Lec 13

Bayesian Est. in LM

$\mathbb{E}Y = X\beta$. $Cov(e) = \sigma^2 I$

70-80s: Freq. vs. Bayesian

$f(y|\theta)$ density; $g(\theta)$ prior density

$\hat{\theta}_{MLE} = \underset{\theta\in\Theta}{\arg\max} \ln L(y|\theta)$, $L(y|\theta) = \prod_{i=1}^{n} f(y_i|\theta)$

Posterior $p(\theta|y) = f(y,\theta)/f(y) \propto f(y|\theta)g(\theta)$

Ex let $Y_1, \dots, Y_n \sim \mathcal{N}(\mu, \sigma^2)$, $\theta = (\mu, \sigma^2)$.

$$L(y|\theta) = \left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\sum_i^n (y_i-\mu)^2}$$

$$= \left(\frac{1}{\sqrt{2\pi}\sigma^2}\right)^{\frac{n}{2}} e^{-\frac{1}{2\sigma^2}\{\sum_i (y_i-\bar{y})^2 + n(\bar{y}-\mu)^2\}}$$

If σ is known,

$$L(y|\mu) \propto e^{-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2}$$

Assume $f(\mu) \propto e^{-\frac{1}{2\sigma_0}(\mu-\mu_0)^2}$

Posterior density is

$$p(\mu|y) \propto e^{-\frac{n}{2\sigma^2}(\bar{y}-\mu)^2} e^{-\frac{1}{2\sigma_0^2}(\mu-\mu_0)^2}$$

$$= \exp\{-\tfrac{n}{2\sigma^2}(\bar{y}-\mu)^2 - \tfrac{1}{2\sigma_0^2}(\mu-\mu_0)^2\}$$

Let's see if $\exists \mu_n$ & σ_n^2 s.t. RHS is $\propto$

$$\exp\{-\tfrac{1}{2\sigma_n^2}(\mu-\mu_n)^2\}$$

Equate coefficients:

$$\mu^2: -\frac{1}{2\sigma_n^2} = -\frac{1}{2\sigma_0^2} - \frac{n}{2\sigma^2}$$

$$\sigma_n^2 = \left(\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}\right)^{-1}$$

$$\mu: \frac{\mu_n}{\sigma_n^2} = \frac{\mu_0}{\sigma_0^2} + \frac{\bar{x}_n}{\sigma^2}$$

$$\Rightarrow \mu_n = \frac{\sigma^2}{\sigma^2 + n\sigma_0^2}\mu_0 + \frac{n\sigma_0^2}{\sigma^2+n\sigma_0^2}\bar{x}$$

posterior density: $\mu|y \propto \mathcal{N}(\mu_n, \sigma_n^2)$

Ex $Y \sim Bin(n, \theta)$, $\mathbb{P}(Y=y) = \binom{n}{y}\theta^y(1-\theta)^{n-y}$

Let $g(\theta)$ be a prior density for θ.

Posterior for $\theta \propto L(y|\theta)g(\theta)$

If $g(\theta) = \frac{1}{B(\alpha,\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$

Posterior is $\propto$ $\underline{\theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1}}$

Relative mean & variance of the Beta-Binomial r.v.

$\mathbb{E}Y = X\beta$, $Var(e) = \sigma^2 I$, $\theta^T = (\underset{p\times1}{\beta^T}, \underset{1\times1}{\sigma^2})$

$$g(\theta) = g(\beta, \sigma^2) = g_1(\beta)g_2(\sigma^2)$$

Preliminaries:

(a) $\int_0^\infty e^{-\frac{k}{x}} x^{-v-1} dx = \frac{1}{k^v \Gamma(v)}$

(b) $\int_0^\infty e^{-\frac{a}{x^2}} x^{-b-1} dx = \frac{\Gamma(b/2)}{2a^{b/2}}$

(c) $\|Y - X\beta\|_2^2 = (n-p)S^2 + \|X(\hat{\beta}-\beta)\|_2^2$

$$Y = X\beta + \varepsilon,\ \varepsilon \sim \mathcal{N}(0, \sigma^2 I)$$

$$f(\theta|y) \propto f(y|\theta)g(\theta)$$

what is $g(\theta)$: $\underbrace{g_1(\beta)}_{\text{constant}}\underbrace{g_2(\sigma^2)}_{\propto\frac{1}{\sigma}}$

$$f(\theta|Y) \propto \frac{1}{\sigma}\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{-\frac{1}{2\sigma^2}\|Y-X\beta\|_2^2}$$

If interest is in β alone,

$$f(\beta|y) = \int_0^\infty \frac{1}{(\sqrt{2\pi})^n}\frac{1}{\sigma^{n+1}} e^{-\frac{1}{2\sigma^2}\|Y-X\beta\|_2^2} d\sigma$$

by (b)

$$= \frac{1}{2}\left(\frac{\|Y-X\beta\|_2^2}{2}\right)^{-\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)$$

$$\Rightarrow f(\beta|y) \propto \|Y - X\beta\|^{-n}$$

by (c)

$$= \left((n-p)S^2 + \|X\hat{\beta} - X\beta\|_2^2\right)^{-\frac{n}{2}}$$

$$\propto \left(1 + \frac{(\hat{\beta}-\beta)^T X^T X(\hat{\beta}-\beta)}{(n-p)S^2}\right)^{-\frac{n}{2}}$$

Multivariate t-distribution

Page 475 (Lee & Seber)

Write $Y \sim t_m(v, \mu, \Sigma)$ if

$$f_Y(y) = \frac{\Gamma(\frac{v+m}{2})}{(\pi v)^{\frac{m}{2}}\Gamma(\frac{v}{2})}\frac{1}{|\Sigma|^{\frac{1}{2}}}\left(1 + \frac{(y-\mu)^T\Sigma^{-1}(y-\mu)}{v}\right)^{-\frac{v+m}{2}}$$

$\mathbb{E}Y = \mu$, $\operatorname{Cov} Y = \Sigma$.

$\mathbb{E}(\mathbb{E}(Y|X)) = Y$; $\operatorname{Var} Y = \mathbb{E}(\operatorname{Var}(Y|X)) + \operatorname{Var}(\mathbb{E}(Y|X))$

Before we worked with improper prior densities. Now work with proper priors

$$\theta^T = (\beta^T, \sigma^2)$$

$$f(\beta, \sigma^2) = \underbrace{f_1(\beta|\sigma^2)}_{N_p(m, \sigma^2 V)} \underbrace{f_2(\sigma^2)}_{\text{Inverted Gamma}}$$

$\boxed{IG}$

$$f(x|\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\beta/x}$$

$$f(\theta|y) \propto f(y|\theta) g(\theta)$$

$$\propto (\sigma^2)^{-\frac{n+d+2+p}{2}} \exp\left(-\frac{1}{2\sigma^2}\left[(y-X\beta)^T(y-X\beta) + (\beta-m)^T V^{-1}(\beta-m) + a\right]\right)$$

where $f(\sigma^2) \propto \frac{1}{(\sigma^2)^{\frac{d+2}{2}}} e^{-a/2\sigma^2}$.

Let $Q = (y-X\beta)^T(y-X\beta) + (\beta-m)^T V^{-1}(\beta-m)$

$$f(\beta|y) \propto \int_0^\infty (\sigma^2)^{-\frac{n+d+2+p}{2}} e^{-\frac{1}{2\sigma^2}(Q+a)} d\sigma^2$$

$$\propto (Q+a)^{-\frac{d+n+p}{2}} \quad \text{(verify)}$$

$$\propto \left(1 + \frac{1}{n+d}(\beta-m_*)^T W_*^{-1}(\beta-m_*)\right)^{-\frac{d+n+p}{2}}$$

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$$\beta|y \sim t_p(n+d, m_*, W_*) \quad \square$$