Biostat 250A HW1

Elvis Cui
Han Cui
Dept. of Biostat
UCLA

Biostat 250A Homework 1 due 10-12-20

1. Fill in all the technical details on the first day handout when we argued that X-Y is a chi-squared random variable with n-m degrees of freedom.
2. We know that if X and Y are random variables and if they are independent, they are uncorrelated. Give an example, where we have uncorrelated random variables but they are not independent.
3. On page 16 of Lecture 1 Notes, show that all claims on the properties of the rank of a matrix are true.
4. Show that if A is a rectangular matrix and has linearly independent columns (i.e. A has full column rank), A'A has an inverse. Deduce a corresponding result for the rows of A.
5. Show that if A has rank r than A can be written as A=BC where B has full column rank and C has full row rank.
6. Do the exercises at the bottom of page 1 in Lecture 2 Notes.

Problem 1: $X \sim \chi^2_n$, $Y \sim \chi^2_m$, $X-Y \perp\!\!\!\perp Y$, find dist. of $X-Y$.

Solution: Since $\chi^2_n \overset{d}{=} Ga(\frac{n}{2}, 2)$

$$\mathbb{E}e^{tX} = \int_{\mathbb{R}^+}\left(\frac{1}{\Gamma(\frac{n}{2})} 2^{-\frac{n}{2}} x^{\frac{n}{2}-1} e^{-\frac{x}{2}}\right)e^{tx}dx$$

$$= \left[\int_{\mathbb{R}^+} x^{\frac{n}{2}-1} e^{-(\frac{1}{2}-t)x}dx\right] C_1$$

where $C_1 = (\Gamma(\frac{n}{2})2^{\frac{n}{2}})^{-1}$

But $[\cdot] = \Gamma(\frac{n}{2})(\frac{1}{2}-t)^{-\frac{n}{2}}$ since the integrand is a kernel of gamma function.

$$\Rightarrow \mathbb{E}e^{tX} = (1-2t)^{\frac{n}{2}}.$$

To make $[\cdot]$ is indeed a definite integral, we need $\frac{1}{2}-t > 0$, or, $t < \frac{1}{2}$

- Next,

$$\mathbb{E}e^{tX} = \mathbb{E}e^{t(X-Y+Y)} = \mathbb{E}e^{t(X-Y)}e^{tY}$$
$$\overset{(*)}{=} \mathbb{E}e^{t(X-Y)}\mathbb{E}e^{tY}$$

where $(*)$ comes from independence assumption.

Then

$$LHS = (1-2t)^{\frac{n}{2}} = \mathbb{E}e^{t(X-Y)}(1-2t)^{\frac{m}{2}} = RHS$$

$$\Rightarrow \mathbb{E}e^{t(X-Y)} = (1-2t)^{\frac{n-m}{2}}$$

Thus, by Lévy's inversion thm, we have

$$X-Y \sim \chi^2_{n-m}$$

Problem 2: Show that $Cov(X,Y)=0 \not\Rightarrow X \perp\!\!\!\perp Y$.

Solution:

Let $X = \sin\Theta$, $Y = \cos\Theta$, $\Theta \sim U(0, 2\pi]$

Then

$$Cov(X,Y) = \mathbb{E}\sin\Theta\cos\Theta - \mathbb{E}\sin\Theta\,\mathbb{E}\cos\Theta$$
$$= \mathbb{E}\tfrac{1}{2}\sin 2\Theta - (\mathbb{E}\sin\Theta)(\mathbb{E}\cos\Theta)$$
$$= 0 \quad \text{(A fact from Calculus I)}$$

However,

$$\mathbb{P}(X > \tfrac{1}{2}, Y > \tfrac{1}{2}) = 0$$

Since $\{X > \frac{1}{2}\} = \{\frac{\pi}{3} < \Theta < \frac{2}{3}\pi\}$

$\{Y > \frac{1}{2}\} = \{\Theta < \frac{\pi}{6}\} \cup \{\frac{11}{6}\pi < \Theta\}$

But $\mathbb{P}(X > \frac{1}{2})\mathbb{P}(Y > \frac{1}{2})$

$$= \left(\frac{\pi}{3} \cdot \frac{\pi}{3}\right)\frac{1}{2\pi}\frac{1}{2\pi} = \frac{1}{36} > 0$$

Problem: Show the following:

(i): $A \in \mathbb{R}^{n \times m}$, then $\text{rank}(A) \leq \min(m, n)$

Pf: WLOG, suppose $m \leq n$ & $\text{rank}(A) = m+1$

Then $\text{Dim}(\text{Span}(a_1, \cdots, a_m)) = m+1$

where a_i is the i^{th} column of A.

But $\forall b \in \text{Span}(a_1, \cdots, a_m)$, $\exists c_1, \cdots, c_m$ s.t. $b = \sum_{i=1}^{m} c_i a_i$, so the dimension can not exceed m, a contradiction.

(ii): $\text{Rank}(A+B) \leq \text{Rank}(A) + \text{Rank}(B)$

Pf: ETS $\forall x \perp \text{Span}(A)$ & $x \perp \text{Span}(B)$

$x \perp A+B$ Since $(x \neq \vec{0})$

$\text{Dim}(\{y+z : y \in \text{Span}(A), z \in \text{Span}(B)\}) \leq$

$\text{Rank}(A) + \text{Rank}(B)$

But $A^T x = 0$ & $B^T x = 0 \Rightarrow$

$(A+B)^T x = 0$.

Thus

$\text{Dim}(\text{Span}(A+B)) \leq \text{Dim}(\{y+z : y \in \text{Span}(A), z \in \text{Span}(B)\})$

(iii): $\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$

Pf: $\forall x \perp \text{Span}(A)$, $x \perp \text{Span}(AB)$

Since $A^T x = 0 \Rightarrow (AB)^T x = 0$

Same argument for $\text{Span}(B) \Rightarrow$

$\text{rank}(AB) \leq \min(\text{rank}(A), \text{rank}(B))$

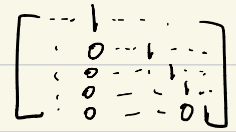
(iv) A invertible iff $\det|A| \neq 0$ (Assume we know $\det|AB| = \det|A| \det|B|$)

Pf: $(\Rightarrow)$: $I = A^{-1}A$ & $n = \det|I| = \det|A^{-1}A|$

$= \det|A^{-1}| \det|A|$

Thus, $\det|A| \neq 0$.

$(\Leftarrow)$: Transform A into its echelon form



, Since $\forall$ elementary matrices, the det is always ± 1.

We have $|\det|A|| = \left|\prod_{i=1}^{n} \text{diag}(\text{echelon})\right|$

Thus the diagonal elements are not 0.

which suggests that A^{-1} exists.

(v): A, B invertible, then $\text{rank}(C) = \text{rank}(AC) = \text{rank}(CB) = \text{rank}(ACB)$

Pf: let $x \perp \text{Span}(CB)$, $(x \neq \vec{0})$

then $B^T C^T x = 0$, Since B is of full rank. we have $C^T x = 0 \Rightarrow x \perp \text{Span}(C)$

Let $x \perp \text{Span}((AC)^T)$,

then $ACx = 0$, but A is of full rank

$\Rightarrow Cx = 0 \Rightarrow x \perp \text{Span}(C^T)$

But $\text{Dim}(\text{Span}(C^T)) = \text{Dim}(\text{Span}(C))$ (*)

$\Rightarrow \text{rank}(C) \leq \text{rank}(AC)$

Similarly,

$\text{rank}(CB) \leq \text{rank}(ACB)$

Since if $x \perp \text{Span}((ACB)^T)$, $ACBx = 0$,

A invertible $\Rightarrow CBx = 0 \Rightarrow x \perp CB$.

Then other 3 directions are trivial. (see (iii)) Q.E.D.

For a proof of (*), see next problem.

(vi): $\mathrm{rank}(A^T) \overset{①}{=} \mathrm{rank}(A) \overset{②}{=} \mathrm{rank}(AA^T) \overset{③}{=} \mathrm{rank}(A^TA)$

Pf: ETS ② Since ③ is implied by ①.

& $\mathrm{rank}(A) = \mathrm{rank}(AA^T) \le \mathrm{rank}(A^T) = \mathrm{rank}(A^TA) \le \mathrm{rank}(A)$

(See problem ③)

②: ETS "$\le$": $\forall x \perp \mathrm{span}(AA^T)$

$AA^Tx = 0 \Rightarrow x^TAA^Tx = 0 \Rightarrow$

$\|A^Tx\|_2^2 = 0 \Rightarrow A^Tx = 0 \Rightarrow x \perp \mathrm{span}(A)$.

(vii): $\mathrm{rank}([A, b]) \ge \mathrm{rank}(A)$

Pf: $\forall x \in \mathrm{Span}(A)$, $\exists\, c_1, \dots, c_n$ s.t.

$x = \sum_i^n c_i a_i$ where a_i is the i^{th} column of A

Thus, $x = \sum_i^n c_i a_i + 0b \Rightarrow$

$x \in \mathrm{Span}([A, b])$

$\Rightarrow \mathrm{rank}([A, b]) \ge \mathrm{rank}(A)$

(viii): $C(A^TA) = C(A^T)$

Pf: see (vi).

(x): $C(ACB) = C(AC)$ if $\mathrm{rank}(CB) = \mathrm{rank}(C)$

Pf: $\mathrm{rank}(CB) = \mathrm{rank}(C) \Rightarrow \mathrm{Span}(CB) = \mathrm{span}(C)$ $(\triangle)$

$\Rightarrow \forall x \in \mathrm{span}(AC)$, $\exists\, b$ s.t.

$x = ACb$, but for $\forall b$, $\exists\, \tilde{b}$ s.t.

$Cb = (CB)\tilde{b}$ by $(\triangle)$, it doesn't have to be unique. $\Rightarrow x = ACB\tilde{b}$

$\Rightarrow x \in \mathrm{span}(ACB)$

Problem 4: If $A \in \mathbb{R}^{m\times n}$ & $\mathrm{rank}(A) = n$, Then A^TA is invertible.

Pf: $\mathrm{rank}(A^TA) = \mathrm{rank}(A) = n$

But $A^TA \in \mathbb{R}^{n\times n}$

$\Rightarrow A^TA$ is invertible.

- If $\mathrm{rank}(A) = m$, then AA^T is invertible.

Problem 5: If $\mathrm{rank}(A) = r$, then $\exists\, B, C$, $A = BC$ where B has full column rank & C has full row rank.

Pf:

Method 1: SVD

$A = U\Lambda V^T = (U\Lambda^{\frac{1}{2}})(\Lambda^{\frac{1}{2}}V^T)$

& $U \in \mathbb{R}^{m\times r}$, $\Lambda \in \mathbb{R}^{r\times r}$, $V \in \mathbb{R}^{n\times r}$

Then by Problem 4 part (v),

$B = U\Lambda^{\frac{1}{2}}$ & $C = \Lambda^{\frac{1}{2}}V^T$.

Method 2: Dr. Wong's method

Let $\{b_1, \dots, b_r\}$ be a basis of $C(A)$

then $a_i = Bc_i$ where $B = [b_1, \dots, b_r]$

$\Rightarrow A = BC$ where $C = [c_1, \dots, c_n]$

&

$r = \mathrm{rank}(A) = \mathrm{rank}(BC) \le \mathrm{rank}(C) \le r$

$\Rightarrow \mathrm{rank}(C) = r$.

Problem 6: Show the following:

Exercises:

1. Show that all columns of an orthogonal matrix are orthogonal and each has length 1. Can the same be said of its rows?
2. Show all eigenvalues of a real symmetric matrix are real.
3. Show that eigenvectors corresponding to distinct eigenvalues of a real symmetric matrix are orthogonal.
4. Show that if Q=x'Ax, then we can always assume A to be symmetric.
5. Show that any matrix A can be written as a sum of a symmetric matrix and a skew matrix, and these summands are unique. $A = S + T,\ S^T = S,\ T^T = -T$

Pf: (1): $A^TA = I$ & $A = [a_1, \dots, a_n]$ $\Rightarrow$

$a_i^Ta_i = 1$ & $a_i^Ta_j = 0\ \forall i \neq j$.

• But AA^T may not be I. Consider

$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$, then $AA^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

(2): I will do this at the end. [Too long]

(3): Assume λ's are real from (2).

Then $Ax_i = \lambda_i x_i$ with $\lambda_i \neq \lambda_j$

$Ax_j = \lambda_j x_j$

$\Rightarrow x_j^T A x_i = \lambda_i x_i^T x_j = \lambda_j x_j^T x_i = x_i^T A^T x_j$

Since A is symmetric

$\Rightarrow (\lambda_i - \lambda_j) x_i^T x_j = 0$

$\Rightarrow x_i \perp x_j$ $\square$.

(4): $x^TAx = x^T \underbrace{\left(\frac{A+A^T}{2}\right)}_{\text{Symmetric}} x$. $\square$.

(5): $M = \frac{A+A^T}{2} \Rightarrow M^T = M$

Existence: $S = \frac{A-A^T}{2} \Rightarrow S^T = -S$

& $A = M + S$

Uniqueness: suppose

$A = M + S = \tilde{M} + \tilde{S}$ &

$\tilde{M} \neq M,\ \tilde{S} \neq S,$ $\Rightarrow$

$\tilde{M}^T = M,\ \tilde{S}^T = -\tilde{S}$

Denote $\Delta_1 = M - \tilde{M},\ \Delta_2 = S - \tilde{S}$, we have

$\Delta_1 + \Delta_2 = 0$ & --- ①

$A = (\Delta_1 + \tilde{M}) + (\Delta_2 + \tilde{S})$

Also, $\Delta_1^T = \Delta_1$ ... ②

$\Delta_2^T = -\Delta_2$ ... ③

② − ③ $= \Delta_1^T - \Delta_2^T = \Delta_1 + \Delta_2 \overset{①}{=} 0$

$\Rightarrow \Delta_1 = \Delta_2 = 0$

$\Rightarrow \tilde{M} = M$ & $S = \tilde{S}$.

(see next page)

- Show if $A^T = A$ then eigen-vals of A are real.

Pf: Let $Ax = (a+bi)x$ where $x \neq 0$.

Then let $B = \left(A - (a+bi)I\right)^T \overline{(A-(a+bi)I)}$

$$\overset{\text{by symmetry}}{\underset{\& A \text{ is real}}{=}} (A-(a+bi)I)(A-(a-bi)I)$$

$$= A^2 - 2aA + (a^2+b^2)I \quad (*)$$

and we have

$$\det|B| = 0 = \det|[A-(a+bi)I]^2|$$

But $(*) = (A-aI)^2 + b^2 I$

$(A-aI)^2$ is p.s.d. implies that $b=0$, otherwise $\det|(*)| > 0$.

Thus, eigen-vals of A are all real.