Biostat 250C HW8

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Q1: Suppose that

$$x_i | x_{(-i)} \sim N\left(\rho \sum_{j=1}^{n} \frac{w_{ij} x_j}{w_{i+}}, \frac{\sigma^2}{w_{i+}}\right)$$

where $\sum_{j=1}^{n} w_{ij} = m_i = w_{i+}$ &

$w_{ii} = 0$

$$w_{ij} = \begin{cases} 1 & \text{if } (i,j) \in \mathcal{E} \\ 0 & \text{if } (i,j) \notin \mathcal{E} \end{cases}$$

Show that (Brook's Lemma)

$$p(x_1, \cdots, x_n) \propto e^{-\frac{1}{2\sigma^2} x^T (D - \rho W) x}$$

where $x = (x_1, \cdots, x_n)^T$, $D = \mathrm{Diag}(w_{1+}, \cdots, w_{n+})$

& W is the binary adjacency matrix.

Sol.

Recall Brook's lemma,

$$p(x_1, x_2, \cdots, x_n) = \frac{p(x_1 | x_2, \cdots, x_n)\, p(x_2 | x_{10}, x_3, \cdots, x_n) \cdots p(x_n | x_{10}, \cdots, x_{(n-1)0})}{p(x_{10} | x_2, \cdots, x_n)\, p(x_{20} | x_{10}, x_3, \cdots, x_n) \cdots p(x_{n0} | x_{10}, \cdots, x_{(n-1)0})} \times p(x_{10}, x_{20}, \cdots, x_{n0})$$

Let $x_{10} = x_{20} = \cdots = x_{n0} = 0$, then

$$\frac{p(x_1, \cdots, x_n)}{p(0)} = \frac{\prod_{i=1}^{n} p(x_i | \cdot)}{\prod_{i=1}^{n} p(0 | \cdot)}$$

I will prove the statement

for special to general.

Step 1: Let $n=2$ & $\begin{cases} w_{1+} = w_{12} \\ w_{2+} = w_{21} \end{cases}$, then

$$\frac{p(x_1, x_2)}{p(0,0)} = \frac{p(x_1 | x_2)\, p(x_2 | 0)}{p(0 | x_2)\, p(0 | 0)} \propto$$

$$\exp\left\{-\frac{w_{1+}}{2\sigma^2}\left(x_1 - \rho \frac{w_{12} x_2 + \overset{0}{\overset{\shortparallel}{w_{11}}} x_1}{w_{1+}}\right)^2\right\} \times$$

$$\exp\left\{-\frac{w_{2+}}{2\sigma^2}\left(x_2 - \frac{w_{22} x_2 + \overset{0}{\overset{\shortparallel}{w_{21}}} 0}{w_{2+}}\right)^2\right\} \times$$

$w_{12} x_2$ by symmetry

$$\exp\left\{\frac{w_{1+}}{2\sigma^2}\left(0 - \rho \frac{\overbrace{w_{12} x_2}^{} + w_{11} \cdot 0}{w_{1+}}\right)^2\right\} \propto$$

$$\exp\left\{-\frac{1}{2\sigma^2}\left(w_{1+} x_1^2 + w_{2+} x_2^2 - 2\rho x_1 w_{12} x_2\right)\right\} =$$

$$\exp\left\{-\frac{1}{2\sigma^2}(x_1, x_2)\left(\underbrace{\begin{pmatrix} w_{1+} & 0 \\ 0 & w_{2+} \end{pmatrix}}_{D} - \rho \underbrace{\begin{pmatrix} 0 & w_{12} \\ w_{21} & 0 \end{pmatrix}}_{W}\right)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\right\}$$

Step 2: General case.

For $i = 1, 2, \cdots, n$, we have

$$p(x_i | x_{(-i)}) \propto e^{-\frac{w_{i+}}{2\sigma^2}\left(x_i - \rho \sum_{j=1}^{n} \frac{w_{ij} x_j}{w_{i+}}\right)^2}$$

$$= e^{-\frac{1}{2\sigma^2}\left(w_{i+} x_i^2 - 2\rho w_i^T x x_i + \frac{\rho^2 (w_i^T x)^2}{w_{i+}}\right)}$$

Where

$$\begin{cases} w_i^T = \{w_{i1}, \cdots, w_{in}\} \\ x = \{x_1, \cdots, x_n\} \\ w_{ii} = 0 \end{cases}$$

Hence, we have

$$\frac{p(x_i \mid x_{10}, \dots, x_{(i-1)0}, x_{i+1}, \dots, x_n)}{p(x_{i0} \mid x_{10}, \dots, x_{(i-1)0}, x_{i+1}, \dots, x_n)}$$

$$\propto \frac{e^{-\frac{1}{2\sigma^2}\left(w_{i+} x_i^2 - 2\rho \widetilde{w}_i^T \widetilde{x} + \frac{\rho^2 (\widetilde{w}_i^T \widetilde{x})^2}{w_{i+}}\right)}}{e^{-\frac{1}{2\sigma^2}\left(\frac{\rho^2 (\widetilde{w}_i^T \widetilde{x})^2}{w_{i+}}\right)}}$$

$$\propto e^{-\frac{1}{2\sigma^2}(w_{i+} x_i^2 - 2\rho \widetilde{w}_i^T \widetilde{x})}$$

where $\begin{cases} \widetilde{w}_i^T = (w_{i(i+1)}, w_{i(i+2)}, \dots, w_{in}) \\ \widetilde{x} = (x_{i+1}, \dots, x_n) \end{cases}$

are the truncated versions.

Next, interestingly, note that

$$2\rho \sum_{i=1}^{n} \widetilde{w}_i^T \widetilde{x}$$

$$= x^T (\rho W) x$$

where $W = \begin{bmatrix} w_1 & w_2 & \cdots & w_n \end{bmatrix}$

and $W = W^T$

(that's why we get factor 2),

we have

$$\frac{p(x_1, \dots, x_n)}{p(0, \dots, 0)} \propto$$

$$e^{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (w_{i+} x_i^2 - 2\rho \widetilde{w}_i^T \widetilde{x})}$$

$$\propto e^{-\frac{1}{2\sigma^2}[① + ②]}$$

where

$$① = \sum_{i=1}^{n} w_{i+} x_i^2 = x^T D x$$

$$D = \mathrm{Diag}(w_{i+})$$

$$② = -x^T (\rho W) x$$

and

① + ② is indeed

$$x^T (D - \rho W) x \qquad \square.$$