Density of a non-central χ^2 - variable.

Write $\chi'^2_n(\delta^2) = \chi^2_{n-1}(0) + \chi_1^2(\delta^2)$

$= u + v \;; \quad u \perp v$

By defn; $F_V(v) = P(V \le v) = P(X^2 \le v)$, $X \sim N(\delta^2, 1)$

$= P(-\sqrt{v} \le X \le \sqrt{v})$

$$= \int_{-\sqrt{v}}^{\sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\delta^2)} dx$$

$\therefore$ The density of V is

$$f_V(v) = \frac{d}{dv} F_V(v) = \frac{d}{dv} \int_{\sqrt{v}}^{\sqrt{v}} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(x-\delta^2)} dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{-\frac{1}{2}(\sqrt{v}-\delta^2)}}{2\sqrt{v}} + \frac{e^{-\frac{1}{2}(-\sqrt{v}-\delta^2)}}{2\sqrt{v}}$$

[Recall Leibniz's rule of differentiating an integral.
If $h(t)$, $g(t)$, $f(t,x)$ are all differentiable,

$$\frac{d}{dt} \int_{g(t)}^{h(t)} \frac{d f(t,x)}{dt} dx + f(h(t),x) \frac{dh}{dt} - f(g(t),x) \frac{dg}{dt} \;]$$

$$\therefore f_V(v) = \frac{1}{2\sqrt{2\pi v}} e^{-\frac{1}{2}(v+\delta^4)} \left\{ e^{\delta^2\sqrt{v}} + e^{-\delta^2\sqrt{v}} \right\}$$

$$= \frac{1}{2\sqrt{2\pi v}} e^{-\frac{1}{2}(v+\delta^4)} \left\{ \sum_{i=0}^{\infty} \frac{(\delta^2\sqrt{v})^{2i}}{(2i)!} \right\}$$

Since $e^x = \sum \frac{x^i}{i!}$

$$= \frac{1}{2\sqrt{2\pi v}} e^{-\frac{1}{2}(v+\delta^4)} \cancel{\sum_{i=0}^{\infty} \frac{(\sqrt{v}\,\delta^2)^{2i}}{(2i)!}} \sum_{i=0}^{\infty} \frac{(v\delta^4)^i}{(2i)!}$$

The joint density of v & u is

$$f_{v,u}(v,u) = f_v(v)\, f_u(u)$$

with $f_u(u) = \dfrac{e^{-\frac{u}{2}}\, u^{\frac{n-3}{2}}}{2^{\frac{n-1}{2}}\, \Gamma(\frac{n-1}{2})}$

Let $Z = u+v$ & $W = u/(u+v)$

The Jacobian[5] for this transformation is given by

$$\left.\begin{aligned} u &= zw \\ v &= z-u = z(1-w) \end{aligned}\right\} \quad J = \left\| \begin{matrix} w & z \\ 1-w & -z \end{matrix} \right\| = |-wz - z + zw| = z$$

So, $f_{Z,W}(z,w) = f_v(z(1-w))\, f_u(zw)\, z$

$$= \frac{1}{2\sqrt{2\pi \underset{z(1-w)}{\overset{}{v}}}} e^{-\frac{1}{2}(z-zw+\delta^4)} \sum_{i=0}^{\infty} \frac{((z-zw)\delta^4)^i}{(2i)!} \cdot z$$

$$\cdot \frac{e^{-zw/2} (zw)^{(n-3)/2}}{2^{\frac{n-1}{2}}\, \Gamma(\frac{n-1}{2})} \cdot z$$

$$= C\, e^{-\frac{1}{2}(z+\delta^4)} z^{\frac{n-2}{2}} (1-w)^{-1/2} \sum_{i=0}^{\infty} (1-w)^i \frac{\delta^{4i} z^i}{(2i)!}$$

$$\therefore f_2(z) = C e^{-\frac{1}{2}(z+\delta^4)} z^{\frac{n-2}{2}} \sum_{i=0}^{\infty} \frac{\delta^{4i} z^i}{(2i)!} \underbrace{\int_0^1 (1-w)^{i-1/2}\, dw}_{= \frac{1}{(i+\frac{1}{2})!}}$$

or

$$f_2(z) = \frac{1}{2\sqrt{2\pi}} \frac{e^{-\frac{1}{2}(z+\delta^4)}}{2^{\frac{n-1}{2}} \Gamma\left(\frac{n-1}{2}\right)} z^{\frac{n-2}{2}} \sum_{i=0}^{\infty} \frac{\delta^{4i} z^i}{(2i)!\,(i+\frac{1}{2})!}$$

$$= \sum_{i=0}^{\infty} \frac{e^{-\lambda/2}\left(\frac{\lambda}{2}\right)^i}{i!} \cdot \frac{1}{\Gamma\left(\frac{n+2i}{2}\right)} \left(\frac{1}{2}\right)^{\frac{n+2i}{2}} z^{\frac{n+2i}{2}-1} e^{-\frac{z}{2}}$$

where $\lambda = \delta^4$ (close enoug

So it has the form

$$\sum_{i=0}^{\infty} q_i \cdot \text{density of a } \chi^2_{n+2i}(0)$$

where q_i's are Poisson weights.