Biostat 250C HW7

Elvis Cui

Dept. of Biostats

UCLA

PhD student

Q1: Show that $p(x_B) = \mathcal{N}(\mu_B, *^{-1})$ where $* = \Sigma_{BB}^{-1}$ is a function of precision matrices Q_{AA}, Q_{AB}, Q_{BB}

Sol. In class, Dr. Banerjee used the Mm-formula to show it.

Here I provide another solution which is based on Schur Complement.

Let $\Sigma = \begin{bmatrix} \Sigma_{AA} & \Sigma_{AB} \\ \Sigma_{BA} & \Sigma_{BB} \end{bmatrix}$

$$Q = \begin{bmatrix} Q_{AA} & Q_{AB} \\ Q_{BA} & Q_{BB} \end{bmatrix}$$

and $\Sigma^{-1} = Q$ or $Q^{-1} = \Sigma$.

Then by 216, 250A or ECE236B, we have

$$\Sigma = Q^{-1} =$$

$$\begin{bmatrix} Q_{AA}^{-1} + Q_{AA}^{-1}Q_{AB}(Q_{BB}-Q_{BA}Q_{AA}^{-1}Q_{AB})^{-1}Q_{BA}Q_{AA}^{-1} & -Q_{AA}^{-1}Q_{AB}(Q_{BB}-Q_{BA}Q_{AA}^{-1}Q_{AB})^{-1} \\ -(Q_{BB}-Q_{BA}Q_{AA}^{-1}Q_{AB})^{-1}Q_{BA}Q_{AA}^{-1} & (Q_{BB}-Q_{BA}Q_{AA}^{-1}Q_{AB})^{-1} \end{bmatrix}$$

where

$$(Q_{BB}-Q_{BA}Q_{AA}^{-1}Q_{AB})^{-1}$$

is the Schur complement of Q_{BB} of Q.

Hence, by integrating out x_A, we have

$$p(x_B) = \mathcal{N}(x_B \mid \mu_B, *^{-1})$$

where

$$* = Q_{BB} - Q_{BA}Q_{AA}^{-1}Q_{AB}.$$

Q2: Without explicitly solving a linear system, can you use the factorization

$$p(x_A, x_B) = p(x_B)\, p(x_A \mid x_B)$$

to express Q_{AA}, Q_{AB}, Q_{BB} in terms of $\Sigma_{AA}, \Sigma_{AB}, \Sigma_{BB}$.

- "without solving a linear system" means that we can't use "Analissa's transformation".

Sol. By 200C and 250A,

$$x_B \sim \mathcal{N}(\mu_B, \Sigma_{BB})$$

$$x_A \mid x_B \sim \mathcal{N}\left(\mu_A + \Sigma_{AB}\Sigma_{BB}^{-1}(x_B - \mu_B),\ \Sigma_{AA} - \Sigma_{AB}\Sigma_{BB}^{-1}\Sigma_{BA}\right)$$

By Q1, $\Sigma_{BB}^{-1} = Q_{BB} - Q_{BA}Q_{AA}^{-1}Q_{AB}$

By 250C lec notes,

$$p(x_A, x_B) \propto e^{-\frac{1}{2}\tilde{x}^T Q \tilde{x}}$$ where

$$\tilde{x}^T Q \tilde{x} = x_A^T Q_{AA} x_A - 2x_A^T Q_{AA}\mu_A + 2x_A^T Q_{AB}(x_B - \mu_B) - 2x_B^T Q_{BA}\mu_A + x_B^T Q_{BB} x_B - 2x_B^T Q_{BB}\mu_B + \text{Const.}$$

Then

$$p(x_A \mid x_B) = \frac{p(x_A, x_B)}{p(x_B)} \propto e^{-\frac{1}{2}\tilde{x}^T Q \tilde{x}}\, e^{\frac{1}{2}(x_B - \mu_B)^T \Sigma_{BB}^{-1}(x_B - \mu_B)}$$

We have

$$\tilde{x}^T Q \tilde{x} - (x_B - \mu_B)^T \Sigma_{BB}^{-1}(x_B - \mu_B)$$

$$= \tilde{x}^T Q \tilde{x} - (x_B - \mu_B)^T Q_{BB}(x_B - \mu_B) + (x_B - \mu_B)^T Q_{BA}Q_{AA}^{-1}Q_{AB}(x_B - \mu_B)$$

$$= x_A^T Q_{AA} x_A - 2x_A^T Q_{AA}\mu_A + 2x_A^T Q_{AB}(x_B - \mu_B) + \text{Const.}$$

$$= (x_A - Mm)^T M^{-1}(x_A - Mm) + \text{Const.}$$

where Const. is independent of x_A

$$\begin{cases} M = Q_{AA}^{-1} \\ Mm = \mu_A - Q_{AA}^{-1}Q_{AB}(x_B - \mu_B) \end{cases}$$

On the other hand, we know

$$p(x_A \mid x_B) \propto e^{-\frac{1}{2}(x_A - \mu_{A|B})^T \Sigma_{A|B}^{-1}(x_A - \mu_{A|B})}$$

where

$$\begin{cases} \mu_{A|B} = \mu_A + \Sigma_{AB}\Sigma_{BB}^{-1}(x_B - \mu_B) \\ \Sigma_{A|B} = \Sigma_{AA} - \Sigma_{AB}\Sigma_{BB}^{-1}\Sigma_{BA} \end{cases}$$

By comparing coefficients, I get

$$\begin{cases} Q_{AA}^{-1} = \Sigma_{A|B} \\ Q_{AA}^{-1}Q_{AB} = \Sigma_{AB}\Sigma_{BB}^{-1} \end{cases}$$

& $Q_{BB} - Q_{BA}Q_{AA}^{-1}Q_{AB} = \Sigma_{BB}^{-1}$

Next page

$$\begin{cases} Q_{AA}^{-1} = \Sigma_{A|B} \\ Q_{AA}^{-1} Q_{AB} = \Sigma_{AB} \Sigma_{BB}^{-1} \\ Q_{BB} - Q_{BA} Q_{AA}^{-1} Q_{AB} = \Sigma_{BB}^{-1} \end{cases}$$

Thus,

$$Q_{AA} = \left(\Sigma_{AA} - \Sigma_{AB} \Sigma_{BB}^{-1} \Sigma_{BA}\right)^{-1}$$

$$\begin{aligned} Q_{AB} &= Q_{AA} \Sigma_{AB} \Sigma_{BB}^{-1} \\ &= \left(\Sigma_{AA} - \Sigma_{AB} \Sigma_{BB}^{-1} \Sigma_{BA}\right)^{-1} \Sigma_{AB} \Sigma_{BB}^{-1} \end{aligned}$$

$$\begin{aligned} Q_{BA} &= Q_{AB}^{T} \\ &= \Sigma_{BB}^{-1} \Sigma_{BA} \left(\Sigma_{AA} - \Sigma_{AB} \Sigma_{BB}^{-1} \Sigma_{BA}\right)^{-1} \end{aligned}$$

$$Q_{BB} = \Sigma_{BB}^{-1} + Q_{BA} Q_{AA}^{-1} Q_{AB}$$

$$= \Sigma_{BB}^{-1} + \Sigma_{BB}^{-1} \Sigma_{BA} \left(\Sigma_{AA} - \Sigma_{AB} \Sigma_{BB}^{-1} \Sigma_{BA}\right)^{-1} \Sigma_{AB} \Sigma_{BB}^{-1}$$

$\square$.

The above can also be derived using block inversion formula.