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Implementacja algorytmów dla systemów Monroe i Chamberlina-Couranta z nieliniową funkcją satysfakcji

Implementation of algorithms for Monroe and Chamberlin-Courant systems under nonlinear satisfaction function

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Oświadczamy, świadomi odpowiedzialności karnej za poświadczenie nieprawdy, że niniejszą pracę dyplomową wykonaliśmy osobiście i samodzielnie (w zakresie wyszczególnionym we wstępie) i że nie korzystaliśmy ze źródeł innych niż wymienione w pracy.

Contents

1. Introduction

We study the effectiveness of algorithms for approximate winner determination under the Monroe [1] and Chamberlin-Courant [2] multiwinner voting rules using nonlinear satisfaction function. The purpose of both these rules is to select a group of candidates that best represent the voters. Having good voting rules and algorithms for them is important, because multiwinner elections are used both in human societies (e.g., for parliament elections) and in particular software systems (e.g., in recommendation systems [3]). Rules studied in this paper are exceptionally interesting because they have two desired features of multiwinner rules: they provide accountability (there is a direct connection between the elected candidates and the voters, so each voter has a representative assigned to her and each candidate knows who she represents) and they represent the views of the voters proportionally.

We assume that candidates participate in the election with multiple winners (a committee with multiple members is selected) and they are elected by voters, each of whom ranks all the candidates (each voter provides a linear order over the set of candidates expressing her preferences). For each voter the Monroe and Chamberlin-Courant rules assign a single candidate as her representative (with some constraints, which are detailed in the further part of the thesis).

The candidates are selected and assigned to the voters optimally, by maximizing the total satisfaction of the voters. The total satisfaction is calculated as a sum of individual satisfactions. We assume that there is a satisfaction function that measures how well a voter is represented by the candidate, depending on which position the voter ranks this candidate. The function is the same for each voter. It is a decreasing function, so a voter is more satisfied if the candidate assigned to her is ranked higher. In this thesis we study cases in which the satisfaction function is a nonlinear one.

Monroe and Chamberlin-Courant systems may be potentially very useful, because they are one of the few multiwinner election systems that provide both accountability and proportionality of the results. Most of the currently used voting rules lack at least one of these properties. For example, D'Hondt method used to elect members of Polish lower house of parliament lacks accountability (specific parliament members are accountable to the political parties, not to the voters), while single-member constituency plurality system used for United Kingdom parliamentary elections lacks proportionality.

The main drawback of the aforementioned rules is that their winner determination problems are NP-hard [4, 3, 5], which makes them hard to use in practice as it would force the use of algorithms that do not provide optimal results for every data set. Therefore, using these systems for real-life elections may rise some difficulties. However, they can be used for the recommendation systems conveniently,

as a good but not optimal recommendation is still useful. Lu and Boutilier [3] and Skowron et al. [6] provide approximation algorithms for both the rules, that return near-optimal results for various test cases (including real-life data and synthetic data), but for linear satisfaction functions only.

In this thesis we provide several algorithms for the Monroe and Chamberlin-Courant rules using nonlinear satisfaction functions and evaluate them empirically against various data sets. For smaller data, results can be easily assessed by comparing them to the optimal ones (calculated with the brute-force algorithm). For bigger data sets, the upper bound of the optimal result must be used for comparison. We implement and evaluate various heuristic algorithms, as well as the existing approximation algorithms for the linear satisfaction function, but applying them to the nonlinear cases.

2. Preliminaries

In the first part of this chapter we explain basic notions regarding multiwinner elections. Next, we present definitions of winner determination problems under Monroe and Chamberlin-Courant (CC) voting rules. Finally, we present some notions regarding approximation algorithms for these tasks.

2.1. Basic Notions

In this section we present essential definitions and notions concerning multiwinner elections.

Definition 1: Preferences [6]. Let's assume we have n agents (representing voters in the elections) and m alternatives (representing candidates in the elections). For each agent i, her *preference order* is a strict linear order $\succ_i$ over all the alternatives that ranks them from the most to the least desirable one. Collection V of the preference orders of all the agents is called a *preference profile*.

If A is the set of all the alternatives and B is a nonempty strict subset of A, then by $B \succ A - B$ we mean that all alternatives in B are preferred to those outside of B for the preference order $\succ$.

Definition 2: Positional scoring function [6]. Let's assume we have m alternatives and function $\alpha^m : \{1, ..., m\} \to \mathbb{N}$ that assigns an integer value to each position in the agent's preference order. If α^m is a decreasing function (for each $i, j \in \{1, ..., m\}$, $i > j \implies \alpha^m(i) < \alpha^m(j)$), it is called a *satisfaction function* and it can represent an agent's satisfaction when an alternative from a particular position in her preference order is elected. For every satisfaction function $\alpha^m(m) = 0$, so an agent is not satisfied at all with her worst alternative. In some cases, we will write α instead of α^m to simplify notation.

A satisfaction function α^m is a *linear satisfaction function* if for each $i \in \{1, ..., m-1\}$ value $\alpha^m(i+1) - \alpha^m(i)$ is constant. Every satisfaction function not satisfying this constraint is a *nonlinear satisfaction function*.

Definition 3: Assignment functions [6]. Let's assume we have n agents and m alternatives. Let B be a set of all the agents ($B = \{1, ..., n\}$). A *K-assignment function* $\Phi : B \to \{a_1, ..., a_m\}$ is a function

that assigns a single alternative to every agent in such way, that no more than K alternatives are selected ($\|\Phi(B)\| \leq K$). It is called a *Monroe K-assignment function* if it additionally satisfies the following constraint: For each alternative a we have that either $\left\lfloor \frac{\|B\|}{K} \right\rfloor \leq \|\Phi^{-1}(a)\| \leq \left\lceil \frac{\|B\|}{K} \right\rceil$ or $\|\Phi^{-1}(a)\| = 0$. It means that for Monroe K-assignment function, agents are assigned to exactly K alternatives and each of the alternatives has about $\frac{\|B\|}{K}$ agents assigned. If we have an assignment function Φ, alternative $\Phi(i)$ is called the *representative* of agent i.

Additionally, if we allow the K-assignment function to assign an empty alternative ($\perp$) to the agents, it is called a *partial K-assignment function*. It is also a *partial Monroe K-assignment function* if it can be extended to a regular Monroe K-assignment function by replacing all the empty alternatives with the regular alternatives ($\{a_1, ..., a_m\}$).

Let S be a set of alternatives. By Φ^S we mean a K-assignment function (or a partial K-assignment function) that assigns agents only to alternatives from S.

Definition 4: Total satisfaction function [6]. We assume that α is a satisfaction function and $pos_i(x)$ represents position of alternative x in the i'th agent's preference order. The following function assigns a positive integer to a given assignment Φ:

$$l^{\alpha}_{sum}(\Phi) = \sum_{i=1}^{n} \alpha(pos_i(\Phi(i))) \tag{2.1}$$

This function combines satisfaction of the agents to assess the quality of the assignment for the entire society. It simply calculates the sum of the individual agents' satisfaction value and is used as the *total satisfaction function*.

For each subset of the alternatives $S \subseteq A$ that satisfies $\|S\| \leq K$, by Φ^S_α we mean the partial K-assignment (or the partial Monroe K-assignment) that assigns agents only to the alternatives from S and such that Φ^S_α maximizes the total satisfaction $l^{\alpha}_{sum}(\Phi^S_\alpha)$.

2.2. Monroe and Chamberlin-Courant Rules

We will now define the problems of winner determination under the Monroe and CC rules. The goal is to find an optimal assignment function, where by the optimal function we understand one that maximizes the total satisfaction.

Definition 5: Chamberlin-Courant and Monroe problems [6]. Let's assume we have n agents, m alternatives, preference profile V, $K \in \mathbb{N}$ ($K < m$) representing the committee size and a satisfaction function α. The goal of the *Chamberlin-Courant problem* is to find a K-assignment function Φ for which the total satisfaction $l^{\alpha}_{sum}(\Phi)$ is maximal under preference profile V. The goal of the *Monroe problem* is

the same, but it searches for a Monroe K-assignment function instead.

Intention of solving these problems is to find a (Monroe) K-assignment function which returns a set of K alternatives, who are viewed as the winners of the given multiwinner election (e.g. elected members of a committee).

2.3. Approximate Solutions

As for many satisfaction functions multiwinner election problems under both Monroe and CC rules are NP-hard [4, 3, 5], we are looking for approximate solutions.

Definition 6: Approximation algorithms [6]. Let r be a real number such that $0 < r \leq 1$, let α be a satisfaction function. An algorithm is an *r-approximation algorithm* for CC or Monroe problem if for every correct input it returns an assignment Φ such that $l^{\alpha}_{sum}(\Phi) \geq r \cdot OPT$, where OPT is the optimal total satisfaction $l^{\alpha}_{sum}(\Phi_{max})$.

2.4. State of the Art

The Chamberlin-Courant and Monroe multiwinner voting rules were introduced by Chamberlin and Courant [2] and Monroe [1] respectively. Despite having desired properties of multiwinner rules when we want to achieve a proportional representation [7], there has not been much research regarding these systems yet.

Complexity of both of these systems was studied in several papers and the conclusion is that they are both NP-hard. Procaccia et al. [5] showed that these systems are NP-hard in the dissatisfaction-based framework in case of approval dissatisfaction function, Lu and Boutilier [3] presented hardness for the Chamberlin-Courant rule under linear satisfaction function, while Betzler et al. [4] studied the parameterized complexity of the rules. There were also papers that studied the complexity for some specific cases where preferences are either single-peaked (Yu et al. [8]) or single-crossing (Skowron et al. [9]).

Lu and Boutilier [3] were the first to study the approximability of the Chamberlin-Courant rules under linear satisfaction function, and they presented an approximation algorithm for this case. Skowron et al. [6] gave more approximation algorithms for both the Monroe and Chamberlin-Courant rules (for the linear satisfaction function) and assessed their effectiveness against various data sets. Faliszewski et al. [10] were the first to show results for nonlinear satisfaction functions other then the approval ones, using algorithms based on voter clustering.

We discuss the algorithms of Skowron et al. [5] and Lu and Boutilier [3] in the next section.

3. Implemented Algorithms

In this chapter we present implemented algorithms for the Monroe and Chamberlin-Courant multiwinner voting rules.

Proposition 1 (Implicit in the paper of Betzler et al. [4]). Let α be a satisfaction function, N be a set of agents, A be a set of alternatives, V be a preference profile of N over A, and S a K-element subset of A (where K divides $\|N\|$). Then there is a polynomial-time-algorithm that computes a (possibly partial) optimal K-assignment Φ_α^S (Monroe K-assignment Φ_α^S) of the agents to the alternatives from S.

3.1. Existing Algorithms

In this section we present algorithms that already exist, but have not been applied to nonlinear satisfaction functions before.

3.1.1. Algorithms A, B and C

Algorithm A was first presented by Skowron et al. [6] and attempts to solve the Monroe problem. It is a greedy algorithm that executes K iterations (where K is a size of the elected committee). In every iteration the algorithm selects an alternative a_i that has not been assigned yet and assigns it to $\frac{N}{K}$ of the remaining agents whose satisfaction of being assigned to a_i is maximal (criterion for picking an alternative in each step is the sum of satisfaction of $\frac{N}{K}$ agents selected this way). This algorithm runs in polynomial time [6]. Pseudocode is presented as Algorithm 1.

Algorithm B is an extension to Algorithm A and was presented in the same paper [6]. In the first step, Algorithm B simply executes Algorithm A. Next, it uses algorithm from Proposition 1 to optimally assign alternatives from the winners set to the agents. As both these algorithms run in polynomial time, so does Algorithm B.

Algorithm C is a further extension of Algorithm B, also presented by Skowron et al. [6]. While Algorithm B only keeps one partial assignment function Φ that is extended in each step until it becomes

Algorithm 1 Algorithm A of Skowron et al. [6]

procedure COMPUTEMONROEPROBLEMSOLUTION
  $\Phi \leftarrow$ a map defining a partial assignment, iteratively built by the algorithm
  $\Phi^{\leftarrow} \leftarrow$ the set of agents for which the assignment is already defined
  $\Phi^{\rightarrow} \leftarrow$ the set of alternatives already used in the assignment
  Φ = {}
  for $i \leftarrow 1$ to K **do**
    $score \leftarrow \{\}$
    $bests \leftarrow \{\}$
    for all $a_i \in A \setminus \Phi^{\rightarrow}$ **do**
      $agents \leftarrow$ sort $N \setminus \Phi^{\leftarrow}$ so that if agent j preceeds agent j' then $pos_j(a_i) \leq pos_{j'}(a_i)$
      $bests[a_i] \leftarrow$ choose first $\frac{N}{K}$ elements from $agents$
      $score[a_i] \leftarrow \sum_{j \in bests[a_i]}(m - pos_j(a_i))$
    $a_{best} \leftarrow argmax_{a \in A \setminus \Phi^{\rightarrow}} score[a]$
    for all $j \in bests[a_{best}]$ **do**
      $\Phi[j] \leftarrow a_{best}$

a complete solution, Algorithm C stores a list of d functions (d is provided as an algorithm parameter). In each step, for each alternative a with no agent assigned and for each Φ of the d functions stored, algorithm computes a greedy extension to Φ that assigns $\frac{N}{K}$ agents (that are not assigned to any other alternative yet) to a (in the same way as in Algorithm A). For the next step, d functions that return the highest satisfaction are used. Finally, after the last iteration, winners are reassigned using algorithm from Preposition 1 in each of the stored functions. Function that gives the highest satisfaction is selected. If $d = 1$, Algorithm C is identical to Algorithm B. Pseudocode is presented in Algorithm 3.

Unlike previous algorithms, Algorithm C can be used for both Monroe and Chamberlin-Courant rules. To adapt it to the Chamberlin-Courant rule, we have to replace the entire first for all loop with the appropriate code, presented in Algorithm 2.

Algorithm 2 Algorithm C - CC for all code replacement

for all $a_i \in A \setminus \Phi^{\rightarrow}$ **do**
  $\Phi' \leftarrow \Phi$
  for all $j \in N$ **do**
    if agent j prefers a_i to $\Phi'(j)$ **then**
      $\Phi'(j) \leftarrow a_i$
  $newPar.push(\Phi')$

Algorithm 3 Algorithm C of Skowron et al. [6]

procedure COMPUTEMONROEPROBLEMSOLUTION
    $\Phi \leftarrow$ a map defining a partial assignment, iteratively built by the algorithm
    $\Phi^{\leftarrow} \leftarrow$ the set of agents for which the assignment is already defined
    $\Phi^{\rightarrow} \leftarrow$ the set of alternatives already used in the assignment
    $Par \leftarrow$ a list of partial representation functions
    $Par = []$
    $Par.push(\{\})$
    for $i \leftarrow 1$ to K **do**
        $newPar = []$
        for $\Phi \in Par$ **do**
            $bests \leftarrow \{\}$
            for all $a_i \in A \setminus \Phi^{\rightarrow}$ **do**
                $agents \leftarrow$ sort $N \setminus \Phi^{\leftarrow}$ (agent j preceeds agent j' implies that $pos_j(a_i) \leq pos_{j'}(a_i)$
                $bests[a_i] \leftarrow$ choose first $\frac{N}{K}$ elements of $agents$
                $\Phi' \leftarrow \Phi$
                for all $j \in bests[a_i]$ **do**
                    $\Phi'[j] \leftarrow a_i$
                $newPar.push(\Phi')$
        sort $newPar$ according to descending order of the total satisfaction of the assigned agents
        $Par \leftarrow$ choose first d elements of $newPar$
    for $\Phi \in Par$ **do**
        $\Phi \leftarrow$ compute the optimal representative function using an algorithm of Betzler et al. [4] for the set of winners $\Phi^{\rightarrow}$
    return the best representative function from Par

3.1.2. Algorithm R

As shown by Skowron et al. [6], algorithms A, B and C are potentially very useful when size of the committee is much lower than the number of alternatives (K is small compared to m), as under this condition they produce results with very high approximation ratio under linear satisfaction function. For the other cases (i.e., relatively large committees), a sampling-based randomized algorithm may be used. We call it Algorithm R. We expect that under nonlinear satisfaction function algorithms should behave similarly in relation to each other as under a linear one.

Algorithm R randomly picks K alternatives and then, using Proposition 1, assigns them to agents optimally. As a single execution of such an algorithm may simply pick only alternatives that are ranked low, random assignment should be computed a given number of times (which is provided as an algorithm parameter k), so there is a greater probability to attain a high quality solution. If size of the committee is only slightly lower than the number of alternatives (K is comparable to m) then the probability of finding a solution that is at least close to optimal is high [6].

3.1.3. Algorithm AR

Algorithm family A-C and Algorithm R are naturally suitable for different cases. Therefore, Skowron et al. [6] proposed to combine algorithms A and R into Algorithm AR. They also showed that under linear satisfaction function, Algorithm AR can achieve approximation ratio of $0.715 - e$ with probability λ. Both e and λ are provided as algorithm parameters. Naturally, for different satisfaction functions approximation ratio may vary, but we decided to test the algorithm in an unchanged version for comparison. Pseudocode is presented in Algorithm 4.

Algorithm 4 Algorithm AR of Skowron et al. [6]

procedure COMPUTEMONROEPROBLEMSOLUTION
  H_j is the j'th harmonic number $H_j = \sum_{i=1}^{j}(\frac{1}{i})$
  $\lambda \leftarrow$ probability of achieving the approximation ratio $0.715 - e$ under linear satisfaction function
  if $\frac{H_K}{K} \geq \frac{e}{2}$ **then**
    compute the optimal solution using an algorithm of Betzler et al. [4] and return
  if $m \leq 1 + \frac{2}{e}$ **then**
    compute the optimal solution using a simple brute force algorithm and return
  $\Phi_1 \leftarrow$ solution computed by Algorithm A
  $\Phi_2 \leftarrow$ solution computed by Algorithm R (sampled $\log(1 - \lambda) \cdot \frac{2+e}{e}$ times)
  return the better assignment among Φ_1 and Φ_2

3.1.4. Algorithm GM

Algorithm GM (greedy marginal improvement) is an algorithm that was introduced by Lu and Boutilier [3] for the Chamberlin-Courant rule only. However, it was later generalized by Skowron et al. [6], so it can be applied to the Monroe rule as well. In the Monroe case, it can be considered as an improvement of Algorithm B.

The algorithm starts with an empty set S. In each iteration of the algorithm an alternative a ($a \notin S$) is selected and added to S. Selected a maximizes the satisfaction value $l^{\alpha}_{sum}(\Phi^{S\cup\{a\}}_{\alpha})$. Iterations are executed until a complete committee is selected, so K iterations are required. For the Monroe case, computing Φ^{S}_{α} is slow (it is achieved by using min-cost/max-flow algorithm mentioned in Proposition 1 [4]), which makes the algorithm execute for a relatively long time, as many computations are required. Pseudocode is presented in Algorithm 5.

Algorithm 5 Algorithm GM of Lu and Boutilier [3]

1: **procedure** COMPUTECCORMONROEPROBLEMSOLUTION
2: Φ^{S}_{α} - the partial assignment that assigns a single alternative to at most $\frac{n}{K}$ agents, that assigns to the agents only the alternatives from S, and that maximizes the satisfaction $l^{\alpha}_{sum}(\Phi^{S}_{\alpha})$
3: $S \leftarrow \emptyset$
4: **for** $i \leftarrow 1$ to K **do**
5: $a \leftarrow argmax_{a\in A\setminus S} l^{\alpha}_{sum}(\Phi^{S\cup\{\alpha\}}_{\alpha})$
6: $S \leftarrow S \cup \{a\}$
7: **return** Φ^{S}_{α}

3.1.5. Algorithm P

Algorithm P can only be applied to the Chamberlin-Courant problem. It was first introduced by Skowron et al. [6]. In the beginning, it computes x (a non-negative integer). Next, it computes an assignment that should maximize the number of agents that have an alternative from the first x spots in their preferences assigned to them. This process is executed greedily. Afterwards, if there are still agents with no alternative assigned, the best alternative is picked from the ones already selected for at least one other agent.

The function $w(x)$ used in the algorithm is Lambert's W-function, defined to be the solution of the equation $x = w(x)e^{w(x)}$. The algorithm runs in polynomial time. Pseudocode of Algorithm P is presented as Algorithm 6.

Algorithm 6 Algorithm P of Skowron et al. [6]

procedure COMPUTECCPROBLEMSOLUTION
  $\Phi \leftarrow$ a map defining a partial assignment, iteratively built by the algorithm
  $\Phi^{\leftarrow} \leftarrow$ the set of agents for which the assignment is already defined
  $\Phi^{\rightarrow} \leftarrow$ the set of alternatives already used in the assignment
  $num\_pos_x(a) \leftarrow \|\{i \in [n] \setminus \Phi^{\leftarrow} : pos_i(a) \leq x\}\|$ - the number of not-yet assigned agents that rank alternative a in one of their first x positions
  $w(\cdot)$ - Lambert's W-function
  $\Phi = \{\}$
  $x = \left\lceil \frac{mw(K)}{K} \right\rceil$
  for $i \leftarrow 1$ to K **do**
    $a_i \leftarrow argmax_{a \in A \setminus \Phi^{\rightarrow}} num\_pos_x(a)$
    for all $j \in [n] \setminus \Phi^{\leftarrow}$ **do**
      if $pos_j(a_i) < x$ **then**
        $\Phi[j] \leftarrow a_i$
  for all $j \in A \setminus \Phi^{\leftarrow}$ **do**
    $a \leftarrow$ such alternative from $\Phi^{\rightarrow}$ that $\forall_{a' \in \Phi^{\rightarrow}} pos_j(a) \leq pos_j(a')$
    $\Phi[j] \leftarrow a$
  return Φ

3.2. New Algorithms

In this section we present algorithms that have been invented and implemented specifically for the purpose of this thesis.

3.2.1. Genetic Algorithm

The idea of our algorithm is loosely based on metaheuristic genetic algorithms [11], such as the Firefly Algorithm [11, 12].

Our Genetic Algorithm starts with an initial set of creatures (each of them presenting a set of possible winners under the Chamberlin-Courant or Monroe rules), which are later mutated and crossed over with each other. The best creatures (ones with the highest total satisfaction) are preferred for further mutation and crossover in order to better investigate the neighbourhood of local maxima. But, on the other hand, the algorithm also produces new creatures by crossing over random existing ones to better explore the entire solution space, not limiting itself to local extrema.

Mutation of the creature is performed by randomly replacing one of the candidates it contains with another one. Crossover of two creatures takes all the candidates from both creatures and randomly

selects K of them (K is the committee size).

In each iteration of the algorithm, the creatures are evaluated (winners are assigned to agents using the algorithm from Preposition 1 and satisfaction is computed). The best creature in terms of total satisfaction is compared with currently best found creature and takes its place if it is better. Half of the evaluated creatures (the best ones) are chosen for further propagation. Each of them is then mutated randomly. Remaining creatures are created by crossing over random creatures from the 'better' half with each other. Resulting set of creatures is used for the next iteration. Number of iterations and number of creatures are the algorithm parameters. Pseudocode is presented in Algorithm 7.

Algorithm 7 Genetic Algorithm

procedure COMPUTECCORMONROEPROBLEMSOLUTION
    I - number of iterations
    c - number of creatures
    Φ_{best} - best creature (preference profile)
    $creatures \leftarrow$ generate initial random set of c creatures
    for $i \leftarrow 1$ to I **do**
        $creaturesSorted \leftarrow$ sort $creatures$ by total satisfaction
        if $satisfaction(creaturesSorted[1]) > satisfaction(\Phi_{best})$ **then**
            $\Phi_{best} = creaturesSorted[1]$
        $bestCreatures \leftarrow$ choose first $c/2$ elements from $creaturesSorted$
        $mutated \leftarrow$ mutate all creatures from $bestCreatures$ randomly
        $crossed \leftarrow$ crossover random creatures from $bestCreatures$ to produce $c/2$-element set
        $newCreatures \leftarrow mutated \cup crossed$
        $creatures \leftarrow newCreatures$
    return Φ_{best}

3.2.2. Simulated Annealing

The Simulated Annealing algorithm is inspired by the physical annealing process, used in metallurgy. It involves heating and cooling a material to make it attain specific physical properties. Using simulated annealing for optimization of a complex function depending on many parameters was proposed by Kirkpatrick et al. [13]. We adapt it to the Chamberlin-Courant and Monroe problems.

In simulated annealing we use a temperature variable, which has a high value at the beginning and gets lower ('cools') during the execution. When the temperature is high, algorithm can accept solutions worse than the current one more frequently, so it is possible to leave a local optimum, as the global one may be in a totally different area of the search space. When the temperature gets lower, algorithm

focuses on the area where solutions close to the optimum may lie.

To decide if the new solution should be accepted, the *acceptance function* is used. If the new solution is better, it is always accepted. If it is worse, acceptance function accepts the new solution with probability p, which is calculated as follows:

$$p = \exp(\frac{E_c - E_n}{T}) \tag{3.1}$$

E_c is the current solution energy, E_n is the new solution energy and T is the temperature. Energy represents quality of the solution and, in our case, is proportional to the total satisfaction value of the solution.

The algorithm proceeds as follows. First, it generates a random initial set of winners. Initial temperature is given as a parameter. Then, in each iteration, a new solution is generated by replacing one of the winners in the current solution with a random alternative that is not a winner in the current solution (every time winners are assigned to agents using the algorithm from Preposition 1). The newly created solution is then evaluated by the acceptance function. If it is accepted, it replaces the current solution. Otherwise, the current solution is kept. For the next iteration, temperature is decreased:

$$T \leftarrow T \cdot (1 - c) \tag{3.2}$$

c is the cooling rate, provided as the algorithm parameter. The algorithm stops when $T \leq 1$. Pseudocode is presented in Algorithm 8.

Algorithm 8 Simulated Annealing

procedure COMPUTECCORMONROEPROBLEMSOLUTION
    T_{start} - initial temperature
    c - cooling rate
    Φ_{curr} - current solution
    T - current temperature
    $E(\Phi)$ - energy of solution Φ
    $\Phi_{curr} \leftarrow$ generate initial random solution
    $T \leftarrow T_{start}$
    while $T > 1$ **do**
        $\Phi_{new} \leftarrow$ perform random replacement on Φ_{curr}
        $p \leftarrow \exp(\frac{E(\Phi_{curr})-E(\Phi_{new})}{T})$
        $r \leftarrow$ generate random number in range $[0; 1)$
        if $E(\Phi_{new}) > E(\Phi_{curr})$ or $p > r$ **then**
            $\Phi_{curr} \leftarrow \Phi_{new}$
        $T \leftarrow T \cdot (1 - c)$
    return Φ_{curr}

4. Evaluation Results

In this chapter we present algorithm evaluation results and conclusions.

All algorithms were run on Lenovo W530 notebook with the Intel Core i7-3740QM (2.7 GHz) CPU, 16 GB of RAM and Windows 7 Enterprise 64-bit operating system. All tests were performed under the "Maximum Performance" setting.

Numerous test cases were generated, varying in problem size, number of winners, generation model and satisfaction function.

Problem size is defined by the number of agents (n) and the number of alternatives (m). We selected three problem sizes for testing:

- Small instances: $n = 30$, $m = 10$
- Medium instances: $n = 400$, $m = 50$
- Large instances: $n = 400$, $m = 300$

For each problem size, we selected two different numbers of winners (K) - the first one to take a small part of alternatives as winners (around 15-20%) and the second one to take about half of alternatives as winners:

- Small instances: $K = 2$ or $K = 5$
- Medium instances: $K = 10$ or $K = 25$
- Large instances: $K = 50$ or $K = 200$

Test data was generated using two different models.

Modified Polya urn model (MPolya) This model assumes that we have an urn with m balls in m colors, each representing an alternative. We generate preferences by drawing random balls from the urn.

First, we draw alternatives for first rank in each agent's preference order, then for second ranks, and so on, until the entire preference profile is drawn. After the ball is drawn, it is returned to the urn and another ball of the same color is added to the urn too, so a probability of drawing the same color in the future is increased ("strong becomes stronger"). The only constraint is that an alternative cannot be drawn for an agent for which it has been already drawn before. This model generates preference profile where some relatively small number of alternatives is preferred to all the others by most agents.

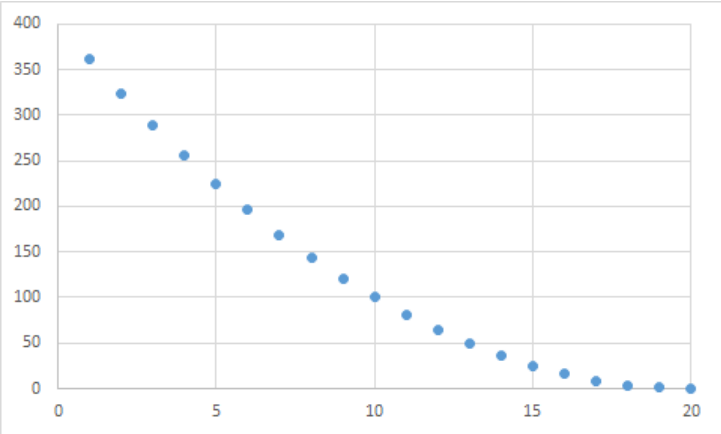
Impartial culture model (IC) In this model, for each agent we generate preference order independently. For each agent, every preference order (every permutation of alternatives) has equal probability of being drawn. This model generates preference profile with no clearly dominating alternatives.

Two different satisfaction functions were used, each of them trying to model real preferences of the voters.

Square function This is simply a square function and it assumes that difference between voters in the first positions of the preference order should be higher than between the last positions (voter is more concerned with his favourite candidates, not the ones at the end of the list). The function is given as follows, where m is the number of alternatives and i is position of the alternative in the agent's preference order:

$$\alpha(i) = (m - i)^2 \tag{4.1}$$

Example of the function for $m = 20$:



Best-and-Worst function This function assumes that for the voter the difference between two candidates from the top is the same as the difference between two candidates from the bottom, while differences between candidates from the middle of the preference order are much less significant. The function is given as follows, where m is the number of alternatives and i is position of the alternative in the agent's preference order:

$$f(x) = \frac{x^2 + x}{2} \tag{4.2}$$

$$h = \frac{m}{2} \tag{4.3}$$

$$d = f(h) \tag{4.4}$$

$$\alpha(i) = \begin{cases} f(h - i + 1) + d - 1 : i < h \\ -f(i - h) + d : i \geq h \end{cases} \tag{4.5}$$

Example of the function for $m = 20$:

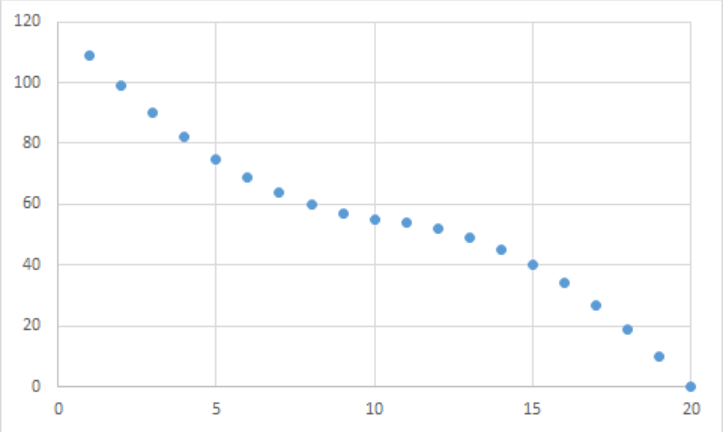


Table below presents all generated test cases.

#	alternatives count	agents count	winners count	generation model	satisfaction function
1	10	30	2	MPolya	Square
2					Best-and-Worst
3				IC	Square
4					Best-and-Worst
5			5	MPolya	Square
6					Best-and-Worst
7				IC	Square
8					Best-and-Worst
9	50	400	10	MPolya	Square
10					Best-and-Worst
11				IC	Square
12					Best-and-Worst
13			25	MPolya	Square
14					Best-and-Worst
15				IC	Square
16					Best-and-Worst
17	300	400	50	MPolya	Square
18					Best-and-Worst
19				IC	Square
20					Best-and-Worst
21			200	MPolya	Square
22					Best-and-Worst
23				IC	Square
24					Best-and-Worst

For each combination of problem size and generation model 100 elections were generated independently. Algorithms were evaluated against all the test files for each test case. We measured execution time and solution quality (total satisfaction) which was compared with brute-force (optimal) result for small instances and the upper bound (total satisfaction if every agent has his top alternative assigned) for the medium and large instances. Execution time and solution quality are presented as an average of 100 results (with standard deviation).

All test cases were used for both Chamberlin-Courant and Monroe problems.

4.1. Chamberlin-Courant Problem Evaluation

In this section we present evaluation results for Chamberlin-Courant problem.

4.1.1. Small Instances

Results for the small instances ($n = 30$, $m = 10$). Results are compared to optimal results.

Algorithms evaluated for the small instances are Algorithm C (in two versions, with $d = 10$ and $d = 15$, where d is the number of functions stored), Algorithm R (with $k = 100$ computations of a random assignment), Algorithm GM, Algorithm P, Genetic Algorithm (with $I = 15$ iterations and $c = 5$ creatures) and Simulated Annealing (with initial temperature $T_{start} = 100$ and cooling rate $c = 0.1$).

Results for 2 winners:

$n = 30$, $m = 10$, $K = 2$, Square function				
algorithm	MPolya		IC	
	time [ms]	quality	time [ms]	quality
C (10)	0.66 ± 0.12	$100.000 \pm 0.000\%$	0.68 ± 0.14	$100.000 \pm 0.000\%$
C (15)	0.74 ± 0.16	$100.000 \pm 0.000\%$	0.74 ± 0.14	$100.000 \pm 0.000\%$
R (100)	0.75 ± 0.14	$99.710 \pm 1.162\%$	0.58 ± 0.05	$99.479 \pm 1.470\%$
GM	0.14 ± 0.03	$99.992 \pm 0.055\%$	0.10 ± 0.02	$99.690 \pm 0.856\%$
P	0.06 ± 0.01	$87.964 \pm 8.825\%$	0.05 ± 0.02	$90.096 \pm 6.482\%$
GA (15, 5)	0.68 ± 0.14	$99.633 \pm 1.261\%$	0.53 ± 0.06	$99.505 \pm 1.555\%$
SA (100, 0.1)	0.37 ± 0.07	$98.603 \pm 2.375\%$	0.28 ± 0.04	$98.693 \pm 2.549\%$

$n = 30$, $m = 10$, $K = 2$, Best-and-Worst function				
algorithm	MPolya		IC	
	time [ms]	quality	time [ms]	quality
C (10)	0.54 ± 0.07	$100.000 \pm 0.000\%$	0.56 ± 0.15	$100.000 \pm 0.000\%$
C (15)	0.56 ± 0.06	$100.000 \pm 0.000\%$	0.55 ± 0.06	$100.000 \pm 0.000\%$
R (100)	0.72 ± 0.15	$99.718 \pm 1.021\%$	0.59 ± 0.07	$99.947 \pm 0.240\%$
GM	0.16 ± 0.03	$99.994 \pm 0.044\%$	0.11 ± 0.03	$99.414 \pm 1.375\%$
P	0.06 ± 0.01	$91.785 \pm 6.058\%$	0.05 ± 0.01	$93.725 \pm 4.245\%$
GA (15, 5)	0.68 ± 0.15	$99.910 \pm 0.383\%$	0.52 ± 0.05	$99.670 \pm 0.780\%$
SA (100, 0.1)	0.32 ± 0.05	$98.820 \pm 2.256\%$	0.27 ± 0.04	$99.144 \pm 1.778\%$

Results for 5 winners:

$n = 30$, $m = 10$, $K = 5$, Square function				
algorithm	MPolya		IC	
	time [ms]	quality	time [ms]	quality
C (10)	1.67 ± 0.30	$100.000 \pm 0.000\%$	1.59 ± 0.13	$99.993 \pm 0.073\%$
C (15)	2.29 ± 0.52	$100.000 \pm 0.000\%$	2.41 ± 0.48	$100.000 \pm 0.000\%$
R (100)	0.95 ± 0.09	$99.389 \pm 0.741\%$	0.94 ± 0.07	$99.218 \pm 0.842\%$
GM	0.16 ± 0.02	$99.980 \pm 0.108\%$	0.16 ± 0.01	$99.534 \pm 0.738\%$
P	0.06 ± 0.01	$91.789 \pm 4.927\%$	0.06 ± 0.01	$92.450 \pm 3.546\%$
GA (15, 5)	0.94 ± 0.04	$99.728 \pm 0.686\%$	0.97 ± 0.05	$99.349 \pm 1.014\%$
SA (100, 0.1)	0.53 ± 0.04	$98.185 \pm 1.550\%$	0.53 ± 0.03	$98.323 \pm 1.464\%$

$n = 30$, $m = 10$, $K = 5$, Best-and-Worst function				
algorithm	MPolya		IC	
	time [ms]	quality	time [ms]	quality
C (10)	1.54 ± 0.13	$100.000 \pm 0.000\%$	1.68 ± 0.45	$100.000 \pm 0.000\%$
C (15)	2.35 ± 0.64	$100.000 \pm 0.000\%$	2.27 ± 0.13	$100.000 \pm 0.000\%$
R (100)	0.93 ± 0.08	$99.331 \pm 0.768\%$	0.94 ± 0.08	$99.425 \pm 0.632\%$
GM	0.16 ± 0.02	$99.982 \pm 0.091\%$	0.16 ± 0.00	$99.672 \pm 0.615\%$
P	0.06 ± 0.01	$90.235 \pm 3.734\%$	0.06 ± 0.01	$94.353 \pm 2.656\%$
GA (15, 5)	0.96 ± 0.10	$99.710 \pm 0.607\%$	0.99 ± 0.06	$99.621 \pm 0.544\%$
SA (100, 0.1)	0.54 ± 0.04	$98.431 \pm 1.350\%$	0.53 ± 0.03	$98.741 \pm 0.959\%$

4.1.2. Medium Instances

Results for the medium instances ($n = 400$, $m = 50$). Results are compared to the upper bound.

Algorithms evaluated for the medium instances are Algorithm C (in two versions, with $d = 10$ and $d = 15$), Algorithm R (with $k = 100$), Algorithm GM, Algorithm P, Genetic Algorithm (in two versions, with $I = 100, c = 20$ and $I = 200, c = 25$) and Simulated Annealing (in two versions, with $T_{start} = 100, c = 0.01$ and $T_{start} = 100, c = 0.005$).

Results for 10 winners:

$n = 400$, $m = 50$, $K = 10$, Square function				
algorithm	MPolya		IC	
	time [ms]	quality	time [ms]	quality
C (10)	251.85 ± 11.73	$96.517 \pm 0.553\%$	267.20 ± 9.65	$89.757 \pm 0.294\%$
C (15)	379.52 ± 9.60	$96.517 \pm 0.553\%$	402.30 ± 9.43	$89.769 \pm 0.286\%$
R (100)	938.14 ± 11.43	$94.589 \pm 0.727\%$	941.29 ± 12.78	$88.540 \pm 0.224\%$
GM	25.45 ± 0.96	$96.508 \pm 0.591\%$	27.24 ± 2.11	$89.494 \pm 0.377\%$
P	2.62 ± 0.07	$89.795 \pm 2.308\%$	2.68 ± 0.22	$86.599 \pm 0.711\%$
GA (100, 20)	696.82 ± 15.81	$96.256 \pm 0.563\%$	761.59 ± 11.25	$89.412 \pm 0.315\%$
GA (200, 25)	1723.27 ± 23.67	$96.404 \pm 0.553\%$	1894.81 ± 19.53	$89.562 \pm 0.292\%$
SA (100, 0.01)	169.32 ± 7.39	$93.589 \pm 0.866\%$	169.55 ± 7.59	$88.132 \pm 0.302\%$
SA (100, 0.005)	335.91 ± 7.52	$94.013 \pm 0.911\%$	334.44 ± 6.37	$88.207 \pm 0.234\%$

$n = 400$, $m = 50$, $K = 10$, Best-and-Worst function				
algorithm	MPolya		IC	
	time [ms]	quality	time [ms]	quality
C (10)	251.14 ± 9.65	$96.733 \pm 0.510\%$	266.23 ± 9.37	$90.719 \pm 0.255\%$
C (15)	380.68 ± 10.18	$96.733 \pm 0.510\%$	404.76 ± 11.29	$90.731 \pm 0.247\%$
R (100)	941.99 ± 16.86	$95.084 \pm 0.624\%$	951.27 ± 15.43	$89.698 \pm 0.188\%$
GM	25.56 ± 1.46	$96.725 \pm 0.516\%$	27.16 ± 1.86	$90.529 \pm 0.323\%$
P	2.61 ± 0.07	$90.769 \pm 1.985\%$	2.68 ± 0.28	$88.062 \pm 0.600\%$
GA (100, 20)	696.42 ± 15.04	$96.518 \pm 0.526\%$	760.18 ± 12.71	$90.423 \pm 0.247\%$
GA (200, 25)	1738.66 ± 27.14	$96.631 \pm 0.515\%$	1922.78 ± 24.20	$90.554 \pm 0.261\%$
SA (100, 0.01)	167.29 ± 5.80	$93.988 \pm 0.739\%$	167.73 ± 7.18	$89.329 \pm 0.292\%$
SA (100, 0.005)	338.42 ± 10.02	$94.450 \pm 0.719\%$	342.32 ± 11.21	$89.450 \pm 0.230\%$

Results for 25 winners:

$n = 400$, $m = 50$, $K = 25$, Square function				
algorithm	MPolya		IC	
	time [ms]	quality	time [ms]	quality
C (10)	541.20 ± 12.76	$99.374 \pm 0.015\%$	569.84 ± 12.51	$97.585 \pm 0.119\%$
C (15)	832.17 ± 16.83	$99.374 \pm 0.015\%$	869.71 ± 14.88	$97.597 \pm 0.118\%$
R (100)	2094.20 ± 15.70	$98.574 \pm 0.204\%$	2142.40 ± 22.10	$97.041 \pm 0.084\%$
GM	51.70 ± 3.53	$99.373 \pm 0.146\%$	54.80 ± 3.58	$97.518 \pm 0.126\%$
P	4.40 ± 0.14	$96.740 \pm 0.855\%$	4.42 ± 0.09	$96.235 \pm 0.255\%$
GA (100, 20)	1652.20 ± 28.08	$99.138 \pm 0.183\%$	1781.69 ± 21.33	$97.335 \pm 0.102\%$
GA (200, 25)	4096.02 ± 60.15	$99.188 \pm 0.179\%$	4483.16 ± 43.37	$97.383 \pm 0.100\%$
SA (100, 0.01)	399.24 ± 8.47	$98.170 \pm 0.302\%$	399.77 ± 8.48	$96.859 \pm 0.113\%$
SA (100, 0.005)	813.16 ± 15.30	$98.326 \pm 0.252\%$	813.26 ± 14.00	$96.898 \pm 0.098\%$

$n = 400$, $m = 50$, $K = 25$, Best-and-Worst function				
algorithm	MPolya		IC	
	time [ms]	quality	time [ms]	quality
C (10)	540.44 ± 12.59	$99.404 \pm 0.138\%$	567.23 ± 12.07	$97.726 \pm 0.108\%$
C (15)	831.26 ± 18.11	$99.404 \pm 0.138\%$	863.77 ± 16.02	$97.732 \pm 0.103\%$
R (100)	2136.59 ± 23.57	$98.667 \pm 0.186\%$	2091.77 ± 16.60	$97.214 \pm 0.085\%$
GM	51.68 ± 2.23	$99.403 \pm 0.138\%$	54.74 ± 3.07	$97.662 \pm 0.118\%$
P	4.39 ± 0.25	$96.939 \pm 0.790\%$	4.45 ± 0.54	$96.472 \pm 0.233\%$
GA (100, 20)	1649.14 ± 28.84	$99.180 \pm 0.179\%$	1769.20 ± 19.84	$97.491 \pm 0.096\%$
GA (200, 25)	4104.78 ± 61.44	$99.227 \pm 0.172\%$	4422.15 ± 27.53	$97.548 \pm 0.099\%$
SA (100, 0.01)	397.18 ± 8.62	$98.280 \pm 0.268\%$	396.13 ± 6.25	$97.034 \pm 0.111\%$
SA (100, 0.005)	799.15 ± 11.14	$98.397 \pm 0.233\%$	799.15 ± 9.75	$97.090 \pm 0.105\%$

4.1.3. Large Instances

Results for the large instances ($n = 400$, $m = 300$). Results are compared to the upper bound.

Algorithms evaluated for the large instances are Algorithm C (in two versions, with $d = 10$ and $d = 15$), Algorithm R (with $k = 100$), Algorithm GM, Algorithm P, Genetic Algorithm (in two versions, with $I = 100, c = 20$ and $I = 200, c = 25$) and Simulated Annealing (in two versions, with $T_{start} = 100, c = 0.01$ and $T_{start} = 100, c = 0.005$).

Results for 50 winners:

$n = 400$, $m = 300$, $K = 50$, Square function				
algorithm	MPolya		IC	
	time [s]	quality	time [s]	quality
C (10)	15.66 ± 0.11	$99.418 \pm 0.051\%$	16.33 ± 0.10	$98.466 \pm 0.056\%$
C (15)	24.12 ± 0.12	$99.420 \pm 0.051\%$	25.04 ± 0.14	$98.472 \pm 0.061\%$
R (100)	0.337 ± 0.011	$97.787 \pm 0.140\%$	0.337 ± 0.011	$97.197 \pm 0.075\%$
GM	1.46 ± 0.02	$99.412 \pm 0.052\%$	1.53 ± 0.02	$98.400 \pm 0.071\%$
P	0.0585 ± 0.0045	$97.029 \pm 0.416\%$	0.0588 ± 0.0049	$96.804 \pm 0.154\%$
GA (100, 20)	6.96 ± 0.11	$99.039 \pm 0.084\%$	8.41 ± 0.05	$97.822 \pm 0.073\%$
GA (200, 25)	16.81 ± 0.29	$99.125 \pm 0.069\%$	$20,88 \pm 0.14$	$97.927 \pm 0.062\%$
SA (100, 0.01)	1.91 ± 0.03	$97.662 \pm 0.213\%$	1.92 ± 0.02	$97.197 \pm 0.087\%$
SA (100, 0.005)	3.81 ± 0.05	$97.794 \pm 0.172\%$	3.85 ± 0.03	$97.221 \pm 0.082\%$

$n = 400$, $m = 300$, $K = 50$, Best-and-Worst function				
algorithm	MPolya		IC	
	time [s]	quality	time [s]	quality
C (10)	15.68 ± 0.16	$99.425 \pm 0.050\%$	16.32 ± 0.10	$98.495 \pm 0.059\%$
C (15)	24.07 ± 0.14	$99.426 \pm 0.050\%$	25.05 ± 0.13	$98.502 \pm 0.060\%$
R (100)	0.329 ± 0.002	$97.838 \pm 0.129\%$	0.329 ± 0.002	$97.264 \pm 0.073\%$
GM	1.42 ± 0.01	$99.419 \pm 0.052\%$	1.49 ± 0.01	$98.438 \pm 0.075\%$
P	0.0575 ± 0.0006	$97.098 \pm 0.400\%$	0.0583 ± 0.005	$96.881 \pm 0.148\%$
GA (100, 20)	6.79 ± 0.10	$99.054 \pm 0.081\%$	8.25 ± 0.03	$97.872 \pm 0.069\%$
GA (200, 25)	16.61 ± 0.23	$99.131 \pm 0.073\%$	20.50 ± 0.42	$97.952 \pm 0.059\%$
SA (100, 0.01)	1.89 ± 0.03	$97.741 \pm 0.197\%$	1.89 ± 0.01	$97.239 \pm 0.092\%$
SA (100, 0.005)	3.77 ± 0.04	$97.840 \pm 0.175\%$	3.78 ± 0.01	$97.286 \pm 0.072\%$

Results for 200 winners:

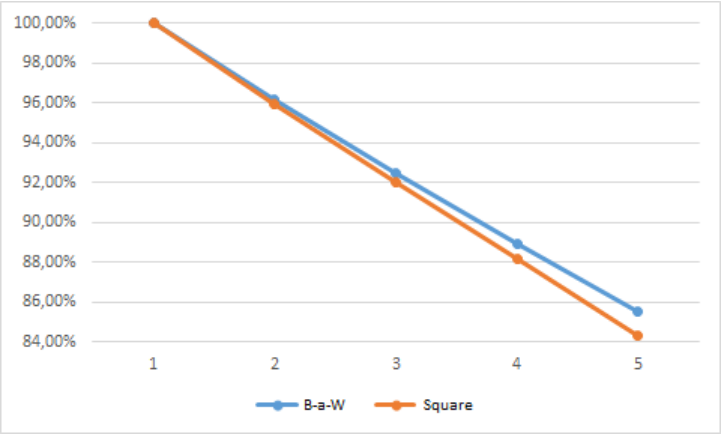
$n = 400$, $m = 300$, $K = 200$, Square function				
algorithm	MPolya		IC	
	time [s]	quality	time [s]	quality
C (10)	50.19 ± 0.38	$100.000 \pm 0.000\%$	50.54 ± 0.27	$99.9536 \pm 0.0099\%$
C (15)	79.44 ± 0.60	$100.000 \pm 0.000\%$	81.67 ± 0.62	$99.9537 \pm 0.0097\%$
R (100)	1.29 ± 0.01	$99.776 \pm 0.016\%$	1.29 ± 0.00	$99.737 \pm 0.011\%$
GM	4.04 ± 0.02	$100.000 \pm 0.000\%$	4.00 ± 0.01	$99.952 \pm 0.010\%$
P	0.134 ± 0.002	$99.669 \pm 0.044\%$	0.134 ± 0.002	$99.672 \pm 0.031\%$
GA (100, 20)	27.20 ± 0.97	$100.000 \pm 0.000\%$	30.67 ± 0.08	$99.950 \pm 0.015\%$
GA (200, 25)	67.72 ± 2.41	$100.000 \pm 0.000\%$	76.45 ± 0.21	$99.953 \pm 0.014\%$
SA (100, 0.01)	7.08 ± 0.04	$99.757 \pm 0.021\%$	7.05 ± 0.02	$99.724 \pm 0.017\%$
SA (100, 0.005)	14.16 ± 0.06	$99.773 \pm 0.024\%$	14.10 ± 0.03	$99.736 \pm 0.014\%$

$n = 400$, $m = 300$, $K = 200$, Best-and-Worst function				
algorithm	MPolya		IC	
	time [s]	quality	time [s]	quality
C (10)	49.33 ± 0.35	$100.000 \pm 0.000\%$	50.04 ± 0.20	$99.9538 \pm 0.0091\%$
C (15)	78.57 ± 0.50	$100.000 \pm 0.000\%$	80.79 ± 0.33	$99.9542 \pm 0.0097\%$
R (100)	1.30 ± 0.00	$99.776 \pm 0.014\%$	1.30 ± 0.00	$99.741 \pm 0.011\%$
GM	4.05 ± 0.01	$100.000 \pm 0.000\%$	4.00 ± 0.01	$99.953 \pm 0.010\%$
P	0.134 ± 0.001	$99.672 \pm 0.043\%$	0.134 ± 0.001	$99.676 \pm 0.031\%$
GA (100, 20)	27.29 ± 0.97	$100.000 \pm 0.000\%$	30.77 ± 0.05	$99.950 \pm 0.015\%$
GA (200, 25)	67.79 ± 2.41	$100.000 \pm 0.000\%$	77.74 ± 0.41	$99.954 \pm 0.014\%$
SA (100, 0.01)	7.11 ± 0.04	$99.755 \pm 0.026\%$	7.24 ± 0.05	$99.727 \pm 0.017\%$
SA (100, 0.005)	14.22 ± 0.07	$99.773 \pm 0.022\%$	14.32 ± 0.11	$99.736 \pm 0.016\%$

4.1.4. Conclusions

These are the final conclusions regarding evaluation of algorithms for Chamberlin-Courant Problem based on all the executed test cases.

1. Solution quality compared to the upper bound is generally a bit better for Best-and-Worst satisfaction function. This is because for the top alternatives Square function decreases faster compared to the first alternative (see the chart below), so algorithms' score is relatively worse when selecting second or third alternative in the preference order.



   However, this is valid for all the algorithms, so differences between them are similar for both functions. It seems that the most important part of the satisfaction function is near the top of the preference order, because the middle and bottom alternatives are rarely selected and differences between them are not very important, and both Square and Best-and-Worst functions are similar for the top alternatives. Results could be much different for some kind of an exponential function.

2. The difference between MPolya and IC is clearly visible for larger instances (MPolya could still be too random for smaller instances to see the difference). It is much easier to find a solution close to the upper bound for an MPolya generation model, as there is a group of alternatives that are ranked high by most agents, while IC is totally random.

3. Algorithm C and Algorithm GM are viable for every tested case. One should use Algorithm C to get solutions with the best quality possible. For many cases, increasing the number of stored functions (d) improves the solution quality (e.g., quality changes from 89.757% to 89.769% when increasing d from 10 to 15 for the medium instances, $K = 10$, Square function, IC), so if we seek the best solution possible, setting higher d is reasonable. Algorithm GM always returns slightly weaker results than Algorithm C (or the same in case of the large instances, 200 winners and MPolya, where both algorithms achieve 100% solution quality), but it runs much faster (e.g., 4.04 seconds vs 50.19 seconds for the large instances, $K = 200$, Square function, MPolya), so it may be used as a compromise between execution time and solution quality.

4. Algorithm P returns substantially weaker solutions than Algorithm GM (e.g., 96.939% vs 99.403% for the medium instances, $K = 25$, MPolya, Best-and-Worst function) but it runs extremely fast (4.39 ms vs 51.68 ms for the same example). It may be used if the solution does not have to be very close to optimal (e.g., in recommendation systems), but there is a need to obtain it as quickly as possible. This algorithm should not be used for small data sets, because the solution may be of very low quality (it has very large standard deviation for the smallest data set, which means that for some instances it returns much better result than average, but for some much worse), but this should not be a problem, because for small instances Algorithms GM runs very fast too. For larger problems, standard deviation becomes much lower and there should be no instances for which the solution quality is much worse than average.

5. For large cases, Algorithm R can provide good results with moderate execution time (e.g., for the large instances, $K = 50$, Best-and-Worst function, IC: Algorithm GM returns 98.438% in 1.49 s, Algorithm R returns 97.264% in 0.329 s and Algorithm P returns 96.881% in 0.0583 s) and it is possible that this algorithm gets relatively better for even larger data sets. However, due to high level of randomization, this algorithm should only be used to supplement another one, so it can possibly find a better solution. For small and medium cases, Algorithm R does not seem to be useful. It executes longer than Algorithm C and returns worse results (e.g., for the medium instances, $K = 25$, Square function, IC: Algorithm C returns 97.585% in 569.8 ms while Algorithm R returns 97.041% in 2142.40 ms).

6. Genetic Algorithm and Simulated Annealing executefor much too long to be useful for the small and medium instances (Algorithm GM is faster and returns better results). However, Genetic Algorithm becomes much better for large data sets where there is a lot of winners. For the large instances and $K = 200$, it becomes similar to Algorithm GM in means of both execution time and solution quality (e.g., for Square function, IC: Algorithm GM returns 99.952% in 4.00 s, while Algorithm R returns 99.737% in 1.29 s). It is possible that this algorithm may be better for even larger data sets, but it requires further research.

7. Test cases from the large instances turned out to be very easy for tested algorithms - all solutions are of very high quality compared to the upper bound. The reason of this is very high number of winners, which allows many agents to have their favourite alternative assigned, even for data generated with IC model. For MPolya and 200 winners, many algorithms always return an optimal solution which has the same total satisfaction as the upper bound - allowing 200 out of 300 candidates to win and using MPolya model allows for every agent to be maximally satisfied in virtually every case.

4.2. Monroe Problem Evaluation

In this section we present evaluation results for Monroe problem.

4.2.1. Small Instances

Results for the small instances ($n = 30$, $m = 10$). Results are compared to optimal results.

Algorithms evaluated for the small instances are Algorithm A, Algorithm B, Algorithm C (in two versions, with $d = 10$ and $d = 15$), Algorithm R (with $k = 100$), Algorithm AR (in two versions, with $e = 0.215$, $\lambda = 0.75$ and $e = 0.015$, $\lambda = 0.9$), Algorithm GM, Genetic Algorithm (with $I = 15, c = 5$) and Simulated Annealing (with $T_{start} = 100, c = 0.1$).

Results for 2 winners:

$n = 30$, $m = 10$, $K = 2$, Square function				
algorithm	MPolya		IC	
	time [ms]	quality	time [ms]	quality
A	0.18 ± 0.03	$95.320 \pm 2.247\%$	0.18 ± 0.02	$94.738 \pm 2.321\%$
B	0.61 ± 0.08	$99.457 \pm 1.296\%$	0.77 ± 0.75	$99.046 \pm 1.710\%$
C (10)	5.26 ± 0.57	$100.000 \pm 0.000\%$	5.36 ± 0.88	$99.9967 \pm 0.0271\%$
C (15)	7.39 ± 0.57	$100.000 \pm 0.000\%$	7.56 ± 0.83	$99.9994 \pm 0.0063\%$
R (100)	44.60 ± 6.59	$99.376 \pm 2.374\%$	43.26 ± 4.16	$99.522 \pm 1.428\%$
AR (0.215, 0.75)	20.75 ± 2.29	$100.000 \pm 0.000\%$	18.90 ± 0.23	$100.000 \pm 0.000\%$
AR (0.015, 0.9)	24.18 ± 4.62	$100.000 \pm 0.000\%$	21.90 ± 5.97	$100.000 \pm 0.000\%$
GM	6.46 ± 1.22	$100.000 \pm 0.000\%$	6.13 ± 0.91	$99.833 \pm 0.640\%$
GA (15, 5)	34.84 ± 3.87	$99.352 \pm 2.259\%$	34.97 ± 4.21	$99.153 \pm 2.040\%$
SA (100, 0.1)	19.11 ± 0.68	$98.053 \pm 3.442\%$	19.07 ± 0.71	$98.482 \pm 2.579\%$

$n = 30$, $m = 10$, $K = 2$, Best-and-Worst function				
algorithm	MPolya		IC	
	time [ms]	quality	time [ms]	quality
A	0.26 ± 0.62	$96.872 \pm 1.604\%$	0.19 ± 0.04	$95.500 \pm 1.679\%$
B	0.61 ± 0.08	$99.581 \pm 1.019\%$	0.61 ± 0.08	$99.214 \pm 1.176\%$
C (10)	5.41 ± 0.77	$100.000 \pm 0.000\%$	5.27 ± 0.39	$99.980 \pm 0.152\%$
C (15)	7.50 ± 0.97	$100.000 \pm 0.000\%$	7.63 ± 1.42	$99.994 \pm 0.0062\%$
R (100)	42.71 ± 3.24	$99.545 \pm 2.078\%$	43.32 ± 1.69	$99.830 \pm 0.625\%$
AR (0.215, 0.75)	22.14 ± 5.08	$100.000 \pm 0.000\%$	22.55 ± 5.37	$100.000 \pm 0.000\%$
AR (0.015, 0.9)	21.90 ± 4.91	$100.000 \pm 0.000\%$	22.62 ± 5.53	$100.000 \pm 0.000\%$
GM	5.94 ± 0.62	$100.000 \pm 0.000\%$	5.96 ± 0.39	$99.847 \pm 0.524\%$
GA (15, 5)	34.90 ± 2.59	$99.662 \pm 1.274\%$	35.40 ± 4.34	$99.546 \pm 1.256\%$
SA (100, 0.1)	19.97 ± 2.94	$98.416 \pm 2.649\%$	19.57 ± 1.23	$99.161 \pm 1.692\%$

Results for 5 winners:

$n = 30$, $m = 10$, $K = 5$, Square function				
algorithm	MPolya		IC	
	time [ms]	quality	time [ms]	quality
A	0.25 ± 0.02	$94.065 \pm 2.307\%$	0.28 ± 0.09	$95.567 \pm 2.081\%$
B	0.91 ± 0.30	$99.296 \pm 1.231\%$	0.81 ± 0.06	$99.017 \pm 1.443\%$
C (10)	7.78 ± 1.64	$99.991 \pm 0.070\%$	7.77 ± 1.01	$99.918 \pm 0.366\%$
C (15)	11.43 ± 1.97	$100.000 \pm 0.000\%$	11.56 ± 1.56	$99.974 \pm 0.112\%$
R (100)	55.91 ± 5.30	$97.678 \pm 2.753\%$	55.28 ± 4.34	$99.014 \pm 1.201\%$
AR (0.215, 0.75)	137.13 ± 5.69	$100.000 \pm 0.000\%$	136.24 ± 6.32	$100.000 \pm 0.000\%$
AR (0.015, 0.9)	154.83 ± 25.07	$100.000 \pm 0.000\%$	153.26 ± 23.78	$100.000 \pm 0.000\%$
GM	11.20 ± 1.23	$99.946 \pm 0.240\%$	11.41 ± 1.94	$99.841 \pm 0.392\%$
GA (15, 5)	44.28 ± 4.15	$99.110 \pm 1.819\%$	45.07 ± 6.11	$99.276 \pm 1.158\%$
SA (100, 0.1)	24.49 ± 1.76	$95.158 \pm 3.961\%$	24.62 ± 2.86	$98.022 \pm 1.707\%$

$n = 30$, $m = 10$, $K = 5$, Best-and-Worst function				
algorithm	MPolya		IC	
	time [ms]	quality	time [ms]	quality
A	0.26 ± 0.01	$96.012 \pm 0.163\%$	0.27 ± 0.03	$96.864 \pm 1.589\%$
B	0.79 ± 0.07	$99.390 \pm 1.017\%$	0.80 ± 0.09	$99.251 \pm 0.997\%$
C (10)	8.18 ± 1.87	$99.994 \pm 0.063\%$	7.83 ± 1.06	$99.939 \pm 0.237\%$
C (15)	11.37 ± 2.00	$99.989 \pm 0.080\%$	11.04 ± 0.48	$99.955 \pm 0.209\%$
R (100)	55.31 ± 4.24	$98.508 \pm 1.602\%$	54.57 ± 4.17	$99.217 \pm 0.849\%$
AR (0.215, 0.75)	160.17 ± 28.16	$100.000 \pm 0.000\%$	156.45 ± 26.68	$100.000 \pm 0.000\%$
AR (0.015, 0.9)	157.34 ± 26.41	$100.000 \pm 0.000\%$	159.08 ± 29.08	$100.000 \pm 0.000\%$
GM	10.96 ± 1.07	$99.971 \pm 0.158\%$	11.15 ± 1.74	$99.921 \pm 0.200\%$
GA (15, 5)	44.08 ± 4.53	$99.458 \pm 1.048\%$	44.45 ± 4.90	$99.601 \pm 0.715\%$
SA (100, 0.1)	24.84 ± 3.14	$96.261 \pm 2.710\%$	24.70 ± 2.93	$98.591 \pm 1.111\%$

4.2.2. Medium Instances

Results for the medium instances ($n = 400$, $m = 50$). Results are compared to the upper bound.

Algorithms evaluated for the medium instances are Algorithm A, Algorithm B, Algorithm C (in two versions, with $d = 10$ and $d = 15$), Algorithm R (with $k = 100$) and Algorithm AR (with $e = 0.215$, $\lambda = 0.75$).

Algorithm GM, Genetic Algorithm and Simulated Annealing have been omitted from results for this case, as their execution time is too long. These are example results for one execution of each algorithm (10 winners, MPolya, Square function):

algorithm	time [s]	quality
GM	207.70	95.467%
GA (100, 20)	$>$ 300 (timeout)	-
SA (100, 0.01)	$>$ 300 (timeout)	-

Results for 10 winners:

$n = 400$, $m = 50$, $K = 10$, Square function				
algorithm	MPolya		IC	
	time [s]	quality	time [s]	quality
A	0.0827 ± 0.0065	$90.515 \pm 0.843\%$	0.0814 ± 0.0041	$83.825 \pm 0.417\%$
B	0.860 ± 0.023	$95.529 \pm 0.526\%$	0.848 ± 0.014	$88.975 \pm 0.400\%$
C (10)	9.91 ± 0.44	$95.684 \pm 0.504\%$	9.76 ± 0.42	$89.390 \pm 0.317\%$
C (15)	14.83 ± 0.62	$95.692 \pm 0.505\%$	14.55 ± 0.53	$89.442 \pm 0.305\%$
R (100)	7.88 ± 0.07	$84.923 \pm 2.476\%$	7.68 ± 0.06	$87.006 \pm 0.400\%$
AR (0.215, 0.75)	11.87 ± 0.07	$90.536 \pm 0.834\%$	11.52 ± 0.07	$87.011 \pm 0.406\%$

$n = 400$, $m = 50$, $K = 10$, Best-and-Worst function				
algorithm	MPolya		IC	
	time [s]	quality	time [s]	quality
A	0.0796 ± 0.0035	$92.016 \pm 0.696\%$	0.0840 ± 0.0077	$86.296 \pm 0.324\%$
B	0.849 ± 0.016	$95.823 \pm 0.498\%$	0.854 ± 0.017	$90.070 \pm 0.343\%$
C (10)	9.86 ± 0.42	$95.966 \pm 0.457\%$	9.70 ± 0.43	$90.402 \pm 0.271\%$
C (15)	14.75 ± 0.51	$95.975 \pm 0.466\%$	14.57 ± 0.55	$90.449 \pm 0.271\%$
R (100)	7.86 ± 0.06	$87.006 \pm 2.160\%$	7.71 ± 0.07	$88.384 \pm 0.325\%$
AR (0.215, 0.75)	11.85 ± 0.07	$92.023 \pm 0.694\%$	11.66 ± 0.09	$88.483 \pm 0.316\%$

Results for 25 winners:

$n = 400$, $m = 50$, $K = 25$, Square function				
algorithm	MPolya		IC	
	time [s]	quality	time [s]	quality
A	0.171 ± 0.005	$93.475 \pm 0.721\%$	0.169 ± 0.006	$93.920 \pm 0.296\%$
B	1.03 ± 0.02	$97.814 \pm 0.363\%$	1.03 ± 0.02	$97.023 \pm 0.139\%$
C (10)	12.00 ± 0.56	$97.957 \pm 0.325\%$	12.04 ± 0.53	$97.165 \pm 0.131\%$
C (15)	18.04 ± 0.72	$97.966 \pm 0.324\%$	18.11 ± 0.69	$97.181 \pm 0.127\%$
R (100)	8.81 ± 0.08	$91.197 \pm 1.900\%$	8.56 ± 0.05	$96.111 \pm 0.150\%$
AR (0.215, 0.75)	13.81 ± 0.72	$93.572 \pm 0.705\%$	13.02 ± 0.08	$96.144 \pm 0.133\%$

$n = 400$, $m = 50$, $K = 25$, Best-and-Worst function				
algorithm	MPolya		IC	
	time [s]	quality	time [s]	quality
A	0.171 ± 0.008	$94.525 \pm 0.597\%$	0.169 ± 0.006	$94.754 \pm 0.225\%$
B	1.04 ± 0.02	$97.936 \pm 0.345\%$	1.04 ± 0.02	$97.200 \pm 0.141\%$
C (10)	12.04 ± 0.56	$98.068 \pm 0.306\%$	12.11 ± 0.55	$97.334 \pm 0.126\%$
C (15)	18.04 ± 0.66	$98.080 \pm 0.306\%$	18.08 ± 0.67	$97.349 \pm 0.121\%$
R (100)	8.82 ± 0.07	$92.322 \pm 1.405\%$	8.58 ± 0.09	$96.345 \pm 0.144\%$
AR (0.215, 0.75)	13.48 ± 0.13	$94.542 \pm 0.583\%$	13.06 ± 0.09	$96.393 \pm 0.141\%$

4.2.3. Large Instances

Results for the large instances ($n = 400$, $m = 300$). Results are compared to the upper bound.

Algorithms evaluated for the large instances are Algorithm A, Algorithm B, Algorithm C (with $d = 5$), Algorithm R (with $k = 10$) and Algorithm AR (with $e = 0.215$, $\lambda = 0.75$).

Parameters of algorithms C and R were adjusted to maintain reasonable execution time.

Results for 50 winners:

$n = 400$, $m = 300$, $K = 50$, Square function				
algorithm	MPolya		IC	
	time [s]	quality	time [s]	quality
A	7.77 ± 0.04	$97.894 \pm 0.120\%$	7.66 ± 0.03	$96.944 \pm 0.127\%$
B	8.81 ± 0.06	$99.103 \pm 0.062\%$	8.67 ± 0.04	$98.170 \pm 0.077\%$
C (5)	45.65 ± 0.50	$99.131 \pm 0.059\%$	45.17 ± 0.40	$98.224 \pm 0.063\%$
R (10)	10.41 ± 0.10	$93.630 \pm 0.915\%$	10.11 ± 0.09	$96.724 \pm 0.089\%$
AR (0.215, 0.75)	23.41 ± 0.14	$97.894 \pm 0.120\%$	22.84 ± 0.19	$96.954 \pm 0.120\%$

$n = 400$, $m = 300$, $K = 50$, Best-and-Worst function				
algorithm	MPolya		IC	
	time [s]	quality	time [s]	quality
A	7.74 ± 0.04	$98.048 \pm 0.104\%$	7.60 ± 0.04	$97.175 \pm 0.106\%$
B	8.78 ± 0.05	$99.118 \pm 0.060\%$	8.58 ± 0.02	$98.205 \pm 0.074\%$
C (5)	45.95 ± 0.58	$99.141 \pm 0.058\%$	45.47 ± 0.57	$98.258 \pm 0.063\%$
R (10)	10.58 ± 0.09	$93.955 \pm 0.670\%$	10.01 ± 0.10	$96.806 \pm 0.087\%$
AR (0.215, 0.75)	23.87 ± 0.19	$98.048 \pm 0.104\%$	22.56 ± 0.13	$97.175 \pm 0.106\%$

Results for 200 winners:

$n = 400$, $m = 300$, $K = 200$, Square function				
algorithm	MPolya		IC	
	time [s]	quality	time [s]	quality
A	26.23 ± 0.11	$98.417 \pm 0.156\%$	25.19 ± 0.08	$99.423 \pm 0.053\%$
B	28.71 ± 0.15	$99.380 \pm 0.056\%$	27.67 ± 0.10	$99.683 \pm 0.019\%$
C (5)	145.10 ± 10.83	$99.395 \pm 0.056\%$	138.29 ± 11.62	$99.693 \pm 0.017\%$
R (10)	26.06 ± 0.25	$97.284 \pm 0.397\%$	26.25 ± 0.41	$99.320 \pm 0.028\%$
AR (0.215, 0.75)	65.38 ± 0.40	$98.417 \pm 0.155\%$	65.00 ± 0.41	$99.425 \pm 0.050\%$

$n = 400$, $m = 300$, $K = 200$, Square function				
algorithm	MPolya		IC	
	time [s]	quality	time [s]	quality
A	25.86 ± 0.11	$98.497 \pm 0.140\%$	24.81 ± 0.06	$99.438 \pm 0.049\%$
B	28.30 ± 0.12	$99.387 \pm 0.057\%$	27.21 ± 0.08	$99.686 \pm 0.019\%$
C (5)	144.59 ± 10.79	$99.402 \pm 0.055\%$	138.02 ± 11.48	$99.696 \pm 0.017\%$
R (10)	25.69 ± 0.26	$97.423 \pm 0.308\%$	25.55 ± 0.21	$99.328 \pm 0.027\%$
AR (0.215, 0.75)	64.05 ± 0.43	$98.497 \pm 0.140\%$	63.16 ± 0.51	$99.439 \pm 0.048\%$

4.2.4. Conclusions

These are the final conclusions regarding evaluation of algorithms for Monroe Problem based on all executed test cases.

1. For most cases algorithms generally return a bit higher satisfaction for the Best-and-Worst function than for the Square function. It is caused by the fact that Square function is decreasing relatively faster for top alternatives (as described in section 4.4.4). However, differences between the algorithms are similar, the same as for Chamberlin-Courant problem. Using much different function, e.g. an exponential one, could potentially change the results.

2. Generally, algorithms achieve higher solution quality for data generated with MPolya (especially with less winners). This is an expected behaviour, as with MPolya it is easier to include more candidates from the top in the solution. The only exception to this rule is Algorithm R. It may be caused by high randomization of IC data, which causes random solutions to be relatively better.

3. Algorithm C seems to be the best and universal option to attain a very high quality solution for most cases. It is sometimes a bit worse than Algorithm AR for the small instances (e.g. 99.994% vs 100% for $K = 2$, Best-and-Worst function, IC), but runs much faster (7.63 ms vs 22.55 ms for the same case). It can be also adjusted to one's needs by modifying number of stored functions

(parameter d) and in many cases it improves the solution quality (e.g. for the medium instances, $K = 10$, Square function, IC, quality changes from 89.390% to 89.442% after increasing d from 10 to 15).

4. Quality of solution generated by Algorithm B is always significantly lower than the one generated by Algorithm C (e.g. 90.070% vs 90.402% for the medium instances, $K = 10$, Best-and-Worst function, IC), but the algorithm itself is much faster (0.854 s vs 9.70 s for the same case). It may be seen as a trade-off between solution quality and execution time.

5. For small and medium cases, Algorithm A may be a good option for some appliances, as it returns result very fast. Its quality is far from optimal, but there may be some systems where it is not as important as execution time. For larger instances, it is better to use Algorithm B instead, as difference in execution time becomes relatively lower when we increase problem size (e.g. 7.60 s vs 8.58 s for the large instances, $K = 50$, Best-and-Worst function, IC), because for larger cases reassignment of agents, which is the only difference between the algorithms, becomes less significant.

6. Algorithm AR gives the best quality of solution among tested algorithms for the small instances, but it is very slow. It seems to be underperforming for larger instances - other algorithms give better solutions faster. Maybe changing parameters could help, but it would further increase execution time.

7. Algorithm R is not a good choice, because for each case there is an algorithm that returns better solution in shorter time.

8. Quality of solutions returned by the algorithms for the large instances is extremely high compared to the upper bound. The reason is that the number of winners is very high compared to the number of agents ($50/400$, $200/400$) and to the number of alternatives ($50/300$, $200/300$). Such situation allows to assign favourite alternatives to most agents, especially with data generated by MPolya.

5. Summary

We have found that for both Chamberlin-Courant and Monroe problems there are algorithms that perform very well under nonlinear satisfaction function. Algorithm C is especially interesting (for both problems), because it attains solutions of very good quality for every tested data set (very often close to optimal or upper bound) and at the same time it can be tuned (by changing its input parameter) to try to find even better solution if we can afford longer execution time.

At the same time, there are excellent algorithms that can be used if we want good but not necessarily best possible solution, but we have limited time. This may be the case for some software systems.

We have shown that heuristic algorithms (Genetic Algorithm and Simulated Annealing) do not fit CC and Monroe problems well. They can provide solutions of good quality, but execution time is too long.

These are main areas of potential further research:

1. Using different satisfaction function could change the results. We have not tested algorithms against any function where differences between top alternatives would be much greater (e.g., exponential function). It would surely make quality of solutions much lower compared to upper bound and could potentially favour different algorithms.

2. Testing algorithms against different combinations of number of alternatives, number of agents and number of winner could also provide some interesting data. As we have seen for the largest tested instance, results vary a lot when ratio between these three parameters is adjusted.

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