1 Let X be a chi-squared distribution with n degrees of freedom. Find $EX^{1/2}$ and hence find an unbiased estimate of σ in the standard linear model EY=Xβ where X is n x p with full column rank and error terms are independent, each with mean 0 and variance σ^2.

Soln: $X \sim \chi^2(n)$. $f(x) = 2^{-n/2}\Gamma(n/2)\, x^{\frac{n}{2}-1} e^{-\frac{x}{2}}$, $x>0$

$$E(X^{1/2}) = \int_0^\infty x^{1/2} f(x)\,dx$$

$$= \int 2^{-\frac{n}{2}} \Gamma(\tfrac{n}{2})^{-1} \cdot x^{\frac{n+1}{2}-1} e^{-\frac{x}{2}}\,dx$$

$$= 2^{\frac{1}{2}} \Gamma(\tfrac{n+1}{2})/\Gamma(\tfrac{n}{2}) \cdot \int 2^{-\frac{n+1}{2}} \Gamma(\tfrac{n+1}{2})^{-1} x^{\frac{n+1}{2}-1} e^{-\frac{x}{2}}\,dx$$

$$= \sqrt{2}\,\frac{\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n}{2})}$$

in MLR $EY = X\beta$, $\frac{(Y-X\hat\beta)'(Y-X\hat\beta)}{\sigma^2} \sim \chi^2(n-p)$

then $E\left(\frac{\sqrt{(Y-X\hat\beta)'(Y-X\hat\beta)}}{\sigma}\right) = \sqrt{2}\,\frac{\Gamma(\frac{n-p+1}{2})}{\Gamma(\frac{n-p}{2})}$

therefore: $\hat S = \frac{\Gamma(\frac{n-p}{2})}{\sqrt{2}\,\Gamma(\frac{n-p+1}{2})} \cdot \sqrt{(Y-X\hat\beta)'(Y-X\hat\beta)}$ has $E(\hat S) = \sigma$

is an unbiased estimator of σ.

2 Let μ belong to V, a p-dimensional vector space of R^n, let y belong to R^n and let P_V be the orthogonal projection matrix onto V.

If $\hat\mu = P_V y$ and $\hat\sigma^2 = ||(I-P_V)y||^2/(n-p)$ show that

i) $E||\hat\mu||^2$ always overestimates $||\mu||^2$,

ii) $E||\hat\mu - \mu||^2/\sigma^2 = p$ and identify the distribution of $||\hat\mu - \mu||^2$,

iii) $E\hat\sigma^4 = (n-p+2)\sigma^4/(n-p)$.

(i) $\mu \in V \subseteq R^n$, $y \sim N(\mu, \sigma^2 I_n)$. $\hat\mu = P_V y$.

then $E||\hat\mu||^2 = E[y' P_V y]$

$$= Ey' P_V Ey + tr(cov(y)\cdot P_V)$$

$$= \mu' P_V \mu + \sigma^2 p$$

$$= \mu'\mu + \sigma^2 p \quad > ||\mu||^2 \qquad (as\ \sigma^2 > 0)$$

$||\hat\mu||^2$ always overestimates $||\mu||^2$

ii) $E||\hat{\mu}-\mu||^2/\sigma^2 = p$ and identify the distribution of $||\hat{\mu}-\mu||^2$,

iii) $E\hat{\sigma}^4 = (n-p+2)\sigma^4/(n-p)$.

(ii) $\|\hat{\mu}-\mu\|^2 = \|P_V(y-\mu)\|^2 = (y-\mu)'P_V^2(y-\mu) = (y-\mu)'P_V(y-\mu)$

$$\frac{\|\hat{\mu}-\mu\|^2}{\sigma^2} = \left(\frac{y-\mu}{\sigma}\right)' P_V \left(\frac{y-\mu}{\sigma}\right)$$

Since $\frac{y-\mu}{\sigma} \sim N(0, I_n)$. P_V is orthogonal projection, $rk(P_V)=p$

$$\Rightarrow \frac{\|\hat{\mu}-\mu\|}{\sigma^2} \sim \chi^2_{(p)}$$

(iii) $\hat{\sigma}^2 = \|(I-P_V)y\|^2/(n-p)$

$$E(\hat{\sigma}^4) = E\left[\frac{(y'(I-P_V)y)^2}{(n-p)^2}\right]$$

$$= \frac{1}{(n-p)^2}\left(E(y'(I-P_V)y)^2 + Var(y'(I-P_V)y)\right)$$

$$= \frac{1}{(n-p)^2}\cdot\left\{(\mu'(I-P_V)\mu + tr(I-P_V)\cdot\sigma^2)^2 + 2\cdot tr(\sigma^2 I_n(I-P_V)\sigma^2 I_n(I-P_V))\right\}$$

$$= \frac{1}{(n-p)^2}\left\{(n-p)^2\sigma^4 + 2(n-p)\sigma^4\right\}$$

$$= \frac{(n-p+2)}{(n-p)}\cdot\sigma^4 \qquad \square$$

3 Recall that if X and Y are random variables with finite means and variances EEX|Y = EX. Use this result to show that if the conditional distribution of V given K is a central chi-squared with p+2K degrees of freedom, then the conditional expectation of 1/V given K is 1/(p-2+2K).

Soln: $V|K \sim \chi^2_{(p+2K)}$, i.e. $V|K=k \sim \chi^2_{(p+2k)}$. need to prove: $E(\frac{1}{V}|K=k) = \frac{1}{p-2+2k}$

$$E(\tfrac{1}{V}|K=k) = \int_0^\infty \frac{1}{v}\, 2^{-\frac{p+2k}{2}}\,\Gamma\left(\frac{p+2k}{2}\right)^{-1} v^{\frac{p+2k}{2}-1} e^{-\frac{v}{2}}\,dv$$

$$= \frac{1}{2}\frac{\Gamma\left(\frac{p+2k-2}{2}\right)}{\Gamma\left(\frac{p+2k}{2}\right)}\int_0^\infty 2^{-\frac{p+2k-2}{2}}\,\Gamma\left(\frac{p+2k-2}{2}\right)^{-1} v^{\frac{p+2k-2}{2}-1} e^{-\frac{v}{2}}\,dv$$

$$= \frac{1}{2}\left(\frac{p+2k}{2}-1\right)^{-1}$$

$$= \frac{1}{p+2k-2} \qquad \square$$

4 Suppose the conditional density of y given μ is univariate normal with mean μ and variance σ^2. If μ is univariate normal with mean μ_0 and variance σ^{*2}, find the conditional density of μ given y. Can you generalize the above setting when you now have a random sample size $y_1, y_2, \ldots, y_n$? Express the conditional density of μ given $y_1, y_2, \ldots, y_n$ in terms of n, μ_0, σ^2 and σ^{*2}. [These resulting distributions are called posterior distributions].

Soln: $y \mid \mu \sim N(\mu, \sigma^2)$. $\mu \sim N(\mu_0, \sigma^{*2})$

$$f(\mu \mid y) = \frac{f(\mu, y)}{f(y)}$$

$$\propto f(y \mid \mu) \cdot f(\mu)$$

$$\propto \exp\left(-\frac{1}{2\sigma^2}(y-\mu)^2\right) \cdot \exp\left(-\frac{1}{2\sigma^{*2}} \cdot (\mu-\mu_0)^2\right)$$

$$\propto \exp\left(-\frac{1}{2\sigma^2\sigma^{*2}}\left(\sigma^{*2}\mu^2 - 2\sigma^{*2} y\mu + \sigma^2\mu^2 - 2\sigma^2\mu_0\mu\right)\right)$$

$$\propto \exp\left(-\frac{(\sigma^{*2}+\sigma^2)}{2\sigma^2\sigma^{*2}} \cdot \left(\mu - \frac{\sigma^{*2}y + \sigma^2\mu_0}{\sigma^{*2}+\sigma^2}\right)^2\right)$$

since pdf and distribution is 1-on-1: $\mu \mid y \sim N\left(\frac{\sigma^{*2}y + \sigma^2\mu_0}{\sigma^{*2}+\sigma^2}, \frac{\sigma^2\sigma^{*2}}{\sigma^{*2}+\sigma^2}\right)$

assuming $y_1 \mid \mu, \ldots y_n \mid \mu \overset{iid}{\sim} N(\mu, \sigma^2)$, $\mu \sim N(\mu_0, \sigma^{*2})$

let $Y = (y_1 \cdots y_n)'$

$$f(\mu \mid Y) = \frac{f(Y \mid \mu) f(\mu)}{f(Y)}$$

$$\propto \prod_{i=1}^{n} f(y_i \mid \mu) \cdot f(\mu)$$

$$\propto \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(\mu-y_i)^2 - \frac{1}{2\sigma^{*2}}(\mu-\mu_0)^2\right)$$

$$\propto \exp\left(-\frac{1}{2\sigma^2}\left(n\mu^2 - 2\left(\sum_{i=1}^{n} y_i\right)\mu\right) - \frac{1}{2\sigma^{*2}}(\mu^2 - 2\mu_0\mu)\right)$$

$$\propto \exp\left(-\frac{1}{2\sigma^2\sigma^{*2}}\left(n\sigma^{*2}\mu^2 + \sigma^2\mu^2 - 2\sigma^{*2}\left(\sum_{i=1}^{n} y_i\right)\mu - 2\sigma^2\mu_0\mu\right)\right)$$

$$\propto \exp\left\{-\frac{(n\sigma^{*2}+\sigma^2)}{2\sigma^2\sigma^{*2}}\left(\mu^2 - 2\,\frac{\sigma^{*2}\left(\sum_{i=1}^{n} y_i\right) + \sigma^2\mu_0}{n\sigma^{*2}+\sigma^2}\mu\right)\right\}$$

$$\propto \exp\left\{-\frac{n\sigma^{*2}+\sigma^2}{2\sigma^2\sigma^{*2}}\left(\mu - \frac{\sigma^{*2}\left(\sum_{i=1}^{n} y_i\right) + \sigma^2\mu_0}{n\sigma^{*2}+\sigma^2}\right)^2\right.$$

therefore: $\mu \mid y_1 \ldots y_n \sim N\left(\frac{\sigma^{*2}\left(\sum_{i=1}^{n} y_i\right) + \sigma^2\mu_0}{n\sigma^{*2}+\sigma^2}, \frac{\sigma^2\sigma^{*2}}{n\sigma^{*2}+\sigma^2}\right)$

5 To obtain Bayesian estimates for the vector of parameters β in the standard linear model, answers to this and the next question are helpful . First recall the gamma function defined by $\Gamma(x) = \int_0^\infty t^{x-1}\exp(-t)dt$.

Show that $\int_0^\infty \exp(-k/x)x^{-v-1}dx = k^{-v}\Gamma(v)$ and $\int_0^\infty \exp(-a/x^2)x^{-b-1}dx = \frac{1}{2}a^{-b/2}\Gamma(b/2)$.

Soln : $Y|\beta,\sigma^2 \sim N(X\beta, \sigma^2 I_n)$, β has locally uniform distribution , i.e. $f(\beta) = C$.

$\sigma > 0$, and $f(\sigma) \propto \frac{1}{\sigma}$; $\beta \perp\!\!\!\perp \sigma$

$$f(\beta,\sigma \mid Y) \propto f(Y \mid \beta,\sigma)\, f(\beta,\sigma)$$

$$\propto (2\pi\sigma^2)^{-\frac{n}{2}} \cdot \exp\left(-\frac{1}{2\sigma^2}(Y-X\beta)'(Y-X\beta)\right)\cdot \sigma^{-1}$$

$$f(\beta \mid Y) \propto \int_0^\infty f(\beta,\sigma^2 \mid Y)\, d\sigma$$

$$\propto \int_0^\infty \sigma^{-(n+1)} \exp\left(-\frac{1}{2}(Y-X\beta)'(Y-X\beta)\cdot\sigma^{-2}\right) d\sigma$$

$$\propto \frac{1}{2}\left(\frac{1}{2}(Y-X\beta)'(Y-X\beta)\right)^{-\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)$$

$$\propto \left[(Y-X\beta)'(Y-X\beta)\right]^{-\frac{n}{2}}$$

$$(Y-X\beta)'(Y-X\beta) = (Y-X\hat\beta)'(Y-X\hat\beta) + (X(\hat\beta-\beta))'(X(\hat\beta-\beta))$$

$$= (n-p)s^2 + \|X\beta - X\hat\beta\|^2$$

$$= (n-p)s^2 + (\beta-\hat\beta)X'X(\beta-\hat\beta)$$

that is : $f(\beta \mid Y) \propto \left((n-p)s^2 + (\beta-\hat\beta)X'X(\beta-\hat\beta)\right)^{-\frac{n}{2}}$

$$\propto \left(1 + \frac{(\beta-\hat\beta)X'X(\beta-\hat\beta)}{(n-p)s^2}\right)^{-\frac{n}{2}}$$

(reference from seber's , p 74)

$\beta \mid Y$ follows p dimensional multivariate t-distribution

with $v = n-p$, $\Sigma = s^2(X'X)^{-1}$, $\mu = \hat\beta$

6 Recall the result from Biostat 250A that if we have conformable matrices A, B, U and V, and the indicated inverses all exist, we have

$$(A+UBV)^{-1} = A^{-1} - A^{-1}UB(B + BVA^{-1}UB)^{-1}BVA^{-1}.$$

Use the above result to show that

$$(y-X\beta)'(y-X\beta) + (\beta-m)'V^{-1}(\beta-m) = (\beta-m^*)'V^{*-1}(\beta-m^*) + (y - Xm)'(I + XVX')^{-1}(y - Xm),$$

where $V^* = (X'X+V^{-1})^{-1}$ and $m^* = V^*(X'y + V^{-1}m)$.

[This is the essence of Theorem 3.7 in Biostat 250A textbook. I urge you to try hard to do the algebra on your own before referring to the book on pages 74-76 that uses much of what we learned from last quarter.]

Soln: $LHS = \beta'X'X\beta - 2\beta'X'y + y'y + \beta'V^{-1}\beta - 2\beta'V^{-1}m + m'V^{-1}m$

$= \beta'(X'X+V^{-1})\beta - 2\beta'(X'y+V^{-1}m) + y'y + m'V^{-1}m$

$= [\beta - (X'X+V^{-1})^{-1}(X'y+V^{-1}m)]'(X'X+V^{-1})[\beta - (X'X+V^{-1})^{-1}(X'y+V^{-1}m)]$

$- (X'y+V^{-1}m)'(X'X+V^{-1})^{-1}(X'y+V^{-1}m)$

$+ y'y + m'V^{-1}m$

$= (\beta-m^*)'V^{*-1}(\beta-m^*) - (X'y+V^{-1}m)'(X'X+V^{-1})^{-1}(X'y+V^{-1}m) + y'y + m'V^{-1}m$

from woodbury: $(X'X+V^{-1})^{-1} = V + VX'(I+XVX')^{-1}XV$

then: $(X'y+V^{-1}m)'(X'X+V^{-1})^{-1}(X'y+V^{-1}m)$

$= (X'y+V^{-1}m)'[V - VX'(I+XVX')^{-1}XV](X'y+V^{-1}m)$

$= y'XVX'y - y'XVX'(I+XVX')^{-1}XVX'y + 2y'X(V - VX'(I+XVX')^{-1}XV)(V^{-1}m)$

$+ m'V^{-1}m - m'X'(I+XVX')^{-1}Xm$

$= y'\{XVX'(I+XVX')^{-1}[I+XVX'-XVX']\}y + 2y'(I - XVX'(I+XVX)^{-1})Xm$

$+ m'V^{-1}m - m'X'(I+XVX')^{-1}Xm$

$RHS = (\beta-m^*)'V^{*-1}(\beta-m^*) + y'\{I - XVX'(I+XVX')^{-1}\}y - 2y'(I - XVX'(I+XVX)^{-1})Xm$

$+ (Xm)'(I+XVX')^{-1}Xm.$

since $I - XVX'(I+XVX')^{-1} = (I+XVX'-XVX')(I+XVX')^{-1} = (I+XVX')^{-1}$

$RHS = (\beta-m^*)'V^{*-1}(\beta-m^*) + (y-Xm)'(I+XVX')^{-1}(y-Xm)$ □