Modelling Krang

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1 Introduction

Unknown parameters:

$$m_1 > 0$$
$$x_1 > 0$$
$$y_1 < 0\,(?)$$
$$m_2 > 0$$
$$x_2 > 0$$
$$y_2 < 0$$

where m_i is the mass and of the i^{th} link and x_i and y_i are the coordinates of the center of mass of the i^{th} link locally within the link. The x-axis is pointing along the link from the bottom to the top. The orientation of coordinate frames is shown in Fig. 1.

The length of the first link is known:

$$\ell_1 = 0.507\,\text{m}$$

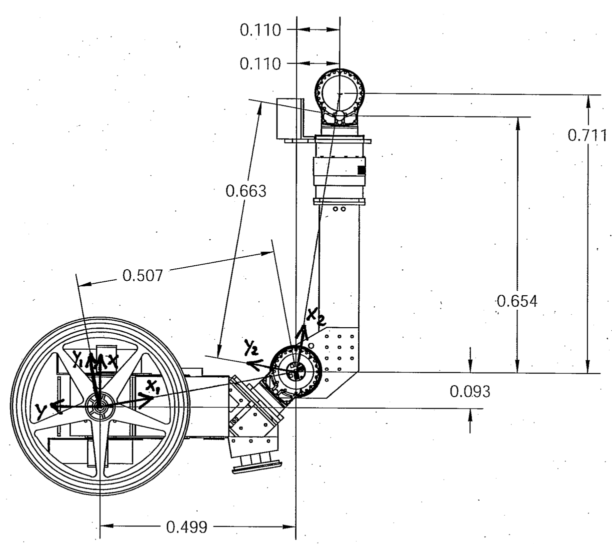


Figure 1: Orientation of the global and local coordinate frames.

2 Center of mass equations

Center of mass of the first link:

$$
\begin{aligned}
\mathrm{cm}_1 &= \begin{bmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{bmatrix} \begin{bmatrix} x_1 \\ y_1 \end{bmatrix} \\
&= \begin{bmatrix} x_1 c_1 - y_1 s_1 \\ x_1 s_1 + y_1 c_1 \end{bmatrix}
\end{aligned}
$$

Center of mass of the second link:

$$
\begin{aligned}
\mathrm{cm}_2 &= \begin{bmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{bmatrix} \begin{bmatrix} \ell_1 \\ 0 \end{bmatrix} + \begin{bmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{bmatrix} \begin{bmatrix} c_2 & -s_2 \\ s_2 & c_2 \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \\
&= \begin{bmatrix} c_1 & -s_1 \\ s_1 & c_1 \end{bmatrix} \begin{bmatrix} \ell_1 \\ 0 \end{bmatrix} + \begin{bmatrix} c_{12} & -s_{12} \\ s_{12} & c_{12} \end{bmatrix} \begin{bmatrix} x_2 \\ y_2 \end{bmatrix} \\
&= \begin{bmatrix} \ell_1 c_1 \\ \ell_1 s_1 \end{bmatrix} + \begin{bmatrix} x_2 c_{12} - y_2 s_{12} \\ x_2 s_{12} + y_2 c_{12} \end{bmatrix}
\end{aligned}
$$

Center of mass of both links together:

$$
\begin{aligned}
\mathrm{cm} &= \frac{m_1 \mathrm{cm}_1 + m_2 \mathrm{cm}_2}{m_1 + m_2} \\
&= \begin{bmatrix} \dfrac{m_1(x_1 c_1 - y_1 s_1) + m_2(x_2 c_{12} - y_2 s_{12} + \ell_1 c_1)}{m_1 + m_2} \\ \dfrac{m_1(x_1 s_1 + y_1 c_1) + m_2(x_2 s_{12} + y_2 c_{12} + \ell_1 s_1)}{m_1 + m_2} \end{bmatrix}
\end{aligned}
$$

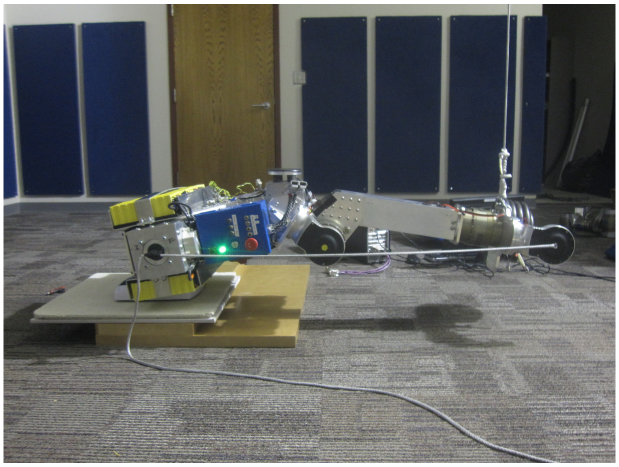
3 Experiment 1: Weighing

$$
\begin{aligned}
m_1 + m_2 &= 112.6\,\mathrm{kg} \\
m_2 &= 31.4 + 2 \cdot 4.0 - 3.6 = 35.8\,\mathrm{kg}\ (\pm 1\,\mathrm{kg}) \\
\Rightarrow m_1 &= 112.6 - 35.8 = 76.8\,\mathrm{kg}
\end{aligned}
$$

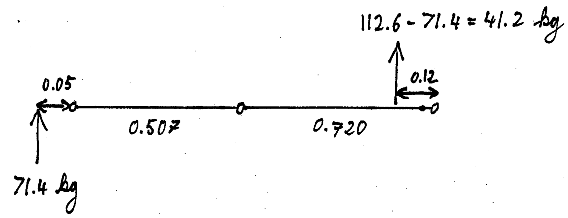
4 Experiment 2: 2-Point Weighing

The setup for the experiment is shown in Fig. 2. The pitch angle is $\theta_1 = -\frac{\pi}{2}$ and the torso angle is $\theta_2 = 0$. From the weight on the scale at one end of the robot we can calculate the y-location of the center of mass. This yields the following equation:

$$
\frac{m_1 x_1 + m_2 x_2 + m_2 \ell_1}{m_1 + m_2} = 0.4230 = \frac{a + c}{m_1 + m_2}
$$



(a) Experiment set-up



(b) Measured values

Figure 2: Experiment 2: 2-Point Weighing

5 Experiment 3: Balancing

When balancing, the y-coordinate of the center of mass is zero.

$$
\begin{aligned}
0 &= m_1(x_1 s_1 + y_1 c_1) + m_2(x_2 s_{12} + y_2 c_{12} + \ell_1 s_1) \\
0 &= (m_1 x_1 + m_2 \ell_1) \cdot s_1 + m_1 y_1 \cdot c_1 + m_2 x_2 \cdot s_{12} + m_2 y_2 \cdot c_{12} \\
0 &= a s_1 + b c_1 + c s_{12} + d c_{12} \qquad (1)
\end{aligned}
$$

with

$$
\begin{aligned}
a &= m_1 x_1 + m_2 \ell_1 \\
b &= m_1 y_1 \\
c &= m_2 x_2 \\
d &= m_2 y_2
\end{aligned}
$$

Equation 1 has infinite solutions. For some solution multiplying the whole equation by a constant factor gives another solution. To eliminate this redundancy, we normalize the equation by assuming a coefficient of 1 in front of c_1, yielding:

$$-c_1 = \frac{a}{b} s_1 + \frac{c}{b} s_{12} + \frac{d}{b} c_{12}$$

Using collected data consisting of pairs (θ_1, θ_2), we estimate the coefficients $\frac{a}{b}$, $\frac{c}{b}$, $\frac{d}{b}$ by minimizing the squared error, yielding the following 3 equations. The result is shown in Fig. 3.

$$
\begin{aligned}
\frac{a}{b} &= \frac{m_1 x_1 + m_2 \ell_1}{m_1 y_1} = -11.5518 \\
\frac{c}{b} &= \frac{m_2 x_2}{m_1 y_1} = -8.8304 \\
\frac{d}{b} &= \frac{m_2 y_2}{m_1 y_1} = -0.2955
\end{aligned}
$$

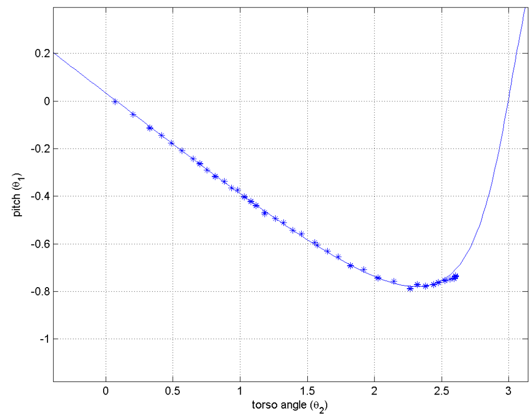


Figure 3: Fitting curve to collected data

6 Experiment 4: Balancing with Additional Weight

We repeat the above experiment with an additional weight attached to the second link. We know the mass and location of the extra weight.

$$
\begin{aligned}
m_2' &= 40\,\mathrm{lbs} + 0.795\,\mathrm{kg} = 18.939\,\mathrm{kg} \\
x_2' &= 0.663\,\mathrm{m} \\
y_2' &= 0.003\,\mathrm{m}
\end{aligned}
$$

This changes the coefficients to

$$
\begin{aligned}
a' &= m_1 x_1 + m_2 \ell_1 + m_2' \ell_1 \\
b &= m_1 y_1 \\
c' &= m_2 x_2 + m_2' x_2' \\
d' &= m_2 y_2 + m_2' y_2'
\end{aligned}
$$

Again, we estimate the new coefficients. The result is shown in Fig. 4. We obtain the following 3 new equations. The 3 equations are not independent. Instead, the results contradict each other.

$$
\begin{aligned}
\frac{a'}{b} &= \frac{m_1 x_1 + m_2 \ell_1 + m_2' \ell_1}{m_1 y_1} = -4.9988 & \Leftrightarrow \quad b &= \frac{m_2' \ell_1}{\frac{a'}{b} - \frac{a}{b}} = 1.4653 \\
\frac{c'}{b} &= \frac{m_2 x_2 + m_2' x_2'}{m_1 y_1} = -4.7620 & \Leftrightarrow \quad b &= \frac{m_2' x_2'}{\frac{c'}{b} - \frac{c}{b}} = 3.0870 \\
\frac{d'}{b} &= \frac{m_2 y_2 + m_2' y_2'}{m_1 y_1} = -0.7189 & \Leftrightarrow \quad b &= \frac{m_2' y_2'}{\frac{d'}{b} - \frac{d}{b}} = 0.3898
\end{aligned}
$$

In order to avoid contradicting equations we reduce the 3 coefficients to one coefficient $\frac{1}{b} = \frac{1}{m_1 y_1}$ given that we know the 3 coefficients $\frac{a}{b}$, $\frac{c}{b}$ and $\frac{d}{b}$ from the previous experiment.

$$
-c_1 = \frac{a}{b} s_1 + \frac{c}{b} s_{12} + \frac{d}{b} c_{12} + \frac{1}{b}(m_2' \ell_1 s_1 + m_2' x_2' s_{12} + m_2' y_2' c_{12})
$$

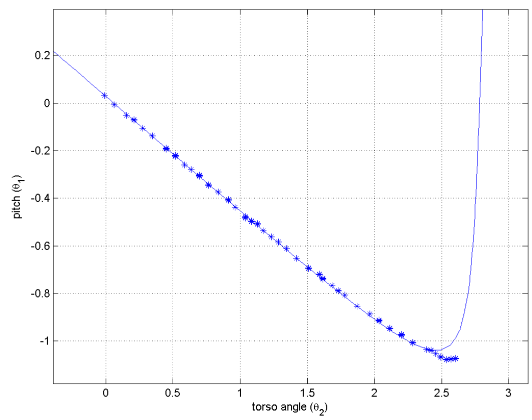
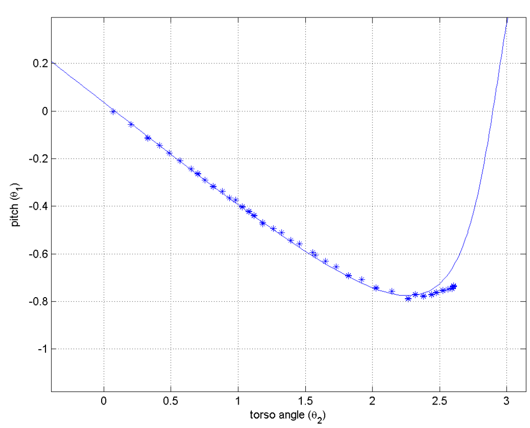
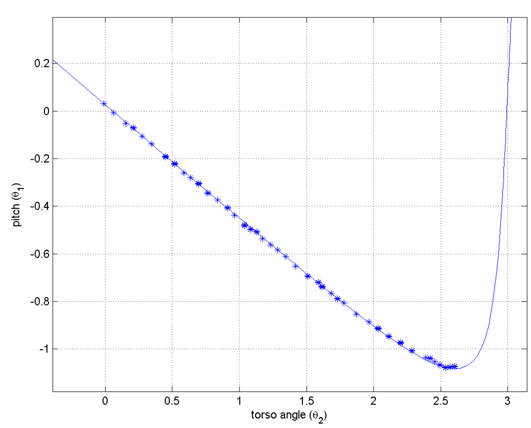


Figure 4: Fitting curve to collected data with an additional weight attached.



(a) without additional weight



(b) with additional weight

Figure 5: Jointly fitted curves

But instead of estimating $\frac{1}{b}$ based on the other 3 coefficients, we estimate all 4 coefficients together by minimizing the squared error over both experiments (with and without additional weight). The results are shown in Fig. 5. This yields the following 4 equations:

$$\begin{aligned}
\frac{a}{b} &= \frac{m_1 x_1 + m_2 \ell_1}{m_1 y_1} = -7.6949 \\
\frac{c}{b} &= \frac{m_2 x_2}{m_1 y_1} = -6.0799 \\
\frac{d}{b} &= \frac{m_2 y_2}{m_1 y_1} = -0.4952 \\
\frac{1}{b} &= \frac{1}{m_1 y_1} = -0.2884
\end{aligned}$$

7 Solution 1

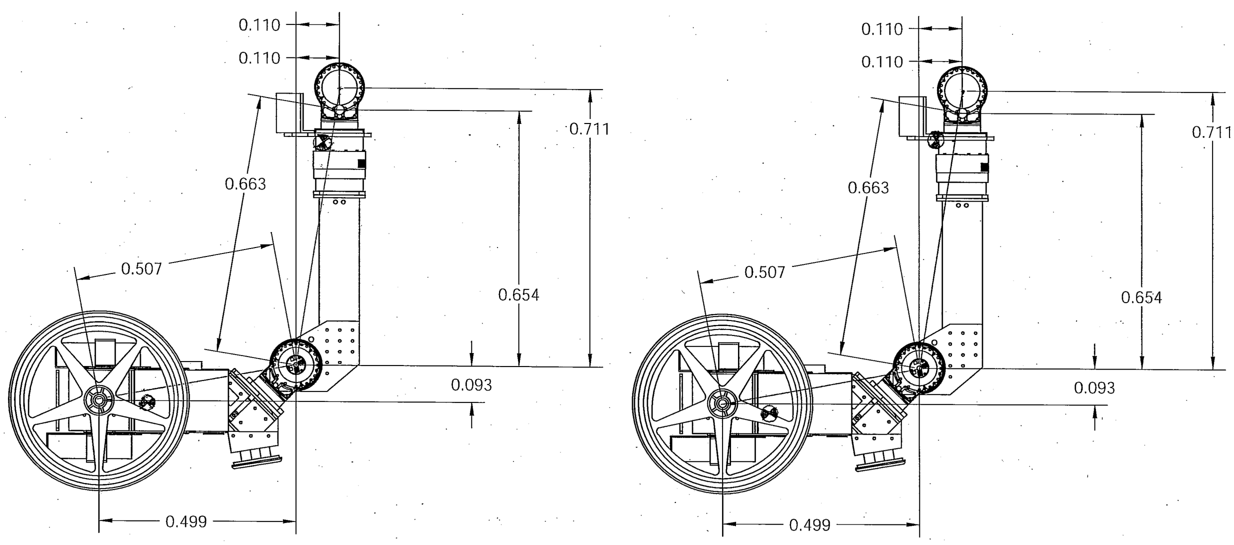
We can solve for the unknown parameters using the equations obtained in experiments 1, 2 and 3. The solution is shown in Fig. 6(a).

$$\begin{aligned}
b &= \frac{\frac{a+c}{m_1+m_2}(m_1 + m_2)}{\frac{a}{b} + \frac{c}{b}} = \frac{0.4230 \cdot (m_1 + m_2)}{-11.5518 + -8.8304} = -2.3368 \\
x_1 &= \frac{\frac{a}{b} b - m_2 \ell_1}{m_1} = 0.1152 \\
y_1 &= \frac{b}{m_1} = -0.0304 \\
x_2 &= \frac{\frac{c}{b} b}{m_2} = 0.5764 \\
y_2 &= \frac{\frac{d}{b} b}{m_2} = 0.0193
\end{aligned}$$

8 Solution 2

We can solve for the unknown parameters using the equations obtained in experiments 1, 3 and 4. The solution is shown in Fig. 6(b).

$$\begin{aligned}
b &= \frac{1}{\frac{1}{b}} = -7.6949 \\
x_1 &= \frac{\frac{a}{b} b - m_2 \ell_1}{m_1} = 0.1111 \\
y_1 &= \frac{b}{m_1} = -0.0451 \\
x_2 &= \frac{\frac{c}{b} b}{m_2} = 0.5889 \\
y_2 &= \frac{\frac{d}{b} b}{m_2} = 0.0480
\end{aligned}$$



(a) Solution 1 (b) Solution 2

Figure 6: Solutions