CSCI203

Algorithms and Data Structures

Algorithm Efficiency

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Analysis of Algorithms

- Issues to be considered
  - Correctness
  - Time efficiency
  - Space efficiency
  - Optimality
- Approaches
  - Theoretical analysis
  - Empirical analysis

Theoretical Analysis of Time Efficiency

- Time efficiency is analyzed by determining the number of repetitions of the *basic operation* as a function of *input size*

- *Basic operation*: the operation that contributes most towards the running time of the algorithm

input size

$$T(n) \approx c_{op} C(n)$$

running time

execution time
for basic operation

Number of times
basic operation is
executed

Input Size and Basic Operations Examples

Problem	Input size measure	Basic operation
Searching for key in a list of n items	Number of list's items, i.e. n	Key comparison
Multiplication of two matrices	Matrix dimensions or total number of elements	Multiplication of two numbers
Checking primality of a given integer n	n'size = number of digits (in binary representation)	Division
Typical graph problem	# vertices and/or edges	Visiting a vertex or traversing an edge

Empirical Analysis of Time Efficiency

- Select a specific (typical) sample of inputs
- Use physical unit of time (e.g., milliseconds)

  or

  Count actual number of basic operation's executions
- Analyze the empirical data

Best-case, average-case, worst-case

- For some algorithms efficiency depends on form of input:
  - Worst case: $C_{worst}(n)$ – maximum over inputs of size *n*
  - Best case: $C_{best}(n)$ – minimum over inputs of size *n*
  - Average case: $C_{avg}(n)$ – "average" over inputs of size *n*
    - Number of times the basic operation will be executed on typical input
    - NOT the average of worst and best case
    - Expected number of basic operations considered as a random variable under some assumption about the probability distribution of all possible inputs

Example: Sequential search

```
ALGORITHM  SequentialSearch(A[0..n − 1], K)
    //Searches for a given value in a given array by sequential search
    //Input: An array A[0..n − 1] and a search key K
    //Output: The index of the first element of A that matches K
    //         or −1 if there are no matching elements
    i ← 0
    while i < n and A[i] ≠ K do
        i ← i + 1
    if i < n return i
    else return −1
```

- Worst case
  - $C_{worst}(n) = n$
- Best case ?
  - $C_{best}(n) = 1$
- Average case
  - $C_{avg}(n) = \frac{p(n+1)}{2} + n(1-p)$

where p is the probability of finding K in A

Types of formulas for basic operation's count

- Exact formula
  - e.g., $C(n) = n(n-1)/2$
- Formula indicating order of growth with specific multiplicative constant
  - e.g., $C(n) \approx 0.5\, n^2$
- Formula indicating order of growth with unknown multiplicative constant
  - e.g., $C(n) \approx cn^2$

Order of Growth

- Most important: Order of growth within a constant multiple as $n \to \infty$
  - e.g. How much longer does it take to solve problem of double input size?

n	$\log_2 n$	n	$n \log_2 n$	n^2	n^3	2^n	$n!$
10	3.3	10^1	$3.3 \cdot 10^1$	10^2	10^3	10^3	$3.6 \cdot 10^6$
10^2	6.6	10^2	$6.6 \cdot 10^2$	10^4	10^6	$1.3 \cdot 10^{30}$	$9.3 \cdot 10^{157}$
10^3	10	10^3	$1.0 \cdot 10^4$	10^6	10^9		
10^4	13	10^4	$1.3 \cdot 10^5$	10^8	10^{12}		
10^5	17	10^5	$1.7 \cdot 10^6$	10^{10}	10^{15}		
10^6	20	10^6	$2.0 \cdot 10^7$	10^{12}	10^{18}		

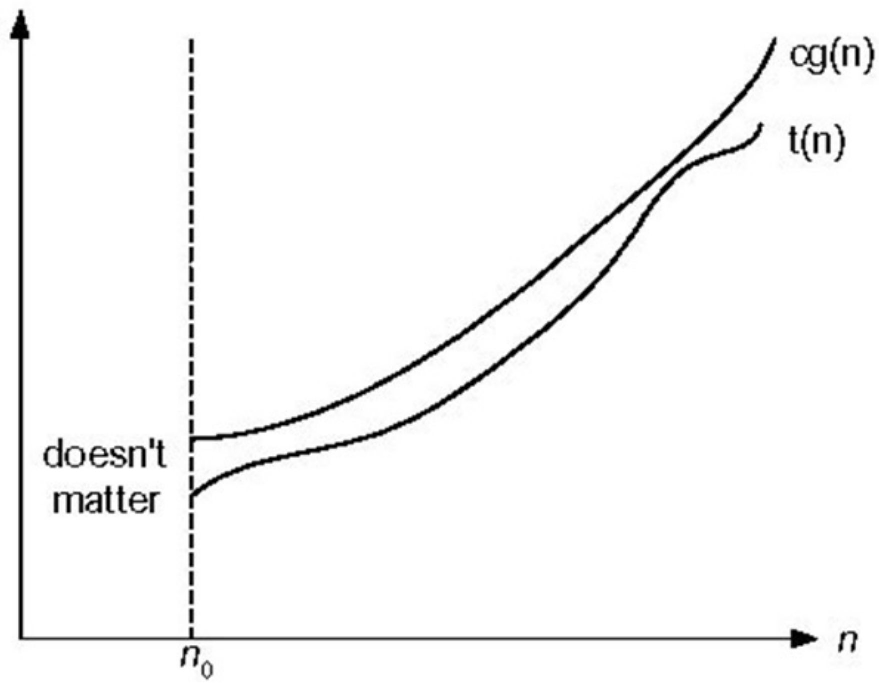
Approximate values of several important functions for analysis of algorithms

Asymptotic Order of Growth

- A way of comparing functions that ignores constant factors and small input sizes

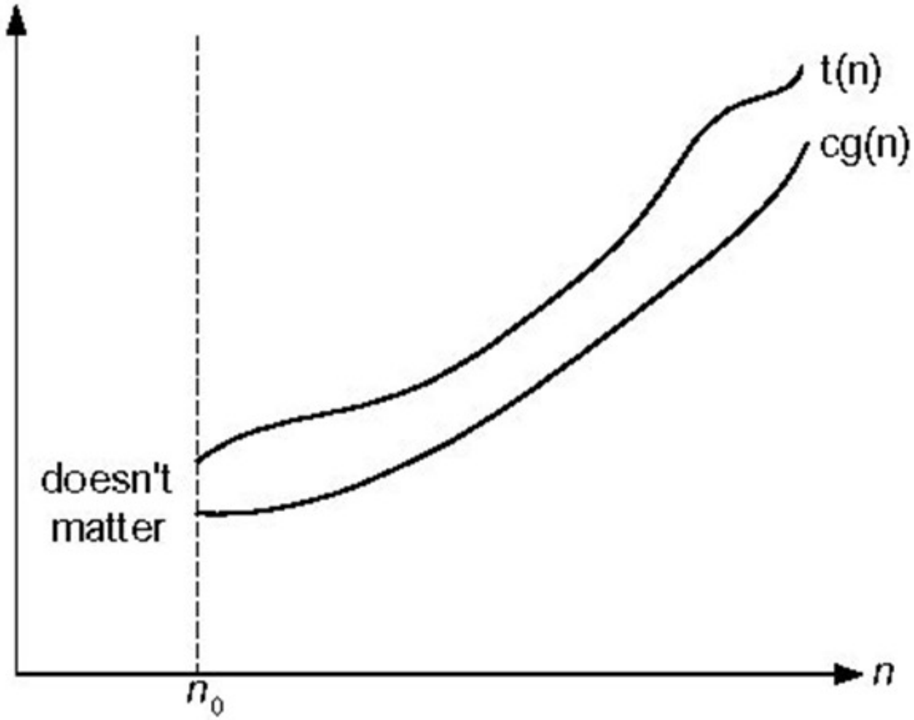
- $O(g(n))$: class of functions $t(n)$ that grow <u>no faster</u> than $g(n)$
  - Related to the worst-case behavior of the algorithm
- $\Omega(g(n))$: class of functions $t(n)$ that grow <u>at least as fast</u> as $g(n)$
  - Related to the best-case behavior of the algorithm

Big-O



Big-O notation: $t(n) \in O(g(n))$

Big-omega



Big-omega notation: $t(n) \in \Omega(g(n))$

Relating O and Ω

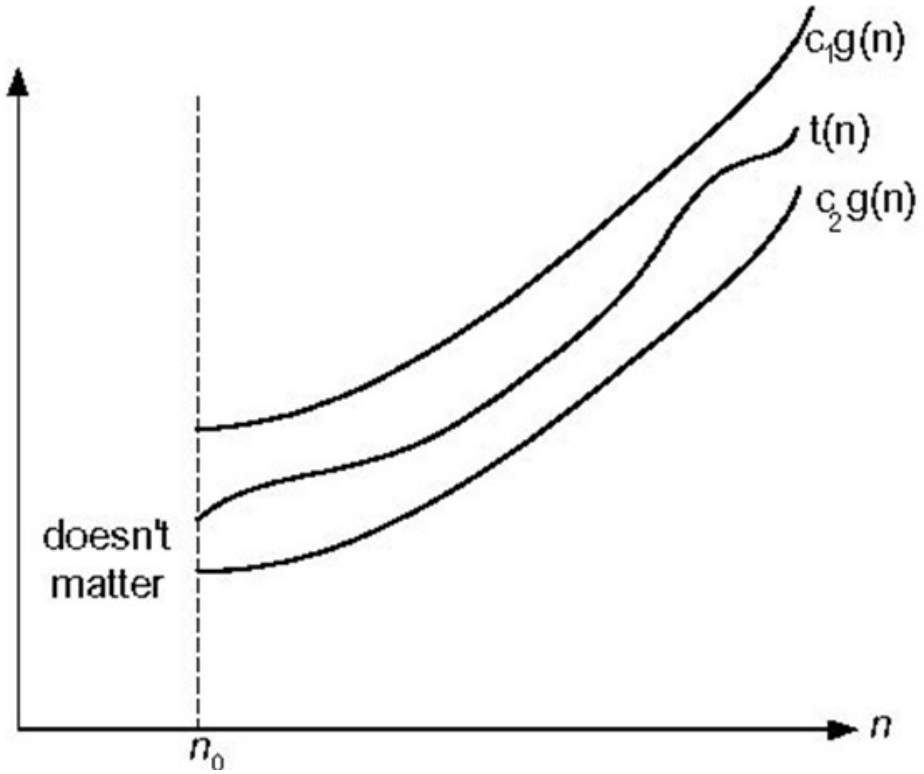
- Both of O and Ω are loose bounds. A function can be in one of these and yet grow much more slowly (or quickly) than the bounding function.
  - E.g. $n^2 \in O(n!)$ and $n! \in \Omega(n^2)$

- A third notation allows us to create a tighter bound for some functions.

Theta, a tighter bound

- Sometimes, we can find a function $g(n)$ so that:
  - $f(n) \in O(\, g(n))$
  - $f(n) \in \Omega(g(n))$
- That is, the same function is both an upper bound and a lower bound.
- In this case we say that $f(n) \in \Theta\, (g(n))$
- $\Theta(g(n))$: class of functions $f(n)$ that grow <u>at same rate</u> as $g(n)$
  - Related to the average or typical case behavior of the algorithm

Big-theta



Big-theta notation: $t(n) \in \Theta(g(n))$

Establishing order of growth using the definition

Definition: $f(n)$ is in $O(g(n))$ if order of growth of $f(n) \leq$ order of growth of $g(n)$ (within constant multiple),

- i.e., there exist positive constant c and non-negative integer n_0 such that

$$f(n) \leq c\, g(n) \text{ for every } n \geq n_0$$

- Examples:
  - $10n$ is $O(n^2)$
  - $5n + 20$ is $O(n)$

Some properties of asymptotic order of growth

- $f(n) \in O(f(n))$
- $f(n) \in O(g(n))$ iff $g(n) \in \Omega(f(n))$
  - *That is to say that f(n) is bounded below by* $g(n)$ *if and only if* $g(n)$ *is bounded above by* $f(n)$.
- If $f(n) \in O(g(n))$ and $g(n) \in O(h(n))$, then $f(n) \in O(h(n))$
- If $f_1(n) \in O\big(g_1(n)\big)$ and $f_2(n) \in O(g_2(n))$, then

$$\mathrm{f}_1(\mathrm{n}) + \mathrm{f}_2(\mathrm{n}) \in O(\max\{g_1(n), g_2(n)\})$$

Establishing order of growth using limits

$$\lim_{n\to\infty} T(n)/g(n) = \begin{cases} 0 & \text{order of growth of } T(n) < \text{order of growth of } g(n) \\ c > 0 & \text{order of growth of } T(n) = \text{order of growth of } g(n) \\ \infty & \text{order of growth of } T(n) > \text{order of growth of } g(n) \end{cases}$$

Examples:

- $\mathrm{T(n)} = 10n$ vs. $\mathrm{g(n)} = n^2$
- $\mathrm{T(n)} = n(n+1)/2$ vs. $\mathrm{g(n)} = n^2$

L'Hôpital's rule and Stirling's formula

- L'Hôpital's rule: If $lim_{n\to\infty} f(n) = limn_{\to\infty} g(n)$ and the derivatives f', g' exist, then

$$\lim_{n\to\infty} \frac{f(n)}{g(n)} = \lim_{n\to\infty} \frac{f'(n)}{g'(n)}$$

Example: $\log n \;\; vs. n$

- Stirling's formula: $n! \approx \sqrt{2\pi n}(n/e)^n$

Example: $2^n \, vs. n!$

An Example

Consider the following algorithm.

```
Algorithm Enigma(A[1⋯n, 1⋯n])
//Input: A matrix of A[1⋯n, 1⋯n] real numbers
for i ← 1 to n − 1 do
    for j ← i + 1 to n do
        if A[i, j] ≠ A[j, i]
            return false
return true
```

- What does this algorithm compute?
- What is its basic operation?
- How many times is the basic operation executed?
- What is the efficiency class of this algorithm?

An Example…

- The algorithm returns "true" if its input matrix is symmetric and "false" if it is not.
- Comparison of two matrix elements.
- $C_{worst}(n) = \sum_{i=1}^{n-1} \sum_{j=i+1}^{n-1} 1 = \sum_{i=1}^{n-1} [(n-1) - (i+1) + 1]$

$$= \sum_{i=1}^{n-1} (n-1-i) = (n-1) + (n-2) + \cdots + 1 = \frac{(n-1)n}{2} = \frac{n^2 - n}{2}$$

- Quadratic $C(n) \in O(n^2)$

Basic asymptotic efficiency classes

1	constant
$\log n$	logarithmic
n	linear
$n \log n$	n-log-n or linearithmic
n^2	quadratic
n^3	cubic
2^n	exponential
$n!$	factorial

Related References

- Introduction to the Design and Analysis of Algorithms, A. Levitin, 3rd Ed., Pearson 2011.
  - Chapters 2.1 & 2.2
- Introduction to Algorithms, T. H. Cormen, 3rd Ed, MIT Press 2009.
  - Chapters 2 & 3