Biostat 250C HW2

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Problem 1: Repeat the algorithm for 3-variables:

$$p(x,y,z) = \frac{?}{?} \times p(x_0,y_0,z_0)$$

where $\frac{?}{?}$ depends upon $p(\cdot|\cdot)$.

Sol. $$p(x_0,y_0,z) = p(x_0,y_0,z_0)\frac{p(z|x_0,y_0)}{p(z_0|x_0,y_0)}$$

$$p(x_0,y,z) = p(x_0,y_0,z)\frac{p(y|x_0,z)}{p(y_0|x_0,z)}$$

$$p(x,y,z) = p(x_0,y,z)\frac{p(x|y,z)}{p(x_0|y,z)}$$

$$\Rightarrow \quad p(x,y,z)$$

$$= p(x_0,y_0,z_0)\frac{p(z|x_0,y_0)}{p(z_0|x_0,y_0)}\frac{p(y|x_0,z)}{p(y_0|x_0,z)}\frac{p(x|y,z)}{p(x_0|y,z)}$$

Problem 2: $$y|\beta,\sigma^2 \sim \mathcal{N}(X\beta, \sigma^2 V_y)$$
$$\beta|\sigma^2 \sim \mathcal{N}(\mu_\beta, \sigma^2 V_\beta)$$
$$\sigma^2 \sim IG(a,b)$$

Show that

$$p(\sigma^2|y) \propto p(0,\sigma^2)p(y|0,\sigma^2)/p(0|\sigma^2,y)$$

Sol. $$p(\sigma^2|y)p(\beta|\sigma^2,y) = p(\beta,\sigma^2|y)$$
$$\propto p(\beta,\sigma^2)p(y|\beta,\sigma^2)$$

$$\Rightarrow \quad p(\sigma^2|y) \propto \frac{p(\beta,\sigma^2)p(y|\beta,\sigma^2)}{p(\beta|\sigma^2,y)}$$

By Problem 1,

$$p(\beta,\sigma^2,y)$$
$$= p(\beta_0,\sigma_0^2,y_0)\frac{p(\sigma^2|y_0,\beta_0)}{p(\sigma_0^2|y_0,\beta_0)}\frac{p(y|\beta_0,\sigma^2)}{p(y_0|\beta_0,\sigma^2)}\frac{p(\beta|\sigma^2,y)}{p(\beta_0|\sigma^2,y)}$$

$$\propto \left[\frac{p(\beta|\sigma^2,y)}{p(\beta_0|\sigma^2,y)}p(y|\beta_0,\sigma^2)\right]\left[\underbrace{\frac{p(\sigma^2|y_0,\beta_0)}{p(y_0|\beta_0,\sigma^2)}}_{(*)}\right]$$

$(*) \propto p(\beta_0,\sigma^2)$

Hence, $$p(\beta,\sigma^2,y) \propto p(\beta_0,\sigma^2)p(y|\beta_0,\sigma^2)\frac{p(\beta|\sigma^2,y)}{p(\beta_0|\sigma^2,y)}$$

Take $\beta_0 = 0$, we have

$$p(\sigma^2|y) \propto \frac{p(0,\sigma^2)p(y|0,\sigma^2)}{p(0|\sigma^2,y)} \qquad \square.$$

After some algebra,

$$p(\sigma^2|y) \propto (\sigma^2)^{a+1+\frac{n}{2}} e^{-\frac{1}{\sigma^2}\left[b+\frac{c^*}{2}\right]}$$

where $c^* = \beta^T V_\beta^{-1}\beta + y^T V_y y - m^T M m$

$M = (V_\beta^{-1} + X^T V_y^{-1} X)^{-1}$ and $m = V_\beta^{-1}\mu_\beta + X^T V_y^{-1} y$.

$$\Rightarrow \sigma^2|y \sim IG\left(a+\frac{n}{2}, b+\frac{c^*}{2}\right)$$