# Biostat 250A HW4


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Read Chapter 2 of text

1 Show that the Moore-Penrose Inverse of a matrix A is unique.

2-3 Ex. 2a: #1, # 3

4-5 Ex. 2b: #3, #9

6 Ex. 2c: #3

7a Determine whether the quadratic form $Q(x,y,z) = 2x^2+2y^2+11z^2+16xy-2xz-2yz$ is non-negative for all real values of x, y and z.

b Write $7x_1^2+4x_1x_2-5x_2^2-6x_2x_3+3x_3^2+6x_1x_3$ in the form $x'Ax$, where $A = A'$ for some $x'=(x_1,x_2,x_3)$. What is the $Ex'Ax$ if x has mean (1,0,-1) and its covariance matrix is $2I_3$?

8-9 Ex 2d: #3, #4

10 Suppose $y' = (y_1,y_2,y_3)$ is distributed as tri-variate normal with mean $m' = (0, 0, 0)$ and define two matrices

$$C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

If C is the covariance matrix of y,

i. what is the correlation of $y_2$ and $y_3$?

ii. write down the moment generating function of y. (No derivation required)

iii. find the mean of $(1+y_1,y_2,y_3-1)'D(1+y_1,y_2,y_3-1)$.

iv. find the condition density of $y_1$, $y_3$ given $y_2 = 1$.

v. are the random variables $y'By$ and $y'Cy$ independent if y has now mean 0 and identity covariance matrix? Justify.

11-13 Miscellaneous Ex 2: #3, #6, #12.

## P1: Show Moore-Penrose is unique.

叶:
$$\begin{cases} AA^+A = A \\ A^+AA^+ = A^+ \\ (A^+A)^T = A^+A \\ (AA^+)^T = AA^+ \end{cases}$$

Existence: $\underset{m\times n}{A} = \underset{m\times r}{L}\ \underset{r\times n}{R}$, $r(L) = r(R) = r$

$\Rightarrow L^TAR^T = L^TLRR^T$ is of full rank

$\Rightarrow A^+ \triangleq R^T(L^TAR^T)^{-1}L^T$

Uniqueness: Let $G$ be another g-inv,

$$\begin{aligned} G &= GAG = (GA)^TG \\ &= A^TG^TG = (AA^+A)^TG^TG \\ &= (A^+A)A^TG^TG = A^+A(GA)^TG \\ &= A^+AG = A^+AA^+AG \\ &= A^+(AA^+)^T(AG)^T = A^+A^{+T}A^TG^TA^T \\ &= A^+A^{+T}(AGA)^T \\ &= A^+A^{+T}A^T = A^+(AA^+)^T \\ &= A^+AA^+ = A^+ \end{aligned}$$

$\square$.

## P2-3: Ex 2a. #1 #3

Sol. (a): $K = (2\pi)^{\frac{n}{2}} \det|\Sigma|^{\frac{1}{2}}$

But $2y_1^2 + y_2^2 + 2y_1y_2 - 22y_1 - 14y_2 + 65$ implies $(*)$

$$\begin{cases} \frac{-2\rho}{\sigma_1\sigma_2} = 2(1-\rho^2) \\ \frac{1}{\sigma_1^2} = 2(1-\rho^2) \\ \frac{1}{\sigma_2^2} = 1-\rho^2 \end{cases} \Rightarrow \begin{cases} \rho = -\frac{\sqrt{2}}{2} \\ \sigma_2 = \sqrt{2} \\ \sigma_1 = 1 \end{cases}$$

$\Rightarrow \Sigma = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \Rightarrow \det|\Sigma|^{\frac{1}{2}} = 1$

$\Rightarrow K = 2\pi$

(b): $\operatorname{Var} Y = \Sigma = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$, $\Sigma^{-1} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$

$$\begin{cases} \frac{\mu_1^2}{\sigma_1^2} - \frac{2\rho\mu_1\mu_2}{\sigma_1\sigma_2} + \frac{\mu_2^2}{\sigma_2^2} = 65(1-\rho^2) \\ -\frac{2\mu_1}{\sigma_1^2} + \frac{2\rho\mu_2}{\sigma_1\sigma_2} = -22(1-\rho^2) \end{cases} \Rightarrow$$

$\begin{cases} \mu_1 = 4 \\ \mu_2 = 3 \end{cases} \Rightarrow E\,Y = \begin{pmatrix} 4 \\ 3 \end{pmatrix}$

$\boxed{3}$ (a): $1 > 0$, $\begin{vmatrix} 1 & \rho \\ \rho & 1 \end{vmatrix} > 0 \Rightarrow 1-\rho^2 > 0$

$\begin{vmatrix} 1 & \rho & \rho \\ \rho & 1 & \rho \\ \rho & \rho & 1 \end{vmatrix} = (1-\rho)^2(1+2\rho) > 0 \Rightarrow$

$\rho > -\frac{1}{2}$.

(b): $\begin{vmatrix} 1-\lambda & \rho \\ \rho & 1-\lambda \end{vmatrix} = 0 \Rightarrow \lambda = \begin{cases} 1-\rho \\ 1+\rho \end{cases}$

$\Rightarrow$ corresponding eig-vecs are

$x_1 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$, $x_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \Rightarrow$

$$\Sigma = (x_1\ x_2)\begin{pmatrix} 1-\rho & \\ & 1+\rho \end{pmatrix}\begin{pmatrix} x_1^T \\ x_2^T \end{pmatrix}$$

$$\Rightarrow \Sigma^{\frac{1}{2}} = (x_1\ x_2)\begin{pmatrix} \sqrt{1-\rho} & \\ & \sqrt{1+\rho} \end{pmatrix}\begin{pmatrix} x_1^T \\ x_2^T \end{pmatrix}.$$

# P4-5 Ex2b #3 #9

#3 Sol. $\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$

$$\Rightarrow \tilde{\mu} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$$

$$\tilde{\Sigma} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 \\ 1 & 3 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 0 & 3 \end{pmatrix}$$

$$\Rightarrow Z \sim \mathcal{N}\left( \begin{pmatrix} 5 \\ 1 \end{pmatrix}, \begin{pmatrix} 10 & 0 \\ 0 & 3 \end{pmatrix} \right)$$

#9 Sol. $\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \underbrace{\begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix}}_{A} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix}$

To show $Y \sim \mathcal{N}_p(0, I_3)$, ETS the matrix $A$ is orthonormal:

$$A^T A = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & 0 & -\frac{2}{\sqrt{6}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\square$.

# P6: 2C #3.

Sol. ET solve:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & \rho \\ 0 & \rho & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 1 & -1 \end{pmatrix} = I_2 \text{ for some } \rho \Rightarrow$$

$$\rho = -\frac{1}{2}$$

# P7:

| | |
|---|---|
| 7a | Determine whether the quadratic form $Q(x,y,z) = 2x^2+2y^2+11z^2+16xy-2xz-2yz$ is non-negative for all real values of x, y and z. |
| b | Write $7x_1^2+4x_1x_2-5x_2^2-6x_2x_3+3x_3^2+6x_1x_3$ in the form $x'Ax$, where $A = A'$ for some $x'=(x_1,x_2,x_3)$. What is the $Ex'Ax$ if x has mean (1,0,-1) and its covariance matrix is $2I_3$? |

Sol. (a): $Q(x,y,z) = a^T \begin{pmatrix} 2 & 8 & -1 \\ 8 & 2 & -1 \\ -1 & -1 & 11 \end{pmatrix} a$, $a = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Since $\det \begin{vmatrix} 2 & 8 \\ 8 & 2 \end{vmatrix} = -60 < 0$. $Q$ is not always non-negative.

(b): $Q = x^T \overbrace{\begin{pmatrix} 7 & 2 & 3 \\ 2 & -5 & -3 \\ 3 & -3 & 3 \end{pmatrix}}^{A} x \Rightarrow$

$$\begin{aligned} \mathbb{E} Q &= \mathbb{E}\, Tr(x^T A x) \\ &= Tr(A\, \mathbb{E}(xx^T)) \\ &= Tr(A[\mu\mu^T + \Sigma]) \\ &= Tr\left(A \begin{pmatrix} 3 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & 3 \end{pmatrix}\right) \\ &= 14 \end{aligned}$$

# P89. 2d #3 #4.

#3 Sol. Use thm 2.7 in textbook:

$a(Y_1-Y_2)^2+b(Y_1+Y_2)^2 \sim \chi_2^2$ iff

$A=\begin{pmatrix} a+b & b-a \\ b-a & a+b \end{pmatrix}$ is of full rank &

$A^2=A \Rightarrow \begin{cases} (a+b)^2 > (b-a)^2; \; a+b>0 \\ (a+b)^2+(b-a)^2 = a+b \end{cases}$

$2(b^2-a^2) = b-a$

$\Rightarrow \begin{cases} a=\frac{1}{2} \\ b=\frac{1}{2} \end{cases}$

#4 Sol.:

$(Y_1-Y_2)^2+(Y_2-Y_3)^2+\dots+(Y_{n-1}-Y_n)^2+(Y_n-Y_1)^2$

$= y^T \underbrace{\begin{pmatrix} 2 & -1 & & -1 \\ -1 & \ddots & \ddots & \\ & \ddots & \ddots & -1 \\ -1 & & -1 & 2 \end{pmatrix}}_{A} y \Rightarrow$

$A^2 = \begin{pmatrix} 6 & -4 & & 4 \\ -4 & 6 & \ddots & \\ & \ddots & \ddots & -4 \\ -4 & & -4 & 6 \end{pmatrix}$ if $n>3$

$= \begin{pmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{pmatrix} = 3A$ if $n=3$ $\Rightarrow$

if $n=3$, then $\left(\frac{A}{3}\right)^2 = \frac{A}{3} = \begin{pmatrix} \frac{2}{3} & -\frac{1}{3} & -\frac{1}{3} \\ -\frac{1}{3} & \frac{2}{3} & -\frac{1}{3} \\ -\frac{1}{3} & -\frac{1}{3} & \frac{2}{3} \end{pmatrix}$

$\Rightarrow r(A) = Tr(A) = 2$

$\Rightarrow y^T\left(\frac{A}{3}\right)y \sim \chi_2^2$

If $n>3$, we need to find $c$ s.t.

$\begin{cases} \frac{6}{c^2} = \frac{2}{c} \\ \frac{-4}{c^2} = \frac{-3}{c} \end{cases}$ which is impossible.

# P10.

10 Suppose $y' = (y_1,y_2,y_3)$ is distributed as tri-variate normal with mean $m' = (0, 0, 0)$ and define two matrices

$$C = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix} \quad \text{and} \quad D = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}.$$

If C is the covariance matrix of y,

i. what is the correlation of $y_2$ and $y_3$?

ii. write down the moment generating function of y. (No derivation required)

iii. find the mean of $(1+y_1,y_2,y_3-1)$ 'D$(1+y_1,y_2,y_3-1)$.

iv. find the condition density of $y_1$, $y_3$ given $y_2 = 1$.

v. are the random variables y'By and y'Cy independent if y has now mean 0 and identity covariance matrix? Justify.

Sol. (i): $Cov(Y_2,Y_3)=1, Var(Y_2)=1, Var(Y_3)=3 \Rightarrow$

$Corr(Y_2,Y_3) = \frac{1}{\sqrt{1}\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

(ii): $Ee^{t^TY} = e^{\mu^Tt+\frac{t^T\Sigma t}{2}} = e^{\frac{1}{2}t^TCt}$

$= \exp\left(\frac{1}{2}(2t_1^2+t_2^2+t_3^2+2t_2t_3)\right)$

(iii): $\begin{pmatrix} 1+Y_1 \\ Y_2 \\ Y_3-1 \end{pmatrix} \sim N_3\left(\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, C\right)$

$\Rightarrow E\,\tilde{y}^TD\tilde{y} = \mu^TD\mu + Tr(DC)$

$= 4+15 = 19$

(iv): $\tilde{\mu} = \mu_{13} + \Sigma_{13,2}\Sigma_2^{-1}(Y_2-\mu_2)$

$= 0 + \begin{pmatrix} 0 \\ 1 \end{pmatrix} 1^{-1} (Y_2 - 0)$

$= \begin{pmatrix} 0 \\ Y_2 \end{pmatrix}$

$\tilde{\Sigma} = \Sigma_{13} - \Sigma_{13\cdot2}\Sigma_{22}^{-1}\Sigma_{2\cdot13}$

$= \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} 1^{-1} (0, 1)$

$= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$, $\det|\tilde{\Sigma}| = 4$

$\Rightarrow$

$f(Y_1,Y_3|Y_2=1) = \frac{1}{4\pi} e^{-\frac{1}{8}(Y_1^2+(Y_3-1)^2)}$.

(v): I Quote the following thm.

$\boxed{\text{THM}}$ $Y \sim \mathcal{N}(0, I)$ Then if $A^2 = A, B^2 = B$,

$\Rightarrow$ $Y^T A Y \perp\!\!\!\perp Y^T B Y$ iff $AB = 0$

- $Y^T D Y \perp\!\!\!\perp Y^T C Y$ iff $CD = 0 = DC$

Given $Y \sim \mathcal{N}(0, I)$

(Stronger version)

Since $C \cdot D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 3 \end{pmatrix}$

$\neq 0$

$\Rightarrow Y^T D Y \not\perp\!\!\!\perp Y^T C Y.$

# P11-13. Miscellaneous 2

## Ex #3 #6 #12.

Sol.

#3: $\bar{Y} = \frac{1}{n} 1^T y$ ; $\sum_{i=1}^{n-1} (Y_i - Y_{i+1})^2 = y^T \underbrace{\begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}}_{A} y$

Since $\frac{1}{n} 1^T \begin{bmatrix} 1 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix} = 0$

we have $\frac{1}{n} 1^T Y \perp\!\!\!\perp A y$

thus $\bar{Y} \perp\!\!\!\perp Y^T A Y$

#6:

$\frac{1}{1-\rho} \sum_i (Y_i - \bar{Y})^2 = y^T (I - \frac{11^T}{n}) y \frac{1}{1-\rho}$ $(\triangle)$

Let $X \sim \mathcal{N}_n(0, I)$

Then $Y \overset{d}{=} \Sigma^{\frac{1}{2}} X + \mu 1$,

But $Y_i - \bar{Y} = Y_i - \mu - (\bar{Y} - \mu)$

implies

$(\triangle) = X^T \Sigma^{\frac{1}{2}} (I - \frac{11^T}{n}) \Sigma^{\frac{1}{2}} X \frac{1}{1-\rho}$

Let $B = \frac{1}{1-\rho} \Sigma^{\frac{1}{2}} (I - \frac{11^T}{n}) \Sigma^{\frac{1}{2}}$

Then $B^2 = \frac{1}{(1-\rho)^2} \Sigma^{\frac{1}{2}} (I - \frac{11^T}{n}) \Sigma (I - \frac{11^T}{n}) \Sigma^{\frac{1}{2}}$

plug-in $\Sigma = (1-\rho) I + \rho 11^T$ $\Rightarrow$

$B^2 = \frac{1}{1-\rho} \Sigma^{\frac{1}{2}} (I - \frac{11^T}{n}) \Sigma^{\frac{1}{2}} = B$,

Moreover,

$r(B) = Tr(B) = \frac{1}{1-\rho} Tr(\Sigma - \frac{\Sigma 11^T}{n})$

$= n-1$

By fundamental thm,

$(\triangle) = X^T B X \sim \chi^2_{n-1}.$

#12: $2(Y_1 Y_2 - Y_2 Y_3 - Y_3 Y_1)$

$= Y^T \underbrace{\begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix}}_{A} Y$ $(*)$

$\mathbb{E} e^{tY^T A Y} = \int \frac{1}{(2\pi)^{\frac{3}{2}}} e^{-\frac{Y_1^2 + Y_2^2 + Y_3^2}{2}} e^{2tY_1Y_2 - 2tY_2Y_3 - 2tY_3Y_1} d\vec{Y}$

$= \int c \cdot e^{-\frac{1}{2} Y^T (I - 2tA) Y} dy$

$= \det |I - 2tA|^{-\frac{1}{2}}$

$\mathbb{E} e^{t(2U_1 - U_2 - U_3)} = \mathbb{E} e^{2tU_1} \mathbb{E} e^{-tU_2} \mathbb{E} e^{-tU_3}$

$= \left[(1-4t)(1-2t)(1-2t)\right]^{-\frac{1}{2}}$

$= \det |I - 2tA|^{-\frac{1}{2}}$ $\square$.