Biostat 250C HW4

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Q1

1. Find expressions for

$$p(\gamma, \sigma^2|y_*) \propto p(\sigma^2|y_*) \times p(\gamma|\sigma^2, y_*)$$

What is $p(\widetilde{y}|\sigma^2, y)$ (Hint: find from $p(\gamma|\sigma^2, y)$)?

Sol. The model is

$$y = X\beta + e_y, e_y \sim \mathcal{N}(0, \sigma^2 I_n)$$
$$\beta = \mu_\beta + e_\beta, e_\beta \sim \mathcal{N}(0, \sigma^2 V_\beta)$$
$$\widetilde{y} = \widetilde{X}\beta + e_{\widetilde{y}}, e_{\widetilde{y}} \sim \mathcal{N}(0, \sigma^2 I_m)$$

$$\underbrace{\begin{bmatrix} y \\ \mu_\beta \\ 0 \end{bmatrix}}_{y_*} = \underbrace{\begin{bmatrix} X_{n\times p} & \mathbf{0}_{n\times m} \\ I_p & \mathbf{0}_{p\times n} \\ \widetilde{X}_{m\times p} & -I_m \end{bmatrix}}_{X_*} \underbrace{\begin{bmatrix} \beta \\ \widetilde{y} \end{bmatrix}}_{\gamma} + \underbrace{\begin{bmatrix} e_y \\ e_\beta \\ e_{\widetilde{y}} \end{bmatrix}}_{e_*}$$

We have

$$e^* \sim \mathcal{N}\left(\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \sigma^2 \overbrace{\begin{pmatrix} I_n & 0 & 0 \\ 0 & V_\beta & 0 \\ 0 & 0 & I_m \end{pmatrix}}^{V_*}\right)$$

$$Y^*|\gamma, \sigma^2 \sim \mathcal{N}(X_*\gamma, V_*)$$
$$\sigma^2 \sim IG(a, b)$$

$\Rightarrow$ $\hat{\gamma}$ solves the normal equation:

$$(X_*^T V_*^{-1} X_*)\hat{\gamma} = X_*^T V_*^{-1} Y_* \qquad (\triangle)$$

$$p(\gamma, \sigma^2 | Y^*) \propto p(Y^*|\gamma, \sigma^2) p(\sigma^2)$$
$$= \left(\frac{1}{\sigma^2}\right)^{a+1} e^{-b/\sigma^2} \left(\frac{1}{\sigma^2}\right)^{\frac{n+p+m}{2}} e^{-\frac{1}{2\sigma^2} Q}$$

where

$$Q = (Y_* - X_*\gamma)^T V_*^{-1} (Y_* - X_*\gamma).$$
$$= (Y_* - X_*\hat{\gamma} + X_*\hat{\gamma} - X_*\gamma)^T V_*^{-1} (Y_* - X_*\hat{\gamma} + X_*\hat{\gamma} - X_*\gamma)$$

→ Right panel

$$Q = \overbrace{(Y_* - X_*\hat{\gamma})^T V_*^{-1} (Y_* - X_*\hat{\gamma})}^{RSS} + 2(\hat{\gamma} - \gamma)^T X_*^T V_*^{-1} (Y_* - X_*\hat{\gamma}) + (\gamma - \hat{\gamma})^T X_*^T V_*^{-1} X_* (\gamma - \hat{\gamma})$$
$$= RSS + (\gamma - \hat{\gamma})^T X_*^T V_*^{-1} X_* (\gamma - \hat{\gamma}).$$

Thus,

$$p(\gamma, \sigma^2 | Y) \propto$$
$$\left(\frac{1}{\sigma^2}\right)^{a + \frac{n}{2} + 1} e^{-\frac{1}{\sigma^2}(b + \frac{1}{2}RSS)} \times$$
$$\left(\frac{1}{\sigma^2}\right)^{\frac{p+m}{2}} e^{-\frac{1}{2\sigma^2}(\gamma - \hat{\gamma})^T X_*^T V_*^{-1} X_* (\gamma - \hat{\gamma})}$$
$$\propto IG(\sigma^2 | a^*, b^*) \mathcal{N}(\gamma | \hat{\gamma}, \sigma^2 (X_*^T V_*^{-1} X_*)^{-1})$$

where $\begin{cases} a^* = a + \frac{n}{2} \\ b^* = b + \frac{1}{2}RSS \end{cases}$

and $\hat{\gamma}$ solves $(\triangle)$,

$$X_*^T V_*^{-1} X_* = \begin{bmatrix} X^T X + V_\beta^{-1} + \widetilde{X}^T \widetilde{X}, & -\widetilde{X}^T \\ -\widetilde{X}, & I \end{bmatrix}$$

Since $\gamma^T = (\beta^T, \widetilde{y}^T)$,

$$p(\widetilde{y} | \sigma^2, y) = \int_{H} \mathcal{N}(\gamma | \hat{\gamma}, \sigma^2 (X_*^T V_*^{-1} X_*)^{-1}) \, d\beta$$

By sol of Linear system,

$$Var(\widetilde{y} | \sigma^2, y) = \left(I - \widetilde{X}(X^T X + \widetilde{X}^T \widetilde{X} + V_\beta^{-1})^{-1} \widetilde{X}^T\right)^{-1}$$

Since we are taking the bottom-right element of $(X_*^T V_*^{-1} X_*)^{-1}$

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For $\hat{Y}$, firstly,

$$X_*^T V_*^{-1} Y_* = \begin{pmatrix} X^T y + V_\beta^{-1}\mu_\beta \\ 0 \end{pmatrix}$$

Again, by sol of linear systems, left bottom element of $(X_*^T V_*^{-1} X_*)^{-1}$ is

$$\tilde{X}(X^T X + V_\beta^{-1})^{-1},$$

$$\Rightarrow E(\tilde{Y} \mid y, \sigma^2) = \tilde{X}(X^T X + V_\beta^{-1})^{-1}(X^T y + V_\beta^{-1}\mu_\beta)$$

To sum up,

$$\tilde{Y} \mid y, \sigma^2 \sim N(\mu, V)$$

$$\mu = \tilde{X}(X^T X + V_\beta^{-1})^{-1}(X^T y + V_\beta^{-1}\mu_\beta)$$

$$V = (I - \tilde{X}(X^T X + \tilde{X}^T \tilde{X} + V_\beta^{-1})^{-1} \tilde{X}^T)^{-1}$$

$$\overset{(*)}{=} I + \tilde{X}\left[X^T X + V_\beta^{-1}\right]^{-1}\tilde{X}^T$$

where $(*)$ is due to Sherman-Woodbury. $\square$

If $M^{-1} \equiv X^T X + V_\beta^{-1}$,
$m \equiv X^T y + V_\beta^{-1}\mu_\beta$, then

$$\mu = \tilde{X} M m \ \& \ V = I + \tilde{X} M \tilde{X}^T \ \square$$

NOTE: Dr. Banerjee's Solution

Re-write:

$$y = X\beta + \epsilon_y, \quad \epsilon_y \sim N(0, \sigma^2 I_n)$$

$$\beta = \mu_\beta + \epsilon_\beta, \quad \epsilon_\beta \sim N(0, \sigma^2 V_\beta)$$

$$\tilde{Y} = \tilde{X}\beta + \tilde{\epsilon}, \quad \tilde{\epsilon} \sim N(0, \sigma^2 I_m)$$

By Mm-formula,

$$\beta \mid y, \sigma^2 \sim N(Mm, \sigma^2 M)$$

$$\Rightarrow \beta = Mm + \epsilon_{\beta|y}, \quad \epsilon_{\beta|y} \sim N(0, \sigma^2 M)$$

Rewrite:

$$\tilde{Y} = \tilde{X}(Mm + \epsilon_{\beta|y}) + \tilde{\epsilon}$$

$$= \tilde{X} M m + \underbrace{\tilde{X}\epsilon_{\beta|y} + \tilde{\epsilon}}_{\epsilon^*}$$

Since $\tilde{Y} \perp\!\!\!\perp y \mid \beta, \sigma^2$, we have

$$\epsilon^* \sim N(0, \sigma^2(\tilde{X} M \tilde{X}^T + I_m))$$

$$\Rightarrow \tilde{Y} \sim N(\tilde{X} M m, \sigma^2(\tilde{X} M \tilde{X}^T + I_m))$$

Q2:

2. Is there a lurking identity related to determinant? Hint: look @

$$p(x,y) = \underbrace{p(x)p(y|x)}_{\frac{1}{|A|^{1/2}|D|^{1/2}}\cdots} = \underbrace{p(y)p(x|y)}_{\frac{1}{|V_y|^{1/2}|M|^{1/2}}\cdots}$$

Then

$$|D + BAB^T| = |A_{p\times p}|^{1/2}|D|^{1/2}|?|_{p\times p}$$

Sol. let $X \sim \mathcal{N}(0, A)$ $(*)$
$Y|X \sim \mathcal{N}(BX, D)$

$\Rightarrow$ joint pdf:

$$p(x,y) = p(x)\,p(y|x)$$
$$= \frac{1}{|A|^{\frac{1}{2}}} e^{-\frac{1}{2}x^T A^{-1} x} \frac{1}{|D|^{\frac{1}{2}}} e^{-\frac{1}{2}(y-Bx)^T D^{-1}(y-Bx)}$$
$$= \frac{1}{|A|^{\frac{1}{2}}|D|^{\frac{1}{2}}} e^{-\frac{1}{2}Q}$$

where

$$Q = x^T A^{-1} x + (y-Bx)^T D^{-1}(y-Bx)$$

On the other hand, we showed in class: given $(*)$, then

$$Y \sim \mathcal{N}(0, D+BAB^T)$$
$$X|Y \sim \mathcal{N}(Mm, M)$$

where

$$M^{-1} = A^{-1} + B^T D^{-1} B$$
$$m = B^{-1} D^T y$$

Thus,

$$p(x,y) = p(y)\,p(x|y)$$
$$= \frac{1}{|D+BAB^T|} e^{-\frac{1}{2}y^T(D+BAB^T)^{-1}y} \times \frac{1}{|M|} e^{-\frac{1}{2}(x-Mm)M^{-1}(x-Mm)}$$
$$= \frac{1}{|M||D+BAB^T|} e^{-\frac{1}{2}Q}.$$

Therefore,

$$\frac{1}{|A|^{\frac{1}{2}}|D|^{\frac{1}{2}}} = \frac{1}{|M||D+BAB^T|}$$

$\Rightarrow$

$$|D + BAB^T| = |A|^{\frac{1}{2}}|D|^{\frac{1}{2}}|M^{-1}|$$

where

$$|M^{-1}| = |A^{-1} + B^T D^{-1} B| \quad \square.$$

Q3: Find M s.t.

$$\underset{n\times n}{|D} - \underset{n\times m}{C}\underset{m\times m}{A^{-1}}\underset{m\times n}{B}| = |D|\times|A|\times|M|$$

Sol.

$$\text{LHS} = |D||I - D^{-1}CA^{-1}B|$$

$$\overset{(*)}{=} |D||I - \underset{m\times n}{B}\underset{n\times n}{D^{-1}}\underset{n\times m}{C}\underset{m\times m}{A^{-1}}|$$

$$= |D||A - BD^{-1}C||A^{-1}|$$

$$= |D||A^{-1}||A - BD^{-1}C|$$

or

$$= |D||A||A^{-1} - A^{-1}BD^{-1}CA^{-1}|$$

Where $(*)$ is from

$$|I - BA| = |I - AB| \text{ for any}$$

conformable matrices A & B.

But I think what Dr. Banerjee wanted is this:

$$\left|\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right| = \left|\begin{bmatrix} A & 0 \\ C & I \end{bmatrix}\right| \left|\begin{bmatrix} I & A^{-1}B \\ 0 & D - CA^{-1}B \end{bmatrix}\right|$$

$$= |A||D - CA^{-1}B|$$

Q4: $\chi^2_{n-p}(0)$ & $Ga(a,b)$?

Sol. By 250A, $\chi^2_{n-p}(0) \overset{d}{=} y^T y$

& $y \sim \mathcal{N}_{n-p}(0, I)$.

mgf of $\chi^2_{n-p}(0)$ is

$$(1-2t)^{-\frac{n-p}{2}}$$

By 202A, mgf of $Ga(a,b)$ is

$$\left(1 - \frac{t}{b}\right)^{-a}$$

$\Rightarrow \chi^2_{n-p}(0)$ is indeed a

$Ga\left(\frac{n-p}{2}, \frac{1}{2}\right)$ variable.