Review:

$y = \underset{n\times p}{X}\beta + \varepsilon$, $r(X) = p$,
$\mathbb{E}\,\varepsilon = 0$, $\text{Var}\,\varepsilon = \sigma^2 V$.

$$\Rightarrow \hat{\beta}_w = (X^T V^{-1} X)^{-1} X^T V^{-1} y$$

- $y = X\beta + Z\gamma + \varepsilon$
$$= (X \;\; Z)\begin{pmatrix}\beta \\ \gamma\end{pmatrix} + \varepsilon$$

γ components are Random.

$\text{Cov}\,\gamma = D$; $\text{Cov}\,\varepsilon = R$
$\text{Cov}(\gamma, \varepsilon) = 0$.

Ex 1-way Anova with 3 total & 2 reps:

$$y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$$

$y = \begin{pmatrix} y_1 \\ \vdots \\ y_6\end{pmatrix}$, $X\mu = \begin{pmatrix}1 \\ \vdots \\ 1\end{pmatrix}\mu = J_6\mu$, $\beta = \mu$.

$$Z = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{pmatrix}; \quad \gamma = \begin{pmatrix}\alpha_1 \\ \alpha_2 \\ \alpha_3\end{pmatrix}$$

We have 1-way balanced ANOVA random effects model in a mat. form.

$$f(y, \gamma) = f_{y|\gamma} f_\gamma$$
$$= \frac{1}{|R|^{\frac{1}{2}}(\sqrt{2\pi})^p} e^{-\frac{1}{2}(y - X\beta - Z\gamma)^T R^{-1}(y - X\beta - Z\gamma)}$$
$$e^{-\frac{1}{2}(\gamma - D)^T D^{-1}(\gamma - D)} \frac{1}{|D|^{\frac{1}{2}}(\sqrt{2\pi})^q}$$

- Verify the MLEs for β & γ are

(1) $X^T R^{-1} X\beta + X^T R^{-1} Z\gamma = X^T R^{-1} y$

(2) $Z^T R^{-1} X\beta + (Z^T R^{-1} Z + D^{-1})\gamma = Z^T R^{-1} y$

Henderson's equations.

Re-parametrize as follows (when appropriate):

$$X^T R^{-1} X\beta + X^T R^{-1} Z D \underbrace{(D^{-1}\gamma)}_{\alpha} = X^T R^{-1} y$$
$$Z^T R^{-1} X\beta + (I + Z^T R^{-1} Z D)\underbrace{D^{-1}\gamma}_{\alpha} = Z^T R^{-1} y$$

Sol's the provide est. for β & $\alpha = D^{-1}\gamma$.

What are sol's $\hat{\beta}$ & $\hat{\gamma}$?

- $\hat{\beta}_w = (X^T V^{-1} X)^{-1} X^T V^{-1} y$, $V = Z D Z^T + R$ ($\text{Cov}(y)$)

Use
$$V^{-1} = R^{-1} Z (D^{-1} + Z^T R^{-1} Z)^{-1} Z^T R^{-1}$$

- $\hat{\gamma} = D Z^T V^{-1}(y - X\hat{\beta})$

Note: $\hat{\beta}_w$ is unbiased for β & $\mathbb{E}\,\hat{\gamma} = 0$.

- **Suppose** u is a RV with mean 0 & finite variance.
- A linear predictor $d + a^T y$ of $c^T\beta + u$ is unbiased iff
$$\mathbb{E}(d + a^T y) = \mathbb{E}(c^T\beta + u) = c^T\beta.$$
or $c^T\beta + u$ is predictable iff
$\exists$ an unbiased linear predictor of $c^T\beta + u$

Claim: $c^T\beta + u$ is predictable iff
$\exists$ a s.t. $c = X^T a$

This implies $d + a^T y$ is a LUP of $c^T\beta + u$ iff $d = 0$, $c = X^T a$.

Recall (MSE) of $\hat{\theta}$ of θ is

$$\mathbb{E}(\hat{\theta}-\theta)^2.$$

$$MSE(\hat{\theta}) = Var(\hat{\theta}) + (Bias(\hat{\theta}))^2$$

Claim: $c^T\hat{\beta}_w + \hat{u}$ has the smallest MSE among all LUE of $c^T\beta + u$,

where $\hat{u} = (\underbrace{Cov(\varepsilon, u)}_{\text{vector}})^T (Var\,\varepsilon)^{-1}(y - X\hat{\beta})$

$$\hat{\beta}_w = (X^TV^{-1}X)^{-1}X^TV^{-1}y$$

Assume that $Cov(\varepsilon, u) = \sigma^2 K$.

K is a known vector.

Goal: predict $c^T\beta + u$ (c known) accurately.

Note that

$$c^T\hat{\beta} + \hat{u} = c^T\hat{\beta} + k^TV^{-1}(y - X\hat{\beta})$$

has expectation

$$c^T\beta + 0 = \mathbb{E}(c^T\beta + u)$$

Further,

$$c^T\hat{\beta} + \hat{u} = (c^T + K^TV^{-1}X)\hat{\beta} + k^TV^{-1}y$$

$$= \underbrace{\left[(c^T - k^TV^{-1}X)(X^TV^{-1}X)^{-1}X^TV^{-1} + k^TV^{-1}\right]}_{b^T} y$$

$$= b^Ty$$

$\Rightarrow$ $c^T\hat{\beta} + \hat{u}$ is Linear in y.

It is unbiased for $c^T\beta + u$, we have

$$c = X^Tb.$$

If a^Ty is any other LUP of $c^T\beta + u$,

$$MSE(a^Ty) = \mathbb{E}(a^Ty - (c^T\beta + u))^2$$
$$= Var(a^Ty - u)$$
$$= Var(a^Ty - b^Ty + b^Ty - u)$$
$$= Var((a-b)^Ty) + Var(b^Ty - u) + 0$$

NOTE: $Cov((a-b)^Ty, b^Ty - u) = 0$.

$$= Var((a-b)^Ty) + Var(b^Ty - (c^T\beta + u))$$
$$= Var((a-b)^Ty) + MSE(b^Ty) \geq MSE(b^Ty)$$

- $Cov((a-b)^Ty, b^Ty - u)$

$$= \sigma^2 (a-b)^T(Vb - k)$$

But $Vb - k = X(X^TV^{-1}X)^{-1}(c - X^TV^{-1}k)$

$(c = X^Tb)$

$\therefore$ Covariance term is

$$\sigma^2(a-b)^TX\underbrace{(X^TV^{-1}X)^{-1}(c - X^TV^{-1}k)}_{\Delta}$$

$$= \sigma^2\{\underset{\substack{\| \\ c^T}}{a^TX} - \underset{\substack{\| \\ c^T}}{b^TX}\}(\Delta) = 0.$$

- $c^T\hat{\beta} + \hat{u}$ is BLUP of $c^T\beta + u$. among all LUE of $c^T\beta + u$.

$$Cov(\varepsilon, u) = \sigma^2 K$$

- $\hat{\gamma} = DZ^TV^{-1}(y - X\hat{\beta})$

$$\begin{pmatrix}\gamma \\ y\end{pmatrix} \sim \mathcal{N}\left(\begin{pmatrix}0 \\ X\beta\end{pmatrix}, \begin{pmatrix}D & ? \\ ? & V\end{pmatrix}\right)$$

$$Cov(\gamma, y) = Cov(\gamma, X\beta + Z\gamma + \varepsilon)$$
$$= DZ^T$$

$$\gamma | y \sim \mathcal{N}_g\left(DZ^TV^{-1}(y - X\beta),\ D - DZ^TV^{-1}ZD\right)$$

$$(\mu_1 + \Sigma_{12}\Sigma_{22}^{-1}(y - X\beta))$$

Ex $Y_{ij} = \mu + \alpha_i + \varepsilon_{ij}$, $i = 1, 2, 3$, $j = 1, 2$

$$y = X\beta + Z\gamma + \varepsilon$$

$$R = Cov\,\varepsilon = \sigma^2 I, \quad J_2 = \begin{pmatrix}1 & 1 \\ 1 & 1\end{pmatrix}$$

$$D = \sigma_\alpha^2 I_3, \quad V = ZDZ^T + R$$

$$\begin{pmatrix}y_1 \\ \vdots \\ y_6\end{pmatrix} = \begin{pmatrix}1 \\ \vdots \\ 1\end{pmatrix}\mu + \begin{pmatrix}\begin{smallmatrix}1\\1\end{smallmatrix} & 0 & 0 \\ 0 & \begin{smallmatrix}1\\1\end{smallmatrix} & 0 \\ 0 & 0 & \begin{smallmatrix}1\\1\end{smallmatrix}\end{pmatrix}\begin{pmatrix}\alpha_1 \\ \alpha_2 \\ \alpha_3\end{pmatrix} + \varepsilon$$

$$\underset{6\times 6}{V^{-1}} = \frac{1}{\sigma^2}\begin{pmatrix}\underset{(2\times 2)}{I_2 - \frac{\sigma_\alpha^2}{2\sigma_\alpha^2 + \sigma^2}J_2} & 0 \\ 0 & \underset{(2\times 2)}{\ddots}\end{pmatrix}$$

$$\Rightarrow BLUP = DZ^TV^{-1}(y - X\hat{\beta})$$

$$= \frac{2\sigma_\alpha^2}{\sigma^2}\begin{pmatrix}\bar{y}_1 - \bar{y} \\ \bar{y}_2 - \bar{y}\end{pmatrix} / (\sigma^2 + 2\sigma_\alpha^2)$$