Biostat 250A Hw 8

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Refer to the textbook:

1 Ex. 4a, Problem 3 (Read subsection 3.8 on Lagrange Multipliers).

2-6 Ex. 4b, Problems 1-5.

7-8 Misc. Ex. 4, Problems 2, 4.

Problem 1:

3. If $\hat{\boldsymbol{\lambda}}_H$ is the least squares estimate of the Lagrange multiplier associated with the constraints $\mathbf{A}\boldsymbol{\beta} = \mathbf{c}$ (cf. Section 3.8), show that

$$\text{RSS}_H - \text{RSS} = \sigma^2 \hat{\boldsymbol{\lambda}}'_H (\text{Var}[\hat{\boldsymbol{\lambda}}_H])^{-1} \hat{\boldsymbol{\lambda}}_H.$$

(This idea is used to construct Lagrange multiplier tests.)

Sol:

$$\hat{\lambda}_H = 2\left\{[A(X^TX)^{-1}A^T]^{-1}(A\hat{\beta} - c)\right\}$$

$$\hat{\beta}_H = \hat{\beta} - \frac{1}{2}(X^TX)^{-1}A^T\hat{\lambda}_H$$

Then $RSS_H - RSS$

$$= \|Y - X\hat{\beta}_H\|_2^2 - \|Y - X\hat{\beta}\|_2^2$$

$$= \hat{\beta}_H^T X^TX\hat{\beta}_H - 2Y^TX\hat{\beta}_H - \hat{\beta}^TX^TX\hat{\beta} + 2Y^TX\hat{\beta}$$

$$= (\hat{\beta}_H - \hat{\beta})^T X^TX(\hat{\beta}_H - \hat{\beta})$$

$$= \frac{1}{4}\hat{\lambda}_H^T A(X^TX)^{-1}A^T\hat{\lambda}_H$$

But $\text{Var}\,\hat{\lambda}_H$

$$= 4(A(X^TX)^{-1}A^T)^{-1}\,\text{Var}(A\hat{\beta})\,(A(X^TX)^{-1}A^T)^{-1}$$

$$= 4\,(A(X^TX)^{-1}A^T)^{-1}$$

Thus, $RSS_H - RSS$

$$= \hat{\lambda}_H^T(\text{Var}\,\hat{\lambda}_H)^{-1}\hat{\lambda}_H$$

Problem 2:

1. Let $Y_i = \beta_0 + \beta_1 x_{i1} + \cdots + \beta_{p-1}x_{i,p-1} + \varepsilon_i$, $i = 1, 2, \ldots, n$, where the ε_i are independent $N(0, \sigma^2)$. Prove that the F-statistic for testing the hypothesis $H: \beta_q = \beta_{q+1} = \cdots = \beta_{p-1} = 0$ $(0 < q \le p-1)$ is unchanged if a constant, c, say, is subtracted from each Y_i.

Sol. $F = \dfrac{(RSS_H - RSS)/(p-q)}{RSS/(n-p)} = \dfrac{Y^T(P - P_H)Y}{Y^T(I-P)Y}\,\dfrac{n-p}{p-q}$

where P, P_H are orthogonal projections.

- The constraint is $\underset{(p-q)\times p}{A}\beta = 0$ where

$\beta^T = [\beta_0, \ldots, \beta_{p-1}]$ and

$$A = \begin{bmatrix} 0 & \cdots & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & 0 & 1 & \ddots & \vdots \\ \vdots & & \vdots & \vdots & \ddots & \ddots & 0 \\ \underbrace{0 \quad \cdots \quad 0}_{q} & & & 0 & \cdots & 0 & 1 \end{bmatrix} = \left[\underset{(p-q)\times q}{O} \;\; \underset{(p-q)\times(p-q)}{I}\right]$$

Thus the restricted model is

$$Y_i = \beta_0 + \beta_1 X_{i1} + \cdots + \beta_{q-1}X_{i(q-1)} + \varepsilon_i, \quad i = 1, \ldots, n$$

The new design matrix is

$$X_H = \begin{bmatrix} 1 & X_{11} & \cdots & X_{1(q-1)} \\ \vdots & \vdots & & \vdots \\ 1 & X_{n1} & \cdots & X_{n(q-1)} \end{bmatrix}.$$

Since both the new design matrix & the unconstrained matrix contain the intercept term, we have

$$(I-P)1 = 0, \quad (I - P_H)1 = 0$$

where $1 = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}$ & P, P_H are Orthogonal proj.

Thus, when a constant C is subtracted from each Y_i, we have

$$(Y - c1)^T(I-P)(Y-c1)$$

$$= Y^T(I-P)Y \quad \text{and}$$

$$(Y-c1)^T(P-P_H)(Y-c1)$$

$$= (Y-c1)([P-I] + [I-P_H])(Y-c1)$$

$$= Y^T(P-P_H)Y, \quad \text{thus,}$$

F statistic remains unchanged.

Problem 3:

2. Let $Y_i = \beta_0 + \beta_1 x_i + \varepsilon_i$, $(i = 1, 2, \ldots, n)$, where the ε_i are independent $N(0, \sigma^2)$.

   (a) Show that the correlation coefficient of $\hat{\beta}_0$ and $\hat{\beta}_1$ is $-n\bar{x}/(n\sqrt{\sum x_i^2})$.

   (b) Derive an F-statistic for testing $H : \beta_0 = 0$.

Sol. (a): Note $Var\begin{pmatrix}\hat\beta_0 \\ \hat\beta_1\end{pmatrix} = \sigma^2 (X^T X)^{-1}$ where

$$X^T = \begin{pmatrix} 1 & \cdots & 1 \\ x_1 & \cdots & x_n \end{pmatrix}$$

Thus, $$r_{\hat\beta_0, \hat\beta_1} = \frac{Cov(\hat\beta_0, \hat\beta_1)}{\sqrt{Var\,\hat\beta_0}\sqrt{Var\,\hat\beta_1}}$$

We have $$Cov(\hat\beta_0, \hat\beta_1) = -\frac{\sum_i x_i}{n(\sum_i (x_i-\bar{x})^2)}\sigma^2$$

$$Var\,\hat\beta_0 = \frac{\sum_i x_i^2}{n(\sum_i (x_i-\bar{x})^2)}\sigma^2$$

$$Var\,\hat\beta_1 = \frac{1}{(\sum_i (x_i-\bar{x})^2)}\sigma^2$$

$$\Rightarrow r_{\hat\beta_0, \hat\beta_1} = -\frac{\sum_i x_i}{\sqrt{n\sum_i x_i^2}} = \frac{-n\bar{x}}{\sqrt{n\sum_i x_i^2}}.$$

(b): Recall from previous problem,

$$F = \frac{RSS_H - RSS}{RSS}\frac{n-p}{q} = \frac{Y^T(P-P_H)Y}{Y^T(I-P)Y}(n-2)$$

The constraint is $[1\ 0]\begin{bmatrix}\beta_0 \\ \beta_1\end{bmatrix} = 0$

& $$RSS_H - RSS = \frac{n\sum_i x_i^2 - (\sum_i x_i)^2}{\sum_i x_i^2}\hat\beta_0^2$$ Theorem 4.1 (in textbook)

$$\frac{RSS}{\sigma^2} = \frac{1}{\sigma^2}S^2(n-p),\ S^2 = \frac{1}{n-p}\sum_i (Y_i - \hat{Y}_i)^2$$

$$\Rightarrow F = \hat\beta_0^2\left(n\sum_i x_i^2 - (\sum_i x_i)^2\right) / S^2 (\sum_i x_i)^2$$

Under H_0, $F \sim F_{1,n-p}(0)$.

Problem 4:

3. Given that $\bar{x} = 0$, derive an F-statistic for testing the hypothesis $H : \beta_0 = \beta_1$ in Exercise No. 2 above. Show that it is equivalent to a certain t-test.

Sol. $H_0 : \beta_0 - \beta_1 = 0$.

① $a^T\beta = 0$ where $a = \begin{pmatrix}1 \\ -1\end{pmatrix}$

$$F = \frac{RSS_H - RSS}{RSS}\frac{n-p}{q} = \frac{Y^T(P-P_H)Y}{Y^T(I-P)Y}(n-2)$$

Note $\bar{x} = 0 \Rightarrow \sum_i x_i = 0$, thus,

$$(RSS_H - RSS) = a^T\hat\beta(a^T(X^TX)^{-1}a)^{-1}a^T\hat\beta$$
$$= (\hat\beta_0 - \hat\beta_1)^2\left[\frac{1}{n} + \frac{1}{\sum_i x_i^2}\right]^{-1}$$

$$\frac{1}{\sigma^2}RSS = \frac{1}{\sigma^2}S^2(n-2),\ S^2 = \frac{1}{n-2}\sum_i (Y_i - \hat{Y}_i)^2$$

$$\Rightarrow F = \frac{(\hat\beta_0 - \hat\beta_1)^2}{S^2}\left(\frac{1}{n} + \frac{1}{\sum_i x_i^2}\right)^{-1}$$

②
$$Var(\hat\beta_0 - \hat\beta_1) = Var\,\hat\beta_0 + Var\,\hat\beta_1 - 2Cov$$
$$= \frac{\sum_i x_i^2}{n\sum_i x_i^2}\sigma^2 + \frac{1}{\sum_i x_i^2}\sigma^2$$
$$= \sigma^2\left(\frac{1}{n} + \frac{1}{\sum_i x_i^2}\right)$$

$\Rightarrow$ a t-statistic can be

$$t = \frac{\hat\beta_0 - \hat\beta_1}{\sqrt{Var(\hat\beta_0 - \hat\beta_1)}}$$
$$= \frac{\hat\beta_0 - \hat\beta_1}{\hat\sigma}\left(\frac{1}{n} + \frac{1}{\sum_i x_i^2}\right)^{-\frac{1}{2}}$$

where $\hat\sigma = (S^2)^{\frac{1}{2}}$, thus,

$$F = t^2.$$

Problem 5:

4. Let

$$\begin{aligned} Y_1 &= \theta_1 + \theta_2 + \varepsilon_1, \\ Y_2 &= 2\theta_2 + \varepsilon_2, \end{aligned}$$

and

$$Y_3 = -\theta_1 + \theta_2 + \varepsilon_2,$$

where the ε_i $(i = 1, 2, 3)$ are independent $N(0, \sigma^2)$. Derive an F-statistic for testing the hypothesis $H : \theta_1 = 2\theta_2$.

Sol: Let $Y^T = (Y_1, Y_2, Y_3)$.

$$X = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ -1 & 1 \end{bmatrix}, \quad \theta^T = (\theta_1, \theta_2), \text{ Then,}$$

$$y = X\theta + \varepsilon, \quad \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{pmatrix}.$$

$H_0: \theta_1 = 2\theta_2$ Let $\alpha^T\theta = 0$ where $\alpha = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

Define $F = \dfrac{RSS_H - RSS}{RSS} \dfrac{n-p}{q}$, $n=3$, $p=2$, $q=1$,

$$\& \quad \hat{\theta} = (X^TX)^{-1}X^Ty = \begin{pmatrix} (Y_1 - Y_3)/2 \\ (Y_1 + 2Y_2 + Y_3)/6 \end{pmatrix}$$

$$(RSS_H - RSS) = a^T\hat{\theta}(a^T(X^TX)^{-1}a)^{-1}a^T\hat{\theta}$$

$$= \frac{6}{7}\left(\frac{Y_1}{6} - \frac{2}{3}Y_2 - \frac{5}{6}Y_3\right)^2$$

$$\frac{1}{6^2}RSS = \frac{1}{6^2}S^2(n-p) = S^2/6^2 \quad \Rightarrow$$

$$F = \frac{6}{7}\frac{\left(\frac{1}{6}Y_1 - \frac{2}{3}Y_2 - \frac{5}{6}Y_3\right)^2}{S^2}$$

where $S^2 = \frac{1}{n-p}\sum_i (Y_i - \hat{Y}_i)^2$

$$= \sum_i (Y_i - \hat{Y}_i)^2 \qquad \square.$$

Under H_0, $F \sim F_{1,1}(0)$.

Problem 6:

5. Given $\mathbf{Y} = \boldsymbol{\theta} + \boldsymbol{\varepsilon}$, where $\boldsymbol{\varepsilon} \sim N_4(\mathbf{0}, \sigma^2\mathbf{I}_4)$ and $\theta_1 + \theta_2 + \theta_3 + \theta_4 = 0$, show that the F-statistic for testing $H : \theta_1 = \theta_3$ is

$$\frac{2(Y_1 - Y_3)^2}{(Y_1 + Y_2 + Y_3 + Y_4)^2}.$$

Sol. $\theta_1 + \theta_2 + \theta_3 + \theta_4 = 0$ implies $\theta_4 = -\theta_1 - \theta_2 - \theta_3$. So the model is

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \end{pmatrix} = \begin{pmatrix} 1 & & \\ & 1 & \\ & & 1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} + \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{pmatrix}$$

$H_0: \theta_1 = \theta_3$: $\alpha^T\theta = 0$ where $\alpha = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

$$F = \frac{RSS_H - RSS}{RSS}\frac{n-p}{q}, \quad n=4, \; p=3, \; q=1$$

$$\& \quad \hat{\theta} = (X^TX)^{-1}X^Ty$$

$$= \begin{pmatrix} \frac{3}{4}Y_1 - \frac{1}{4}(Y_2 + Y_3 + Y_4) \\ \frac{3}{4}Y_2 - \frac{1}{4}(Y_1 + Y_3 + Y_4) \\ \frac{3}{4}Y_3 - \frac{1}{4}(Y_1 + Y_2 + Y_4) \end{pmatrix}$$

$$RSS_H - RSS = a^T\hat{\theta}(a^T(X^TX)^{-1}a)^{-1}a^T\hat{\theta}$$

$$= \frac{1}{2}(\hat{\theta}_1 - \hat{\theta}_3)^2$$

$$= \frac{1}{2}(Y_1 - Y_3)^2$$

$$RSS = Y^T(I - X(X^TX)^{-1}X^T)Y$$

$$= \frac{1}{4}(Y_1 + Y_2 + Y_3 + Y_4)^2$$

$$\Rightarrow \quad F = \frac{2(Y_1 - Y_3)^2}{(Y_1 + Y_2 + Y_3 + Y_4)^2}$$

Problem 7:

2. Given the two regression lines

$$Y_{ki} = \beta_k x_i + \varepsilon_{ki} \qquad (k = 1, 2;\ i = 1, 2, \ldots, n),$$

show that the F-statistic for testing $H : \beta_1 = \beta_2$ can be put in the form

$$F = \frac{(\hat\beta_1 - \hat\beta_2)^2}{2S^2 \left(\sum_i x_i^2\right)^{-1}}.$$

Obtain RSS and RSS_H and verify that

$$\text{RSS}_H - \text{RSS} = \frac{\sum_i x_i^2 (\hat\beta_1 - \hat\beta_2)^2}{2}.$$

Sol. Rewrite:

$$\underbrace{\begin{pmatrix} Y_{11} \\ \vdots \\ Y_{1n} \\ Y_{21} \\ \vdots \\ Y_{2n} \end{pmatrix}}_{Y} = \underbrace{\begin{pmatrix} X_1 & 0 \\ \vdots & \vdots \\ X_n & 0 \\ 0 & X_1 \\ \vdots & \vdots \\ 0 & X_n \end{pmatrix}}_{X} \underbrace{\begin{pmatrix} \beta_1 \\ \beta_2 \end{pmatrix}}_{\beta} + \underbrace{\begin{pmatrix} \varepsilon_{11} \\ \vdots \\ \varepsilon_{1n} \\ \varepsilon_{21} \\ \vdots \\ \varepsilon_{2n} \end{pmatrix}}_{\epsilon}$$

$H_0 : \beta_1 = \beta_2$, or $\alpha^T \beta = 0$ where $\alpha = (1, -1)$.

Then $F = \dfrac{RSS_H - RSS}{RSS}(n-2)$

Let $\hat\beta = (X^T X)^{-1} X^T Y$, then

$$RSS_H - RSS = a^T \hat\beta (a^T (X^T X)^{-1} a)^{-1} a^T \hat\beta$$
$$= \frac{1}{2}\left(\Sigma_i X_i^2\right)(\hat\beta_1 - \hat\beta_2)^2$$

$RSS = (n-2) S^2 \Rightarrow$

$$F = (\hat\beta_1 - \hat\beta_2)^2 \frac{\Sigma_i X_i^2}{2S^2}.$$

Problem 8:

4. A series of $n+1$ observations Y_i $(i = 1, 2, \ldots, n+1)$ are taken from a normal distribution with unknown variance σ^2. After the first n observations it is suspected that there is a sudden change in the mean of the distribution. Derive a test statistic for testing the hypothesis that the $(n+1)$th observation has the same population mean as the previous observations.

Sol: Write the model as

$$\underbrace{\begin{pmatrix} Y_1 \\ \vdots \\ Y_n \\ Y_{n+1} \end{pmatrix}}_{Y} = \underbrace{\begin{pmatrix} 1 & 0 \\ \vdots & \vdots \\ 1 & 0 \\ 0 & 1 \end{pmatrix}}_{X} \underbrace{\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}}_{\mu} + \underbrace{\begin{pmatrix} \varepsilon_1 \\ \vdots \\ \varepsilon_n \\ \varepsilon_{n+1} \end{pmatrix}}_{\varepsilon}$$

$H_0 : \mu_1 = \mu_2$

or $\alpha^T \mu = 0$, $\alpha = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Let $\hat\mu = (X^T X)^{-1} X^T y = \begin{pmatrix} \frac{1}{n}\Sigma_i Y_i \\ Y_{n+1} \end{pmatrix}$.

& $F = \dfrac{RSS_H - RSS}{RSS}(n-1)$

- $RSS_H - RSS = \alpha^T \hat\mu (\alpha^T (X^T X)^{-1} \alpha)^{-1} \alpha^T \hat\mu$
$$= \frac{n}{n+1}\left(\frac{1}{n}\Sigma_i Y_i - Y_{n+1}\right)^2$$
- $RSS = (n-1) S_n^2 = \Sigma_i (Y_i - \bar Y)^2$ where $\bar Y = \frac{1}{n}\Sigma_i Y_i$.

$$\Rightarrow F = \frac{1}{S^2} \frac{n}{n+1}\left(\frac{1}{n}\Sigma_i Y_i - Y_{n+1}\right)^2$$

Under H_0,

$$F \sim F_{1, n-1}(0).$$