Biostat 250C

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Q1: $A_{3\times 3}$. Fill out a table of all perm. of (1,2,3). Then find $|A|$.

Sol.

π	term	sign
(1,2,3)	$a_{11}a_{22}a_{33}$	+1
(1,3,2)	$a_{11}a_{23}a_{32}$	−1
(2,1,3)	$a_{12}a_{21}a_{33}$	−1
(3,2,1)	$a_{13}a_{22}a_{31}$	−1
(2,3,1)	$a_{12}a_{23}a_{31}$	+1
(3,1,2)	$a_{13}a_{21}a_{32}$	+1

$$\Rightarrow \det |A| = \det\left|\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}\right|$$

$$= a_{11}a_{22}a_{33} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} - a_{11}a_{23}a_{32} - a_{12}a_{21}a_{33} - a_{13}a_{22}a_{31}.$$

$\square$

Q2: $A_{p\times p}$ & $D_{n\times n}$. Suppose $|A| \neq 0$ and $|D| \neq 0$. Prove

$$\det\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \det A \det D$$

Sol. Let $M = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix}$

$$\det M = \sum_{\pi} (-1)^{\text{sign}(\pi)} m_{1\pi_1} \cdots m_{(p+n)\pi_{p+n}} \quad (\triangle)$$

If π maps any element from

$\{p+1, \dots, p+n\}$ (say, i)

to

$\{1, \dots, p\}$ (say, j)

Then $m_{ij} = 0$ implies the whole term associated with π is 0.

Note there are $(p+n)!$ terms in total.

In other words, summation over π can be replaced with

$$\left\{\begin{array}{l} \text{permutation w.r.t. } \{1, \dots, p\} \\ \qquad + \\ \text{permutation w.r.t. } \{p+1, \dots, p+n\} \end{array}\right.$$

Since other choices will result in a "0" in the summation.

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Let π^A & π^D be the perm. w.r.t. to $\{1,\cdots,p\}$ & $\{p+1,\cdots,p+n\}$ respectively.

Then

$$(\Delta) = \sum_{\pi_A,\pi_D} (-1)^{\mathrm{Sign}(\pi_A)} (-1)^{\mathrm{Sign}(\pi_D)} \times \left(m_{1\pi_1^A}\cdots m_{p\pi_p^A}\right)\left(m_{p+1\,\pi_{p+1}^D}\cdots m_{p+n\,\pi_{p+n}^D}\right)$$

$$= \left(\sum_{\pi_A}(-1)^{\mathrm{Sign}(\pi_A)} m_{1\pi_1^A}\cdots m_{p\pi_p^A}\right) \times \left(\sum_{\pi_D}(-1)^{\mathrm{Sign}(\pi_D)} m_{p+1\,\pi_{p+1}^D}\cdots m_{p+n\,\pi_{p+n}^D}\right)$$

But the first term is

$$\det A$$

and the second term is

$$\det D$$

To sum up,

$$\det M = \det\begin{pmatrix} A & B \\ 0 & D\end{pmatrix} = \det A \det D$$

Q3: SWM for $|U+VWX|$

$U_{n\times n}$, $W_{p\times p}$, $V_{n\times p}$, $X_{p\times n}$, $|U|,|W|\neq 0$

Sol.

$\boxed{\text{method 1}}$: By 250A, $|I+AB|=|I+BA|$

Since U, W non-singular, thus

$$|U+VWX|$$

$$= |U|\,|I+U^{-1}VWX|$$

$$\overset{250A}{=} |U|\,|I+XU^{-1}VW|$$

$$= |U|\,|W|\,|W^{-1}+XU^{-1}V|$$

$\boxed{\text{method 2}}$ Let $M = \begin{bmatrix} W^{-1} & X \\ V & U\end{bmatrix}$

Then $\begin{bmatrix} I & XU^{-1} \\ 0 & I\end{bmatrix} M = \begin{bmatrix} W^{-1}+XU^{-1}V & 0 \\ V & U\end{bmatrix}$

$$\begin{bmatrix} I & 0 \\ VW & I\end{bmatrix} M = \begin{bmatrix} W^{-1} & X \\ 0 & U+VWX\end{bmatrix}$$

Thus, by Q2, we have

$$\begin{vmatrix} I & 0 \\ VW & I\end{vmatrix} = \begin{vmatrix} I & XU^{-1} \\ 0 & I\end{vmatrix} = 1$$

and

$$|M| = |U|\,|W^{-1}+XU^{-1}V| = |W^{-1}|\,|U+VWX|$$

$$\Rightarrow |U+VWX| = |W|\,|U|\,|W^{-1}+XU^{-1}V|$$