- **Mixed linear model**
  - $Y = X\beta + Z\gamma + \varepsilon$

    $\mathbb{E}\varepsilon = 0$, $\mathrm{Cov}\,\varepsilon = R$
  - β: unknown vector of fixed effect.
  - γ : a vector of random effects

    $\mathbb{E}\gamma = 0$, $\mathrm{Cov}\,\gamma = D$,

    X, Z are known, $\mathrm{Cov}(\gamma, \varepsilon) = 0$.

$\mathbb{E}\,Y = X\beta$; $\mathrm{Cov}\,Y = ZDZ^T + R \equiv V$

To find MLEs, assume

$\gamma \sim \mathcal{N}(0, D)$, $\varepsilon \sim \mathcal{N}(0, R)$

Find MLEs for β & γ.

Write down likelihood function for y & γ in two steps.

$Y|\gamma \sim \mathcal{N}(X\beta + Z\gamma, R)$

$\gamma \sim \mathcal{N}(0, D)$

$$f_{Y,\gamma}(y,\gamma) = f_{Y|\gamma}(y) f_\gamma(\gamma)$$

$$\propto \frac{1}{|R|^{\frac{1}{2}}} e^{-\frac{1}{2}(y - X\beta - Z\gamma)^T R^{-1}(y - X\beta - Z\gamma)}$$

$$\frac{1}{|D|^{\frac{1}{2}}} e^{-\frac{1}{2}\gamma D^{-1}\gamma}$$

Dif. $\ln f_{y,\gamma}(y,\gamma)$ w.r.t. β & γ

results

$$\begin{bmatrix} X^T R^{-1} X & X^T R^{-1} Z \\ Z^T R^{-1} X & D^{-1} + Z^T R^{-1} Z \end{bmatrix} \begin{pmatrix} \beta \\ \gamma \end{pmatrix} = \begin{bmatrix} X^T R^{-1} y \\ Z^T R^{-1} y \end{bmatrix}$$

This is the set of Henderson's equations.

Can I Solve for $\hat\beta$ & $\hat\gamma$?

$$V = ZDZ^T + R = \mathrm{Cov}(y)$$

$$V^{-1} = (ZDZ^T + R)^{-1} \quad \text{Woodbury}$$

$$= R^{-1} - R^{-1} Z (D^{-1} + Z^T R^{-1} Z)^{-1} Z^T R^{-1}$$

$$\begin{pmatrix} \hat\beta \\ \hat\gamma \end{pmatrix} = \begin{pmatrix} \cdot & \cdot \\ \cdot & \cdot \end{pmatrix}^{-1} \begin{pmatrix} X^T R^{-1} y \\ Z^T R^{-1} y \end{pmatrix}$$

2^{nd}:

- $Z^T R^{-1} X \hat\beta + (D^{-1} + Z^T R^{-1} Z)\hat\gamma = Z^T R^{-1} y$

Express $\hat\gamma$ in terms of $(y - X\hat\beta)$

1^{st}:

- $X^T R^{-1} X \hat\beta + X^T R^{-1} Z \hat\gamma = X^T R^{-1} y$.

Notice:

$$\hat\gamma = \left(D\{D^{-1} + Z^T R^{-1} Z\} - DZ^T R^{-1} Z\right) \times$$
$$(D^{-1} + Z^T R^{-1} Z)^{-1} Z^T R^{-1} (y - X\hat\beta)$$
$$= \cdots =$$
$$D Z^T V^{-1} (y - X\hat\beta)$$

(Ex) MLE's for dif. scenerios in the attached notes.

BLUP: best linear unbiased predictor (of u)

To be a BLUP, the following are requirements.

① $\hat{u}$ is a linear fct. of Y.

② $\hat{u}$ is unbiased for u:

$$\mathbb{E}(\hat{u}-u)=0.$$

③ $Var(\hat{u}-u) \le Var(v-u)$ & v is any linear unbiased predictor.

How does $\hat{u}$ look like?

$$\gamma \sim \mathcal{N}(0, D)$$

$$Y \sim \mathcal{N}(X\beta, V), \quad V = ZDZ^T + R$$

Then,

$$\begin{pmatrix} \gamma \\ Y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ X\beta \end{pmatrix}, \begin{pmatrix} D & DZ^T \\ ZD & V \end{pmatrix}\right)$$

$$Cov(\gamma, Y) = Cov(\gamma, X\beta + Z\gamma + \varepsilon)$$

$$= DZ^T$$

$$\mathbb{E}\gamma | Y = DZ^T V^{-1}(Y - X\beta)$$

$$\mathbb{E}(\mathbb{E}(\gamma | Y)) - \mathbb{E}\gamma = 0.$$