J-S Estimator

$\underset{n\times p}{\mu = X}\beta \in C(X)$, $Y \sim \mathcal{N}_n(\mu, \sigma^2 I)$, $\sigma^2 > 0$.

$\hat{\mu} = P_X Y$, $\hat{\sigma}^2 = \dfrac{Y^T Q_X Y}{n-p}$

⓪ $\mathbb{E}\|\hat{\mu}-\mu\|_2^2 = \mathbb{E}\|P_X Y - \mu\|_2^2 = \mathbb{E}\|P_X(Y-\mu)\|_2^2 = \mathrm{Tr}\,\mathbb{E}\,P_X(Y-\mu)(Y-\mu)^T = \mathrm{Tr}\,P_X \,\mathrm{Var}\,Y = \mathrm{Tr}\,P_X = p.$

① $U \sim \mathcal{N}_p(\theta, I)$. $K \sim \mathrm{Poi}\left(\frac{\|\theta\|^2}{2}\right)$

$\Rightarrow$ (a): $\mathbb{E}\dfrac{1}{\|U\|_2^2} = \mathbb{E}\dfrac{1}{p-2+2K}$ (b): $\mathbb{E}\dfrac{U^T(U-\theta)}{\|U\|_2^2} = \mathbb{E}\dfrac{p-2}{p-2+2K}$.

Pf: (a): $V|K \sim \chi^2_{p+2K}(0) \Rightarrow V \sim \chi^2_p(\|\theta\|_2^2)$

$p(v) = \int p(v|k)\,\mathbb{P}(dk) \Rightarrow \mathbb{E}\dfrac{1}{\|U\|_2^2} = \mathbb{E}\left(\mathbb{E}\left(\frac{1}{V}\,\Big|\,K\right)\right) = \mathbb{E}\dfrac{1}{p-2+2K}$

(b) Tricky.

$a(\hat{\mu}, \hat{\sigma}^2) = 1 - c\,\hat{\sigma}^2 / \|\hat{\mu}\|_2^2$