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*Implementacja algorytmów dla systemów Monroe i  
Chamberlina-Couranta z nieliniową funkcją satysfakcji*

*Implementation of algorithms for Monroe and Chamberlin-Courant  
systems under nonlinear satisfaction function*

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*Oświadczamy, świadomi odpowiedzialności karnej za poświadczenie nieprawdy, że niniejszą pracę dyplomową wykonaliśmy osobiście i samodzielnie (w zakresie wyszczególnionym we wstępie) i że nie korzystaliśmy ze źródeł innych niż wymienione w pracy.*

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# 1. Introduction

We study the effectiveness of algorithms for approximate winner determination under the Monroe and Chamberlin-Courant multiwinner voting rules using nonlinear satisfaction function. Both of the rules aim to select a group of candidates that best represent the voters. Having good voting rules and algorithms for them is important, as multiwinner elections are used both in human societies (e.g. parliament elections) and software systems (e.g. recommendation systems). Rules studied in this paper are particularly interesting because they have both desired features of multiwinner rules: they provide accountability (there is connection between the elected candidates and the voters, so each voter has a representative assigned to her and each candidate knows who she represents) and proportional representation of the voters' views.

We assume that candidates participate in the election with multiple winners (a committee with multiple members is selected) and they are elected by voters, each of whom ranks all the candidates (each voter provides a linear order over the set of candidates expressing their preferences). For each voter the Monroe and Chamberlin-Courant rules assign a single candidate as their representative (with some constraints, which are detailed in the further part of the thesis).

The candidates are selected and assigned to the voters optimally, either by maximizing the total satisfaction of all voters, or by minimizing the total dissatisfaction of all voters. The total satisfaction is calculated as a sum of individual satisfactions of the voters. We assume that there is a satisfaction function that measures how well a voter is represented by the candidate. The function is the same for each voter. It is a decreasing function, so a voter is more satisfied if the candidate assigned to her is ranked higher. The dissatisfaction is calculated in the similar way, except that the function is an increasing one. In this paper I study cases in which the satisfaction function is a nonlinear one.

The main drawback of the aforementioned rules is that election winner determination is NP-hard under each of them [1] which makes them hard to use in practice, as it would force the use of algorithms that don't provide an optimal result for every data set. Therefore, using these systems for real-life elections may rise some difficulties. However, they can be used for the recommendation systems conveniently, as a good but not optimal recommendation is still useful. Skowron et al. [2] provided approximation algorithms for both the rules that return near-optimal results for various test cases (including real-life data and synthetic data), but for linear satisfaction function only.

In this paper we focus on providing several algorithms for the Monroe and Chamberlin-Courant rules using nonlinear satisfaction function and evaluating them empirically against various data sets. For

smaller data, results can be easily assessed by comparing them to the optimal result (calculated with the brute-force algorithm). For bigger data, the upper bound of the optimal result must be used for comparison. We implement and evaluate various heuristic algorithms, as well as the existing approximation algorithms for the linear satisfaction function, but applying them to the nonlinear cases.

## 2. Motivation and Goals

Monroe and Chamberlin-Courant systems may be potentially very useful, because they are one of the few multiwinner election systems that provide both accountability of candidates to the voters (each voter has one particular representative in the elected committee) and proportional results. Most of the currently used voting rules lack at least one of these properties. For example, D'Hondt method used to elect members of Polish lower house of parliament lacks accountability (voters are accountable to political parties rather than specific parliament members), while single-member constituency plurality system used for United Kingdom parliament elections lacks proportionality.

As finding optimal solution for both of the aforementioned systems is NP-hard [1], there is a need to provide good algorithms which can compute result which is suboptimal but still as close to optimal as possible. Using such result in real-life elections (e.g. for parliament) is disputable. However, for some software applications, e.g. recommendation systems, it can be used seamlessly.

Skowron et. al [2] have already provided approximation algorithms for these systems, but only under linear satisfaction function. In this thesis we focus on providing algorithms for non-linear satisfaction functions, as they can better reflect real preferences of the voters.

### 3. Preliminaries

In the first part of the chapter we explain basic notions regarding multiwinner election problem. Then we present formal definitions of winner determination problem under Monroe and Chamberlin-Courant voting rules. Finally, we present some notions regarding computational complexity.

#### 3.1. Basic notions

**Preferences.** For each  $n \in \mathbb{N}$ , by  $[n]$  we mean  $\{1, \dots, n\}$ . We assume that there is a set  $N = [n]$  of *agents* and a set  $A = \{a_1, \dots, a_m\}$  of *alternatives*. Each agent  $i$  has a *preference order*  $\succ_i$  over  $A$ , i.e., a strict linear order of the form  $a_{\pi(1)} \succ_i a_{\pi(2)} \succ_i \dots \succ_i a_{\pi(m)}$  for some permutation  $\pi$  of  $[m]$ . For an alternative  $a$ , by  $pos_i(a)$  we mean the position of  $a$  in the  $i$ 'th agent's preference order. For example, if  $a$  is the most preferred alternative for  $i$  then  $pos_i(a) = 1$ , and if  $a$  is the least preferred one then  $pos_i(a) = m$ . A collection  $V = (\succ_1, \dots, \succ_n)$  of agents' preference orders is called a *preference profile*.

Subsets of the alternatives are often included in the descriptions of preference orders. If  $A$  is the set of all the alternatives and  $B$  is a nonempty strict subset of  $A$ , then by  $B \succ A - B$  we mean that for the preference order  $\succ$  all alternatives in  $B$  are preferred to those outside of  $B$ .

**Positional scoring function.** A *positional scoring function* (PSF) is a function  $\alpha^m : [m] \rightarrow \mathbb{N}$ . A PSF  $\alpha^m$  is an *increasing positional scoring function* (IPSF) if for each  $i, j \in [m]$ , if  $i < j$  then  $\alpha^m(i) < \alpha^m(j)$ . Analogously, a PSF  $\alpha^m$  is a *decreasing positional scoring function* (DPFS) if for each  $i, j \in [m]$ , if  $i < j$  then  $\alpha^m(i) > \alpha^m(j)$ .

If  $\beta^m$  is an IPSF then  $\beta^m(i)$  can represent the *dissatisfaction* of an agent if she is assigned to an alternative ranked  $i$ 'th in her preference order. We assume that for each IPSF  $\beta^m(1) = 0$  (an agent is not dissatisfied with her best alternative). Analogously, a DPSF  $\gamma^m$  can represent an agent's *satisfaction*. We assume that for each DPSF  $\gamma^m(m) = 0$  (an agent is completely not satisfied with her worst alternative). In some cases, we will write  $\alpha$  instead of  $\alpha^m$  to simplify notation.

An IPSF (DPSF)  $\alpha^m$  is considered a *linear dissatisfaction (satisfaction) function* if for each  $i \in [m - 1]$  value  $\alpha^m(i + 1) - \alpha^m(i)$  is constant. Every IPSF (DPSF) not satisfying that constraint is

considered a *nonlinear dissatisfaction (satisfaction) function*.

We define a family  $\alpha$  of IPSFs (DPSFs) of the form  $\alpha = (\alpha^m)_{m \in \mathbb{N}}$ , where  $\alpha^m$  is a PSF on  $[m]$ , such that:

1. For a family of IPSFs it holds that  $\alpha^{m+1}(i) = \alpha^m(i)$  for all  $m \in \mathbb{N}$  and  $i \in [m]$ .
2. For a family of DPSFs it holds that  $\alpha^{m+1}(i+1) = \alpha^m(i)$  for all  $m \in \mathbb{N}$  and  $i \in [m]$ .

We build families of IPSFs (DPSFs) by appending (prepending) values to functions with smaller domains. To simplify notation, we will refer to such families of IPSFs (DPSFs) as *normal* IPSFs (DPSFs).

**Assignment functions.** A *K-assignment function* is any function  $\Phi : N \rightarrow A$ , such that  $\|\Phi(N)\| \leq K$  (it assigns agents to at most  $K$  alternatives). A *Monroe K-assignment function* is an assignment function that additionally satisfies the following constraint: For each alternative  $a \in A$  we have that either  $\left\lfloor \frac{\|N\|}{K} \right\rfloor \leq \|\Phi^{-1}(a)\| \leq \left\lceil \frac{\|N\|}{K} \right\rceil$  or  $\|\Phi^{-1}(a)\| = 0$ .

A *partial K-assignment function* is defined in the same way as a regular one, except that it may assign a null alternative,  $\perp$ , to some of the agents. It is convenient to think that for each agent  $i$  we have  $pos_i(\perp) = m$ . A *partial Monroe K-assignment* is a partial K-assignment that can be extended to a regular Monroe K-assignment. If he have an assignment function  $\Phi$ , for each agent  $i$  we refer to alternative  $\Phi(i)$  as the *representative* of  $i$ .

Having a normal IPSF (DPSF)  $\alpha$ , we may consider the following three functions, each assigning a positive integer to a given assignment  $\Phi$ :

$$l_{sum}^\alpha(\Phi) = \sum_{i=1}^n \alpha(pos_i(\Phi(i))), \quad (3.1)$$

$$l_{max}^\alpha(\Phi) = \max_{i=1}^n \alpha(pos_i(\Phi(i))), \quad (3.2)$$

$$l_{min}^\alpha(\Phi) = \min_{i=1}^n \alpha(pos_i(\Phi(i))). \quad (3.3)$$

These functions aggregate individual dissatisfaction (satisfaction) values of the agents to measure the quality of the assignment for the entire society. In the utilitarian framework (which is used in this thesis), we use the first function as the *total dissatisfaction function* in the IPSF case and as the *total satisfaction function* in the DPSF case. We use the second and the third functions, respectively, as the total dissatisfaction and satisfaction functions for IPSF and DPSF cases in the egalitarian framework.

## 3.2. Monroe and Chamberlin-Courant rules

We will now define the problems of winner determination under the Monroe and Chamberlin-Courant (abbreviated as CC) rules. In both cases the goal is to find an optimal assignment function,



where the optimality is relative to one of the total dissatisfaction or satisfaction functions introduced earlier. The former is to be minimized and the latter is to be maximized.

**Definition 1.** Let  $\alpha$  be a normal IPSF. An instance of  $\alpha$ -CC-DisWinner problem consists of a set of agents  $N = [n]$ , a set of alternatives  $A = \{a_1, \dots, a_m\}$ , a preference profile  $V$  of the agents, and positive integer  $K$ . We ask for a K-assignment function  $\Phi$  such that  $l_{sum}^\alpha(\Phi)$  is minimized. The problem  $\alpha$ -Monroe-DisWinner is defined in the same way but we additionally require  $\Phi$  to be a Monroe K-assignment function.

**Definition 2.** Let  $\alpha$  be a normal DPSF. The problem  $\alpha$ -CC-SatWinner is defined in the same way as  $\alpha$ -CC-DisWinner, except that we seek a K-assignment  $\Phi$  such that  $l_{sum}^\alpha(\Phi)$  is maximized. The problem  $\alpha$ -Monroe-SatWinner is defined in the same way but we additionally require  $\Phi$  to be a Monroe K-assignment function.

In terms of solutions dissatisfaction-based problems are equivalent to the satisfaction-based ones. One can always transform an DPSF to an equivalent IPSF and vice versa.

The goal in our problems is to compute a particular (Monroe) K-assignment function. Such a function defines a set of  $K$  alternatives, who are viewed as the winners of the given multiwinner election.

For each subset of the alternatives  $S \subseteq A$  such that  $\|S\| \leq K$ , we write  $\Phi_\alpha^S$  to denote the partial (Monroe) K-assignment that assigns agents only to the alternatives from  $S$  and such that  $\Phi_\alpha^S$  maximizes the utilitarian satisfaction  $l_{sum}^\alpha(\Phi_\alpha^S)$ .

### 3.3. Computational complexity

As for many normal IPSFs our problems are NP-hard [1], we seek approximate solutions.

**Definition 3.** Let  $r$  be a real number such that  $r \geq 1$  ( $0 < r \leq 1$ ), let  $\alpha$  be a normal IPSF (DPSF), and let  $R$  be either Monroe or CC. An algorithm is an  $r$ -approximation algorithm for  $\alpha$ -R-DisWinner problem ( $\alpha$ -R-SatWinner problem) if on each instance  $I$  it returns a feasible assignment  $\Phi$  such that  $l_{sum}^\alpha(\Phi) \leq r \cdot OPT$  (such that  $l_{sum}^\alpha(\Phi) \geq r \cdot OPT$ ), where  $OPT$  is the optimal total dissatisfaction (satisfaction)  $l_{sum}^\alpha(\Phi_{OPT})$ .

## **4. State of the Art**

## 5. Implemented Algorithms

In this chapter we present implemented algorithms for the utilitarian versions of Monroe and Chamberlin-Courant multiwinner voting rules in the satisfaction-based framework.

**Proposition 1 (Implicit in the paper of Betzler et al. [3]).** Let  $\alpha$  be a normal DFSF,  $N$  be a set of agents,  $A$  be a set of alternatives,  $V$  be a preference profile of  $N$  over  $A$ , and  $S$  a  $K$ -element subset of  $A$  (where  $K$  divides  $\|N\|$ ). Then there is a polynomial-time-algorithm that computes a (possibly partial) optimal  $K$ -assignment  $\Phi_\alpha^S$  (Monroe  $K$ -assignment  $\Phi_\alpha^S$ ) of the agents to the alternatives from  $S$ .

### 5.1. AlgorithmA

Algorithm A was first presented by Skowron et al. [2] and tries to solve  $\alpha$ -Monroe-SatWinner. It builds a solution iteratively (greedily). In each step we pick some not-yet-assigned alternative  $a_i$  (using some criterion) and assign it to those  $\frac{N}{K}$  agents that are not assigned to any other alternative yet and whose satisfaction of being matched with  $a_i$  is maximal.

### 5.2. AlgorithmB

### 5.3. AlgorithmCC

### 5.4. Genetic Algorithm

**Algorithm 1** Algorithm A

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1: procedure COMPUTEMONROESATWINNER
2:    $\Phi \leftarrow$  a map defining a partial assignment, iteratively built by the algorithm
3:    $\Phi^{\leftarrow} \leftarrow$  the set of agents for which the assignment is already defined
4:    $\Phi^{\rightarrow} \leftarrow$  the set of alternatives already used in the assignment
5:   if  $K \leq 2$  then
6:     compute the optimal solution using an algorithm of Betzler et al. [2] and return
7:    $\Phi = \{\}$ 
8:   for  $i \leftarrow 1$  to  $K$  do
9:      $score \leftarrow \{\}$ 
10:     $bests \leftarrow \{\}$ 
11:    for all  $a_i \in A \setminus \Phi^{\rightarrow}$  do
12:       $agents \leftarrow$  sort  $N \setminus \Phi^{\leftarrow}$  so that if agent  $j$  preceeds agent  $j'$  then  $pos_j(a_i) \leq pos_{j'}(a_i)$ 
13:       $bests[a_i] \leftarrow$  choose first  $\frac{N}{K}$  elements from  $agents$ 
14:       $score[a_i] \leftarrow \sum_{j \in bests[a_i]} (m - pos_j(a_i))$ 
15:       $a_{best} \leftarrow \operatorname{argmax}_{a \in A \setminus \Phi^{\rightarrow}} score[a]$ 
16:      for all  $j \in bests[a_{best}]$  do
17:         $\Phi[j] \leftarrow a_{best}$ 

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## **6. Evaluation Results**

## **7. Summary**

## References

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