}_2 = \beta[1,-3,2]^{\mathrm{T}}$; $\lambda_3 = 2$, $\mathbf{e}_3 = \gamma[1,-2,1]^{\mathrm{T}}$ so $[x(t),y(t),z(t)]^{\mathrm{T}} = \alpha[1,-4,5]^{\mathrm{T}} + \beta[1,-3,2]^{\mathrm{T}}e^t + \gamma[1,-2,1]^{\mathrm{T}}e^{2t}$ or $x(t) = \alpha + \beta e^t + \gamma e^{2t}$, $y(t) = -4\alpha - 3\beta e^t - 2\gamma e^{2t}$, $z(t) = 5\alpha + 2\beta e^t + \gamma e^{2t}$ **25.** (a) $\det(\mathbf{A} - \lambda\mathbf{B}) = (5-8\lambda)^2 - (1+4\lambda)^2 = 0$ so $\lambda = 1/3, 3/2$: $\lambda_1 = 1/3$, $\mathbf{e}_1 = \alpha[1,-1]^{\mathrm{T}}$; $\lambda_2 = 3/2$, $\mathbf{e}_2 = \beta[1,1]^{\mathrm{T}}$ **27.** (a) $\lambda = \pm 2$, saddle (d) $\lambda = -2, -4$, stable node

## Section 11.3

**1.** (a) $\lambda_1 = 0$, $\mathbf{e}_1 = \alpha[1,-1]^{\mathrm{T}}$; $\lambda_2 = 2$, $\mathbf{e}_2 = \beta[1,1]^{\mathrm{T}}$. Orthogonal basis: $\{[1,-1]^{\mathrm{T}}, [1,1]^{\mathrm{T}}\}$ (d) $\lambda_1 = -2$, $\mathbf{e}_1 = \alpha[1,-1,0]^{\mathrm{T}} + \beta[0,0,1]^{\mathrm{T}}$; $\lambda_2 = 2$, $\mathbf{e}_2 = \gamma[1,1,0]^{\mathrm{T}}$. Orthogonal basis: $\{[1,-1,0]^{\mathrm{T}}, [0,0,1]^{\mathrm{T}}, [1,1,0]^{\mathrm{T}}\}$
(g) Orthogonal basis: $\{[1,0,0,1]^{\mathrm{T}}, [0,1,1,0]^{\mathrm{T}}, [1,0,0,-1]^{\mathrm{T}}, [0,1,-1,0]^{\mathrm{T}}\}$ **2.** (a) $\lambda_1 = 0$, $\mathbf{e}_1 = \alpha[1,-3]^{\mathrm{T}}$; $\lambda_2 = 1$, $\mathbf{e}_2 = \beta[0,1]^{\mathrm{T}}$. No. **7.** (a) $\alpha = 5/2$, $\phi_1 = \pi/2$, $\beta = -1/2$, $\phi_2 = \pi/2$ **8.** (b) $\lambda = \omega^2$. $\lambda_1 = 2-\sqrt{2} = 0.586$, $\omega_1 = 0.765$, $\mathbf{e}_1 = \alpha[1,\sqrt{2},1]^{\mathrm{T}}$; $\lambda_2 = 2$, $\omega_2 = 1.414$, $\mathbf{e}_2 = \beta[1,0,-1]^{\mathrm{T}}$; $\lambda_3 = 2+\sqrt{2} = 3.414$, $\omega_3 = 1.848$, $\mathbf{e}_3 = \gamma[1,-\sqrt{2},1]^{\mathrm{T}}$.
$x_1(t) = \alpha\sin(0.765t + \phi_1) + \beta\sin(1.414t + \phi_2) + \gamma\sin(1.848t + \phi_3)$
$x_2(t) = \sqrt{2}\,\alpha\sin(0.765t + \phi_1) + 0\beta\sin(1.414t + \phi_2) - \sqrt{2}\,\gamma\sin(1.848t + \phi_3)$
$x_3(t) = \alpha\sin(0.765t + \phi_1) - \beta\sin(1.414t + \phi_2) + \gamma\sin(1.848t + \phi_3)$
The natural frequencies are $\omega_1, \omega_2, \omega_3$ (above) and their mode shapes are given by their corresponding eigenvectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$.
(c) Low mode: $x_1(0) = 1, x_2(0) = \sqrt{2}, x_3(0) = 1, x_1'(0) = x_2'(0) = x_3'(0) = 0$
Middle mode: $x_1(0) = 1, x_2(0) = 0, x_3(0) = -1, x_1'(0) = x_2'(0) = x_3'(0) = 0$
High mode: $x_1(0) = 1, x_2(0) = -\sqrt{2}, x_3(0) = 1, x_1'(0) = x_2'(0) = x_3'(0) = 0$
**9.** Partial answer: lowest natural frequency $= 0.518$, corresponding mode shape $= \alpha[1,\sqrt{3},2,\sqrt{3},1]^{\mathrm{T}}$.
**11.** (c) $[1,0,0]^{\mathrm{T}}$ is an eigenvector, corresponding to $\lambda = 2$, so the values of $R(\mathbf{x})$ for $\mathbf{x} = [0.4,0.3,0.3]^{\mathrm{T}}$, $[0.6,0.2,0.2]^{\mathrm{T}}$, $[0.8,0.1,0.1]^{\mathrm{T}}$, $[0.96,0.02,0.02]^{\mathrm{T}}$, $[1,0,0]^T$, should converge to 2. In fact, they are $-0.65$, 1.09, 1.85, 1.996, 2. Similarly, $[0,1,1]^{\mathrm{T}}$ is an eigenvector corresponding to $\lambda = -3$, so the values of $R(\mathbf{x})$, for $\mathbf{x} = [0,1,1.4]^{\mathrm{T}}, [0,1,1.1]^{\mathrm{T}}, [0,1,1]^{\mathrm{T}}$, should converge to $-3$. In fact, they are $-3.08$, $-3.04$, $-3$. **13.** (a) $\mathbf{x}^{(0)} = [1,0,0]^{\mathrm{T}}$, $\mathbf{x}^{(1)} = [2,1,-1]^{\mathrm{T}}$, $\mathbf{x}^{(2)} = [6,3,-3]^{\mathrm{T}}$ which is an exact multiple of $\mathbf{x}^{(1)}$, namely, 3. Thus, one eigenpair is $\lambda = 3$, $\mathbf{e} = [2,1,-1]^{\mathrm{T}}$. Repeat with $\mathbf{x}^{(0)} = [0,1,0]^{\mathrm{T}}$. Then $\mathbf{x}^{(1)} = [1,4,3]^{\mathrm{T}}$, $\mathbf{x}^{(2)} = [3,26,23]^{\mathrm{T}}$, $\mathbf{x}^{(3)} = [9,176,167]^{\mathrm{T}}$, $\mathbf{x}^{(4)} = [27,1214,1187]^{\mathrm{T}}$ so $\lambda \approx \mathbf{x}^{(3)T}\mathbf{A}\mathbf{x}^{(3)}/\mathbf{x}^{(3)T}\mathbf{x}^{(3)} = \mathbf{x}^{(3)T}\mathbf{x}^{(4)}/\mathbf{x}^{(3)T}\mathbf{x}^{(3)} = 412136/58946 = 6.992$ and $\mathbf{e} \approx [27,1214,1187]^{\mathrm{T}}$. Finally, repeating with $\mathbf{x}^{(0)} = [0,0,1]^{\mathrm{T}}$ gives convergence to the same eigenpair as that obtained from $\mathbf{x}^{(0)} = [0,1,0]^{\mathrm{T}}$. Since $\mathbf{x}^{(0)} = [1,0,0]^{\mathrm{T}}$ gave convergence to the subdominate eigenvalue $\lambda = 3$, $[1,0,0]^{\mathrm{T}}$ must be orthogonal to the eigenvector corresponding to the dominate eigenvalue. Indeed it is, because the exact values are $\lambda_1 = 7$, $\mathbf{e}_1 = [0,1,1]^{\mathrm{T}}$; $\lambda_2 = 3$, $\mathbf{e}_2 = [2,1,-1]$; $\lambda_3 = 0$, $\mathbf{e}_3 = [1,-1,1]^{\mathrm{T}}$.
**14.** (a) $\mathbf{A}^4 = \begin{bmatrix}54 & 27 & -27\\27 & 1214 & 1187\\-27 & 1187 & 1214\end{bmatrix}$, $\mathbf{x}^{(0)} = [0,1,0]^{\mathrm{T}}$, $\mathbf{x}^{(1)} = [27,1214,1187]^{\mathrm{T}}$,

$\mathbf{x}^{(2)} = [2187, 2883493, 2881307]^{\mathrm{T}}$, $\lambda \approx \mathbf{x}^{(1)T}\mathbf{x}^{(2)}/\mathbf{x}^{(1)T}\mathbf{x}^{(1)} = 2400.12$. But this is the fourth power of $\mathbf{A}$'s $\lambda$, so $\lambda = (2400.12)^{1/4} = 6.9994$, with $\mathbf{e} \approx [2187, 2883493, 2881307]^{\mathrm{T}}$. Convergence is more rapid using $\mathbf{A}^4$ because its eigenvalues are $7^4 = 2401$, $3^4 = 81$, $0^4 = 0$, so the largest eigenvalue of $\mathbf{A}^4$ is more dominant than the largest eigenvalue of $\mathbf{A}$; i.e., $(7/3)^4 \gg 7/3$. **15.** (a) For $\mathbf{A}$, $\lambda_1 = 7$, $\mathbf{e}_1 = [0,1,1]^{\mathrm{T}}$; $\lambda_2 = 3$, $\mathbf{e}_2 = [2,1,-1]^{\mathrm{T}}$; $\lambda_3 = 0$, $\mathbf{e}_3 = [1,-1,1]^{\mathrm{T}}$. Thus, $\mathbf{c} = [1,2,3]^{\mathrm{T}} = c_1\mathbf{e}_1 + c_2\mathbf{e}_2 + c_3\mathbf{e}_3$ where (because the $\mathbf{e}_j$'s are orthogonal) $c_1 = \mathbf{c}\cdot\mathbf{e}_1/\mathbf{e}_1\cdot\mathbf{e}_1 = 5/2$, $c_2 = 1/6$, $c_3 = 2/3$. Then $a_j = c_j/(\lambda_j - \Lambda)$ gives $a_1 = 1/2$, $a_2 = 1/6$, $a_3 = -1/3$, hence the unique solution $\mathbf{x} = \frac{1}{2}\mathbf{e}_1 + \frac{1}{6}\mathbf{e}_2 - \frac{1}{3}\mathbf{e}_3 = [0,1,0]^{\mathrm{T}}$. (b) This time $\Lambda = 3$ coincides with $\lambda_2$. $\mathbf{c}\cdot\mathbf{e}_2 = 6 \neq 0$. Hence, no solution. (c) Again $\Lambda = 3$ coincides with $\lambda_2$, but this time $\mathbf{c}\cdot\mathbf{e}_2 = 0$. Hence, nonunique solution. $c_1 = 2$, $c_2 = 0$, $c_3 = 1$ so $a_1 = c_1/(\lambda_1 - \Lambda) = 1/2$, $a_2 = \alpha$ (arbitrary), $a_3 = c_3/(\lambda_3 - \Lambda) = -1/3$ and $\mathbf{x} = \frac{1}{2}\mathbf{e}_1 + \alpha\,\mathbf{e}_2 - \frac{1}{3}\mathbf{e}_3 = [2\alpha - 1/3, \alpha + 5/6, -\alpha + 1/6]^{\mathrm{T}}$. **17.** (b) $\lambda_1 = 0$, $\mathbf{e}_1 = [1,-1]^{\mathrm{T}}$; $\lambda_2 = 5/6$, $\mathbf{e}_2 = [2,3]^{\mathrm{T}}$. $\mathbf{e}_1\cdot\mathbf{M}\mathbf{e}_2 = [1,-1]^{\mathrm{T}}\cdot[6,6]^{\mathrm{T}} = 0$ and $\mathbf{e}_2\cdot\mathbf{M}\mathbf{e}_1 = 0$ too.

## Section 11.4

**1.** (a) $\lambda_1 = 2$, $\mathbf{e}_1 = [1,0]^{\mathrm{T}}$; $\lambda_2 = 0$, $\mathbf{e}_2 = [3,2]^{\mathrm{T}}$. Thus, $\mathbf{Q} = \begin{bmatrix} 1 & 3 \\ 0 & 2 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$ (d) $\lambda_1 = 0$, $\mathbf{e}_1 = [1,-1,0]^{\mathrm{T}}$; $\lambda_2 = -1$, $\mathbf{e}_2 = [1,1,-1]^{\mathrm{T}}$; $\lambda_3 = 2$, $\mathbf{e}_3 = [1,1,2]^{\mathrm{T}}$. Can use these $\mathbf{e}_j$'s as columns of $\mathbf{Q}$ but since we are asked to compute $\mathbf{Q}^{-1}$ it is best to use the *normalized* $\mathbf{e}_j$'s instead, so that $\mathbf{Q}^{-1}$ is simply $\mathbf{Q}^T$ (since the $\mathbf{e}_j$'s are ON). Then $\mathbf{Q} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ -1/\sqrt{2} & 1/\sqrt{3} & 1/\sqrt{6} \\ 0 & -1/\sqrt{3} & 2/\sqrt{6} \end{bmatrix}$ and $\mathbf{D} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ **2.** (a) $x(t) = Ae^{2t} + B$, $y(t) = Ae^{2t} - B$

**5.** (a) The ODE's are $x'' + \frac{1}{4}x - \frac{\sqrt{3}}{4}y = 0$ and $y'' - \frac{\sqrt{3}}{4}x + \frac{39}{4}y = 0$. $\omega_1 = \sqrt{\lambda_1} = \sqrt{0.2303} = 0.4799$, $\mathbf{e}_1 = [21.98, 1]^{\mathrm{T}}$; $\omega_2 = \sqrt{\lambda_2} = \sqrt{9.770} = 3.126$, $\mathbf{e}_2 = [0.04548, -1]^{\mathrm{T}}$. **8.** (a) $\lambda_1 = 4$, $\mathbf{e}_1 = [1,1,1]^{\mathrm{T}}$; $\lambda_2 = \lambda_3 = -2$, $\mathbf{e} = \alpha[1,-1,0]^{\mathrm{T}} + \beta[1,0,-1]^{\mathrm{T}}$, from which we can form the orthogonal pair $\mathbf{e}_2 = [1,-1,0]^{\mathrm{T}}$ and $\mathbf{e}_3 = [1,1,-2]^{\mathrm{T}}$. Normalize these so $\mathbf{Q}^{-1} = \mathbf{Q}^{\mathrm{T}}$. Then

$$\mathbf{A}^{1000} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 4^{1000} & 0 & 0 \\ 0 & (-2)^{1000} & 0 \\ 0 & 0 & (-2)^{1000} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix}$$

$$= \frac{1}{3}\begin{bmatrix} (4^{1000} + 2^{1001}) & (4^{1000} - 2^{1000}) & (4^{1000} - 2^{1000}) \\ (4^{1000} - 2^{1000}) & (4^{1000} + 2^{1001}) & (4^{1000} - 2^{1000}) \\ (4^{1000} - 2^{1000}) & (4^{1000} - 2^{1000}) & (4^{1000} + 2^{1001}) \end{bmatrix}$$

**9.** (a) $I_{xx} = \sigma$, $I_{xy} = 9\sigma/4$, $I_{yy} = 9\sigma$, $I_{zz} = 10\sigma$, $I_{xz} = I_{yz} = 0$ so $I = \sigma\begin{bmatrix} 1 & -9/4 & 0 \\ -9/4 & 9 & 0 \\ 0 & 0 & 10 \end{bmatrix}$. $\lambda_1 = 9.589\sigma$, $\mathbf{e}_1 = [1, -3.818, 0]^{\mathrm{T}}$; $\lambda_2 = 0.411\sigma$, $\mathbf{e}_2 = [1, 0.262, 0]^{\mathrm{T}}$; $\lambda_3 = 10\sigma$, $\mathbf{e}_3 = [0,0,1]^{\mathrm{T}}$ so $I_{x'x'} = 9.589\sigma$, $I_{y'y'} = 0.411\sigma$, $I_{z'z'} = 10\sigma$ where $\hat{\mathbf{e}}_1$, $\hat{\mathbf{e}}_2$, $\hat{\mathbf{e}}_3$ are unit vectors in the $x', y', z'$ coordinate directions.

## Section 11.5

**1.** (a) $\mathbf{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $\mathbf{f}(t)\begin{bmatrix} 0 \\ e^{3t} \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Then $\lambda_1 = 2$, $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\lambda_2 = 0$, $\mathbf{e}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{D} = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$, $\mathbf{Q} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$, $\mathbf{Q}^{-1} = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$, $\mathbf{x}(t) = \mathbf{0} + \int_0^t \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} e^{2(t-\tau)} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}\begin{bmatrix} 0 \\ e^{3\tau} \end{bmatrix} d\tau$ gives $x(t) = \frac{1}{3}e^{3t} - \frac{1}{2}e^{2t} + \frac{1}{6}$, $y(t) = \frac{2}{3}e^{3t} - \frac{1}{2}e^{2t} - \frac{1}{6}$. (i) Let $x_1(t) \equiv x(t)$, $x_2(t) \equiv x'(t)$. Then $x(t) = x_1(t) = \frac{1}{6}e^t - \frac{1}{2}e^{-t} + \frac{1}{3}e^{-2t}$ [and $x'(t) = x_2(t) = \frac{1}{6}e^t + \frac{1}{2}e^{-t} - \frac{2}{3}e^{-2t}$]. **3.** (a) $\lambda_1 = 1$, $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$; $\lambda_2 = 3$, $\mathbf{e}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$; $\mathbf{Q} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, $\mathbf{Q}^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$,

$\mathbf{D} = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$, $e^{\mathbf{A}} = \mathbf{Q}e^{\mathbf{D}}\mathbf{Q}^{-1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e & 0 \\ 0 & e^3 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} e & e^3 - e \\ 0 & e^3 \end{bmatrix}$.
(d) $e^{\mathbf{A}} = \dfrac{e}{2}\begin{bmatrix} e^2+1 & e^2-1 \\ e^2-1 & e^2+1 \end{bmatrix}$.

## Section 11.6

**1.** (a) $\mathbf{A} = \begin{bmatrix} 2 & 1/2 \\ 1/2 & 4 \end{bmatrix}$ (d) $\mathbf{A} = \begin{bmatrix} 1 & 3/2 & 0 \\ 3/2 & 0 & 0 \\ 0 & 0 & -4 \end{bmatrix}$ **2.** (a) $2x_1^2 + 4x_2^2 + x_1x_2 = \lambda_1\tilde{x}_1^2 + \lambda_2\tilde{x}_2^2 = \frac{3+\sqrt{5}}{2}\tilde{x}_1^2 + \frac{3-\sqrt{5}}{2}\tilde{x}_2^2$ where $\mathbf{x} = \mathbf{Q}\tilde{\mathbf{x}}$ and $\mathbf{Q} = \begin{bmatrix} a & b \\ (2+\sqrt{5})a & (2-\sqrt{5})b \end{bmatrix}$ where $a = 1/\sqrt{10+4\sqrt{5}}$ and $b = 1/\sqrt{10-4\sqrt{5}}$. $\lambda_1 > 0$, $\lambda_2 > 0$ so positive definite. (g) $3x_1x_2 = \frac{3}{2}\tilde{x}_1^2 - \frac{3}{2}\tilde{x}_2^2$ where $\mathbf{x} = \mathbf{Q}\tilde{\mathbf{x}}$ and $\mathbf{Q} = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$. Neither positive nor negative definite. **4.** (a) $3x^2 + 2y^2 - 2xy = \frac{5+\sqrt{5}}{2}\tilde{x}^2 + \frac{5-\sqrt{5}}{2}\tilde{y}^2 = 6$ is an ellipse with intercepts at $\pm\sqrt{12/[5+\sqrt{5}]}$ on $\tilde{x}$ axis and $\pm\sqrt{12/[5-\sqrt{5}]}$ on $\tilde{y}$ axis. The positive $\tilde{x}$ and $\tilde{y}$ axes are oriented in the $x, y$ plane in the directions of the orthogonal eigenvectors $\mathbf{e}_1 = [2, 1-\sqrt{5}]^{\mathrm{T}}$ and $\mathbf{e}_2 = [2, 1+\sqrt{5}]^{\mathrm{T}}$.
(d) $x^2 + y^2 - 10xy = 6\tilde{x}^2 - 4\tilde{y}^2 = 4$ is an hyperbola with intercepts at $\pm 2/\sqrt{6}$ on $\tilde{y}$ axis. The positive $\tilde{x}$ and $\tilde{y}$ axes are oriented in the $x, y$ plane in the directions of the orthogonal eigenvectors $\mathbf{e}_1 = [1, -1]^{\mathrm{T}}$ and $\mathbf{e} = [1, 1]^{\mathrm{T}}$.

# Chapter 12

## Section 12.2

**1.** (a) $\hat{\mathbf{u}} = \left[\frac{1}{\sqrt{2}}, \frac{i}{\sqrt{2}}\right]$ **2.** (a) The set is a basis for $\mathbb{C}^3$. **4.** (a) $\mathbf{u} = \frac{4-5i}{2}\mathbf{e}_1 + \frac{8i}{9}\mathbf{e}_2 - \frac{9-4i}{18}\mathbf{e}_3$ **5.** $(\alpha\mathbf{x}\cdot\mathbf{y}) = \sum_1^n \alpha x_j\bar{y}_j = \alpha\sum_1^n x_j\bar{y}_j = \alpha(\mathbf{x}\cdot\mathbf{y})$. $\mathbf{x}\cdot(\alpha\mathbf{y}) = \sum_1^n x_j\overline{\alpha y}_j = \bar{\alpha}\sum_1^n x_j\bar{y}_j = \bar{\alpha}(\mathbf{x}\cdot\mathbf{y})$

## Section 12.3

**1.** (a) $\mathbf{A}^{-1} = \dfrac{8+i}{65}\begin{bmatrix} 4 & -i \\ -1 & 2 \end{bmatrix}$ **3.** (a) $\lambda_1 = 3 + \sqrt[4]{2}e^{\pi i/8}$, $\mathbf{e}_1 = \alpha\left[1, \sqrt[4]{2}\sin\frac{\pi}{8} - (1+\sqrt[4]{2}\cos\frac{\pi}{8})i\right]^{\mathrm{T}}$, $\lambda_2 = 3 - \sqrt[4]{2}e^{\pi i/8}$, $\mathbf{e}_2 = \beta\left[1, -\sqrt[4]{2}\sin\frac{\pi}{8} + (-1+\sqrt[4]{2}\cos\frac{\pi}{8})i\right]^{\mathrm{T}}$ **5.** (b) $\lambda_1 = 2+i$, $\mathbf{e}_2 = \alpha\begin{bmatrix} 1 \\ i \end{bmatrix}$; $\lambda_2 = 2-i$, $\mathbf{e}_1 = \beta\begin{bmatrix} 1 \\ -i \end{bmatrix}$ **6.** $(a-d)^2 + 4bc < 0$ **7.** (a) $\mathbf{A} = \begin{bmatrix} 2 & 1+i \\ 1-i & 3 \end{bmatrix}$, $\lambda_1 = 1$, $\mathbf{e}_1 = \alpha\begin{bmatrix} 1+i \\ -1 \end{bmatrix}$, $\lambda_2 = 4$, $\mathbf{e}_2 = \beta\begin{bmatrix} 1 \\ 1-i \end{bmatrix}$ so $F = |x_1'|^2 + 4|x_2'|^2$, where $\mathbf{x} = \mathbf{U}\mathbf{x}' = \frac{1}{\sqrt{3}}\begin{bmatrix} 1+i & 1 \\ -1 & 1-i \end{bmatrix}\mathbf{x}'$, and $\mathbf{x}' = \mathbf{U}^{-1}\mathbf{x} = \mathbf{U}^*\mathbf{x} = \frac{1}{\sqrt{3}}\begin{bmatrix} 1-i & -1 \\ 1 & 1+i \end{bmatrix}\mathbf{x}$ **8.** (b) $\mathbf{A}^{1000} = \dfrac{1}{5}\begin{bmatrix} 4+6^{1000} & 2i(1-6^{1000}) \\ 2i(6^{1000}-1) & 4(6^{1000})+1 \end{bmatrix}$ **12.** Neither **15.** (a) $c_2 = c_1$ (b) None

# Chapter 13

## Section 13.2

**1.** (a) $\sqrt{21}$ (d) $5\sqrt{2}$ **2.** Yes, because $d(P, P_0) = 0.36 < r$ **6.** (a) Connected, neither (f) Not connected, closed (k) Connected, open (n) Connected, closed **12.** (a) The points $x = 0, 1, -2$ (c) The lines $x + y = n\pi/2$ $(n = \pm 1, \pm 3, \ldots)$

## Section 13.3

**1.** (a) $f_x = 3x^2y^5$, $f_y = 5x^3y^4$, $f_{xy} = f_{yx} = 15x^2y^4$, $f_{xx} = 6xy^5$, $f_{yy} = 20x^3y$ **4.** $\alpha = 1 - n/2$: for $n = 2$, $\alpha = 0$; for $n = 3$, $\alpha = -1/2$; etc.

## Section 13.4

**1.** $F'(t) = [4x^3\cos(x^4 + 3y)](5) + [3\cos(x^4 + 3y)](2t) = (2500t^3 + 6t)\cos(625t^4 + 3t^2 + 3)$ **2.** (a) $F'(t) = (ye^{xy})(2t) + (xe^{xy})(3\cos 3t)$ **3.** (a) $G'(s) = (\sin s\cos s + 3\sin 6s)/\sqrt{\sin^2 s - \cos 6s}$ **4.** (a) $R'(2) = 0.3553$

## Section 13.5

**1.** (a) $\text{TS}\, e^{-2x}\big|_{x=0} = 1 - 2x + 2x^2 - \frac{4}{3}x^3 + \cdots$ (d) $\text{TS}\, \ln x\big|_{x=2} = \ln 2 + \frac{1}{2}(x-2)^2 - \frac{1}{8}(x-2)^2 + \frac{1}{24}(x-2)^3 - \cdots$ **2.** (a) $1 + x^5 + x^{10} + x^{15} + \cdots$ **9.** (a) $0 = f(x, y, z) = x^3yz + 6 = 0 - 18(x-1) - 2(y-3) + 3(z-2) + \cdots$ gives the tangent plane $18x + 2y - 3z = 30$. **11.** (a) First, expand in $x$, say: $x^5y^4 - y = [y^4 - y] + [5y^4](x-1) + \frac{1}{2!}[20y^4](x-1)^2 + \frac{1}{3!}[60y^4](x-1)^3 + \cdots$. Next, expand each square-bracketed coefficient in $y$: $x^5y^4 - y = [14 + 31(y-2) + 24(y-2)^2 + 8(y-2)^3 + \cdots] + [80 + 160(y-2) + 120(y-2)^2 + \cdots](x-1) + \frac{1}{2}[320 + 640(y-2) + \cdots](x-1)^2 + \frac{1}{6}[960 + \cdots](x-1)^3 + \cdots$. Arrange terms in ascending order:

$x^5y^4 - y = 14 + 80(x-1) + 31(y-2) + 160(x-1)^2 + 160(x-1)(y-2) + 24(y-2)^2 + 160(x-1)^3 + 320(x-1)^2(y-2) + 120(x-1)(y-2)^2 + 8(y-2)^3 + \cdots$. NOTE: Clearly, this series converges for all $x, y$, because it contains only a finite number of terms (since all $x$ derivatives of $x^5y^4 - y$ above fifth–order and all $y$ derivatives above fourth–order are zero); i.e., the series *terminates*.

## Section 13.6

**1.** (a) $f$ is $C^1$ in the entire plane, but $f_y(2, 1) = 0$ so the conditions are not met. Indeed, the graph of $x^2 + (y-1)^2 = 1$ is a circle of radius 1 centered at $(0, 1)$, with vertical tangent at $(1, 1)$. NOTE: Although the relation does not imply an implicit function $y(x)$ through $(1, 1)$ it does imply an implicit function $x(y)$ through that point.
(d) $y(x) = 1 + \frac{3}{2}(x+2) + \frac{48-9\pi^2}{32}(x+2)^2 + \cdots$ **2.** (a) $y' = y/(3y^2 - x)$, $y'' = -2xy/(3y^2 - x)^3$
**3.** (a) $y_x = -[y + \cos(x+z)]/x$, $z_x = -[y + \cos(x+z)]/[2z - \cos(x+z)]$ **4.** (a) Conditions not met because $\frac{\partial(f,g)}{\partial(u,v)} = 0$ at $(0, 0, 0, 0)$. NOTE: The conditions are stated to be sufficient, not necessary. In this example the conditions are not met so the theorm gives no information. **5.** (a) $\partial/\partial y$ gives $-1 + 2uu_y + 2vv_y = 0$ and $1 + 3u^2e^vu_y + u^3e^vv_y = 0$ so $u_y = (u^3 + 2ve^{-v})/(2u^4 - 6u^2v)$. **6.** (a) $\frac{\partial(f,g)}{\partial(u,v)} = \begin{vmatrix} f_u & f_v \\ g_u & g_v \end{vmatrix} = \begin{vmatrix} 3v^2 & 6uv \\ 2u & -2v \end{vmatrix} = -6v^3 - 12u^2v$

**12.** (b) They are true. Since there are only the three independent variables $p, T, v$ in (12.1), both $\frac{\partial v}{\partial p}$ and $\frac{\partial p}{\partial v}$ are computed with $T$ fixed and are therefore numerical inverses of each other. Or, working them out, $\partial/\partial p$ and $\partial/\partial v$ of (12.1) give $f_p + f_v v_p = 0$ and $f_v + f_p p_v = 0$. Thus, $v_p = -f_p/f_v$ and $p_v = -f_v/f_p = 1/v_p$.
**14.** (a) $\frac{\partial}{\partial x} = \frac{1}{4}\frac{\partial}{\partial u} + \frac{1}{2}\frac{\partial}{\partial v}$ and $\frac{\partial}{\partial y} = \frac{1}{4}\frac{\partial}{\partial u} - \frac{1}{2}\frac{\partial}{\partial v}$ so $T_{xx} + T_{yy} = \left(\frac{1}{4}\frac{\partial}{\partial u} + \frac{1}{2}\frac{\partial}{\partial v}\right)\left(\frac{1}{4}T_u + \frac{1}{2}T_v\right) + \left(\frac{1}{4}\frac{\partial}{\partial u} - \frac{1}{2}\frac{\partial}{\partial v}\right)\left(\frac{1}{4}T_u - \frac{1}{2}T_v\right) = \frac{1}{8}T_{uu} + \frac{1}{2}T_{vv} = 0$ or $T_{uu} + 4T_{vv} = 0$ (e) $T_{uu} + \frac{1}{u}T_u + \frac{1}{u^2}T_{vv} = 0$

## Section 13.7

**1.** (a) $f(x) = 0 + 0(x-1) + )(x-1)^2 + \frac{6}{3!}(x-1)^3 + \cdots$ so horizontal inflection point (c) maximum (e) minimum
**2.** (a) $x = 0$: $f(x) = 1 - x^3 + \cdots$ so horizontal inflection point (c) $x = 2$: $f(x) = e^{16} - 6e^{16}(x-2)^2 + \cdots$ so maximum;
$x = -2$: $f(x) = e^{-16} - 6e^{-16}(x-2)^2 + \cdots$ so minimum **5.** (a) $f_x = 4x + y = 0$ and $f_y = x + 2y - 7 = 0$ gives $x = 1, y = -4$. At $(1, -4)$, $f_{xx} = 4$, $f_{xy} = 1$, $f_{yy} = 2$ so

$\mathbf{A} = \begin{pmatrix} 4 & 1 \\ 1 & 2 \end{pmatrix}$ with $\lambda = -1, 7$. Mixed signs, hence saddle. (g) $f_x = (2x + y)/(x^2 + xy + y^2 + 4) = 0$ and $f_y = (x + 2y)/(x^2 + xy + y^2 + 4) = 0$ give $x = y = 0$. At $(0, 0)$, $f_{xx} = 1/2$, $f_{xy} = 1/4$, $f_{yy} = 1/2$

so $\mathbf{A} = \begin{pmatrix} 1/2 & 1/4 \\ 1/4 & 1/2 \end{pmatrix}$ with $\lambda = 1/4, 3/4$. Both positive, hence minimum. **6.** (a) $f_x$, $f_y$, $f_z$ are 0 at (0,0,0); $\lambda = 2, 1 \pm \sqrt{10}$ are of mixed sign, hence saddle. (d) $f_x$, $f_y$, $f_z$ are 0 at (0,0,0); $\lambda = 1, 1, 1$, hence minimum. **8.** $|\alpha| \le 10$ **13.** (a) $x = 2/5, y = 4/5$ **16.** (a) $x = 13/5, y = -6/5$ **17.** (a) $x = 3, y = 1, z = 0$ **18.** $x = (\sqrt{7}-1)/3, y = [(\sqrt{7}-1)/3]^{3/2}$ **19.** max. at $(1/\sqrt{2}, 0, 1/\sqrt{2})$, min. at $(-1/\sqrt{2}, 0, -1/\sqrt{2})$

## Section 13.8

**1.** (a) $\int_0^{t^2} x^2 \cos(tx^2)dx + 2t\sin(t^5)$ (b) $\int_{-2\alpha^2}^{-\alpha} x^3 e^{\alpha x^3} dx + (-1)e^{\alpha(-\alpha)^3} - (-4\alpha)e^{\alpha(-2\alpha^2)^3}$ **2.** (a) $0 + x + 0x^2 + \cdots$ (d) $1 + \frac{5}{3}x - \frac{19}{10}x^2 + \cdots$ **9.** We can evaluate $I(t)$ by using the change of variables $\sqrt{x} = u$ ($x = u^2$), and find that $I(t) = 2e^{\sqrt{t}} - 2$. Thus, $I'(t) = e^{\sqrt{t}}/\sqrt{t}$. On the other hand, the Leibniz rule gives $I'(t) = e^{\sqrt{t}}/\sqrt{t}$, which is the same result. **11.** (a) Liebniz gives $u_{xx} = \frac{1}{4\sqrt{\pi}\alpha^3 t^{3/2}} \int_0^1 f(\xi)\left[-1 + \frac{(x-\xi)^2}{2\alpha^2 t}\right] e^{-(x-\xi)^2/(4\alpha^2 t)} d\xi$, $u_t = -\frac{1}{4\alpha\sqrt{\pi}t^{3/2}} \int_0^1 f(\xi) e^{-(x-\xi)^2/(4\alpha^2 t)} d\xi + \int_0^1 \frac{f(\xi)}{2\alpha\sqrt{\pi t}} \frac{(x-\xi)^2}{4\alpha^2 t^2} e^{-(x-\xi)^2/(4\alpha^2 t)} d\xi$ so $\alpha^2 u_{xx}$ does not equal $u_t$.

# Chapter 14

## Section 14.3

**1.** (a) $2\mathbf{u} - \mathbf{v} = 3\hat{\mathbf{i}} - 3\hat{\mathbf{j}} - 5\hat{\mathbf{k}}$, $\mathbf{u}\cdot\mathbf{v} = 4$, $\mathbf{v}\cdot\mathbf{u} = 4$, $\mathbf{u}\times\mathbf{v} = 4\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$, $\mathbf{v}\times\mathbf{u} = -4\hat{\mathbf{i}} + \hat{\mathbf{j}} - 3\hat{\mathbf{k}}$, $\|\mathbf{u}\times\mathbf{v}\| = \sqrt{26}$ (h) $\mathbf{u}\times\mathbf{v} = 4\hat{\mathbf{i}} - \hat{\mathbf{j}} + 3\hat{\mathbf{k}}$ is perpendicular to $\mathbf{u}$ and $\mathbf{v}$. Check: $\mathbf{u}\cdot(\mathbf{u}\times\mathbf{v}) = 8 + 1 - 9 = 0$, $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{v}) = 4 - 1 - 3 = 0$. (k) Area $= \sqrt{122}/2$, $\alpha(-8\hat{\mathbf{i}} - 7\hat{\mathbf{j}} + 3\hat{\mathbf{k}})$ for any $\alpha \ne 0$ **2.** (a) $\mathbf{M} = -3\hat{\mathbf{i}} - 3\hat{\mathbf{j}} + \hat{\mathbf{k}}$ (d) $\mathbf{M} = -4\hat{\mathbf{i}} - \hat{\mathbf{j}}$ **3.** (a) No (c) Yes (e) No **4.** (a) $\sqrt{161}$ (d) $\sqrt{2873}$ **5.** (a) $\pm(2\hat{\mathbf{i}} + \hat{\mathbf{j}} - \hat{\mathbf{k}})/\sqrt{6}$ **8.** $\mathbf{x}\cdot\mathbf{w} = \{a[1,1,2] + b[3,2,-1]\}\cdot[2,4,3] = 0$ gives $12a + 11b = 0$ so $b = -12a/11$. Let $a = 11$; then $b = -12$ and $\mathbf{x} = [-25, -13, 34]$ (times an arbitrary nonzero scale factor)

## Section 14.4

**1.** (a) $\mathbf{u}\cdot\mathbf{v}\times\mathbf{w} = \mathbf{u}\times\mathbf{v}\cdot\mathbf{w} = -6$ **3.** (a) LHS $= \hat{\mathbf{i}} \times (\hat{\mathbf{j}} \times \hat{\mathbf{k}}) = \hat{\mathbf{i}} \times \hat{\mathbf{i}} = \mathbf{0}$; RHS $= (\hat{\mathbf{i}}\cdot\hat{\mathbf{k}})\hat{\mathbf{j}} - (\hat{\mathbf{i}}\cdot\hat{\mathbf{j}})\hat{\mathbf{k}} = \mathbf{0}$ **4.** (a) 3 (d) 30 **12.** (a) No (d) Yes

## Section 14.5

**1.** (a) $\mathbf{u} = 2\tau\mathbf{B}$, $\mathbf{u}'' = 2\mathbf{B}$, $\|\mathbf{u}''\| = 2\|\mathbf{B}\|$ (d) $\mathbf{u}' = -\sin\tau\,\hat{\mathbf{i}} + \cos\tau\,\hat{\mathbf{j}} + \cos\tau\,\hat{\mathbf{k}}$, $\mathbf{u}'' = -\cos\tau\,\hat{\mathbf{i}} - \sin\tau\,\hat{\mathbf{j}} - \sin\tau\,\hat{\mathbf{k}}$, $\|\mathbf{u}''\| = \sqrt{1 + \sin^2\tau}$ **2.** (a) $(\mathbf{u}\cdot\mathbf{v})' = (2\tau^2\cos\tau + 3\tau^2)' = 4\tau\cos\tau - 2\tau^2\sin\tau + 6\tau$, $\mathbf{u}\cdot\mathbf{v}' + \mathbf{u}'\cdot\mathbf{v} = -2\tau^2\sin\tau + 3\tau + 4\tau\cos\tau + 3\tau$ (checks) (b) $(\mathbf{u}\times\mathbf{v})' = 2\tau\,\hat{\mathbf{i}} + (3\tau^2 - 6\cos\tau + 6\tau\sin\tau)\hat{\mathbf{j}} + (2\cos\tau - 2\tau\sin\tau)\hat{\mathbf{k}}$, $\mathbf{u}\times\mathbf{v}' + \mathbf{u}'\times\mathbf{v} = \tau\,\hat{\mathbf{i}} + (\tau^2 + 6\tau\sin\tau)\hat{\mathbf{j}} - 2\tau\sin\tau\,\hat{\mathbf{k}} + \tau\,\hat{\mathbf{i}} + (2\tau^2 - 6\cos\tau)\hat{\mathbf{j}} + 2\cos\tau\,\hat{\mathbf{k}}$ (checks) **7.** (a) $(\mathbf{u}\cdot\mathbf{v})'' = (\mathbf{u}'\cdot\mathbf{v} + \mathbf{u}\cdot\mathbf{v}')' = (\mathbf{u}'\cdot\mathbf{v})' + (\mathbf{u}\cdot\mathbf{v}')' = \mathbf{u}''\cdot\mathbf{v} + \mathbf{u}'\cdot\mathbf{v}' + \mathbf{u}'\cdot\mathbf{v}' + \mathbf{u}\cdot\mathbf{v}'' = \mathbf{u}''\cdot\mathbf{v} + 2\mathbf{u}'\cdot\mathbf{v}' + \mathbf{u}\cdot\mathbf{v}''$

## Section 14.6

**1.** (a) $r = 2\sqrt{2}$, $\theta = \tan^{-1}(2/2) = \pi/4$ and $5\pi/4$, $z = 3$. Discard $5\pi/4$ so $\theta = \pi/4$ rad $= 45°$. (d) $r = 0$, $\theta$ not defined, $z = 5$ (g) $r = \sqrt{26}$, $\theta = \tan^{-1}(5/-1) = 1.768$ rad $= 101.3°$ **2.** (a) $\rho = 2$, $\phi = \pi/2$ rad $= 90°$, $\theta = 0$ (d) $\rho = \sqrt{14}$, $\phi = \cos^{-1}(3/\sqrt{14}) = 0.641$ rad $= 36.70°$, $\theta = \tan^{-1}(2/1) = 1.107$ rad $= 63.43°$ **3.** (a) Since $\hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ is an ON basis for 3- space we can write $\hat{\mathbf{e}}_\rho = (\hat{\mathbf{e}}_\rho\cdot\hat{\mathbf{i}})\hat{\mathbf{i}} + (\hat{\mathbf{e}}_\rho\cdot\hat{\mathbf{j}})\hat{\mathbf{j}} + (\hat{\mathbf{e}}_\rho\cdot\hat{\mathbf{k}})\hat{\mathbf{k}}$ and similarly for $\hat{\mathbf{e}}_\phi$ and $\hat{\mathbf{e}}_\theta$. Then $\hat{\mathbf{e}}\cdot\hat{\mathbf{i}} = (1)(1)\cos(\hat{\mathbf{e}}_\rho, \hat{\mathbf{i}}) = x/\rho = \sin\phi\cos\theta$. Similarly, $\hat{\mathbf{e}}_\rho\cdot\hat{\mathbf{j}} = y/\rho = \sin\phi\sin\theta$ and $\hat{\mathbf{e}}_\rho\cdot\hat{\mathbf{k}} = z/\rho = \cos\phi$. To obtain $\hat{\mathbf{e}}_\phi$ it is simplest to observe that it can be obtained from $\hat{\mathbf{e}}_\rho$ by adding $\pi/2$ to $\phi$: $\hat{\mathbf{e}}_\phi = \sin(\phi + \pi/2)(\cos\theta\,\hat{\mathbf{i}} + \sin\theta\,\hat{\mathbf{j}}) + \cos(\phi + \pi/2)\hat{\mathbf{k}}$. And since the spherical coordinate base vector $\hat{\mathbf{e}}_\theta$ is the same

as the cylindrical coordinate base vector $\hat{\mathbf{e}}_\theta$ we have $\hat{\mathbf{e}}_\theta = -\sin\theta\,\hat{\mathbf{i}} + \cos\theta\,\hat{\mathbf{j}}$.
**4.** (a) $d\mathbf{A}/dt = (3\cos 3t + 27t^4)\,\hat{\mathbf{e}}_r + (3\sin 3t - 36t^3)\,\hat{\mathbf{e}}_\theta + 2t\,\hat{\mathbf{e}}_z$ (d) $d\mathbf{A}/dt = 2t\,\hat{\mathbf{e}}_\rho + 2t^2\,\hat{\mathbf{e}}_\phi + 3t^2\sin 2t\,\hat{\mathbf{e}}_\theta$
**5.** (a) $\mathbf{v} = \dot{x}\,\hat{\mathbf{e}} + \dot{y}\,\hat{\mathbf{j}} + \dot{z}\,\hat{\mathbf{k}} = 2t\,\hat{\mathbf{i}} = 2t(\cos\theta\,\hat{\mathbf{e}}_r - \sin\theta\,\hat{\mathbf{e}}_\theta) = 2t\left(\frac{x}{r}\,\hat{\mathbf{e}}_r - \frac{y}{r}\,\hat{\mathbf{e}}_\theta\right) = \frac{2t}{\sqrt{t^4+4}}(t^2\,\hat{\mathbf{e}}_r - 2\,\hat{\mathbf{e}}_\theta)$, $\mathbf{a} = \dot{\mathbf{v}} = 2\,\hat{\mathbf{i}} = \frac{2}{\sqrt{t^4+4}}(t^2\,\hat{\mathbf{e}}_r - 2\,\hat{\mathbf{e}}_\theta)$ **6.** (a) $\mathbf{v} = \dot{x}\,\hat{\mathbf{j}} + \dot{y}\,\hat{\mathbf{j}} + \dot{z}\,\hat{\mathbf{k}} = 2t\,\hat{\mathbf{i}} = 2t(\sin\phi\cos\theta\,\hat{\mathbf{e}}_\rho + \cos\phi\cos\theta\,\hat{\mathbf{e}}_\phi - \sin\theta\,\hat{\mathbf{e}}_\theta) = 2t\left(\frac{x}{\rho}\,\hat{\mathbf{e}}_\rho + \frac{y}{\rho}\,\hat{\mathbf{e}}_\phi + \frac{z}{\rho}\hat{\mathbf{e}}_\theta\right) = \frac{2}{\sqrt{t^4+4}}\left(t^2\,\hat{\mathbf{e}}_\rho + 2\,\hat{\mathbf{e}}_\phi\right)$, $\mathbf{a} = \dot{\mathbf{v}} = 2\,\hat{\mathbf{i}} = \frac{2}{\sqrt{t^4+4}}\left(t^2\,\hat{\mathbf{e}}_\rho + 2\,\hat{\mathbf{e}}_\phi\right)$ **9.** (c) $\rho = s = Vt$, $\theta = \Omega t$, $\phi = \pi/4$ so (30) and (31) give $\mathbf{v}(t) = V\,\hat{\mathbf{e}}_\rho + \frac{V}{\sqrt{2}}\,\hat{\mathbf{e}}_\theta$, $\mathbf{a}(t) = -\frac{1}{2}V\Omega^2 t(\,\hat{\mathbf{e}}_\rho + \hat{\mathbf{e}}_\phi) + \sqrt{2}\,V\Omega\,\hat{\mathbf{e}}_\theta$

# Chapter 15

## Section 15.2

**1.** (a) Gauss elimination gives the solution set as $z = \alpha$, $y = (3\alpha - 8)/5$, $x = (\alpha + 4)/5$ so we have a parametrization $x = (\tau + 4)/5$, $y = (3\tau - 8)/5$, $z = \tau$ $(-\infty < \tau < \infty)$ (f) $x^2 + y^2 = 4$ suggests letting $x = 2\cos\tau$, $y = 2\sin\tau$ $(0 \le \tau < 2\pi)$. Then $x + y + 2z = 5$ gives $z = (5 - x - y)/2 = \frac{5}{2} - \cos\tau - \sin\tau$. **2.** (a) $x = 5 - 3\tau$, $y = -1 + \tau$, $z = 2 + 4\tau$ $(0 \le \tau \le 1)$ **3.** (a) $R'(t) \cdot R'(t) = 1 + 4\tau^2$ so $s(\tau) = \int_0^\tau \sqrt{1 + 4t^2}\,dt = \frac{1}{2}\tau\sqrt{1 + 4\tau^2} + \frac{1}{4}\ln(2\tau + \sqrt{1 + 4\tau^2})$ **11.** (a) Det = 0 so the curve is a plane curve.

## Section 15.3

**1.** (a) $\int_0^1 \int_0^y y^2\,dxdy = \int_0^1 \int_x^1 y^2\,dydx = 1/4$
(d) $\int_1^2 \int_0^y \sin(x - y)\,dxdy = \int_0^1 \int_1^2 \sin(x - y)\,dydx + \int_1^2 \int_x^1 \sin(x - y)\,dydx = \sin 2 - \sin 1 - 1$
**4.** (a) $M = \sigma ab/2$, $x_c = 2a/3$, $y_c = b/3$ (b) $M = 4\sigma$, $x_c = 14/15$, $y_c = 11/5$ **5.** (a) $I_x = 3\sigma/4$, $I_y = 65\sigma/12$ (b) $I_x = 46\sigma/3$, $I_y = 10\sigma/3$ **10.** (a) $-14$ **11.** 1/3

## Section 15.4

**8.** (a) $x = u$, $y = v$ and $x = u$, $y = u + v$ $(-\infty < u < \infty, -\infty < v < \infty)$ (d) $z = u$, $y = v$, $x = 2 - \frac{1}{3}u + \frac{2}{3}v$ $(-\infty < u < \infty, -\infty < v < \infty)$ **11.** (a) (25) gives $\hat{\mathbf{n}} = \pm(4\hat{\mathbf{i}} + 2\hat{\mathbf{j}} - \hat{\mathbf{k}})/\sqrt{21}$ (d) $\hat{\mathbf{n}} = \pm(\hat{\mathbf{i}} + 3\hat{\mathbf{j}} + \hat{\mathbf{k}})/\sqrt{11}$
(g) $\mathbf{n} = \mathbf{R}_u \times \mathbf{R}_v = \hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}}$, $\hat{\mathbf{n}} = \pm(\hat{\mathbf{i}} - \hat{\mathbf{j}} + 4\hat{\mathbf{k}})/\sqrt{18}$

## Section 15.5

**1.** (a) With $E = 1$, $F = 0$, $G = 1$, (5) gives $dA = dudv$ so $A = \int_0^{2\pi} \int_0^{1+\sin v} dudv = 2\pi$. (c) With $E = 2$, $F = 0$, $G = u^2$, (5) gives $dA = \sqrt{2}\,u\,dudv$ so $A = \int_0^{2\pi} \int_0^h \sqrt{2}\,u\,dudv = \sqrt{2}\pi h^2$. **6.** (a) $\pi/4$ **10.** (a) With $z = 1 + y$, (18) gives $dA = \sqrt{1 + 0 + 1}\,dxdy = \sqrt{2}\,dxdy$. Hence, $\int_0^1 \int_0^1 (1 + x)\sqrt{2}\,dxdy = 3/\sqrt{2}$. (c) $2\pi h$
(e) With $x = 3\sin v\cos u$, $y = 3\sin v\sin u$, $z = 3\cos v$, we obtain $\int_0^\pi \int_{-\pi/2}^{\pi/2} (1 + 3\sin v\cos u)9\sin v\,dudv = 45\pi$.
(g) $\sqrt{2}\pi h^2$

## Section 15.6

**3.** $\mathbf{F}(0, 0, 0) = \pi RG\sigma\hat{\mathbf{k}}$ **4.** (a) $z_c = 3h/4$ **8.** (a) $\mathbf{F} = 2\pi\sigma Gh\hat{\mathbf{k}}$ on $z = 0$, $\mathbf{F} = -2\pi\sigma Gh\hat{\mathbf{k}}$ on $z = h$

# Chapter 16

## Section 16.2

**4.** (a) $\mathbf{w}(2, 3, -1) = 3\hat{\mathbf{i}} - 2\hat{\mathbf{j}} - \hat{\mathbf{k}}$ so we want $\frac{y}{3} = \frac{-x}{-2} = \frac{z}{-1}$. Setting $x = \tau$, say, gives the parametric equations of the desired curve as $x = \tau$, $y = 3\tau/2$, $z = -\tau/2$ $(-\infty < \tau < \infty)$. (d) $x = 2$, $y = \tau$, $z = -3/\tau$ $(0 < \tau < \infty)$. To

understand our choice of the $\tau$ interval, note that $z\tau = -3$ gives a hyperbola in the $\tau, z$ plane, with one branch in the fourth quadrant and one in the second. The one in the fourth quadrant passes through the given point $z = -1$, so we choose that branch, for which $0 < \tau < \infty$.

## Section 16.3

**1.** (a) div $\mathbf{v} = 0$ everywhere (d) div $\mathbf{v} = yz + xz - 3xy = 17$ at $(3, -1, 4)$

## Section 16.4

**1.** (a) grad $u = 6\hat{\mathbf{i}}$ everywhere (e) grad $u = yz\hat{\mathbf{i}} + xz\hat{\mathbf{j}} + xy\hat{\mathbf{k}} = -4\hat{\mathbf{i}} - 9\hat{\mathbf{j}} + 36\hat{\mathbf{k}}$ **2.** (a) $du/ds = \nabla u \cdot \hat{\mathbf{v}} = (2x\hat{\mathbf{i}} + 2y\hat{\mathbf{j}} + 2z\hat{\mathbf{k}}) \cdot \hat{\mathbf{i}} = 2x = 4$ at (2,1,5) (d) $11/\sqrt{3}$ **4.** (a) $\nabla V = (6xy - z)\hat{\mathbf{i}} + 3x^2\hat{\mathbf{j}} - x\hat{\mathbf{k}} = 37\hat{\mathbf{i}} + 12\hat{\mathbf{j}} - 2\hat{\mathbf{k}}$ is in the direction of maximum rate of *increase*, so the charge will move in the direction $-37\hat{\mathbf{i}} - 12\hat{\mathbf{j}} + 2\hat{\mathbf{k}}$.
**6.** (a) $dT/dt = 200t + (20xt\hat{\mathbf{i}} - 10y\hat{\mathbf{j}}) \cdot (4x\hat{\mathbf{i}} + 4y\hat{\mathbf{j}}) = 2,040$ at $(2, -1, 3, 4)$

## Section 16.5

**1.** (a) curl $\mathbf{v} = \mathbf{0}$ everywhere (d) curl $\mathbf{v} = x(z - y)\hat{\mathbf{i}} - y(z - x)\hat{\mathbf{j}} + z(y - x)\hat{\mathbf{k}} = -15\hat{\mathbf{i}} + 16\hat{\mathbf{j}} - \hat{\mathbf{k}}$ at $(3, 4, -1)$

## Section 16.6

**1.** (a) $0, \mathbf{0}$ (d) $2y^3 + 6x^2y, \mathbf{0}$ **2.** (a) $0, \mathbf{0}$ (d) $0, 2\hat{\mathbf{j}}$ **7.** (d) $\nabla \times \mathbf{H} = \mathbf{J}$ **11.** (a) $\nabla^2\mathbf{v} = \nabla^2(xz^2)\hat{\mathbf{i}} + \nabla^2(y \sin z)\hat{\mathbf{k}} = 2x\hat{\mathbf{i}} - y \sin z\hat{\mathbf{k}}$

## Section 16.7

**1.** (a) $\nabla u = \hat{\mathbf{e}}_r$, $\nabla^2 u = 1/r$, $\nabla \cdot \mathbf{v} = 3$ [which makes sense because $\mathbf{v} = r\hat{\mathbf{e}}_r + z\hat{\mathbf{e}}_z$ is the position vector to the point and is therefore expressible in Cartesian coordinates as $x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + z\hat{\mathbf{k}}$, with divergence $\partial(x)/\partial x + \partial(y)/\partial y + \partial(z)/\partial z = 3$], $\nabla \times \mathbf{v} = \mathbf{0}$. NOTE: In using (16) and (17) we have $v_r = r$, $v_\theta = 0$, $v_z = z$.
(d) $\nabla u = z^3\hat{\mathbf{e}}_r + 3z^2r\hat{\mathbf{e}}_z$, $\nabla^2 u = 6zr + z^3/r$. With $v_r = v_z = 0$ and $v_\theta = 1$, (16) and (17) give $\nabla \cdot \mathbf{v} = 0$ and $\nabla \times \mathbf{v} = (1/r)\hat{\mathbf{e}}_z$. **2.** (a) $1/r$ (d) $-(2 \sin\theta \cos\theta)/r$ **8.** (a) $3\hat{\mathbf{e}}_r$ **11.** (a) $\nabla u = \hat{\mathbf{e}}_\rho$, $\nabla^2 u = 2/\rho$, $\nabla \cdot \mathbf{v} = 3$, $\nabla \times \mathbf{v} = \mathbf{0}$ (d) $\nabla u = 2\rho \sin\theta\hat{\mathbf{e}}_\rho + \rho(\cos\theta/\sin\phi)\hat{\mathbf{e}}_\theta$, $\nabla^2 u = \left[6 - (1/\sin^2\phi\right] \sin\theta$, $\nabla \cdot \mathbf{v} = (2 + \cot\phi)/\rho$, $\nabla \times \mathbf{v} = (\cot\phi\,\hat{\mathbf{e}}_\rho - \hat{\mathbf{e}}_\phi + \hat{\mathbf{e}}_\theta)/\rho$

## Section 16.8

**1.** (a) Each is 0 (d) Each is 1/3 (g) Each is 1/2 **4.** (a) Each is $2ab$ **5.** (a) Use $\mathbf{v} = (x^3yz/3)\hat{\mathbf{i}}$, for instance
**12.** (a) Each is $1920\pi$ (g) Each is $2\pi a^5$

## Section 16.9

**2.** (a) Each is 1/3 (e) Each is $-1/6$ (i) Each is 1/12 **3.** (a) Each is -2/15 (e) Each is 0 **11.** (a) Each is $-63$
**16.** (a) Each is $2\pi a^2\omega$

## Section 16.10

**1.** (a) All of 3-space (e) Everywhere except along the $z$ axis ($x = y = 0$) **2.** (a) $\Phi = x - 2y - 8z$; everywhere (f) $\Phi = 2ye^x - 3z$ in $x < 0$ (k) Irrotational nowhere **3.** (a) No **5.** (c) $\Phi = 2x^{5/2}e^{2y} + \frac{5}{3}y^3$, $I = -\frac{10}{3}$
**6.** (a) $\Phi = (x^2 + y^2 + z^2)/2$, $I = \pi^2(\pi^2 + 9)/2$ **11.** (a) $\nabla \cdot \mathbf{v} = 0 + 0 + 0 = 0$. With $x_0 = y_0 = 0$, say, (10.3) gives $\mathbf{w} = cx\hat{\mathbf{j}} + (ay - bx)\hat{\mathbf{k}}$, to which we can add the gradient of an arbitrary function $f$. (d) $\mathbf{w} = -xyz\,\hat{\mathbf{j}}$, to which we can add the gradient of an arbitrary function $f$. **13.** (a) $\Phi = rz$, $I = 2$ (e) $\Phi = \rho^6/6$, $I = 62/3$

(h) $\Phi = \sin\phi + \cos\theta$, $I = 2(\sqrt{5}-5)/5$

# Chapter 17

## Section 17.2

**5.** (a) $f_e(x) = 2$, $f_o(x) = -5x$ (e) $f_e(x) = x^2/(x^2-4)$, $f_o(x) = 2x/(4-x^2)$ **9.** $f_e(x) + f_o(x) = g_e(x) + g_o(x)$ and, changing $x$ to $-x$, $f_e(-x) + f_o(-x) = g_e(-x) + g_o(-x)$ or $f_e(x) - f_o(x) = g_e(x) - g_o(x)$. Adding these two equations gives $2f_e(x) = 2g_e(x)$, and subtracting gives $2f_o(x) = 2g_o(x)$. Hence $f_e(x) = g_e(x)$ and $f_o(x) = g_o(x)$. **12.** (a) No (g) Yes, $\pi$ (m) Yes, $2\pi/3$ (s) No (Sketch its graph) **13.** (a) $2\pi$ (e) $2\pi$

## Section 17.3

**1.** (a) Yes, in fact it is continuous (d) No, its limit does not exist as $x \to 0$ **4.** (a) $\text{FS}f = 2\sum_1^\infty \left[(-1)^{n+1}/n\right] \sin nx$ converges to $f(x)$ at all $x$ except at $\pm\pi$, $\pm 3\pi$, $\pm 5\pi$, $\ldots$, where $f(x)$ is $\pi$ but the series converges to the average value 0 (d) $\text{FS}f = 50$ is identical to $f(x)$ for all $x$ (g) $\text{FS}f = \frac{2}{\pi} - \frac{4}{\pi}\sum_1^\infty \left[1/(4n^2-1)\right] \cos 2nx$ converges to $f(x)$ for all $x$ **5.** (a) $\text{FS}f = \pi^2/3 + \sum_1^\infty \left[(-1)^n/n^2\right] \cos nx$ **12.** (a) $l = 3$, $p(3) = 3$, $p'(3) = 1$, $p''(3) = \cdots = 0$, $b_n = 2(3)(-1)^{n+1}/(n\pi)$ so $\text{FS}f = \frac{6}{\pi}\sum_1^\infty \left[(-1)^{n+1}/n\right] \sin\frac{n\pi x}{3}$ (d) $\text{FS}f = -\frac{112}{15} + \dfrac{768}{\pi^4}\displaystyle\sum_1^\infty \left[(-1)^{n+1}/n^4\right] \cos\frac{n\pi x}{2}$

**16.** (a) $\text{FS}f = 25 + \frac{50}{\pi}\sum_{-\infty}^\infty \frac{1}{n} \sin\frac{n\pi}{2} e^{in\pi x/2}$ (d) $\text{FS}f = 6(e^{ix} - e^{-ix})/2i = 3ie^{-ix} - 3ie^{ix}$ (a 2-term series)

**18.** (a) $x(t) = 50 + \frac{800}{\pi}\sum_{n=1,3,\ldots}^\infty \dfrac{1}{n(4-n^2\pi^2)} \sin\dfrac{n\pi t}{2}$ (d) $x(t) = \frac{5}{2} - \dfrac{20}{\pi^2}\displaystyle\sum_{n=1,3,\ldots}^\infty \frac{1}{n^2(1-n^2\pi^2)} \cos n\pi t$

## Section 17.4

**2.** (a) HRC: $f(x) = \frac{25}{2} + \frac{50}{\pi}\sum_1^\infty \frac{1}{n} \sin\frac{n\pi}{2} \cos\frac{n\pi x}{2}$, HRS: $f(x) = \frac{50}{\pi}\sum_1^\infty \frac{1}{n}(1 - \cos\frac{n\pi}{2}) \sin\frac{n\pi x}{2}$, QRC: $f(x) = \frac{100}{\pi}\sum_{1,3,\ldots}^\infty \frac{1}{n} \sin\frac{n\pi}{4} \cos\frac{n\pi x}{4}$, QRS: $f(x) = \frac{100}{\pi}\sum_{1,3,\ldots}^\infty \frac{1}{n}(1 - \cos\frac{n\pi}{4}) \sin\frac{n\pi x}{4}$

## Section 17.5

**2.** (a) $|e^{-nx}\sin nx| < e^{-2n}$ on $2 < x < 5$, and $\sum_1^\infty e^{-2n}$ is convergent [by the ratio test or by writing it as the geometric series $\sum_1^\infty (e^{-2})^n$, which converges because $|e^{-2}| < 1$]. Hence, the given series is uniformly convergent on $2 < x < 5$ by Theorem 17.5.1. **3.** (a) On $1 < x_0 < x < \infty$, $\left|\dfrac{1}{1+x^n}\right| \le \dfrac{1}{1+x_0^n} \sim \dfrac{1}{x_0^n} = (1/x_0)^n$, and $\sum_0^\infty (1/x_0)^n$ is a convergent geometric series if $x_0 > 1$. Hence, the series converges uniformly on $x_0 < x < \infty$ for any $x_0 > 1$, by Theorem 17.5.1. (d) On $|x| < x_0$ for any $x_0 < 1/5$

**4.** (a) $\frac{d}{dx}\sum_1^\infty \dfrac{\sin 2nx}{n^4} = \displaystyle\sum_1^\infty \frac{d}{dx}\left(\frac{\sin 2nx}{n^4}\right) = 2\sum_1^\infty \frac{\cos 2nx}{n^3}$ according to Theorem 17.5.2 because the latter series converges uniformly by Theorem 17.5.1, with $M_n = 1/n^3$. (Recall from the calculus that the **p-series** $\sum_1^\infty 1/n^p$ converges if $p > 1$; in this case $p = 3 > 1$.) **5.** (a) Expanding $f(t)$, let us call $x'' + x' + x = \frac{2}{\pi} - \frac{4}{\pi}\sum_1^\infty \dfrac{1}{4n^2-1} \cos 2nt$ equation (A). A particular solution of (A) due to the $2/\pi$ is $x_p = 2/\pi$, and a particular solution of $x'' + x' + x = \cos 2nt$ is $x_p(t) = [(1-4n^2)^2 \cos 2nt + 2n \sin 2nt]/[(1-4n^2)^2 + 4n^2]$ so, by superposition, a particular solution of (A) is $x_p(t) = \frac{2}{\pi} - \frac{4}{\pi}\sum_1^\infty \dfrac{1}{4n^2-1} \dfrac{(1-4n^2)\cos 2nt + 2n\sin 2nt}{(1-4n^2)^2 + 4n^2}$. The latter does satisfy (A), as can be verified by substitution, provided that we can justify the termwise differentiations of the series needed for $x'$ and $x''$. Differentiating termwise gives $x_p'(t) = -\frac{4}{\pi}\sum_1^\infty \dfrac{1}{4n^2-1} \dfrac{-2n(1-4n^2)\sin 2nt + 4n^2\cos 2nt}{(1-4n^2)^2 + 4n^2} \equiv -\dfrac{4}{\pi}\displaystyle\sum_1^\infty a_n(t)$. Using

$|\sin 2nt| \le 1$ and $|\cos 2nt| \le 1$, $|a_n(t)| \le \dfrac{2n + 8n^3 + 4n^2}{(4n^2-1)\,[(1-4n^2)^2 + 4n^2]} \sim \dfrac{1}{8n^3}$ as $n \to \infty$, so there will exist some constant $C$ such that $|a_n(t)| \le C/n^3$ for all $n = 1, 2, \ldots$. Since $\sum_1^\infty (C/n^3) = C\sum_1^\infty (1/n^3)$ converges, the termwise differentiation is justified by Theorem 17.5.1. Similarly for $x''$.

## Section 17.6

**2.** (a) $\|\mathbf{E}\|^2 = \dfrac{2\pi^3}{3} - \pi\left[\dfrac{\pi^2}{2} + \dfrac{4}{\pi^2}\sum_1^k [(-1)^n - 1]^2/n^4\right]$ so $\|\mathbf{E}\| = 0.27, 0.27, 0.11, 0.11, 0.06, 0.06, 0.04, 0.04$ for $k = 1, 2, \ldots, 8$.

## Section 17.7

**1.** (a) $\lambda_n = [(2n-1)\pi/2L]^2$, $\phi_n(x) = \sin[(2n-1)\pi x/2L]$ for $n = 1, 2, \ldots, 100 = \frac{400}{\pi}\sum_1^\infty \frac{1}{2n-1}\sin\frac{2n-1}{2L}\pi x$ (c) $\lambda_n = (n\pi/L)^2$, $\phi_n(x) = \cos(n\pi x/L)$ for $n = 0, 1, \ldots$, $f(x) = \frac{1}{2} + \frac{2}{\pi}\sum_1^\infty \dfrac{(-1)^{n+1}}{2n-1}\cos[(2n-1)\pi x/L]$ **4.** (a) $\sigma(x) = x^4$, $(x^5y')' + \lambda x^5 y = 0$, $p(x) = x^5$, $q(x) = 0$, $w(x) = x^5$ **9.** (a) $\lambda_n = n^2$, $\phi_n(x) = \cos nx$ for $n = 0, 1, 2, \ldots$, $f(x) = a_0 + \sum_1^\infty a_n \cos nx$, $a_0 = \langle f, 1\rangle/\langle 1, 1\rangle = \int_0^2 x^4\,dx/\pi = 32/(5\pi)$, $a_n = \langle f, \cos nx\rangle/\langle \cos nx, \cos nx\rangle = (2/\pi)\int_0^2 x^4 \cos nx\,dx = [32/(n^5\pi)][(2n^4 - 6n^2 + 3)\sin n\cos n + (4n^3 - 6n)\cos^2 n - 2n^3 + 3n]$ **17.** (a) $\langle L[u], v\rangle = \int_0^1 u'v\,dx = uv|_0^1 - \int_0^1 uv'\,dx$ so $L^*[v] = -v'$ (i.e., $L^* = -d/dx$), $v(1) = 0$, not Hermitian (c) $L^* = d^2/dx^2$, $v(1) = v'(1) = 0$, not Hermitian because the boundary conditions associated with $L^*$ are different from those associated with $L$ (e) $L^* = d^2/dx^2 + 3$, $v'(0) = v'(1) = 0$, Hermitian

## Section 17.8

**1.** (a) $\lambda_0 = 0$, $\phi_0(x) = 1$; $\lambda_n = (n\pi/2)^2$, $\phi_n(x) = \cos(n\pi x/2)$ and $\sin(n\pi x/2)$ for $n = 1, 2, \ldots$, $H(x-2) = \frac{1}{2} - \frac{2}{\pi}\sum_{1,3,\ldots}^\infty \frac{1}{n}\sin\frac{n\pi x}{2}$ (c) $\lambda_0 = 0$, $\phi_0(x) = 1$; $\lambda_n = (2n\pi/\ln 2)^2$, $\phi_n(x) = \cos\left(2n\pi\frac{\ln x}{\ln 2}\right)$ for $n = 1, 2, \ldots$; expansions of the form $f(x) = a_0 + \sum_1^\infty a_n \cos\left(2n\pi\frac{\ln x}{\ln 2}\right)$, so if $f(x) = 6$ then $a_0 = 6$ and $a_n = 0$ for $n \ge 1$ by inspection. Perhaps that result is not obvious because the weight function is $w(x) = 1/x$, but $a_0 = \langle f(x), 1\rangle/\langle 1, 1\rangle = \int_1^2 (6)(1)x^{-1}\,dx/\int_1^2 (1)^2 x^{-1}\,dx = 6$ and the two integrals in the evaluation of $a_n$ look difficult but are readily evaluated using the substitution $u = 2n\pi(\ln x)/\ln 2$. (e) See Section 4.4. $\lambda_n = n(n+1)$, $\phi_n(x) = P_n(x)$ for $n = 0, 2, 4, \ldots$; expansions of the form $f(x) = \sum_{0,2,4,\ldots}^\infty a_n P_n(x)$. If $f(x) = x$, then $a_0 = \langle x, 1\rangle/\langle 1, 1\rangle = 1/2$, $a_2 = \langle x, P_2(x)\rangle/\langle P_2(x), P_2(x)\rangle = 5/8$, $a_4 = \langle x, P_4(x)\rangle/\langle P_4(x), P_4(x)\rangle = -3/16$ **4.** From Section 4.4 we see that $\phi_n(x) = P_{2n}(x)$ for $n = 0, 1, 2, \ldots$, so $1 - x = \sum_0^\infty a_n P_{2n}(x)$ where $a_n = \langle 1 - x, P_{2n}(x)\rangle/\langle P_{2n}(x), P_{2n}(x)\rangle$. From (18) in Section 4.4, $\int_0^1 P_{2n}^2(x)\,dx = 1/(2n+1)$ so $\langle P_{2n}(x), P_{2n}(x)\rangle = \int_0^1 P_{2n}^2(x)\,dx = 1/(4n+1)$ and $a_n = (4n+1)\int_0^1 (1-x)P_{2n}(x)\,dx$ : $a_0 = 1/2$, $a_1 = -5/8$, $a_2 = 9/48, \ldots$

## Section 17.9

**2.** (a) $f(x) = (100/\pi)\int_0^\infty [\sin 2\omega\cos\omega x + (1 - \cos 2\omega)\sin\omega x]\,d\omega/\omega$ for all $x$ except $x = 0$ and $x = 2$: $f(0) = f(2) = 100$ but the Fourier integral of $f$ converges to the average value 50. As a check, application of the *Maple* int command gives these values for the Fourier integral, integrating from 0 to 5000: $200Si(5000)/\pi$ at $x = 1$ and $-100[Si(10{,}000) - Si(20{,}000)]/\pi$ at $x = 4$. Following the int command with evalf(") evaluates these quantities as 99.96 and $-0.0043$, respectively. (d) $f(x) = (1/\pi)\int_0^\infty [(\cos 5\omega + 5\omega\sin 5\omega - 1)\cos\omega x + (5\omega\cos 5\omega - \sin 5\omega)\sin\omega x]\,d\omega/\omega^2$ for all $x$ except $x = -5$: $f(-5) = 0$ but the Fourier integral of $f$ converges to 2.5.

## Section 17.10

**4.** (a) $f(x) = a/[\pi(a^2 + x^2)]$ (b) $f(x) = \sin ax/(\pi x)$ **6.** (a) $F\left\{4x^2 e^{-3|x|}\right\} = 4i^2\frac{d^2}{d\omega^2}F\left\{e^{-3|x|}\right\}$ (entry 17) $= -4\frac{d^2}{d\omega^2}\left(\frac{6}{\omega^2+9}\right)$ (entry 4) $= 48\left[\frac{1}{(\omega^2+9)^2} - \frac{4\omega^2}{(\omega^2+9)^3}\right]$ (c) With $c = 3$, $a = 1$, $f(x) = 1/(x^2 + 2)$ in entry 13,

and $\hat{f} = (\pi/\sqrt{2})\exp(-\sqrt{2}|\omega|)$, we have $F\left\{\frac{\cos 3x}{x^2+2}\right\} = \frac{1}{2}[\hat{f}(\omega-3)+\hat{f}(\omega+3)] = \frac{\pi}{2\sqrt{2}}(e^{-\sqrt{2}|\omega-3|}+e^{-\sqrt{2}|\omega+3|})$ (e) $F\left\{\frac{3}{2x^2+1}-5e^{-|x|}\right\} = \frac{3}{2}F\left\{\frac{1}{x^2+1/2}\right\} - 5F\left\{e^{-|x|}\right\}$ (entry 18) $= \frac{3}{2}\pi\sqrt{2}e^{-|\omega|/\sqrt{2}} - 5\frac{2}{\omega^2+1}$ (entries 1, 4) $= \frac{3\pi}{\sqrt{2}}e^{-|\omega|/\sqrt{2}} - \frac{10}{\omega^2+1}$ (g) $F^{-1}\left\{\frac{4\sin\omega}{\omega} - \frac{1}{\sqrt{|\omega|}}\right\} = 2F^{-1}\left\{\frac{2\sin\omega}{\omega}\right\} - \frac{1}{\sqrt{2\pi}}F^{-1}\left\{\sqrt{\frac{2\pi}{|\omega|}}\right\}$ (entry 18) $= 2[H(x+1) - H(x-1)] - \frac{1}{\sqrt{2\pi|x|}}$ (entries 9, 7) (i) $F^{-1}\left\{\frac{9}{2\omega+i}\right\} = -\frac{9i}{2}F^{-1}\left\{\frac{1}{1/2-i\omega}\right\}$ (entry 18) $= -\frac{9i}{2}H(-x)e^{x/2}$ (entry 3) (k) $F^{-1}\left\{e^{-|\omega|}\cos\omega\right\} = \frac{1}{2\pi}F^{-1}\left\{2(\pi e^{-|\omega|})\cos\omega\right\}$ (entry 18) $= \frac{1}{2\pi}\left[\frac{1}{(x+1)^2+1} + \frac{1}{(x-1)^2+1}\right]$ (entries 15, 1) (m) $F^{-1}\left\{\frac{1}{\omega^2+i\omega+2}\right\} = F^{-1}\left\{\frac{1}{3i}\left(\frac{1}{\omega-i} - \frac{1}{\omega+2i}\right)\right\} = F^{-1}\left\{\frac{1}{3}\left(\frac{1}{1+i\omega} - \frac{1}{-2+i\omega}\right)\right\} = \frac{1}{3}F^{-1}\left\{\frac{1}{1+i\omega} + \frac{1}{2-i\omega}\right\}$ (entry 18) $= \frac{1}{3}[H(x)e^{-x} + H(-x)e^{2x}]$ (entries 2, 3)

## Section 17.11

**9.** (a) Because $u(0)$ is given, use the sine transform. Obtain $\hat{u}_S = -50\frac{\omega}{(\omega^2+9)^2}$ and consider entry 2S with $a = 3$. Since $\mathrm{Im}(3+i\omega)^2 = 6\omega$, write $\hat{u}_S = -\frac{50}{6}\frac{6\omega}{(\omega^2+9)^2} = -\frac{25}{3}\frac{\mathrm{Im}(3+i\omega)^2}{(\omega^2+3^2)^2}$ so $u(x) = -\frac{25}{3}xe^{-3x}$.

# Chapter 18

## Section 18.2

**2.** (a) $L = \nabla^2+k^2$ is linear because $L[\alpha u+\beta v] = (\nabla^2+k^2)(\alpha u+\beta v) = \alpha(\nabla^2+k^2)u+\beta(\nabla^2+k^2)u = \alpha L[u]+\beta L[v]$ (e) The operator defined by $L[u] = u_{xx}+u_{yy}-e^u$ is nonlinear because $L[\alpha u+\beta v] - \alpha L[u] - \beta L[v] = (\alpha u+\beta v)_{xx}+(\alpha u+\beta v)_{yy} - e^{(\alpha u+\beta v)} - \alpha(u_{xx}+u_{yy}-e^u) - \beta(v_{xx}+v_{yy}-e^v) = \alpha e^u+\beta e^v - e^{\alpha u}e^{\beta v} \neq 0$ in general. For example, if $\alpha = 2$, $\beta = 0$, $u(x,y) = x$, $v(x,y) = y$, then the latter is $2e^x - e^{2x}$, which is not identically zero (on any $x, y$ domain of interest). **3.** (a) $A = 1$, $B = 1/2$, $C = 0$ so $B^2 - AC = 1/4 > 0$, so hyperbolic (everywhere in $x, y$ plane). Note that $D$, $E$, $F$, and $f$ are irrelevant insofar as this classification. (d) $B^2 - AC = 0 + x(\sin^2 y + 1)$, so hyperbolic in the half plane $x > 0$, elliptic in the half plane $x < 0$, parabolic on the line $x = 0$.

## Section 18.3

**4.** (a) $X''T = XT' + 3XT$, $\frac{X''}{X} = \frac{T'}{T} + 3 = -\kappa^2$, $X'' + \kappa^2 X = 0$ and $T' + (\kappa^2+3)T = 0$ so $X = A\cos\kappa x + B\sin\kappa x$ and $T = C\exp(-\kappa^2-3)t$ for $\kappa \neq 0$, and $X = D + Ex$ and $T = Fe^{-3t}$ for $\kappa = 0$. Thus, $u(x,t) = (D+Ex)Fe^{-3t} + (A\cos\kappa x + B\sin\kappa x)C\exp(-\kappa^2-3)t = (G+Hx)e^{-3t} + (I\cos\kappa x + J\sin\kappa x)\exp(-\kappa^2-3)t$, say (c) $\frac{X''}{X} + 2\frac{X'T'}{XT} = \frac{T'}{T}$, which cannot be separated because of the mixed term $X'T'/XT$ **6.** (a) $u(x,t) = 20 + 3x + \sum_{1,3,\ldots}^{\infty} A_n \sin\frac{nx}{2}\, e^{-(n\alpha/2)^2 t}$ is the solution, where (with the help of a quarter-range sine expansion) $A_n = (2/\pi)\int_0^\pi(-20-3x)\sin\frac{nx}{2}\,dx = \frac{2}{\pi}\left[\frac{40+6\pi}{n}\cos\frac{n\pi}{2} - \frac{12}{n^2}\sin\frac{n\pi}{2}\right]$. $u_s(x) = 20+3x$. (d) $u(x,t) = \sum_{1,3,\ldots}^{\infty} A_n \sin\frac{n\pi x}{4}\, e^{-(n\pi\alpha/4)^2 t}$, where the $A_n$'s are found from the initial condition $u(x,0) = 5\sin(\pi x/4) - 12\sin(5\pi x/4) = \sum_{1,3,\ldots}^{\infty} A_n \sin\frac{n\pi x}{4}$ (on $0 < x < 2$). The quarter-range sine formula gives $A_n = \int_0^2\left(5\sin\frac{\pi x}{4} - 12\sin\frac{5\pi x}{4}\right)\sin\frac{n\pi x}{4}\,dx = 10\left\{\frac{\sin[(n-1)\pi/2]}{(n-1)\pi} - \frac{\sin[(n+1)\pi/2]}{(n+1)\pi}\right\} - 24\left\{\frac{\sin[(n-5)\pi/2]}{(n-5)\pi} - \frac{\sin[(n+5)\pi/2]}{(n+5)\pi}\right\}$ as found using the *Maple* int command or tables. Observe that each sine term is zero because $n$ $(= 1, 3, \ldots)$ is odd, so $A_n$ appears to be zero for each $n = 1, 3, \ldots$. However, for $n = 1$ the $n-1$ denominator vanishes and for $n = 5$ the $n-5$ denominator vanishes so the latter expression is indeterminate for $n = 1$ and $n = 5$ (of the form 0/0). For those caes l'Hôpital's rule gives $A_1 = 10\left\{\frac{1}{2}\right\} = 5$ and $A_5 = -24\left\{\frac{1}{2}\right\} = -12$, with $A_n = 0$ for $n = 3, 7, 9, 11, 13, \ldots$. Thus, the series solution reduces to the two-term result $u(x,t) = 5\sin\frac{\pi x}{4}\, e^{-(\pi\alpha/4)^2 t} - 12\sin\frac{5\pi x}{4}e^{-(5\pi\alpha/4)^2 t}$. Observe that this result is obtained more readily, from the initial condition, merely by matching terms: $5 = A_1$, $-12 = A_5$, and $0 = A_n$ for all other $n$'s $(3, 7, 9, 11, \ldots)$. $u_s(x) = 0$ (g) $u(x,t) = A + \sum_1^\infty B_n \cos nx\, e^{-(n\alpha)^2 t}$ and

$u(x,0) = 300 = A + \sum_1^\infty B_n \cos nx$ gives (half-range cosine formulas) $A = (1/\pi)\int_0^\pi 300\,dx = 300$, $B_n = (2/\pi)\int_0^\pi 300\cos nx\,dx = (600\sin n\pi)/(n\pi) = 0$ for each $n$. Thus, the solution is simply $u(x,t) = 300$. With a little experience, one could anticipate this result by inspection because the ends of the rod are insulated ($u_x = 0$ at each end) and the initial temperature distribution is uniform, so why should the temperature change? Or, mathematically, observe that $u(x,y) = 300$ does satisfy the PDE, the boundary conditions, and the initial condition.

**7.** Consider, for example, $t = 1000$. Running the *Maple* sum command for the sum on the right-hand side of (45), with $n = 1..20$, gives $-43.28600284$. Running it again, with $n = 1..30$, gives the same result so it is reasonable to assume that the result is correct to that many decimal places. However, understand that this reasoning is only heuristic. (In fact, it is interesting to run it for $n = 1..2$, $n = 1..4$, $n = 1..6$, and so on. We find that the sum settles down to 10 significant figures at $n = 1..8$; i.e., the series converges rapidly for this choice of $x$ and $t$.) Finally, the command evalf(100 + ") gives the value 56.71399716 for $u(0,1000)$, which appears to agree with the value obtained (crudely) from Fig. 6.

**10.** (a) $u_s(x) = u_1 + Q_2 x$ (d) $u_s(x) = (u_2 - u_1\cosh\beta L)\frac{\sinh\beta x}{\sinh\beta L} + u_1\cosh\beta x$, where $\beta = \sqrt{H}/\alpha$ (h) $u_s(x) = A + Be^{Vx/\alpha^2}$, $A = (u_2 - u_1 e^{VL/\alpha^2})/(1 - e^{VL/\alpha^2})$, $B = (u_1 - u_2)/(1 - e^{VL/\alpha^2})$ **14.** (a) $a_0(t) = A_0$, $a_n(t) = A_n e^{-(2n\pi\alpha/L)^2 t}$, $b_n(t) = B_n e^{-(2n\pi\alpha/L)^2 t}$, where $u(x,0) = f(x) = A_0 + \sum_1^\infty \left(A_n\cos\frac{2n\pi x}{L} + B_n\sin\frac{2n\pi x}{L}\right)$ gives $A_0 = \frac{1}{L}\int_0^L f(x)\,dx$, $A_n = \frac{1}{L}\int_0^L f(x)\cos\frac{2n\pi x}{L}\,dx$, $B_n = \frac{1}{L}\int_0^L f(x)\sin\frac{2n\pi x}{L}\,dx$ **16.** (a) $u(x,t) = u_s(x) + \sum_1^\infty A_n \sin\frac{n\pi x}{L}\,e^{-(n\pi\alpha/L)^2 t}$, where $u_s(x) = Fx(L-x)/(2\alpha^2)$ and $A_n = -\frac{2}{L}\int_0^L u_s(x)\sin\frac{n\pi x}{L}\,dx$

**17.** (b) $u(x,t) = \frac{4}{\pi}\sum_{1,3,\ldots}^\infty \frac{1}{n}\frac{1}{(n\pi\alpha/L)^2 - 1}\sin\frac{n\pi x}{L}\left(e^{-t} - e^{-(n\pi\alpha/L)^2 t}\right)$

**30.** (a) $u(x,t) = 50 + e^x\sum_{1,3,\ldots}^\infty A_n \sin\frac{n\pi x}{L}\,e^{-[(n\pi/L)^2+1]t}$. Then $u(x,0) = 0 = 50 + \sum_{1,3,\ldots}^\infty A_n e^x \sin\frac{n\pi x}{L}$. The eigenfunctions of the Sturm–Liouville problem $X'' - 2X' + \kappa^2 X = 0$ on $0 < x < L$ with $X(0) = 0$ and $X(L) = 0$ are $e^x\sin\frac{n\pi x}{L}$, so we satisfy $-50 = \sum_{1,3,\ldots}^\infty A_n e^x \sin\frac{n\pi x}{L}$ by setting $A_n = \langle -50, e^x\sin\frac{n\pi x}{L}\rangle / \langle e^x\sin\frac{n\pi x}{L}, e^x\sin\frac{n\pi x}{L}\rangle$. To obtain the weight function for the inner product write $\sigma X'' - 2\sigma X' + \kappa^2\sigma X = 0$, require that $-2\sigma = \sigma'$ and obtain $\sigma(x) = e^{-2x}$, which is also the weight function. Thus, $A_n = \int_0^L (-50)e^x\sin\frac{n\pi x}{L}\,e^{-2x}\,dx / \int_0^L \left(e^x\sin\frac{n\pi x}{L}\right)^2 e^{-2x}\,dx = -\frac{100}{L}\int_0^L e^{-x}\sin\frac{n\pi x}{L}\,dx$. NOTE We wrote $X(0) = 0$ and $X(L) = 0$ rather than $X(0) = 50$ and $X(L) = 50$, because the $u_s(x) = 50$ term in the solution satisfies the nonhomogeneous boundary conditions, so that the transient part of the solution satisfies a homogeneous, or "homogenized," version of those boundary conditions, which point was discussed in Example 3. Remember, a Sturm–Liouville problem is an eigenvalue problem so its boundary conditions *must* be homogeneous.

## Section 18.4

**8.** (b) We can use either transform but the Laplace transform is a bit simpler because the transformed equation $\alpha^2\bar{w}_{xx} - s\bar{w} = -\bar{F}(s)$ has only a constant forcing function [i.e., $\bar{F}(s)$ does not vary with the active variable $x$] whereas if we use the Fourier transform then the transformed equation $-\alpha^2\omega^2\hat{w} - \hat{w}_t = -F(t)$ has a nonconstant forcing function. Then $\bar{w}(x,s) = Ae^{\sqrt{s}x/\alpha} + Be^{-\sqrt{s}x/\alpha} + \bar{F}(s)/s$. Reasoning as in (29)–(31), we require that $\bar{w}(x,s) \to 0$ as $x \to \pm\infty$ so $A = B = 0$ and $w(x,t) = L^{-1}\{\bar{F}(s)/s\} = \int_0^t F(\tau)\,d\tau$ (which, as we could have anticipated from the beginning, varies with $t$ but not with $x$). **15.** At $x = 0.5$ Hastings' formula gives erf(0.5) $\approx 0.5204876$, whereas tables give erf(0.5) $= 0.5204999$. The difference, 0.0000123, is indeed within the claimed accuracy.

## Section 18.5

**8.** (a) Yes (d) No, due to the $u^2$ (g) Yes (j) Yes (m) No; $L = x^3\partial^2/\partial x^2 + 3x^2\partial/\partial x - \partial^2/\partial t^2 + 1$ and $x^3$ is not even, nor is $3x^2$ odd

## Section 18.6

**5.** Here are a few: $U_{11} = 0.2$, $U_{21} = 0.2$, and $U_{33} = 0.44896$ **7.** Here are a few: $U_{11} = 76$, $U_{21} = 92$, and $U_{33} = 38.0096$, where we have used the average values $U_{00} = 50$ and $U_{40} = 50$ at the corners.

**10.** In the left half $r = r_L = 0.225$; in the right half $r = r_R = 0.025$. Thus, in the left half ($j = 1,2$) use

$U_{j,k+1} = 0.225U_{j-1,k} + 0.55U_{jk} + 0.225U_{j+1,k}$ and in the right half ($j = 4, 5$) use $U_{j,k+1} = 0.025U_{j-1,k} + 0.95U_{jk} + 0.025U_{j+1,k}$. For the interface at $x = 6$ use $K_L(U_{3k} - U_{2k})/\Delta x = K_R(U_{4k} - U_{3k})/\Delta x$ or $U_{3k} = (K_L U_{2k} + K_R U_{4k})/(K_L + K_R) = 0.893U_{2k} + 0.107U_4$. Thus, $U_{11} = 11.25$, $U_{21} = U_{31} = U_{41} = U_{51} = 0$; $U_{12} = 28.69$, $U_{22} = 2.53$, $U_{32} = 0.893(2.53) + 0.107(0) = 2.26$, $U_{42} = U_{52} = 0$; $U_{13} = 38.85$, $U_{23} = 8.36$, then $U_{43} = 0.06$, then $U_{33} = 0.893(8.36) + 0.107(0.06) = 7.47$, $U_{53} = 0$; $U_{14} = 45.75$, $U_{24} = 15.02$, then $U_{44} = 0.24$, then $U_{34} = 0.893(15.02) + 0.107(0.24) = 13.44$, $U_{54} = 0.00$ **11.** (a) With exact values in parentheses, $U_{11} = U_{31} = 57.46\,(58.04)$, $U_{21} = 81.25\,(82.09)$, $U_{12} = U_{32} = 46.69\,(47.65)$, $U_{22} = 66.02\,(67.38)$, $U_{31} = U_{33} = 37.93\,(39.11)$, $U_{23} = 53.65\,(55.31)$

# Chapter 19

## Section 19.2

**1.** (a) $y(5,1) = -3/5$ (d) $y(5,4) = -3/5$ (g) $y(5,20) = 1$ **2.** (a) $y(x,t) = \frac{50L}{\pi c}\sin\frac{\pi x}{L}\sin\frac{\pi ct}{L}$. NOTE: The initial condition $y_t(x,0) = 50\sin\frac{\pi x}{L} = \sum_1^\infty \frac{n\pi c}{L}S_n \sin\frac{n\pi x}{L}$ gives, by the half-range sine formulas, $\frac{n\pi c}{L}S_n = \frac{2}{L}\int_0^L 50\sin\frac{\pi x}{L}\sin\frac{n\pi x}{L}\,dx$, which equals 50 for $n = 1$ and 0 for $n \geq 2$. However, it is much simpler to satisfy the initial condition (stated above) merely by matching coefficients of the $\sin(n\pi x/L)$ terms on the left- and right-hand sides.
**5.** (b) $y(x,t) = \sum_{1,3,\ldots}^\infty \sin\frac{n\pi x}{2L}\left(A_n\cos\frac{n\pi ct}{2L} + B_n\sin\frac{n\pi ct}{2L}\right)$. Then, $y(x,0) = f(x)$ gives $A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{2L}\,dx$ and $y_t(x,0) = 0$ gives $B_n = 0$, both by quarter-range sine series. **6.** $y(x,t) = e^{-at/2}\sum_1^\infty \sin\frac{n\pi x}{L}(A_n\cos\omega_n t + B_n\sin\omega_n t)$, where $\omega_n = \sqrt{(n\pi c/L)^2 - (a/2)^2}$. Then, $y(x,0) = f(x)$ gives $A_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$ and $y_t(x,0)$ gives the $B_n$'s in terms of the $A_n$'s as $B_n = \frac{a}{2\omega_n}A_n$. The $ay_y$ term in (6.1) causes these changes in the solution, all of which make sense physically: first, the $e^{-at/2}$ factor in the solution causes the motion to damp out as $t \to \infty$; second, $a > 0$ causes a reduction in the frequencies, from $\omega_n = n\pi c/L$ to $\omega_n = \sqrt{(n\pi c/L)^2 - (a/2)^2}$ (i.e., the damped system is more "sluggish"); third, $a > 0$ causes a phase shift, for if $a = 0$ then $B_n = 0$, and as $a$ increases $B_n$ increases. **8.** $y(x,t) = \frac{g}{2c^2}x(x-L) + \sum_1^\infty A_n\sin\frac{n\pi x}{L}\cos\frac{n\pi ct}{L}$, and $y(x,0) = f(x)$ gives $A_n = \frac{2}{L}\int_0^L[f(x) - \frac{g}{2c^2}x(x-L)]\sin\frac{n\pi x}{L}\,dx$. Note that even if $y(x,0) = f(x) = 0$ the solution is nonzero.
**12.** (b) $u(x,t) = \frac{8s_0L}{\pi^2E}\sum_{1,3,\ldots}^\infty \frac{1}{n^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{2L}\cos\frac{n\pi ct}{2L}$ (c) $s(0,t) = E\epsilon(0,t) = Eu_x(0,t)$ $= \frac{4s_0}{\pi}\sum_{1,3,\ldots}^\infty \frac{1}{n}\sin\frac{n\pi}{2}\cos\frac{n\pi ct}{2L}$

## Section 19.3

**1.** (a) $w(x,y,t)$ is given by (16a), where the $H_{mn}$'s are found from (17): $8\sin 2x\sin 2y = \cdots + H_{24}\sin\frac{2\pi x}{\pi}\sin\frac{4\pi y}{2\pi} + \cdots$ so, matching coefficients [which is much simpler than using (22)], we see that all the $H_{mn}$'s are zero except for $H_{24}$, which is 8. Thus, $w(x,y,t) = 8\sin 2x\sin 2y\cos 2\sqrt{2}t$. Nodal lines: $x = \pi/2$, $y = \pi/2$, $\pi$, $3\pi/2$
(d) Matching coefficients in $\sin 3x\sin y - \sin x\sin 3y = \cdots + H_{32}\sin\frac{3\pi x}{\pi}\sin\frac{2\pi y}{2\pi} + H_{16}\sin\frac{1\pi x}{\pi}\sin\frac{6\pi y}{2\pi} + \cdots$ gives $H_{32} = 1$, $H_{16} = -1$ and all other $H_{mn}$'s are zero. Thus, $w(x,y,t) = (\sin 3x\sin y - \sin x\sin 3y)\cos\sqrt{10}\,t$. Nodal curve(s) are defined by the relation $\sin 3x\sin y - \sin x\sin 3y = 0$. **2.** $w(x,y,t) = 20\sin 3\pi x\sin 4\pi y\cos 5\pi t - 8\sin 5\pi x\sin 12\pi y\cos 13\pi t$. If the period is $T$, then for any integers $p$ and $q$ we have $5\pi T = 2p\pi$ and $13\pi T = 2q\pi$, so $T = 2p/5$ and $T = 2q/13$. To find the *fundamental* period we seek the smallest $p$ and $q$ such that $2p/5 = 2q/13$ or $p/q = 5/13$. They are $p = 5$ and $q = 13$, so the fundamental period is $T = 2$. **3.** You better believe it.

## Section 19.4

**2.** (a) The change of variables reduces the PDE to the canonical form $u_{\xi\eta} = 0$, with general solution $u(\xi,\eta) = F(\xi) + G(\eta)$ or $u(x,y) = F(x-y) + G(3x-y)$, where $F$ and $G$ are arbitrary twice-differentiable functions. Comparing the PDE with (10) in Section 18.2 gives $A = 1$, $B = 2$, $C = 3$, so $B^2 - AC = 4 - 3 > 0$ and the PDE is

hyperbolic. **3.** (a) The PDE becomes $(a^2+8ab+12b^2)u_{\xi\xi}+(2ac+8ad+8bc+24bd)u_{\xi\eta}+(c^2+8cd+12d^2)u_{\eta\eta}=0$. Setting $a^2+8ab+12b^2=0$ gives $a/b=-2$ or $-6$, and setting $c^2+8cd+12d^2=0$ gives $c/d=-2$ or $-6$ (the same as $a/b$ by coincidence). Set $a=-2$, $b=1$, $c=-6$, $d=1$, say. Then $u_{\xi\eta}=0$ and $u(\xi,\eta)=F(\xi)+G(\eta)$ so $u(x,y)=F(-2x+y)+G(-6x+y)$. Comparing the PDE with (10) in Section 18.2 gives $B^2-AC=16-12>0$ so the PDE is hyperbolic. **7.** (a) It is easy to construct particular solutions – such as $-Kx^2/(2c^2)$ and $Kt^2/2$. Choosing the former, we can write the general solution as $y(x,t)=-Kx^2/(2c^2)+F(x-ct)+G(x+ct)$. (b) Imposing the initial conditions, we can solve for $F$ and $G$ but it is simpler to set $y(x,t)=-Kx^2/(2c^2)+v(x,t)$. Then $v$ satisfies the problem $c^2v_{xx}=v_{tt}$; $v(x,0)=p(x)+Kx^2/(2c^2)$, $v_t(x,0)=q(x)$ and we can use the d'Alembert solution (17) to solve for $v(x,t)$. The result is $y(x,t)=-Kx^2/(2c^2)+\frac{1}{2}[P(x-ct)+P(x+ct)]+\frac{1}{2c}\int_{x-ct}^{x+ct}q(x')\,dx'$, where $P(x)=p(x)+Kx^2/(2c^2)$. **8.** $y(x,t)=H(t-\frac{x}{c})h(t-\frac{x}{c})$. NOTE: $y(x,t)=F(x-ct)+G(x+ct)$ gives $y(x,0)=0=F(x)+G(x)$ and $y_t(x,0)=0=-cF'(x)+cG'(x)$. Solving these gives $F(x)=\text{constant}\equiv A$, say, and $G(x)=-A$, so $y(x,t)=A-A=0$. The key step is to realize that the two boundary condition equations hold only for $x>0$, so $F(x)=0$ (we can let $A=0$ without loss) and $G(x)=0$ hold only for $x>0$. Thus, $F(x-ct)=0$ for $x>ct$ (the wedge $0<\theta<\pi/4$ in the first quadrant of the $x,t$ plane) and $G(x+ct)=0$ for $x>-ct$ (the wedge $0<\theta<3\pi/4$), so $y(x,t)=0$ in $0<\theta<\pi/4$ and $y(x,y)=F(x-ct)$ in $\pi/4<\theta<\pi/2$. Then the boundary condition $y(0,t)=h(t)=F(-ct)$ gives $F(arg)=h\left(\frac{arg}{-c}\right)$, where "arg" denotes the argument of $F$, so $y(x,t)=F(x-ct)=h\left(\frac{x-ct}{-c}\right)$ holds in the wedge $\pi/4<\theta<\pi/2$.

**16.** (b) $(Y_{i0}-Y_{i,-1})/\Delta t=g(i\Delta x)$ gives $Y_{i,-1}=Y_{i0}-g(i\Delta x)\Delta t$. Hence, $Y_{1,-1}=0-\left(\sin\frac{\pi}{4}\right)(0.02)=-0.0141$, $Y_{2,-1}=0-\left(\sin\frac{\pi}{2}\right)(0.02)=-0.02$, $Y_{3,-1}=0-\left(\sin\frac{3\pi}{4}\right)(0.02)=-0.0141$. Then $Y_{11}=0.64Y_{00}+0.72Y_{10}+0.64Y_{20}-Y_{1,-1}=0.0141$, $Y_{21}=0.64Y_{10}+0.72Y_{20}+0.64Y_{30}-Y_{2,-1}=0.02$, $Y_{31}=0.64Y_{20}+0.72Y_{30}+0.64Y_{40}-Y_{3,-1}=0.0141$. Similarly, $Y_{12}=0.64Y_{01}+0.72Y_{11}+0.64Y_{21}-Y_{10}=0.0230$, $Y_{22}=0.64Y_{11}+0.72Y_{21}+0.64Y_{31}-Y_{20}=0.0324$, and $Y_{32}=0.64Y_{21}+0.72Y_{31}+0.64Y_{41}-Y_{30}=0.023$

# Chapter 20

## Section 20.2

**1.** (a) $u(x,y)=50\sin(\pi x/3)\left[\cosh(\pi y/3)-\coth(2\pi/3)\sinh(\pi y/3)\right]$. NOTE: Our results can sometimes be simplified using identities such as $\sinh(A\pm B)=\sinh A\cosh B\pm\cosh A\sinh B$ and $\cosh(A\pm B)=\cosh A\cosh B\pm\sinh A\sinh B$. Using the former, we can express our solution, alternatively, as $u(x,y)=50\sin\frac{\pi x}{3}\,\frac{\sinh[(2-y)\pi/3]}{\sinh(2\pi/3)}$.
(d) $u(x,y)=\sum_1^\infty A_n\sin\frac{n\pi x}{3}\left(\sinh\frac{n\pi y}{3}-\tanh\frac{2n\pi}{3}\cosh\frac{n\pi y}{3}\right)$, where (by half-range sine formulas) $u(x,0)=\sum_1^\infty -A_n\tanh\frac{2n\pi}{3}\sin\frac{n\pi x}{3}$ gives $-A_n\tanh\frac{2n\pi}{3}=\frac{2}{3}\int_0^3 u(x,0)\sin\frac{n\pi x}{3}\,dx$ or
$A_n=-\frac{2}{3}\coth\frac{2n\pi}{3}\int_2^3 50\sin\frac{n\pi x}{3}\,dx=\frac{100}{n\pi}\coth\frac{2n\pi}{3}[(-1)^n-\cos\frac{2n\pi}{3}]$
(h) $u(x,y)=\sum_{1,3,\ldots}^\infty A_n\left(\sinh\frac{n\pi x}{4}-\tanh\frac{3n\pi}{4}\cosh\frac{n\pi x}{4}\right)\sin\frac{n\pi y}{4}$, where (by quarter-range sine formulas) $A_n=-\frac{4}{n\pi}\cos\frac{n\pi}{4}\coth\frac{3n\pi}{4}$ (k) $u(x,y)=\frac{5}{3\pi}(\sinh 3\pi x-\tanh 9\pi\cosh 3\pi x)\sin 3\pi y$ **2.** (a) $u(2.5,1)=6.07$ (4 terms), $u(2.5,0.5)=17.43$ (10 terms), $u(2.5,0.2)=33.87$ (27 terms), $u(2.5,0.1)=41.60$ (51 terms). Why are more terms needed as $y\to 0$? When $y=0$ then $u(x,0)=50H(x-2)$ is discontinuous so its Fourier series will converge very slowly indeed. As $y$ increases from 0 the solution $u(x,y)$ becomes a more gradually varying function of $x$, so the series becomes more rapidly convergent. We urge you to use the series to evaluate $u(2.5,0)$, i.e., right *on* the boundary. How many terms are needed to achieve two-decimal-place accuracy? **4.** $u(x,y)=u_1+(u_2-u_1)\frac{y}{b}+\sum_1^\infty(A_n\sinh\frac{n\pi x}{b}+B_n\cosh\frac{n\pi x}{b})\sin\frac{n\pi y}{b}$, where $B_n$ is given by $B_n=\frac{2}{b}\int_0^b[p(y)-u_1-(u_2-u_1)\frac{y}{b}]\sin\frac{n\pi y}{b}\,dy$ and then $A_n$ is given by $A_n\sinh\frac{n\pi a}{b}+B_n\cosh\frac{n\pi a}{b}=\frac{2}{b}\int_0^b[f(y)-u_1-(u_2-u_1)\frac{y}{b}]\sin\frac{n\pi y}{b}\,dy$. Of course, to complete the evaluation we need to specify $p(y)$ and $f(y)$. **6.** Using $+\kappa^2$, in the first part of the solution, your solution should be identical to the solution, given above, to Exercise 4 but with $p(y)$ changed to $u_3$ and $f(y)$ changed to $u_4$.
**10.** (a) $u(x,y)=10-\frac{40}{\pi}\sum_{1,3,\ldots}^\infty\frac{1}{n}\sin\frac{n\pi y}{2}e^{-n\pi x/2}$ **13.** $u(x,y)=20+6x+\sum_1^\infty\sin\frac{n\pi x}{5}(A_ne^{n\pi y/5}+B_ne^{-n\pi y/5})$, where $A_n+B_n=\frac{2}{5}\int_0^5[f(x)-20-6x]\sin\frac{n\pi x}{5}\,dx$. The latter is one equation in the two unknowns $A_n$ and $B_n$, so we

can choose $A_n$ *arbitrarily* and use the equation to solve for $B_n$. And if $A_n \neq 0$ then the $A_n e^{n\pi y/5}$ term is unbounded on the semi-infinite strip. **15.** (a) $u(x,y)$ is given by (15.2), where $U(x,y) = -\frac{fa}{2}x + \sum_1^\infty \sin\frac{n\pi x}{a}(A_n \cosh\frac{n\pi y}{a} + B_n \sinh\frac{n\pi y}{a})$; $A_n = \frac{f}{a}\int_0^a (ax - x^2)\sin\frac{n\pi x}{a}\,dx$ and $B_n = \frac{1-\cosh(n\pi b/a)}{\sinh(n\pi b/a)}A_n$
**19.** (a) $u(x,y) = (A+Bx)(C+Dy) + (E\cosh\kappa x + F\sinh\kappa x)(G\cosh\kappa y + H\sinh\kappa y)$, so $u(x,0) = 0$ gives $C = G = 0$ so $u(x,y) = (P+Qx)y + (R\cosh\kappa x + S\sinh\kappa x)\sin\kappa y$. Then $u(x,3) + 5u_y(x,3) = 0$ gives $P = Q = 0$ and $\sin 3\kappa + 5\kappa\cos 3\kappa = 0$ with $R$ and $S$ remaining arbitrary. Thus, $u(x,y) = \sum_1^\infty (R_n\cosh\kappa_n x + S_n\sinh\kappa_n x)\sin\kappa_n y$, where $\kappa_n$'s are the roots of $\tan\kappa = -5\kappa$. Finally, $u(0,y) = 0$ gives $R_n = 0$ and $u(4,y) = 100$ gives $S_n\sinh 4\kappa_n = \frac{\langle 100, \sin\kappa_n y\rangle}{\langle \sin\kappa_n y, \sin\kappa_n y\rangle}$ so $S_n = \frac{400}{\sinh 4\kappa_n}\frac{1-\cos 3\kappa_n}{6\kappa_n - \sin 6\kappa_n}$. [NOTE: The Sturm–Liouville problem is $Y'' + \kappa^2 Y = 0$ on $(0,3)$ with $Y(0) = 0$ and $Y(3) + 5Y'(3) = 0$, so the inner product weight function is 1.] The *Maple* fsolve command gives $\kappa_1 = 0.6266$, $\kappa_2 = 1.6119$, $\kappa_3 = 2.6432$, $\kappa_4 = 3.6833$, $\kappa_5 = 4.7265$ so the first five terms of the series solution are $u(2,1) = 18.6150 + 1.4048 + 0.0650 - 0.0056 - 0.0012 + \cdots = 20.078$. [Drawing the domain and boundary conditions, this value seems reasonable. If, on the other hand, we obtained $u(2,1) = 83$ or 2.7, say, we would surely understand the calculation to be erroneous.]

## Section 20.3

**2.** (a) $u(r,\theta) = 100\theta/\pi + \sum_1^\infty (A_n r^n + B_n r^{-n})\sin n\theta$, where $A_n = \frac{200}{n\pi}\frac{2^{-n}+(-1)^{n+1}}{2^{-n}-2^n}$ and $B_n = \frac{200}{n\pi}\frac{(-1)^n - 2^n}{2^{-n}-2^n}$
(c) $u(r,\theta) = 100\theta/\pi + \frac{200}{\pi}\sum_1^\infty \frac{1}{n}\frac{(-1)^n}{2^n+2^{-n}}(r^n + r^{-n})\sin n\theta$ (e) $u(r,\theta) = 100\frac{\ln r}{\ln 2}$
(g) $u(r,\theta) = 100 - \frac{400}{\pi}\sum_{1,3,\ldots}^\infty \frac{\sin(n\pi/2)}{n}\left(\frac{r}{3}\right)^{n/3}\cos\frac{n\theta}{3}$ NOTE: Applying the condition $u(3,\theta) = 0$ last gives $-100 = \sum_{1,3,\ldots}^\infty A_n 3^{n/3}\cos\frac{n\theta}{3}$. Re-expressing the $\cos(n\theta/3)$ as $\cos\frac{n\pi\theta}{3\pi}$ we can identify "$2L$" $= 3\pi$, so "$L$" $= 3\pi/2$ in the quarter-range cosine formulas. **3.** (a) $u(r,\theta) = 50 + 20r\cos\theta$ **4.** Boundedness and the conditions at $\theta = 0$ and $\theta = 2\pi$ give $u(r,\theta) = A + \sum_1^\infty B_n b^{n/2}\cos\frac{n\theta}{2}$. Finally, $u(r,\theta) = 50 + 50\sin\theta = A + \sum_1^\infty B_n b^{n/2}\cos\frac{n\theta}{2}$ gives (by the half-range cosine formulas) $A$ and $B_n$. The result is $u(r,\theta) = 50 + \frac{400}{\pi}\sum_{1,3,\ldots}^\infty \frac{1}{4-n^2}\left(\frac{r}{b}\right)^{n/2}\cos\frac{n\theta}{2}$. No, the solution does not depend on the material. The unsteady diffusion equation $\alpha^2\nabla^2 u = u_t$ does contain the diffusivity $\alpha^2$ of the material but in steady state $u_t = 0$ and the $\alpha^2$ cancels out. **16.** (a) $u(r,z) = (A + B\ln r)(E + Fz) + [CJ_0(\kappa r) + DY_0(\kappa r)](Ge^{\kappa z} + He^{-\kappa z})$. Boundedness gives $u(r,z) = P + QJ_0(\kappa r)e^{-\kappa z}$, and $u(b,z) = 0$ gives $P = 0$ and $J_0(\kappa b) = 0$ or $\kappa = z_n/b \equiv \kappa_n$ $(n = 1, 2, \ldots)$, where $z_n$ is the $n$th positive root of $J_0(x) = 0$. Finally, $u(r,z) = \sum_1^\infty Q_n J_0(\kappa_n r)e^{-\kappa_n z}$ where $Q_n = \frac{2}{b^2[J_1(\kappa_n b)]^2}\int_0^b f(r)J_0(\kappa_n r)r\,dr$. (b) $u(r,z) = (A + B\ln r)(E + Fz) + [CI_0(\kappa r) + DK_0(\kappa r)](G\cos\kappa z + H\sin\kappa z)$. Boundedness as $r \to 0$ gives $B = D = 0$ and boundedness as $z \to \pm\infty$ gives $F = 0$, so $u(r,z) = P + I_0(\kappa r)(Q\cos\kappa z + R\sin\kappa z)$. We can superimpose any number of product solutions for different $\kappa$'s so let us anticipate the Fourier series expansion of $u(b,z)$ and write $u(r,z) = P + \sum_1^\infty I_0\left(\frac{n\pi r}{L}\right)\left(Q_n\cos\frac{n\pi z}{L} + R_n\sin\frac{n\pi z}{L}\right)$. Then $u(b,z) = P + \sum_1^\infty I_0\left(\frac{n\pi b}{L}\right)\left(Q_n\cos\frac{n\pi z}{L} + R_n\sin\frac{n\pi z}{L}\right)$ gives (from the formulas for the Fourier series of a periodic function, with period $2L$) $P, Q_n$, and $R_n$. The result is $u(r,z) = 50 + \frac{200}{\pi}\sum_{1,3,\ldots}^\infty \frac{1}{n}\frac{I_0(n\pi r/L)}{I_0(n\pi b/L)}\sin\frac{n\pi z}{L}$. **19.** (a) $u(\rho,\phi) = \sum_0^\infty A_n\rho^n P_n(\cos\phi)$, $u(\rho,\pi/2) = 0 = \sum_0^\infty A_n\rho^n P_n(0)$ so we need $A_n P_n(0) = 0$ for each $n = 0, 1, 2, \ldots$. But $P_n(0) = 0$ for $n = 1, 3, \ldots$ so we learn that $A_0, A_2, A_4, \ldots$ are zero while $A_1, A_3, \ldots$ remain arbitrary. Thus far, then, $u(\rho,\phi) = \sum_{1,3,\ldots}^\infty A_n\rho^n P_n(\cos\phi)$. Finally, $u(c,\phi) = 100 = \sum_{1,3,\ldots}^\infty A_n c^n P_n(\mu)$ so $A_n c^n = \frac{\langle 100, P_n(\mu)\rangle}{\langle P_n(\mu), P_n(\mu)\rangle} = \frac{\int_0^1 100P_n(\mu)\,d\mu}{\int_0^1 P_n^2(\mu)\,d\mu} = 100(2n+1)\int_0^1 P_n(\mu)\,d\mu$. Evaluating these integrals gives $A_1 = 300/(2c)$, $A_3 = -700/(8c^3)$, $A_5 = 1100/(16c^5)$, $A_7 = -1500[5/(128c^7)]$, $A_9 = 1900[7/(256c^9)], \ldots$, so $u(\rho,\mu) = 150\left(\frac{\rho}{c}\right)P_1(\mu) - \frac{175}{2}\left(\frac{\rho}{c}\right)^3 P_3(\mu) + \frac{275}{4}\left(\frac{\rho}{c}\right)^5 P_5(\mu) - \frac{1875}{32}\left(\frac{\rho}{c}\right)^7 P_7(\mu) + \frac{3325}{64}\left(\frac{\rho}{c}\right)^9 P_9(\mu) - \cdots$. As a partial check let us compute $u$ on the $z$ axis (so $\phi = 0$, $\mu = 1$) at $\rho = c/2$. Since $P_n(1) = 1$, we have $u(c/2, \mu = 1) = 75 - 10.94 + 2.15 - 0.46 + 0.10 - \cdots \approx 65.85$ which looks reasonable since the point is roughly midway between the flat bottom (on which $u = 0$) and the hemispherical top (on which $u = 100$), and the surface area of the hemisphere is twice that of the flat bottom. (b) $u(\rho,\phi) = \sum_0^\infty A_n\rho^n P_n(\cos\phi)$. On $\phi = \pi/2$ the unit outward normal is $\hat{\mathbf{n}} = \hat{\mathbf{e}}_\phi$ so $u_n = \nabla u\cdot\hat{\mathbf{n}} = \nabla u\cdot\hat{\mathbf{e}}_\phi = \frac{1}{\rho}\frac{\partial u}{\partial\phi}$. Thus, $u_n = 0$ on $\phi = \pi/2$ gives $u_\phi = 0$ there, so $u_\phi(\rho,\pi/2) = 0 = \sum_0^\infty A_n\rho^n P_n'(\cos\frac{\pi}{2})(-\sin\frac{\pi}{2}) = -\sum_0^\infty A_n P_n'(0)\rho^n$. Since

$P_n'(0) = 0$ for even $n$'s and $P_n'(0) \neq 0$ for odd $n$'s, it follows that $A_1 = A_3 = \cdots = 0$ with $A_0, A_2, A_4, \ldots$ remaining arbitrary, so $u(\rho, \phi) = \sum_{0,2,\ldots}^{\infty} A_n \rho^n P_n(\cos\phi)$. Finally, $u(c, \phi) = 100 = \sum_{0,2,\ldots}^{\infty} A_n c^n P_n(\mu)$ gives $A_n = 100 \frac{2n+1}{c^n} \int_0^1 P_n(\mu)\, d\mu$. Integration gives $A_0 = 100$, $A_2 = A_4 = \cdots = 0$ so the solution is simply the leading term, $u(\rho, \phi) = 100$, which does indeed satisfy the PDE and boundary conditions.

## Section 20.4

**1.** (a) $u(x, y) = (100y/\pi) \int_0^\infty d\xi/[(\xi - x)^2 + y^2] = 50 - \frac{100}{\pi} \tan^{-1}\left(-\frac{x}{y}\right)$ where $\tan^{-1}(\,)$ is to be taken as the "principal value," which is an odd function of its argument and which varies continuously from $-\pi/2$ at $(\,) = -\infty$, through 0 at $(\,) = 0$, to $+\pi/2$ at $(\,) = +\infty$. Check: If $x > 0$ and $y \to 0$ through positive values then $u \to 50 - (100/\pi)\tan^{-1}(-\infty) = 50 + 50 = 100$, and if $x < 0$ and $y \to 0$ through positive values then $u \to 50 - (100/\pi)\tan^{-1}(\infty) = 0$, in agreement with the boundary conditions. NOTE: This problem is solved most easily using separation of variables in polar coordinates (the region being $0 < r < \infty$, $0 < \theta < \pi$) for in the solution form $u(r, \theta) = (A + B \ln r)(C + D\theta) + (Er^\kappa + Fr^{-\kappa})(G \cos\kappa\theta + H \sin\kappa\theta)$ boundedness as $r \to 0$ and as $r \to \infty$ gives $B = E = F = 0$ so $u = P + Q\theta$. Then $u(r, 0) = 100 = P$ and $u(r, \pi) = 0 = P + \pi Q$ give $u(r, \theta) = 100(1 - \frac{\theta}{\pi})$. If we write the latter as $u(r, \theta) = 100(1 - \frac{1}{\pi}\tan^{-1}\frac{y}{x})$ it doesn't look the same as our earlier solution but it must be remembered that $\tan^{-1}$ in the first solution is the branch of $\tan^{-1}$ lying between $-\pi/2$ and $\pi/2$, whereas $\tan^{-1}$ in the second solution is the branch lying between 0 and $\pi$ (because $0 < \theta < \pi$).

## Section 20.5

**1.** (a) $\alpha = \beta = \gamma = \delta = 1$ except at $c$, where $\alpha = \sqrt{12} - 3$ and $\delta = 3/4$; at $e$, where $\alpha = \sqrt{8} - 2$; and at $g$, where $\delta = 3/4$. Results: $u_a = 14.77$, $u_b = 18.24$, $u_c = 19.76$, $u_d = 13.12$, $u_e = 17.97$, $u_f = 14.58$ (d) Partial results: $u_a = 37.02$, $u_b = 65.83$, $u_c = 64.89$ (g) Partial results: $u_a = 48.26$, $u_b = 57.45$, $u_c = 45.71$. NOTE: At $a$ we use the average value $U_W = (0 + 50)/2 = 25$. Similarly, at $e$ we use $U_S = 25$. **2.** (b) The solution is symmetric about the lines $x = 0$, $y = 0$, $y = x$, and $y = -x$, so the unknowns reduce to $u(0, 0.8)$, $u(0.2, 0.8)$, $u(0.4, 0.8)$, $u(0.6, 0.8)$, and $u(0.8, 0.8)$, which values are found to be 50.30, 50.60, 52.08, 57.74, and 78.87, respectively. NOTE: From a sketch of the region, the grid, and the boundary conditions, these results look correct. **3.** (b) At $a$, $h = 0.5$, $\alpha = \beta = \gamma = \delta = 1$; at $b$, $h = 1$, $\alpha = 1$, $\beta = \gamma = \delta = 0.5$; at $c$, $h = 1$, $\alpha = \gamma = 1$, $\beta = \delta = 0.5$; and so on. $u_d = 5.051$, $u_c = 0.511$, $u_b = 0.060$, $u_a = 0.016$, $u_e = 0.004$, $u_g = 0.001$, $u_h = u_e$, $u_i = u_a$, $u_j = u_b$, $u_k = u_c$, $u_l = u_d$. Observe that the boundary condition input $u = 50$ at the ends are "felt," to any appreciable degree, only within a couple of end-widths of those ends; $u_c$ is already only 0.511, and $u_b, u_c, \ldots, u_g$ are even smaller. This numerical result further illustrates the idea stated in Comment 3 of Example 1 (Section 20.2). **4.** (a) Partial results: Denoting the centerline points $(0.25, 0.5), (0.5, 0.5), \ldots, (2.5, 0.5)$ as $a, b, c, d, e, f$, respectively, the values obtained there are 53.13, 26.17, 12.40, 5.48, 1.11, 0.08. (b) By comparison, the exact values at those points are 54.47, 26.10, 12.03, 5.50, 1.14, 0.05. **6.** (a) The Taylor expansions of $u(x, y)$ about $x_j, y_k$, first with a step $h$ to the right and then with a step $h$ to the left, are

$$u(x_{j+1}, y_k) = u(x_j, y_k) + u_x(x_j, y_k)h + \frac{1}{2}u_{xx}(x_j, y_k)h^2 + \frac{1}{6}u_{xxx}(x_j, y_k)h^3 + \frac{1}{24}u_{xxxx}(x_j, y_k)h^4 + \cdots,$$
$$u(x_{j-1}, y_k) = u(x_j, y_k) - u_x(x_j, y_k)h + \frac{1}{2}u_{xx}(x_j, y_k)h^2 - \frac{1}{6}u_{xxx}(x_j, y_k)h^3 + \frac{1}{24}u_{xxxx}(x_j, y_k)h^4 - \cdots,$$

Addition gives $u_{xx}(x_j, y_k) = [u(x_{j+1}, y_k) - 2u(x_j, y_k) + u(x_{j-1}, y_k)]/h^2$ plus terms of order $h^2$ and higher. Similarly for $u_{yy}(x_j, y_k)$. When we drop those higher-order terms, in deriving (5), we incur a truncation error that is of order $O(h^2)$. (c) $h = 1/4$ gives $u_{\text{center}} = 21.339$, $h = 1/6$ gives $u_{\text{center}} = 20.570$, and the exact solution is $u_{\text{center}} = 19.927$. Then $19.927 - 21.339 = C\,(1/4)^p$ and $19.927 - 20.570 = C\,(1/6)^p$, which give $p = 1.94$.

# Chapter 21

## Section 21.2

**9.** (a) $2-11i$ (e) $-\frac{26}{125}-\frac{18}{125}i$ **11.** (a) $|z_1+z_2|=|6+2i|=\sqrt{40}=6.32$, $|z_1|+|z_2|=\sqrt{13}+\sqrt{17}=7.73$

## Section 21.3

**2.** (a) $2<u<3$, $1<v<2$ (f) $\frac{u^2}{4}-1<v<1-\frac{u^2}{4}$, $-2<u<0$ **9.** (a) $-e^2$ (d) $\sin 3\cosh\pi+i\cos 3\sinh\pi$ (g) $\frac{2\cosh 1\sin 1}{\cosh 2-\cos 2}+i\frac{2\cos 1\sinh 1}{\cosh 2-\cos 2}$
**16.** (a) $\int_0^\infty e^{-x}\sin\omega x\,dx=\operatorname{Im}\int_0^\infty e^{(i\omega-1)x}\,dx=\operatorname{Im}e^{(i\omega-1)x}/(i\omega-1|_{x=0}^{x=\infty}=\operatorname{Im}[1/(1-i\omega)]=\omega/(1+\omega^2)$
**18.** (a) $x_p(t)=\operatorname{Im}\frac{F_0e^{i\omega t}}{(k-m\omega^2)+i\omega c}=\operatorname{Im}\frac{F_0e^{i\omega t}}{(k-m\omega^2)+i\omega c}\frac{(k-m\omega^2)-i\omega c}{(k-m\omega^2)-i\omega c}=F_0\frac{(k-m\omega^2)\sin\omega t-\omega c\cos\omega t}{(k-m\omega^2)^2+\omega^2c^2}$
(d) $x_p(t)=-\frac{10}{13}(5\sin 5t+\cos 5t)$

## Section 21.4

**1.** (a) $r=3$, $\theta_0=-\pi/2$ rad (f) $r=2\sqrt{37}$, $\theta_0=-1.406$ rad **4.** (a) $(-1+i)^{10}=32e^{i3\pi/2}$ (polar) $=-32i$ (Cartesian); $(-1+i)^{20}=1024e^{15\pi i}$ (polar) $=-1024$ (Cartesian) **5.** (a) $i^{1/2}=e^{i\pi/4},e^{i5\pi/4}$; $i^{1/5}=e^{i\pi/10},e^{i\pi/2},e^{i9\pi/10},e^{i13\pi/10},e^{i17\pi/10}$ **6.** (a) $\log(-2)=\ln 2+(2k+1)\pi i$ $(k=0,\pm1,\pm2,\ldots)$ **8.** (a) $(2i)^{2/3}=\sqrt[3]{4}e^{\pi i/3},\sqrt[3]{4}e^{\pi i},\sqrt[3]{4}e^{5\pi i/3}$; $(2i)^{3/2}=\sqrt{8}e^{3\pi i/4},\sqrt{8}e^{7\pi i/4}$; $(2i)^\pi=e^{\pi\ln 2}e^{i(1+4k)\pi^2/2}$ $(k=0,\pm1,\pm2,\ldots)$
**9.** (a) $(2i)^i=e^{-(\pi/2+2k\pi)}[\cos(\ln 2)+i\sin(\ln 2)]$; $(2i)^{1-i}=e^{\ln 2+\pi/2+2k\pi}[\cos\left(\frac{\pi}{2}+2k\pi-\ln 2\right)$ $+i\sin\left(\frac{\pi}{2}+2k\pi-\ln 2\right)]$ $(k=0,\pm1,\pm2,\ldots)$ **11.** (a) $\log(-3i)=\ln 3-i\frac{\pi}{2}$, $\sqrt{-3i}=\sqrt{\frac{3}{2}}-\sqrt{\frac{3}{2}}\,i$

## Section 21.5

**10.** (a) $f'(z)=-\sin z$ **11.** (d) $f'(z)=-(2z+3i)/(z^2+3iz-2)^2$ for all $z\neq -i,-2i$ so $f(z)$ is analytic for all $z\neq -i,-2i$ **15.** (a) Harmonic for all $z$: $v(x,y)=e^x\sin y+c$; $f(z)=e^z+\text{constant}$

# Chapter 22

## Section 22.2

**5.** (a) $e^z$ analytic at $z=0$ and $\frac{d}{dz}e^z=e^z\neq 0$ there, so yes, conformal at $z=0$ (d) $iz^2$ analytic at $z=0$ but $\frac{d}{dz}(iz^2)=2iz=0$ there, so no, not conformal at $z=0$

## Section 22.3

**9.** (a) Let $z_1=i$, $z_2=0$, $z_3=-i$ and $w_1=2i$, $w_2=2$, $w_3=-2i$. Then (6.1) gives $w=\frac{2z+2}{-z+1}$. Note that this mapping must send the half plane $x<0$ (rather than $x>0$) into the interior $|w|<2$ because as we walk along the boundary curve $x=0$ from $z_1$ to $z_2$ to $z_3$ the region is on our right, so when we walk from $w_1$ to $w_2$ to $w_3$ the image region will again be our right. To obtain a second such mapping we could change $z_1$ to $5i$, say, and keep $z_2,z_3,w_1,w_2,w_3$ unchanged. (b) See the answer to part (a). To map onto the exterior $|w|>2$ we can keep the $z_j$'s unchanged and reverse the sequence of $w_j$'s: $w_1=-2i$, $w_2=2$, $w_3=2i$. **10.** (a) $z=0\to w=i$, $z=1\to w=1+i$, $z=\infty\to w=1$ so the image is the interior of the circle through those three $w$ points. To see that it is the interior and not the exterior we can use the idea presented in Exercise 8 or we can simply check one point: e.g., $z=i$ is outside of $\mathcal{D}$, so its image $w=\infty$ must be outside of $\mathcal{D}'$. (e) The image is the interior of a "crescent" with vertices at $w=1$ and $w=i$. One arc of the crescent passes through $w=1+i$ and the other passes through $w=2+i$. **11.** (a) The image $\mathcal{D}'$ is bounded by a straight line from $0+i\infty$ down to $i$, then a semicircle from $i$ to $(1+i)/2$ to 0, then a straight line from 0 to $0-i\infty$ and is the region to the right of that boundary. (d) $\mathcal{D}'$ is the $90^\circ$ sector with vertex at $w=2$, with one edge running from 2 to $2-i\infty$ and the other edge running from 2

to $-\infty$. **14.** (a) The mapping $w = (z-4)/(z+4)$ sends $z = -4$ to $w = \infty$ and $z = 4$ to $w = 0$. The image $\mathcal{D}'$ is the wedge with vertex at $w = 0$ with one edge from $w = 0$ to infinity through $w = (3+4i)/5$ and with the other edge from $w = 0$ to infinity through $w = (3-4i)/5$. Then $\Psi = A\phi + B$. $\Psi = 0$ on $\phi = -\alpha$ and $\Psi = 100$ on $\phi = +\alpha$ (where $\alpha = \tan^{-1}\frac{4}{3}$) gives $\Psi = 50(1 + \frac{\phi}{\alpha})$. But $\phi = \tan^{-1}(v/u) = \tan^{-1}[8y/(x^2+y^2-16)]$ because $u = (x^2+y^2-16)/[(x+4)^2+y^2]$ and $v = 8y/[(x+4)^2+y^2]$, so $\psi(x,(x,y) = 50\left(1 + \frac{\tan^{-1}[8y/(x^2+y^2-16)]}{\tan^{-1}(4/3)}\right)$, where $\tan^{-1}$'s are between $-\pi/2$ and $\pi/2$. Let us check as follows: At $x = 0$ and $y = 10$ we have $\psi(0,10) = 50(1 + \frac{0.761}{0.927}) = 91.0$, which seems reasonable. Further, as we approach infinity along any ray $y = mx$, $\psi \to 50$, which is also correct. (b) In entry 4, let $x_2 = 2$ and let $x_1 \to +\infty$. Then $a \sim (2x_1 + x_1\sqrt{3})/x_1 \to 2+\sqrt{3}$ and $R \sim (2x_1 - x_1\sqrt{3})/x_1 \to 2-\sqrt{3}$. $\Psi = A + B\ln\rho$ and the boundary conditions give $\Psi = 20 + 10\frac{\ln\rho}{\ln R}$ or $20 + 5\frac{\ln\rho^2}{\ln R}$. We can find $u(x,y)$ and $v(x,y)$ and then put $\rho^2 = u^2(x,y) + v^2(x,y)$ but it is simpler to write $\rho^2 = |w|^2 = \left|\frac{z-a}{az-1}\right|^2 = \left|\frac{(x-a)+iy}{(ax-1)+iay}\right|^2 = \frac{|(x-a)+iy|^2}{|(ax-1)+iay|^2}$, so $\psi(x,y) = 20 + \frac{5}{\ln R}\ln\frac{(x-a)^2+y^2}{(ax-1)^2+a^2y^2}$, where $R = 2-\sqrt{3}$ and $a = 2+\sqrt{3}$. As a check, let us compute $\psi(1.5,0) = 25.49$, which looks good. [Do you see why $\psi(1.5,0)$ should be just be slightly greater than 25?] (c) $w = \frac{z+3}{z-3}$ will map $\mathcal{D}$ into the strip $-1 < u < 0$.

## Section 22.4

**4.** (a) $\mathcal{D}'$ is the upper half plane and $\Psi = 0$ on the entire $u$ axis except on $0 < u < 1$, where $\Psi = 20$. Then $\Psi(u,v) = \frac{v}{\pi}\int_0^1 \frac{20\,du'}{(u'-u)^2+v^2} = \frac{20}{\pi}\left(\tan^{-1}\frac{u}{v} - \tan^{-1}\frac{u-1}{v}\right)$. But $w = u + iv = e^z = e^x(\cos y + i\sin y)$ so $u = e^x\cos y$, $v = e^x\sin y$, and $\psi(x,y) = \frac{20}{\pi}\left[\tan^{-1}(\cot y) - \tan^{-1}\left(\frac{e^x\cos y - 1}{e^x\sin y}\right)\right]$ where the $\tan^{-1}$'s are between $-\pi/2$ and $+\pi/2$. As a check, observe that on the midline of the strip $\psi(x,\pi/2) = \frac{20}{\pi}[\tan^{-1}0 - \tan^{-1}(-e^{-x})] = \frac{20}{\pi}\tan^{-1}e^{-x}$ does tend to 10 as $x \to -\infty$ (as it should) and to 0 as $x \to +\infty$ (as it should). (c) (12) in Section 20.4 gives $\Psi(u,v) = \frac{35}{2} - \frac{35}{\pi}\tan^{-1}(\frac{u+4}{v})$ so $\psi(x,y) = \frac{35}{2} - \frac{35}{\pi}\tan^{-1}(\frac{x^2-y^2+4}{2xy})$ where $-\pi/2 < \tan^{-1}(\,) < \pi/2$. $\psi(2,2) = 12.33$ (which looks reasonable). (g) $\Psi(u,v) = 200 + \frac{400}{\pi}\tan^{-1}(\frac{u-1}{v})$ so $\psi(x,y) = 200 - \frac{400}{\pi}\tan^{-1}(\frac{\cos\pi x\cosh\pi y + 1}{\sin\pi x\sinh\pi y})$ where $-\pi/2 < \tan^{-1}(\,) < \pi/2$. Note that $\psi(0.5,y) \to 200$ as $y \to \infty$, as it should. Also, $\psi(0.5,0.1) = 39.4$.
(j) $\psi(x,y) = 30 - 5(x^2-y^2)/2$; $\psi(1,0) = 27.5$, $\psi(0.6,-0.4) = 29.5$

## Section 22.5

**3.** (a) $\mathcal{D}'$ is the wedge $0 < \phi < 3\pi/2$ (i.e., the first three quadrants of the $w$ plane); $\Psi(0,v) = \sin[\ln(-v)]$; $\Psi_N(u,0) = \frac{1}{|\exp(z)|}\psi_n(x,0) = (1/e^x)(-5) = -5/u$. Since $\mathcal{D}'$ is better suited to the polar coordinates $\rho,\phi$, let us restate the mapped boundary conditions in terms of $\rho,\phi$: $\Psi(\rho,3\pi/2) = \sin(\ln\rho)$ along $\phi = 3\pi/2$, and $\Psi_\phi(\rho,0) = 5$ along $\phi = 0$. To obtain the former, recall the directional derivative formula $\Psi_N = \hat{\mathbf{e}}_N\cdot\nabla\Psi$. On $\phi = 0$, $\hat{\mathbf{e}}_N$ is $-\hat{\mathbf{e}}_\phi$ so $\Psi_N = -\hat{\mathbf{e}}_\phi\cdot\nabla\Psi = -\frac{1}{\rho}\Psi_\phi(\rho,0)$ and setting this equal to $-5/\rho$ gives $\Psi_\phi(\rho,0) = 5$. (d) $\mathcal{D}'$ is the quarter of the unit disk $0 < \rho < 1$, $0 < \phi < \pi/2$. On $\phi = 0$, $\Psi = 5e^x = 5u$; on $\rho = 1$, $\Psi_N = 2y = 2\cos^{-1}u$; on $\phi = \pi/2$, $\Psi_N = 5/e^x = 5/v$. Or, in terms of polar variables $\rho,\phi$, $\Psi(\rho,0) = 5\rho$, $\Psi_\rho(1,\phi) = 2\phi$, $\Psi_\phi(\rho,\pi/2) = 5$. (g) $\mathcal{D}'$ is the unit disk $\rho < 1$. The negative $y$ axis maps to the lower semicircle so, in terms of the polar coordinates $\rho,\phi$, $\Psi(1,\phi) = 50$ on $\pi < \phi < 2\pi$. The positive $y$ axis maps to the upper semicircle so, since $y = 2v/[(1-u)^2+v^2] = v/(1-u) = \sin\phi/(1-\cos\phi)$, $\Psi(1,\phi) = 50\exp[-\sin\phi/(1-\cos\phi)]$ on $0 < \phi < \pi$. (j) $\mathcal{D}'$ is bounded below by $v = 0$ $(-\infty < u < 1)$ and on the right by $u = 1$ $(0 < v < \infty)$. The mapped boundary conditions are: $\Psi(u,0) = \sin\sqrt{-u}$ on $-\infty < u < 0$; $\Psi_v(u,0) = -1/(2\sqrt{u})$ on $0 < u < 1$; $\Psi(1,v) = 2\exp[-\sqrt{\sqrt{1+v^2} - 1/\sqrt{2}}]$ since $v = 2\sqrt{1+y^2}\,y$ gives $y = \sqrt{\sqrt{1+v^2} - 1/\sqrt{2}}$ by squaring and using the quadratic formula.

## Section 22.6

**2.** (a) The mapping $\zeta = z^2$ gives the flow $W = U\zeta^2$ in the $\zeta$ plane and, hence, $w = U(z^2)^2 = Uz^4$ in the $z$ plane. $u(x,y) = 4Ux(x^2-3y^2)$, $v(x,y) = 4Uy(y^2-3x^2)$, $\phi(x,y) = U[(x^2-y^2)^2 - 4x^2y^2]$, $\psi(x,y) = 4Uxy(x^2-y^2)$.

$u(0,0) = v(0,0) = 0$ so the corner is a stagnation point. (b) The mapping $\zeta = z^{2/3}$ gives the horizontal flow $dW/d\zeta = U$, $W = U\zeta$, $w = Uz^{2/3}$, $dw/dz = u - iv$ gives $u(r,\theta) = \frac{2U}{3r^{1/3}}\cos\frac{\theta}{3}$, $v(r,\theta) = \frac{2U}{3r^{1/3}}\sin\frac{\theta}{3}$. Rather than the origin being a stagnation point, $u \to \infty$ and $v \to \infty$ as $r \to 0$. (More generally, inside corners are stagnation points; outside corners are singular points.) **4.** (a) It is the flow given in Example 1, for any $U_0 > 0$.

# Chapter 23

## Section 23.2

**1.** (a) $\frac{2}{3}(1+i)$ (d) $-\pi i$ **3.** (a) Maximum $|z|$ on $C$ is $\sqrt{8}$, and $L = \sqrt{5}$, so $|I| \le (\sqrt{8})^5\sqrt{5} = 128\sqrt{10}$ (b) $|e^z| = |e^{x+iy}| = |e^x||e^{iy}| = e^x \le e$ on $C$, and $L = 3\sqrt{2}$, so $|I| \le 3\sqrt{2}e$ (c) $|e^{-z}| = e^{-x} \le e^2$ on $C$, and $L = 3\sqrt{2}$, so $|I| \le 3\sqrt{2}e^2$ (d) $\max|1/z| = 1/\min|z| = 1/(4-\sqrt{5})$, and $L = 8\pi$, so $|I| \le 8\pi/(4-\sqrt{5})$

## Section 23.3

**4.** (a) $\oint_{C_1} x\,dz = \int x\,d(e^{i\theta}) = i\int xe^{i\theta}\,d\theta = i\int_0^{2\pi}\cos\theta(\cos\theta + i\sin\theta)\,d\theta = \pi i$ (d) 0 by Cauchy's theorem since $\pm\sqrt{3}$ lie outside of $C_3$ (g) $-2\pi i/5$
(j) $\oint_{C_3}\frac{dz}{\sqrt{x^2+y^2}} = \int_{-1}^{1}\frac{i\,dy}{\sqrt{1+y^2}} + \int_1^{-1}\frac{dx}{\sqrt{x^2+1}} + \int_1^{-1}\frac{i\,dy}{\sqrt{1+y^2}} + \int_{-1}^{1}\frac{dx}{\sqrt{x^2+1}} = 0$
**6.** (a) Since $z^{20}$ is analytic everywhere we can deform the contour to a straight line. Thus, $\int_C z^{20}\,dz = \int_0^1 x^{20}\,dx = 1/21$ **9.** (a) $I = \oint_C \frac{dz}{z-1} - \oint_C \frac{dz}{z} = 2\pi i - 2\pi i = 0$ (d) $I = -\oint_C \frac{dz}{z-1} + 2\oint_C \frac{dz}{z-2} = -2\pi i + 2(2\pi i) = 2\pi i$

## Section 23.4

**2.** (a) $z^2/2$, $z^2/2+6$, $z^2/2+3-7i$ **3.** (a) $\int_0^i z\,dz = \frac{z^2}{2}\big|_0^i = -\frac{1}{2}$ (d) $\int_i^0 \cos 3z\,dz = \frac{\sin 3z}{3}\big|_i^0 = -\frac{\sin 3i}{3} = -\frac{i}{3}\sinh 3$ (g) $(e^{-9}-1)/2$ **4.** $(2n+\frac{1}{2})\pi i$

## Section 23.5

**1.** (a) $2\pi i$ (d) $-4\pi i\sin 1$ (g) $\pi i(\sin 3 - 3\cos 3)$ (j) 0

# Chapter 24

## Section 24.2

**5.** (a) Ratio test: $L = 1/\sqrt{5} < 1$ so convergent (d) $c_n \to 1$ as $n \to \infty$ so (Theorem 24.2.2) divergent (g) $|c_n| = 1$ for all $n$, so $c_n$ does not tend to 0 as $n \to \infty$; hence, divergent (Theorem 24.2.2) **6.** (a) $|z| < 1$ (d) $|z+i| < 1/e$ (g) $|z| < 1$ **8.** (a) The series converges in $|z-1| < 1$ so (Theorem 24.2.8) it is the Taylor series of the sum function, and it represents the sum function, in that disk. (d) Yes, it is the Taylor series of $1+z^3$ (about $z=0$) and it represents the function in $|z| < \infty$. **9.** (a) 1 (b) $\sqrt{10}$ **10.** (a) $\sqrt{4-\sqrt{6}}$, namely, the distance from the origin to the closer of the two roots of $z^2 - 2z + 3i + 1 = 0$ **11.** (a) $\sin z = z - \frac{1}{3!}z^3 + \frac{1}{5!}x^5 - \cdots, R = \infty$ (b) $\sin z = \sin a + (\cos a)t - \frac{\sin a}{2!}t^2 - \frac{\cos a}{3!}t^3 + \frac{\sin a}{4!}t^4 + \cdots, R = \infty$, where $t = z - a$, $\sin a = \sin(2-i) = \sin 2\cos i - \sin i\cos 2 = \sin 2\cosh 1 - i\sinh 1\cos 2$, $\cos a = \cos 2\cosh 1 + i\sin 2\sinh 1$ (d) Let $z^6 = w$. $e^{z^6} = e^w = 1 + w + \frac{1}{2!}w^2 + \frac{1}{3!}w^3 + \cdots = 1 + z^6 + \frac{1}{2!}z^{12} + \frac{1}{3!}z^{18} + \cdots, R = \infty$ **14.** (a) $f = z^{1/2}$, $f' = \frac{1}{2}z^{-1/2}$, $f'' = -\frac{1}{4}z^{-3/2}$, $f''' = \frac{3}{8}z^{-5/2}, \ldots$ so $z^{1/2} = 1^{1/2} + \frac{1}{2}1^{-1/2}(z-1) - \frac{1}{4(2!)}1^{-3/2}(z-1)^2 + \frac{3}{8(3!)}1^{-5/2}(z-1)^3 + \cdots = 1^{1/2}\left[1 + \frac{1}{2}(z-1) - \frac{1}{4(2!)}(z-1)^2 + \frac{3}{8(3!)}(z-1)^3 - \cdots\right]$ (in $|z-1| < 1$), where, according to the branch cut, the $1^{1/2}$ factor is $+1$. (c) $z^{1/2} = i^{1/2} + \frac{1}{2}i^{-1/2}(z-1) - \frac{1}{4(2!)}i^{-3/2}(z-1)^2 + \frac{3}{8(3!)}i^{-5/2}(z-1)^3 - \cdots = i^{1/2}\left[1 + \frac{1}{2i}(z-1) - \frac{1}{4(2!)i^2}(z-1)^2 + \frac{3}{8(3!)i^3}(z-1)^3 - \cdots\right] = \frac{1+i}{\sqrt{2}}\left[1 - \frac{i}{2}(z-1) + \frac{1}{4(2!)}(z-1)^2\right.$

$+\frac{3i}{8(3!)}(z-1)^3 - \cdots\Big]$ in $|z-i|<1$ **16.** (a) $\tan z = z + \frac{1}{3}z^3 + \frac{2}{15}z^5 + \frac{17}{315}z^7 + \cdots$ in $|z| < \pi/2$ (d) $1 - z - z^2 + 5z^3 - 7z^4 - z^5 + 23z^6 - \cdots$ in $|z| < 1/\sqrt{3}$ (g) $\frac{1}{2} + \frac{1}{4}z + \frac{1}{8}z^2 + \frac{1}{48}z^3 - \frac{1}{96}z^4 - \frac{13}{960}z^5 - \cdots$. To determine the region of convergence, we seek the root of $2 - \sin z = 0$ that is closest to the origin. We could write $2 - \sin(x+iy) = 0$, put in $\sin(x+iy) = \sin x \cos iy + \sin iy \cos x = \sin x \cosh y + i \sinh y \cos x$, equate real and imaginary parts to 0 and solve the resulting two equations for $x$ and $y$. But it is simpler to write $(e^{iz} - e^{-iz})/(2i) = 2$ or $t - 1/t = 4i$ where $t = e^{iz}$. Then $t^2 - 4it - 1 = 0$ gives $t = (2 \pm \sqrt{3})i$, so $iz = \log[(2 \pm \sqrt{3})i] = \ln(2 \pm \sqrt{3}) + i\left(\frac{\pi}{2} + 2n\pi\right)$ for $n = 0, \pm 1, \pm 2, \ldots$. Hence, $z = \frac{\pi}{2} + 2n\pi - i\ln(2 \pm \sqrt{3}) = \frac{\pi}{2} + 2n\pi \pm 1.317i$. The closest of these to the origin corresponds to $n = 0$ and gives the radius of convergence as $\sqrt{(\pi/2)^2 + (1.317)^2} = 2.05$.

## Section 24.3

**4.** (a) $\frac{1}{z} = \sum_0^\infty \frac{(-i)^n}{(z-i)^{n+1}}$ (d) $\frac{1}{e^z - 1} = \frac{1}{z} - \frac{1}{2} + \frac{1}{12}z + \cdots$ (h) $\frac{1}{z^2} = \sum_0^\infty \frac{(n+1)i^n}{(z+1)^{n+2}}$ **5.** (a) $\sin\frac{1}{z} = \sum_0^\infty \frac{(-1)^n}{(2n+1)!}\frac{1}{z^{2n+1}}$ in $0 < |z| < \infty$ (b) Taylor series in $|z+2| < 2$: $\frac{1}{z} = -\frac{1}{2}\sum_0^\infty \left(\frac{z+2}{2}\right)^n$. Laurent series in $2 < |z+2| < \infty$: $\frac{1}{z} = \sum_0^\infty \frac{2^n}{(z+2)^{n+1}}$ **6.** $f(z) = \frac{1}{z^2}\left(1 + \frac{1}{z} + \frac{1}{z^2} + \cdots\right) = \frac{1}{z^2}\frac{1}{1-1/z} = \frac{1}{z^2 - z}$ so $f(2i) = -\frac{1}{5} + \frac{i}{10}$.

## Section 24.4

**2.** (a) First-order zeros at $z = 0$, $z = 1$ (d) First-order zero at $z = 0$, second-order zeros at $z = (2n+1)\frac{\pi}{2}$ for $n = 0, \pm 1 \pm 2, \ldots$ **3.** (a) Second-order pole at $z = 0$ (d) First-order pole at $z = -2$ (g) Second-order pole at $z = 0$, third-order poles at $z = n\pi$ for $n = \pm 1, \pm 2, \ldots$ (j) Clearly analytic for all $z \neq 0$. For $z = 0$ expand $\sinh t$ in a Taylor series in $t$, $\sinh t = t + \frac{1}{3!}t^3 + \frac{1}{5!}t^5 + \cdots$ for $|t| < \infty$, so $\sinh\frac{1}{z} = \frac{1}{z} + \frac{1}{3!}\frac{1}{z^3} + \frac{1}{5!}\frac{1}{z^5} + \cdots$ in $|1/z| < \infty$, namely, in the annulus $0 < |z| < \infty$. Essential singularity at $z = 0$. **4.** (a) No, essential singularity there (d) Yes (g) No, essential singularity there (j) Yes **5.** (a) First-order pole. The key is to realize that the series does not converge in $0 < |z| < R$ for some $R$, but in $1 < |z| < \infty$. Thus, we cannot use the series to classify the singularity (if any) at $z = 0$. Rather, proceed as in Exercise 6 of Section 24.3.

## Section 24.5

**1.** (a) 0 (d) $12\pi i$ (e) $\pi i$ **2.** (a) $\pi\sqrt{2}/(4a^3)$ (d) $\pi/4$ (g) $3\pi/(4e^2)$ **3.** (a) $\pi/2$ (d) $\pi/4$ (g) $5\pi/16$ (j) $\pi/\sqrt{2}$ **6.** (a) $\pi/\sin(\pi a)$ (c) $3\pi/(4\sqrt{2})$ (d) 0 **9.** (a) $2\pi/(3\sqrt{3})$

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