LEC14 Linear mixed effect model

One-way Anova with Random Effects

$$Y_{ij} = \mu + \tau_i + \varepsilon_{ij},$$
$$\varepsilon_{ij} \sim NID(0, \sigma^2)$$
τ_i = effect of trtmt i.

$H_0: \tau_1 = \tau_2 = \cdots = \tau_k$

H_1: They are not the same.

When interest is in more than k trtmts, i.e., interest is in a population of trtmt, then sample K of them.

$\Rightarrow \tau_i \sim NID(0, \sigma_\tau^2)$ Random effect.

$\varepsilon_{ij} \sim NID(0, \sigma^2)$

& $\tau_i \perp \varepsilon_{ij} \ \forall\ i \,\&\, j$.

Hypothesis of interest is

$$H_0: \sigma_\tau^2 = 0 ; \ H_1: \sigma_\tau^2 > 0.$$

we have data Y_{ij}, $i = 1, \cdots, k$, $j = 1, 2, \cdots, n$

$$\sum_i \sum_j (Y_{ij} - \bar{Y})^2 = \sum_i \sum_j (\bar{Y}_{i.} - \bar{Y} + Y_{ij} - \bar{Y}_{i.})^2$$
$$= \sum_i \sum_j (\bar{Y}_{i.} - \bar{Y})^2 + \sum_i \sum_j (Y_{ij} - \bar{Y}_{i.})^2$$
$$= SST_r + SSE$$

$\bar{Y}_{i.} = \mu + \alpha_i + \bar{\varepsilon}_{i.}$; $\bar{Y} = \mu + \bar{\alpha} + \bar{\varepsilon}$

(α & τ are the same thing)

$$SST_r = \sum_i^K \sum_j^n (\alpha_i - \bar{\alpha} + \bar{\varepsilon}_{i.} - \bar{\varepsilon})^2$$
$$\mathbb{E}\, SST_r = n\mathbb{E}\left[\sum_i \left((\alpha_i - \bar{\alpha})^2 + (\bar{\varepsilon}_{i.} - \bar{\varepsilon})^2\right)\right]$$
$$= n\sigma_\alpha^2 (k-1) \quad \left(Y_i \sim N(\mu, \sigma^2) \Rightarrow \frac{SSE}{\sigma^2} \sim \chi^2_{n-1} \right)$$
$$+ \sigma^2 (k-1)$$

$$\mathbb{E}\, SST_r / (k-1) = n\sigma_\alpha^2 + \sigma^2$$

$$\mathbb{E}(SSE) = \mathbb{E}\, n \sum_{i=1}^k (Y_{ij} - \bar{Y}_{i.})^2$$
$$= n\,\mathbb{E} \sum_{i=1}^k (\varepsilon_{ij} - \bar{\varepsilon}_{i.})^2$$
$$= n(k-1)\sigma^2$$

Since $\sum_{i=1}^k (\varepsilon_{ij} - \bar{\varepsilon}_{i.})^2 \sim \chi^2_{k-1} \cdot \sigma^2$

$$\mathbb{E}\, MST_r = n\sigma_\tau^2 + \sigma^2$$

$$\Rightarrow \mathbb{E}\, \frac{MST_r - MSE}{n} = \sigma_\tau^2$$

$$\Rightarrow \hat{\sigma}_\tau^2 = \frac{MST_r - MSE}{n} \qquad \square.$$

C.I. for σ^2:

$$\frac{SSE}{\sigma^2} \sim \chi^2_{k(n-1)}$$

$\Rightarrow 100(1-\alpha)\%$ CI for σ^2

$$\frac{SSE}{\chi^2_{k(n-1), 1-\frac{\alpha}{2}}} < \sigma^2 < \frac{SSE}{\chi^2_{k(n-1), \frac{\alpha}{2}}}$$

To find CI for $\sigma_\tau^2 / \sigma^2 \equiv \theta$

$$\frac{(k-1) MST_r}{n\sigma_\tau^2 + \sigma^2} \sim \chi^2_{k(n-1)}$$

Since $MSE \perp MST_r$, we

$$\frac{MST_r}{MSE} \frac{\sigma^2}{n\sigma_\alpha^2 + \sigma^2} \sim F_{k, (n-1)k}$$

A $100(1-\alpha)\%$ C.I. for θ is

$$\left(\frac{1}{\underbrace{F}_{\text{upper } F_{\alpha/2}}} \frac{MST_r}{MSE} - 1 \right) \frac{1}{n} \le \theta \le \left(\frac{1}{\underbrace{F}_{\text{lower } F_{\alpha/2}}} \frac{MST_r}{MSE} - 1 \right) \frac{1}{n}$$

Intraclass correlation Coef.

$$\rho = \frac{\sigma_\alpha^2}{\sigma_\alpha^2 + \sigma^2}$$

So $100(1-\alpha)\%$ C.I. for ρ is

$$\frac{L}{1+L} \le \rho \le \frac{U}{1+U}$$

where L & U are conf. limits for θ.

& noting that $\rho = \frac{\theta}{\theta+1}$

ANOVA – Two factor:

$$y_{ijk} = \mu + \tau_i + \beta_j + (\tau\beta)_{ij} + \varepsilon_{ijk}$$

Three factors:

$$y_{ijkl} = \mu + \tau_i + \beta_j + \gamma_k + (\tau\beta)_{ij} + (\tau\gamma)_{ik} + (\beta\gamma)_{jk} + (\tau\beta\gamma)_{ijk} + \varepsilon_{ijkl}$$

- $i = 1, \dots, a$, $j = 1, \dots, b$, $k = 1, \dots, c$

l = # of replicates $= 1, 2, \dots, n$.

Fixed effects: $\sum_{i=1}^{a} \tau_i = 0$ for example.

Random effects: $\tau_i \sim NID(0, \sigma_\tau^2)$. Ex

Rule 1: fixed effects (say factor) associate $\sum_i \tau_i^2 / (a-1)$ with its mean sq. otherwise σ_τ^2 if factor is random.

$$y_{ijk} = \mu + \underbrace{\tau_i}_{A} + \underbrace{\beta_j}_{B} + (\tau\beta)_{ij} + \varepsilon_{ijk}$$

Case (i): A, B both fixed.

Factor	a F i	b F j	n R k	EMS
τ_i	0	b	n	*
β_j	a	0	n	**
$(\tau\beta)_{ij}$	0	0	n	***
ε_{ijk}	1	1	1	Δ

(*) $E(MSA) = bn \frac{\sum_i \tau_i^2}{a-1} + \sigma^2$

(**) $E(MSB) = an \frac{\sum_j \beta_j^2}{b-1} + \sigma^2$

(***) $E(MSAB) = n \frac{\sum_i \sum_j (\tau\beta)_{ij}^2}{(a-1)(b-1)} + \sigma^2$

(Δ) $E(MSE) = \sigma^2$

Case (ii) A & B are random

Factor	a R i	b R j	n R k	EMS
τ_i	1	b	n	$bn\sigma_\tau^2 + n\sigma_{\tau\beta}^2 + \sigma^2$
β_j	a	1	n	$an\sigma_\beta^2 + n\sigma_{\tau\beta}^2 + \sigma^2$
$(\tau\beta)_{ij}$	1	1	n	$n\sigma_{\tau\beta}^2 + \sigma^2$
ε_{ijk}	1	1	1	σ^2

Test $\sigma_\tau^2 = 0 \Rightarrow \frac{MSA}{MSAB}$

$\sigma_\beta^2 = 0 \rightarrow \frac{MSB}{MSAB}$

$\sigma_{\tau\beta}^2 = 0 \rightarrow \frac{MSAB}{MSE}$

Case (iii) One R One F

	a	b	n	EMS
	F	R	R	
	i	j	k	
τ_i	0	b	n	$bn\frac{\sum_i \tau_i^2}{a-1} + n\sigma_{\tau\beta}^2 + \sigma^2$
β_j	a	1	n	$an\sigma_\beta^2 + \sigma^2$
$(\tau\beta)_{ij}$	1	1	n	$n\sigma_{\tau\beta}^2 + \sigma^2$
ε_{ijk}	1	1	1	σ^2

$H_{0A}: \tau_1 = \tau_2 = \cdots = \tau_a : \frac{MSA}{MSAB}$

$H_{0B}: \sigma_\beta^2 = 0 : \frac{MSB}{MSE}$

$H_{0C}: \sigma_{\tau\beta}^2 = 0 : MSAB/MSE$

Satterwaithes' Approximations.

$\Rightarrow$ Approx. F-test.