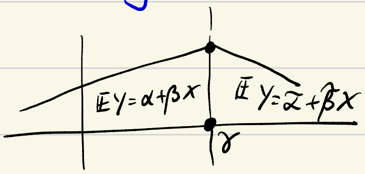
Two phase regression & Fieller's theorem



$$\mathbb{E}\, Y = \begin{cases} \alpha_1+\beta_1 x, & x \le \gamma \\ \alpha_2+\beta_2 x, & x \ge \gamma \end{cases}$$

Continuity assump.

$$\alpha_1+\beta_1\gamma = \alpha_2+\beta_2\gamma$$

$$\hat{\gamma} = \frac{\hat{\alpha}_1-\hat{\alpha}_2}{\hat{\beta}_2-\hat{\beta}_1} \quad \text{point estimate}$$

$y = X\beta + \varepsilon$ for a two phase reg.

Calibration: $Y, X, \hat{\beta}$ are known.

- Observe Y_0 but no info. on (x_0) was provided. Q: How to estimate x_0?

$$Y_0 = \hat{\alpha} + \hat{\beta}(x_0 - \bar{x})$$

$$\Rightarrow x_0 = \frac{Y_0-\hat{\alpha}}{\hat{\beta}} + \bar{x} = \frac{Y_0-\bar{Y}}{\hat{\beta}} + \bar{x}$$

$\boxed{\text{Fieller's Theorem}}$ Suppose u, v are uncorrelated r.v.s,

$$u \sim N(\mu_u, \sigma_u^2),\ \sigma_u^2 = \sigma^2 a_1$$

$$v \sim N(\mu_v, \sigma_v^2),\ \sigma_v^2 = \sigma^2 a_2$$

$\hat{\sigma}_u = S a_1$. $\hat{\sigma}_v = S a_2$.

Define $g = t^2_{m,\alpha/2} S^2 a_2^2 / v^2$, m: df associated with S^2 for estimating σ^2.

$$\sigma_u^2 = \sigma^2 a_1, \quad \sigma_v^2 = \sigma^2 a_2$$

If $g<1$. a $100(1-\alpha)\%$ C.I. for $\frac{\mu_u}{\mu_v}$ is:

$$\frac{1}{1-g}\left(\frac{u}{v} \pm \frac{t_{m,\frac{\alpha}{2}} S}{v}\sqrt{(1-g)\, a_1^2 + \frac{u^2 a_2^2}{v^2}}\right)$$

Pf: $\theta \equiv \frac{\mu_u}{\mu_v}$ and $W = u - \theta v$

$$\mathbb{E}\, W = \mu_u - \theta\mu_v = 0$$

$$\mathrm{Var}\, W = \sigma^2 a_1 + \theta^2 \sigma^2 a_2$$

$$\widehat{\mathrm{Var}}\, w = S^2 a_1 + S^2 a_2 \frac{u}{}$$

$$u - \theta v \sim N(0, \sigma^2 a_1^2 + \theta^2 \sigma^2 a_2^2)$$

Also $\frac{mS^2}{\sigma^2} \sim \chi^2_m$

$$\frac{(u-\theta v)/\sigma\sqrt{a_1^2+\theta^2 a_2^2}}{\sqrt{\frac{mS^2}{\sigma^2}/m}} \sim t_m$$

$$(u-\theta v)^2 = t^2_{m,\frac{\alpha}{2}} S^2 (a_1^2 + \theta^2 a_2^2)$$

Verify equation simplifies to

$$(1-g)\theta^2 - \frac{2u}{v}\theta + \frac{u^2}{v^2} - g\frac{a_1^2}{a_2^2} = 0$$

$$\theta = \frac{1}{2a}\left(-b \pm \sqrt{b^2-4ac}\right)$$

Verify

$$\theta \in \frac{1}{1-g}\left(\frac{u}{v} \pm \frac{t_{m,\frac{\alpha}{2}} S}{v}\sqrt{(1-g)a_1^2 + \frac{u^2 a_2^2}{v^2}}\right)$$

$u \leftarrow Y_0 - \bar{Y}$; $v \leftarrow \hat{\beta}$

$$\mathrm{Var}\ u = \sigma^2 + \frac{\sigma^2}{n} = \sigma^2\left(1+\frac{1}{n}\right)$$

$$\mathrm{Var}\ v = \sigma^2\left(\frac{1}{\sum_i (x_i-\bar{x})^2}\right)$$

Verify $100(1-\alpha)\%$ C.I. for x_0 is

$$\bar{x} + \frac{Y_0-\bar{Y}}{\hat{\beta}(1-g)} \pm \frac{t_{n-2,\frac{\alpha}{2}} S}{\hat{\beta}(1-g)}\sqrt{\frac{1-g}{n}(1+n) + \frac{(Y_0-\bar{Y})^2}{\hat{\beta}\sum_i (x_i-\bar{x})^2}}$$

Ex $\mathbb{E}Y = \beta_0 + \beta_1(X - \bar{X})$

If $y=0$: $X_0 - \bar{X} = -\frac{\beta_0}{\beta_1}$

$\hat{X}_0 = \hat{\beta}_0/\hat{\beta}_1$. Fieller's thm applies.

Ex Prognosting factors interaction assessment in a 2-trtment trial with trtmt T & C, along with a prognostic factor.

$$\mathbb{E}Y_? = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2$$

- $X_1 = \begin{cases} 1 & \text{if trtmt } T \\ 0 & \text{if } C \end{cases}$
- X_2 some prognostic factor

$$\mathbb{E}Y_T = \beta_0 + \beta_1 + \beta_2 X_2 + \beta_3 X_2$$
$$\mathbb{E}Y_C = \beta_0 + \beta_2 X_2$$

- $\mathbb{E}(Y_T - Y_C) = \beta_1 + \beta_3 X_2$

Q: WTK X_2 for which $\mathbb{E}(Y_T - Y_C) > 0$.

i.e. WT estimate $X_2 = -\frac{\beta_1}{\beta_3}$

$$\hat{\beta} = (X^T X)^{-1} X^T y = \begin{pmatrix} \hat{\beta}_0 \\ \hat{\beta}_1 \\ \hat{\beta}_2 \\ \hat{\beta}_3 \end{pmatrix}; \quad \operatorname{cov}(\hat{\beta}) = \sigma^2 (X^T X)^{-1}$$

$$\tilde{\beta} = \begin{pmatrix} \hat{\beta}_1 \\ \hat{\beta}_3 \end{pmatrix}; \quad \operatorname{cov}(\tilde{\beta}) = V = \begin{pmatrix} V_{11} & V_{13} \\ V_{13} & V_{33} \end{pmatrix}$$

Let $w = \tilde{\beta}_1 + \theta \tilde{\beta}_3$, $\theta = -\frac{\beta_1}{\beta_3}$

then $\mathbb{E}\,W = 0$

$$\operatorname{Var} W = (V_{11} + \theta^2 V_{33} + 2\theta V_{13})\,\sigma^2$$

So $\frac{(n-4)\hat{\sigma}^2}{\sigma^2} \sim \chi^2_{n-4}$; $\frac{W}{\sqrt{\operatorname{Var} W}} \sim N(0,1)$

If σ^2 is known,

$$(\hat{\beta}_1 + \theta \hat{\beta}_3)^2 = Z^2_{\frac{\alpha}{2}} (V_{11} + \theta^2 V_{33} + 2\theta V_{13})\sigma^2$$

CI for θ are

$$\theta = \frac{-w_2 \pm \sqrt{w_2^2 - 4w_1 w_2}}{2w_2}.$$

$$w_1 = \hat{\beta}_1^2 - V_{11} Z^2_{\frac{\alpha}{2}}.$$
$$w_2 = 2\hat{\beta}_1 \hat{\beta}_3 - 2V_{13} Z^2_{\frac{\alpha}{2}}$$
$$w_3 = \hat{\beta}_3^2 - V_{33} Z^2_{\frac{\alpha}{2}}$$

Note: $\beta_1 = 0$ iff $\frac{\hat{\beta}_1}{\sqrt{V_{11}}} > Z_{\alpha/2}$ iff $w_1 > 0$.

$\beta_3 = 0$ iff $\frac{\hat{\beta}_3}{\sqrt{V_{33}}} > Z_{\frac{\alpha}{2}}$ iff $w_3 > 0$

Ex Health economics.

Cost effectiveness ratio

$$R = \frac{\mu_{CT} - \mu_{CC}}{\mu_{ET} - \mu_{EC}}$$

$$\hat{R} = \frac{\bar{C}_T - \bar{C}_C}{\bar{E}_T - \bar{E}_C}$$

C.I. for R. Define

$$W = (\bar{C}_T - \bar{C}_C) - \theta(\bar{E}_T - \bar{E}_C)$$

Var W (estimate)

Extensions to correlated variables in the ratio.

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix} \mu_X \\ \mu_Y \end{pmatrix}, \sigma^2 \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix}\right)$$

WT find C.I. for $\theta = \mu_X / \mu_Y$

Let $W = X - \theta Y$

$$\mathbb{E}W = 0, \quad \operatorname{Var} W = \sigma^2 (V_{11} + V_{22}\theta^2 - 2\theta V_{12})$$
$$= \sigma^2 \cdot \sigma_w^2$$

$$\frac{w}{\sqrt{\sigma^2 \cdot \sigma_w^2}} \sim \mathcal{N}(0,1)$$

$$\frac{r S_r^2}{\sigma^2} \sim \chi_r^2 \quad S_r^2 : \text{unbiased estimator of } \sigma^2$$

Then $$\frac{x - \theta y}{S_r \sqrt{V_{11} + \theta^2 V_{22} - 2\theta V_{12}}} \sim t_r$$

Verify the limits for θ are

$$\frac{1}{1-g}\left\{ \frac{X}{Y} + \frac{V_{12} g}{V_{22}} \pm \frac{t_s}{Y}\sqrt{r} \right\} \text{ where } r \text{ is}$$

$$V_{11} + \frac{2XV_{12}}{Y} + \frac{X^2}{Y^2}V_{22} - g\left(V_{11} - \frac{V_{12}^2}{V_{22}}\right)$$

& $g = \dfrac{t^2 s^2 V_{22}}{Y^2}$ [Wiki...

Ex Two phase reg.

$\hat{r}$ = estimated threshold when change occurs

(seber, p.161)

$$= \frac{\hat{\alpha}_1 - \hat{\alpha}_2}{\hat{\beta}_1 - \hat{\beta}_2}$$

$$w = (\hat{\alpha}_1 - \hat{\alpha}_2) + r(\hat{\beta}_1 - \hat{\beta}_2) \Rightarrow$$

$$\mathbb{E}W = 0.$$