Scheffé's Method

$\mathbb{E}Y = X\beta$. $A \in \mathbb{R}^{g\times p}$, $r(A) = g$,

WT find an interval estimate for $A\beta$:

$$\hat\beta \equiv (X^TX)^{-1}X^T\beta.$$

$$A\hat\beta \sim \mathcal{N}_g\left(A\beta, \sigma^2 A(X^TX)^{-1}A^T\right)$$

$$\frac{1}{\sigma^2}(A\hat\beta - A\beta)^T\left[A(X^TX)^{-1}A^T\right]^{-1}(A\hat\beta - A\beta) \sim \chi^2_g$$

& $S^2 = \dfrac{\hat\sigma^2(n-p)}{\sigma^2} \sim \chi^2_{n-p}$, $p = \dim(\beta)$

so $$\frac{(A\hat\beta - A\beta)^T\left(A(X^TX)^{-1}A^T\right)^{-1}(A\hat\beta - A\beta)/g}{S^2} \sim F_{g,n-p}$$

so $1-\alpha = \mathbb{P}(F_{g,n-p} \le F_{g,n-p,\alpha})$.

Let $\phi \equiv A\beta$. $L \equiv A(X^TX)^{-1}A^T$

Then

$$1-\alpha = \mathbb{P}\Big(\overbrace{(\hat\phi-\phi)}^{b}{}^T L^{-1}(\hat\phi-\phi) \le \overbrace{g\cdot S^2 F_{g,n-p,\alpha}}^{m}\Big)$$

$$= \mathbb{P}(b^TL^{-1}b \le m)$$

$$= \mathbb{P}\left(\max_{h\ne 0}\frac{(h^Tb)^2}{h^TLh} \le m\right) \text{ [KEY STEP]}$$

$$= \mathbb{P}\left(\forall_{h\ne 0} h,\ (h^Tb)^2 \le m\cdot h^TLh\right)$$

$$= \mathbb{P}\left(\forall h,\ |h^Tb| \le \sqrt{m\, h^TLh}\right)$$

$$= \mathbb{P}\left(\forall h: h^T\phi \in h^T\hat\phi \pm \sqrt{m\, h^TLh}\right)$$

Consider multiple comparison tests for one-way ANOVA

Def: Let $Z_1, \dots, Z_k$ & U are indept. RV with $Z_i \sim \mathcal{N}_1(0,1)$ & $U \sim \chi^2_m(0)$.

define $$q \equiv \max_{i\ne j}\frac{|Z_i - Z_j|}{\sqrt{U/m}}$$

we call q has a studentized range dist. with k & m dfs & write $q \sim q_{k,m}$

Lemma: In a one-way ANOVA

$$Y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$$

$j = 1, 2, \dots, n$; $i = 1, 2, \dots, k$

$\bar Y_{i\cdot} = \mu + \alpha_i + \bar\varepsilon_{i\cdot}$; $\bar Y_{j\cdot} = \mu + \alpha_j + \bar\varepsilon_{j\cdot}$

Verify $$\max_{i\ne j}\frac{|\bar Y_{i\cdot} - \bar Y_{j\cdot} - (\alpha_i - \alpha_j)|}{\hat\sigma} \sim q_{k,\underbrace{k(n-1)}_{\text{df error for a 1-way ANOVA}}}$$

It follows

$$\mathbb{P}\left(\alpha_i - \alpha_j \in \bar Y_{i\cdot} - \bar Y_{j\cdot} \pm \frac{\hat\sigma}{\sqrt n} q_{k,k(n-1),\alpha} \text{ for } i\ne j\right)$$

$$= \mathbb{P}\left(\frac{\sqrt n(\bar Y_{i\cdot} - \bar Y_{j\cdot} - (\alpha_i - \alpha_j))}{\hat\sigma} \le q_{k,k(n-1),\alpha} \text{ for } i \ne j\right)$$

$= 1-\alpha$. (Tukey's pair-comparison)

Q: How to construct sets of C.I. for contrasts in a 1-way ANOVA.

Lemma: Let $a_1, a_2, \dots, a_k$ be numbers.

Then $|a_i - a_j| \le b \ \forall i, j$

$$\iff \left|\sum_{i=1}^k c_i a_i\right| \le \frac{b}{2}\left(\sum_{i=1}^k |c_i|\right)$$

$\forall c$ s.t. $\sum_{i=1}^k c_i = 0$ (s.t. $c^T1 = 0$)

Goal: Construct CI for $\sum_i c_i\alpha_i$ $\forall c$

$$\mathbb{P}\left(\sum_i c_i\alpha_i \in \sum_i c_i\bar Y_{i\cdot} \pm \frac{\hat\sigma}{\sqrt n} q_{k,k(n-1),\alpha}\left(\sum_i \frac{|c_i|}{2}\right), \forall c\right)$$

$$= \mathbb{P}\left(\left|\sum_i c_i(\bar Y_{i\cdot} - \alpha_i)\right| \le \frac{\hat\sigma}{\sqrt n} q_{k,k(n-1),\alpha}\left(\sum_i \frac{|c_i|}{2}\right) \forall c\right)$$

$$\overset{\text{lemma}}{=} \mathbb{P}\left(|\bar Y_{i\cdot} - \alpha_i - (\bar Y_{j\cdot} - \alpha_j)| \le \frac{\hat\sigma}{\sqrt n} q_{k,n(k-1),\alpha}\ \forall i,j\right)$$

$= 1-\alpha$. $\square$