Syllabus

Review of 250A final

Q1

(a)(i) $Y \sim N(0, I_p)$

$Y \mid 1^T Y = 0 \sim ?$

Sol. Define $Z = \begin{pmatrix} Y \\ 1^T Y \end{pmatrix} = \begin{pmatrix} I_p \\ 1^T \end{pmatrix} Y$

$\operatorname{Cov} Z = \begin{pmatrix} I_p & 1 \\ 1^T & p \end{pmatrix}$, $Z = \begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix}$

$\Rightarrow Z_1 \mid Z_2 = 0 \sim N(0, I - \frac{11^T}{p})$

$x^T (I - \frac{11^T}{p}) x = \sum_i x_i^2 - p\bar{x}^2$

$= \sum_i (x_i - \bar{x})^2 \geq 0$

(ii) $Y^T I Y \mid a^T Y = 0 \sim ?$ $(a \neq 0)$

Sol. Let $\underset{p\times 1}{Z} = \underset{p \times p}{\begin{pmatrix} \frac{a^T}{\|a\|_2} \\ Q_{(p-1)\times p} \end{pmatrix}} \underset{p\times 1}{Y}$, $QQ^T = I$

$= AY = b = \begin{pmatrix} b_1^T Y \\ b_2^T Y \\ \vdots \\ b_p^T Y \end{pmatrix}$

$Z^T Z = Y^T A^T A Y = b^T b = \sum_{i=2}^{p} (b_i^T Y)^2$ $(\triangle)$

$b_i^T Y \sim N(0, \underset{1}{\underset{\shortparallel}{b_i^T b_i}})$

$\Rightarrow (\triangle) \sim \chi^2_{p-1}(0)$

$\Rightarrow E(\triangle) = p-1$.

(b) $Y \sim N(0, \Sigma)$, $\Sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{pmatrix}$.

$Q = Y^T \Sigma^{-1} Y - \frac{Y_1^2}{\sigma_1} \sim ?$

Sol. $Y^T \underbrace{(\Sigma^{-1} - \begin{pmatrix} \frac{1}{\sigma_{11}} & 0 \\ 0 & 0 \end{pmatrix})}_{A} Y$

ETS $A^2 = A$ since $A^T = A$ by Fundamental Thm.

Verify $AV = AVAV$, $V = \Sigma$

Then $\operatorname{rank}(AV) = \operatorname{Tr}(AV) = 1$

$\Rightarrow Q \sim \chi^2_1(0)$

Q2

(a): $\hat{Y} = PY = \left(\sim \right) \begin{pmatrix} Y_1 \\ \vdots \\ Y_8 \end{pmatrix}$

$\hat{\sigma}^2 = \frac{400}{8-4} = 100$

$e = Y - \hat{Y} = (I - P) Y$

Sol. $\widehat{\operatorname{Var} e_2} = (1 - P_{22}) \hat{\sigma}^2 = 62.6$

$\widehat{\operatorname{Var} e_3} = 68.2$

$\widehat{\operatorname{Var} e_1} = ???$ $(Q = I - P)$

$\widehat{\operatorname{Cov}(e_1, e_2)} = \hat{\sigma}^2 Q_{12} = \hat{\sigma}^2 Q_{21}$

$= 100 (0 - (-0.242))$

$= 24.2$

$\widehat{\operatorname{Cov}(e_1, e_3)} = \hat{\sigma} Q_{31} = 6.1$

$\widehat{\operatorname{Cov}(e_2, e_3)} = -32.9$

(b): $P^2 = P$, $(P^2)_{ii} = (P)_{ii}$

$$P_{ii} = \sum_k P_{ik} P_{ki} = \sum_k P_{ik}^2 \quad (\triangle)$$

If r replicates

$$(\triangle) \geq r P_{ii}^2 \Rightarrow \frac{1}{r} \geq P_{ii}$$

$\frac{1}{3} \geq P_{22}$ if 2 other rows

are the same as row 2.

So the answer is "NO"!

(c): (i): $P_{ij} = \sum_k P_{ik} P_{kj} \Rightarrow$

$$(P_{ij})^2 = (\sum_k P_{ik} P_{kj})^2 \quad \text{(C-S ineq.)}$$

$$\leq \underbrace{(\sum_k P_{ik}^2)}_{P_{ii}} \underbrace{(P_{kj}^2)}_{P_{jj}}$$

(ii): $P^2 = P \Rightarrow (I-P)^2 = I-P$

Let $Q = I-P$, then:

$q_{ij} \leq q_{ii} q_{jj}$ but

$q_{ij} = -P_{ij}$ & $q_{ii} = 1 - P_{ii}$,
$q_{jj} = 1 - P_{jj}$.

Q3: $y = \alpha 1 + X\beta + \varepsilon$, $\mathbb{E}\varepsilon = 0$

$\mathbb{E}\varepsilon\varepsilon^T = \mathrm{Cov}\,\varepsilon = V^{-1}$.

Find $\hat{\alpha}$!

Sol. $V^{\frac{1}{2}} y = \alpha \underbrace{V^{\frac{1}{2}} 1} + \underbrace{V^{\frac{1}{2}} X}\beta + V^{\frac{1}{2}}\varepsilon$

$\overset{\triangle}{\tilde{y}} = \alpha V^{\frac{1}{2}} 1 + (P+Q) V^{\frac{1}{2}} X\beta + V^{\frac{1}{2}}\varepsilon$

$P = P_{C(V^{\frac{1}{2}}1)} = V^{\frac{1}{2}} 1 1^T V^{\frac{1}{2}} / 1^T V 1$,

$Q = I - P$.

$$\Rightarrow \mathbb{E}\tilde{y} = V^{\frac{1}{2}} 1 \overbrace{\left(\alpha + \frac{1^T V^{\frac{1}{2}} V^{\frac{1}{2}} X\beta}{1^T V 1}\right)}^{\gamma} + Q V^{\frac{1}{2}} X\beta$$

$$= \overbrace{V^{\frac{1}{2}} 1}^{\otimes}\gamma + \overbrace{Q V^{\frac{1}{2}} X}^{\odot}\beta \qquad \odot \perp \otimes\,!$$

$$\hat{\gamma} = \frac{1^T V^{\frac{1}{2}}}{1^T V 1} V^{\frac{1}{2}} y = \frac{1^T V y}{1^T V 1}$$

$$= \hat{\alpha} + \frac{1^T V X\hat{\beta}}{1^T V 1}$$

$$\Rightarrow \hat{\alpha} = -\frac{1^T V X\hat{\beta}}{1^T V 1} + \frac{1^T V y}{1^T V 1}$$

$$\hat{\beta} = (X^T V^{\frac{1}{2}} Q V^{\frac{1}{2}} X)^{-1} X^T V^{\frac{1}{2}} Q V^{\frac{1}{2}} y$$

Verify

$$\hat{\alpha} = \frac{1^T (V - VX(X^T W^{-1} X)^{-1} X^T W^{-1}) y}{1^T V 1}.$$