Biostat 250C HW9

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Q1: Complete the (?) part of the following.

(1): $Vec(AB) = (?)\, Vec(B) = (?)\, Vec(A)$

(2): $Vec(ABC) = (?)\, Vec(C) = (?)\, Vec(A)$

Sol. (1): Let $A \in \mathbb{R}^{m\times n}$, $B \in \mathbb{R}^{n \times q}$ &

$$A = \begin{bmatrix} a_{1*}^T \\ \vdots \\ a_{m*}^T \end{bmatrix}, \quad B = \begin{bmatrix} b_{*1} & \cdots & b_{*q} \end{bmatrix}$$

$$\Rightarrow Vec(AB) = \begin{bmatrix} Ab_{*1} \\ Ab_{*2} \\ \vdots \\ Ab_{*q} \end{bmatrix}$$

$$= \begin{bmatrix} \overbrace{A \quad\quad\quad\quad}^{q \text{ of them}} & & \\ & A & \\ & & \ddots & \\ & & & A \end{bmatrix} \begin{bmatrix} b_{*1} \\ b_{*2} \\ \vdots \\ b_{*q} \end{bmatrix}$$

$$= (I_q \otimes A)\, vec(B)$$

Next,

$$Vec(AB) = Vec\Big(\sum_{i=1}^{n} \underset{\text{cols of } A}{a_{*i}}\, \underset{\text{rows of } B}{b_{i*}^T}\Big)$$

$$= \sum_{i=1}^{n} Vec(a_{*i} b_{i*}^T)$$

$$= \sum_{i=1}^{n} b_{i*} \otimes a_{*i}$$

$$= (B^T \otimes I_m)\, Vec(A)$$

Since
$$b_{i*} \otimes a_{*i} = \begin{bmatrix} b_{i1} a_{*i} \\ b_{i2} a_{*i} \\ \vdots \\ b_{iq} a_{*i} \end{bmatrix}$$

$$= \begin{bmatrix} b_{i1} I_m \\ \vdots \\ b_{iq} I_m \end{bmatrix} a_{*i}$$

(2): Let $C \in \mathbb{R}^{q \times p}$ and

$$C = \begin{bmatrix} c_{1*}^T \\ \vdots \\ c_{q*}^T \end{bmatrix} = \begin{bmatrix} c_{*1} & \cdots & c_{*p} \end{bmatrix}$$

- By (1),

$$Vec(ABC) = (I_p \oplus AB)\, Vec(C)$$

$$Vec(ABC) = ((BC)^T \oplus I_m)\, vec(A)$$

Next, let $B = \sum_{i=1}^{q} b_{*i} e_i^T$ where

$$e_i^T = [0 \;\; 0 \;\; \cdots \;\; \underset{\text{ith position}}{1} \;\; \cdots \;\; 0]$$

$$Vec(ABC) = Vec\Big(\sum_{i=1}^{q} Ab_{*i} e_i^T C\Big)$$

$$= \sum_{i=1}^{q} Vec[(Ab_{*i})(e_i^T C)]$$

$$= \sum_{i=1}^{q} (C^T e_i) \otimes (Ab_{*i})$$

$$\overset{(\Delta)}{=} \sum_{i=1}^{q} [C^T \otimes A]\, e_i \otimes b_{*i}$$

$$= \sum_{i=1}^{q} (C^T \otimes A)\, Vec(B)$$

(Δ) comes from a lemma in Lecture:

$$(A \otimes B)(C \otimes D) = (AC \otimes BD)$$