Biostat 250A
HW3

Elvis Cui
Han Cui
Dept. of Biostat
UCLA

Biostat 250A Homework 3 due 10/26/20

Read Appendix A.1-A.13 and Chapter 1 in the text.

1 Let A and B be two n x n positive definite matrices. If A-B is positive definite, we frequently write A>B. Answer true or false for each of the following questions below, with justification.

(a) tr A > tr B? (b) $A^{-1}>B^{-1}$? (c) |A|>|B|?

2 Show that the sets of nonzero eigenvalues for AB and BA are the same for any two conformable matrices.

4 Show that if A, B, C and D are conformable matrices, and all indicated inverses exist,

$$(A+BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1} DA^{-1}.$$

Hence or otherwise, show that $(A+ab')^{-1} = A^{-1}$ - .

5 If C= is positive definite and, both A and D are square submatrices, show that A and D are also positive definite matrices. Express the determinant of C in terms of A. Which of these results hold if C is no longer symmetric? Justify your answer.

6 If A and D are symmetric matrices such that the below inverses exist, show that

$$^{-1} =$$

where $E = D - B'A^{-1}B$ and $F = A^{-1}B$. Hence or otherwise, find the inverse of .

7. Let $y_i = b_0 + b_1x_{1,i} + ... + b_{p-1}x_{p-1,i} + e_i$ and assume e_i has mean 0 and variance c^2, i = 1, 2, ..., n . Put this equation in a matrix form E(y) = Xb, where X is the design matrix, $b' = (b_0, b_1, ... , b_{p-1})$ and $y' = (y_1, .., y_n)$.

Suppose now we make the following partition

$$X'X = \begin{pmatrix} n & n\bar{x}' \\ n\bar{x} & X'_1 X_1 \end{pmatrix}.$$

Show that

$$(X'X)^{-1} = ,$$

where the elements of the (p-1)x(p-1) symmetric matrix V are the corrected sums of squares and products of the x's and ' = ($_1$, $_2$, .., $_{p-1}$). What can you say about the variance of the estimated intercept term? Can you express the variance of the estimated b_{p-1} in terms of the determinant of X'X? If yes, how?

Problem 1:

1 Let A and B be two n x n positive definite matrices. If A-B is positive definite, we frequently write A>B. Answer true or false for each of the following questions below, with justification.

(a) tr A > tr B? (b) $A^{-1}>B^{-1}$? (c) |A|>|B|?

Sol. (a): True. $Tr(A)=\sum_i^n \lambda_i^A$ & $Tr(B)=\sum_i^n \lambda_i^B$

$A-B\succ 0 \Rightarrow \lambda_i^{A-B}>0 \ \forall i=1,\cdots,n \Rightarrow$

$\sum_i \lambda_i^{A-B}>0 \Rightarrow Tr(A-B)=Tr(A)-Tr(B)>0$

(b): False. $\exists C$ s.t. $C^TAC=I$ & $C^TBC=\Lambda$

$\Rightarrow \begin{cases} A=C^{-T}C^{-1} \\ B=C^{-T}\Lambda C^{-1}\end{cases} \Rightarrow \begin{cases} A^{-1}=CC^T \\ B^{-1}=C\Lambda^{-1}C^T\end{cases}$

$\Rightarrow \begin{cases} A^{-1}-B^{-1}=C(I-\Lambda^{-1})C^T \\ A-B=C^{-T}(I-\Lambda)C^{-1}\end{cases}$

- $A-B\succ 0 \Rightarrow I-\Lambda\succ 0$

$\Rightarrow I-\Lambda^{-1}\prec 0$ (Since $\lambda_i^{-1}>1$)

$\Rightarrow A^{-1}-B^{-1}\prec 0$ or $A^{-1}\prec B^{-1}$.

(c): True. Again, using C in (b):

$\det|A|=|C^{-1}|^2$

$\det|B|=|C^{-1}|^2|\Lambda|$

But $A-B\succ 0 \Rightarrow |\Lambda|<1 \Rightarrow$

$\det|A|>\det|B|$.

Problem 2:

2 Show that the sets of nonzero eigenvalues for AB and BA are the same for any two conformable matrices.

Sol. Lemma: $\begin{vmatrix} A_{11} & A_{12} \\ A_{21} & A_{22}\end{vmatrix}=|A_{22}|\,|A_{11}-A_{12}A_{22}^{-1}A_{21}|$

Pf: Let $Z=\begin{pmatrix} I & -A_{12}A_{22}^{-1} \\ 0 & I\end{pmatrix}$, then we have

$|ZA|=|Z||A|=|A|$, but LHS also equals to

$\begin{vmatrix} A_{11}-A_{12}A_{22}^{-1}A_{21} & 0 \\ A_{21} & A_{22}\end{vmatrix}=|A_{22}||A_{11}-A_{12}A_{22}^{-1}A_{21}|$.

Then WLOG, let $A\in\mathbb{R}^{n\times p}$, $B\in\mathbb{R}^{p\times n}$.

we have

$\begin{vmatrix} I_p & \underset{p\times n}{B} \\ \underset{n\times p}{A} & \lambda I_n\end{vmatrix}=|\lambda I_n||I_p-\frac{BA}{\lambda}|$

$=\lambda^{n-p}|I_n||\lambda I_p-BA|$

$=\lambda^{n-p}|\lambda I_p-BA|$

$=\lambda^{n-p}|\lambda I_n-AB|$

So $\forall\lambda$ is a non-zero eig-val of BA, it is an eig-val of AB & vice versa.

NOTE: I did this before watching recording, here is the fastest way:

$|AB-\lambda I_n|=|BA-\lambda I_p|$. □

Problem 3:

4 Show that if A, B, C and D are conformable matrices, and all indicated inverses exist,

$$(A+BCD)^{-1} = A^{-1} - A^{-1}B(C^{-1} + DA^{-1}B)^{-1} DA^{-1}.$$

Hence or otherwise, show that $(A+ab')^{-1} = A^{-1} - \frac{A^{-1}ab'A^{-1}}{1+b'A^{-1}a}$.

Pf: (i): left-multiply by $(A+BCD)$:

$$(A+BCD)(A^{-1} - A^{-1}B(C^{-1}+DA^{-1}B)^{-1}DA^{-1}) \quad (*)$$

$$= I - B(C^{-1}+DA^{-1}B)^{-1}DA^{-1} + BCDA^{-1}$$
$$- BCDA^{-1}\underbrace{B(C^{-1}+DA^{-1}B)^{-1}}_{\parallel}DA^{-1}$$
$$BC(I+DA^{-1}BC)^{-1}$$

$$= I - BC[\underbrace{(I+DA^{-1}BC)^{-1} - I + DA^{-1}BC(I+DA^{-1}BC)^{-1}}_{\parallel}]DA^{-1}$$

$$(I+DA^{-1}BC)(I+DA^{-1}BC)^{-1} - I = 0$$

$$\Rightarrow (*) = I \qquad \square.$$

ii) Let $C=1$, $A=A$, $B=a$, $D=b^T$

$$\Rightarrow (A+ab^T)^{-1} = A^{-1} + A^{-1}a(1+b^TA^{-1}a)^{-1}b^TA^{-1}$$
$$= RHS. \qquad \square.$$

Problem 4:

5 If $C=\begin{pmatrix} A & B \\ B' & D \end{pmatrix}$ is positive definite and, both A and D are square submatrices, show that A and D are also positive definite matrices. Express the determinant of C in terms of A. Which of these results hold if C is no longer symmetric? Justify your answer.

Sol. WLOG, let $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{m\times m}$ so that $C \in \mathbb{R}^{(n+m)\times(n+m)}$

Then $(I_n \; O_{n\times m})\begin{pmatrix} A & B \\ B^T & D \end{pmatrix}\begin{pmatrix} I_n \\ O_{n\times m} \end{pmatrix}$

$$= (A \;\; B)\begin{pmatrix} I_n \\ O \end{pmatrix} = A$$

But $C \succ 0 \Rightarrow y^TCy > 0 \;\; \forall y$

Let $y = \begin{pmatrix} I_n \\ O_{m\times n} \end{pmatrix} x$, $x \in \mathbb{R}^n$

then $y^TCy > 0$ iff $x^TAx > 0$

Thus, A is positive definite.

Replace y by $\begin{pmatrix} O_{m\times n} \\ I_m \end{pmatrix}$ then $D \succ 0$ as well.

- $\det|C| = |A||D - B^TA^{-1}B|$ by Problem 3.

if $C^T \neq C$, still we have $A, D \succ 0$ since the choice of y stays the same.

Problem 5:

6 If A and D are symmetric matrices such that the below inverses exist, show that

$$\begin{pmatrix}A & B\\ B' & D\end{pmatrix}^{-1} = \begin{pmatrix}A^{-1}+FE^{-1}F' & -FE^{-1}\\ -E^{-1}F' & E^{-1}\end{pmatrix}$$

where $E = D - B'A^{-1}B$ and $F = A^{-1}B$. Hence or otherwise, find the inverse of $\begin{pmatrix}1&0&0&3\\0&1&0&4\\0&0&1&2\\3&4&2&5\end{pmatrix}$.

Sol. ETS $\begin{pmatrix}A & B\\ B^T & D\end{pmatrix}\begin{pmatrix}A^{-1}+FE^{-1}F^T & -FE^{-1}\\ -E^{-1}F^T & E^{-1}\end{pmatrix}=I$

① $A(A^{-1}+FE^{-1}F^T)-BE^{-1}F^T$

$= I + AFE^{-1}F^T - BE^{-1}F^T$

$= I + B(D-B^TA^{-1}B)^{-1}F^T - BE^{-1}F^T = I$

② $-AFE^{-1}+BE^{-1} = -BE^{-1}+BE^{-1}=0$

③ $-B^TFE^{-1}+DE^{-1} = -B^TA^{-1}BE^{-1}+DE^{-1}$

$= EE^{-1} = I$ $\square$.

Let $A=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$, $B=\begin{pmatrix}3\\4\\2\end{pmatrix}$ $D=(5)$

$\Rightarrow E = D - B^TA^{-1}B = -24$

$F = A^{-1}B = \begin{pmatrix}3\\4\\2\end{pmatrix}$

$\Rightarrow -FE^{-1} = \begin{pmatrix}\frac{1}{8}\\ \frac{1}{6}\\ \frac{1}{12}\end{pmatrix}$

$E^{-1} = -\frac{1}{24}$

$A^{-1}+FE^{-1}F^T = I - \begin{pmatrix}\frac{1}{8}\\ \frac{1}{6}\\ \frac{1}{12}\end{pmatrix}\left(\frac{1}{8}, \frac{1}{6}, \frac{1}{12}\right)$

$= \begin{pmatrix}\frac{63}{64} & -\frac{1}{48} & -\frac{1}{96}\\ -\frac{1}{48} & \frac{35}{36} & -\frac{1}{72}\\ -\frac{1}{96} & -\frac{1}{72} & \frac{143}{144}\end{pmatrix} \quad (*)$

$\Rightarrow \begin{pmatrix}A & B\\ B^T & D\end{pmatrix}^{-1} = \begin{pmatrix}(*) & \begin{matrix}\frac{1}{8}\\ \frac{1}{6}\\ \frac{1}{12}\end{matrix}\\ \begin{matrix}\frac{1}{8} & \frac{1}{6} & \frac{1}{12}\end{matrix} & -\frac{1}{24}\end{pmatrix}$

Problem 6:

7. Let $y_i = b_0 + b_1x_{1,i} + \dots + b_{p-1}x_{p-1,i} + e_i$ and assume e_i has mean 0 and variance c^2, $i = 1, 2, \dots, n$. Put this equation in a matrix form $E(y) = Xb$, where X is the design matrix, $b' = (b_0, b_1, \dots, b_{p-1})$ and $y' = (y_1, \dots, y_n)$.

Suppose now we make the following partition

$$X'X = \begin{pmatrix}n & n\bar{x}'\\ n\bar{x} & X'_1X_1\end{pmatrix}.$$

Show that

$$(X'X)^{-1} = \begin{pmatrix}\frac{1}{n}+\bar{x}'V^{-1}\bar{x} & -\bar{x}'V^{-1}\\ -V^{-1}\bar{x} & V^{-1}\end{pmatrix},$$

where the elements of the (p-1)x(p-1) symmetric matrix V are the corrected sums of squares and products of the x's and $\bar{x}' = (\bar{x}_1, \bar{x}_2, \dots, \bar{x}_{p-1})$. What can you say about the variance of the estimated intercept term? Can you express the variance of the estimated b_{p-1} in terms of the determinant of X'X? If yes, how?

Sol. ①: $A = n$, $B = n\bar{x}^T$, $D = X_1^TX_1$,

$\Rightarrow A^{-1} = \frac{1}{n}$, $F = \bar{x}^T$,

$E = D - B^TA^{-1}B = X_1^TX_1 - n\bar{x}\bar{x}^T$

$\Rightarrow (X^TX)^{-1} = \begin{pmatrix}\frac{1}{n}+\bar{x}^TE^{-1}\bar{x} & -\bar{x}^T(E^{-1})\\ -E^{-1}\bar{x} & E^{-1}\end{pmatrix}$

② $\hat{b} = (X^TX)^{-1}X^Ty$

$= \begin{pmatrix}\frac{1}{n}+\bar{x}^TE^{-1}\bar{x} & -\bar{x}^TE^{-1}\\ -E^{-1}\bar{x} & E^{-1}\end{pmatrix}\begin{pmatrix}1\\ X_1^T\end{pmatrix}y$

$Var\,\hat{b} = c^2(X^TX)^{-1}$

$\Rightarrow Var(\hat{b}_0) = c^2\left(\frac{1}{n}+\bar{x}^TE^{-1}\bar{x}\right)$

③ $Var\,\hat{\beta}_{p-1} = c^2[(X^TX)^{-1}]_{pp}$

$X^TX = \begin{pmatrix}\tilde{X}^T\\ x_p^T\end{pmatrix}\begin{pmatrix}\tilde{X} & x_p\end{pmatrix} = \begin{pmatrix}\tilde{X}^T\tilde{X} & \tilde{X}^Tx_p\\ x_p^T\tilde{X} & x_p^Tx_p\end{pmatrix}$

$\Rightarrow (X^TX)^{-1} = \begin{pmatrix}A & B\\ B^T & \tilde{E}^{-1}\end{pmatrix}$, where

$\tilde{E}^{-1} = (x_p^Tx_p - x_p^T\tilde{X}(\tilde{X}^T\tilde{X})^{-1}\tilde{X}^Tx_p)^{-1}$

$= (x_p^T(I - P_{\tilde{X}})x_p)^{-1}$, $P_{\tilde{X}} = \tilde{X}(\tilde{X}^T\tilde{X})^{-1}\tilde{X}^T$

Further, $\det|(X^TX)^{-1}| = \tilde{E}^{-1} - \det|\Sigma_{11\cdot2}|$ where $\Sigma_{11\cdot2} = A - B\tilde{E}^{-1}B^T \Rightarrow$

$Var\,\hat{\beta}_{p-1} = c^2\tilde{E}^{-1} = c^2\left(\det|(X^TX)^{-1}| + \det|\Sigma_{11\cdot2}|\right).$

Problem 7:

8-9 Ex. 1b: #1, #2 and #5a, b from text.

1. Suppose that X_1, X_2, and X_3 are random variables with common mean μ and variance matrix

$$\mathrm{Var}[\mathbf{X}] = \sigma^2 \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{4} \\ 0 & \frac{1}{4} & 1 \end{pmatrix}.$$

Find $E[X_1^2 + 2X_1X_2 - 4X_2X_3 + X_3^2]$.

2. If $X_1, X_2, \ldots, X_n$ are independent random variables with common mean μ and variances $\sigma_1^2, \sigma_2^2, \ldots, \sigma_n^2$, prove that $\sum_i (X_i - \overline{X})^2/[n(n-1)]$ is an unbiased estimate of $\mathrm{var}[\overline{X}]$.

5. Let $X_1, X_2, \ldots, X_n$ be independently distributed as $N(\mu, \sigma^2)$. Define

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$$

Sol. ① $\mathbb{E}X_1^2 = \sigma^2 + \mu^2$ $\quad \mathbb{E}X_3^2 = \sigma^2 + \mu^2$

$\mathbb{E}\, 2X_1X_2 = 0$

$\mathbb{E}\, 4X_2X_3 = \sigma^2 \quad \Rightarrow$

The result is $2(\mu^2 - \sigma^2)$.

② $\mathrm{Var}(\bar{x}) = \frac{1}{n^2}(\sum_i^n \sigma_i^2)$.

$(X_i - \bar{x})^2 = (X_i - \mu)^2 + (\mu - \bar{x})^2 - 2(X_i - \mu)(\bar{x} - \mu)$

$\Rightarrow \sum_i (X_i - \bar{x})^2 = \sum_i (X_i - \mu)^2 - n(\bar{x} - \mu)^2$

$\Rightarrow \mathbb{E}(\sum_i (X_i - \bar{x})^2) = \sum_i \sigma_i^2 - \frac{1}{n}\sum_i^n \sigma_i^2$

$\Rightarrow \frac{1}{n(n-1)}\mathbb{E}(\sum_i (X_i - \bar{x})^2) = \frac{1}{n^2}(\sum_i \sigma_i^2)$

③ By Biostat 202A, S^2 is indept. of $\bar{X}$. (there are many proofs. one of them will be using orthogonal transform of MVN).

$A^T = \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ 1 & 1 & \cdots & 1 \end{bmatrix} \Rightarrow A^TA = I \Rightarrow$

$\mathrm{Var}(A^TX) = \sigma^2 A^TA = \sigma^2 I \Rightarrow$

$(X_i - \bar{x}) \perp\!\!\!\perp \bar{x} \ \forall i \Rightarrow S^2 \perp\!\!\!\perp \bar{X} \Rightarrow$

let $\tilde{S}^2 = (n-1)S^2$

MGF of $\tilde{S}^2$:

$\mathbb{E}e^{t(\sum_i (X_i - \mu)^2)}$

$= \mathbb{E}\, e^{t(\tilde{S}^2 - n(\bar{x} - \mu)^2)}$

$= \mathbb{E}e^{t\tilde{S}^2}\ \mathbb{E}e^{-nt(\bar{x} - \mu)^2}$

But LHS $= \frac{1}{(1 - 2\sigma^2 t)^{\frac{n}{2}}}$

& $\mathbb{E}e^{-nt(\bar{x} - \mu)^2} = \frac{1}{(1 - 2\sigma^2 t)^{\frac{1}{2}}}$

$\Rightarrow \mathbb{E}e^{t\tilde{S}^2} = \frac{1}{(1 - 2\sigma^2 t)^{\frac{n-1}{2}}}$

$\Rightarrow \frac{\tilde{S}^2}{\sigma^2} \sim \chi^2_{n-1}$

$\Rightarrow \mathrm{Var}(\frac{\tilde{S}^2}{\sigma^2}) = 2(n-1)$

$\Rightarrow \mathrm{Var}\,\tilde{S}^2 = 2\sigma^4(n-1)$

$\Rightarrow \mathrm{Var}\,S^2 = \frac{2\sigma^4}{n-1}$ $\quad \square$

Problem 8:

10-11 Miscellaneous Ex. 1: \$2, #3 from text.

Sol.

(2): $\operatorname{Var} X = \begin{pmatrix} 5 & 2 & 3 \\ 2 & 3 & 0 \\ 3 & 0 & 3 \end{pmatrix}$

(a): $\operatorname{Var} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}^T X = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}^T \begin{pmatrix} 5 & 2 & 3 \\ 2 & 3 & 0 \\ 3 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}$

$= 18$

(b): $Y = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} X$

$\Rightarrow \operatorname{Var} Y = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 5 & 2 & 3 \\ 2 & 3 & 0 \\ 3 & 0 & 3 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 0 & 1 \end{pmatrix}$

$= \begin{pmatrix} 12 & 15 \\ 15 & 21 \end{pmatrix}$

(3): $\mathbb{E} X_i = \mu,\ i=1,\dots,n \quad \operatorname{Cov}(X_i, X_{i+k}) = 0$

$\forall k = 2, 3, \dots$, Define

$Q_2 = (X_1 - X_2)^2 + \dots + (X_{n-1} - X_n)^2 + (X_n - X_1)^2$

$Q_1 = \sum_i^n (X_i - \bar{X})^2.$

- $\operatorname{Var}(\bar{X}) = \mathbb{E}((\bar{X} - \mu)^2)$

$Q_1 = \sum_i^n (X_i - \mu)^2 - n(\bar{X} - \mu)^2$

$Q_2 = [(X_1-\mu)^2 + (X_2-\mu)^2 - 2(X_1-\mu)(X_2-\mu)] +$
$[(X_2-\mu)^2 + (X_3-\mu)^2 - 2(X_2-\mu)(X_3-\mu)] +$
$\dots +$
$[(X_{n-1}-\mu)^2 + (X_n-\mu)^2 - 2(X_{n-1}-\mu)(X_n-\mu)] +$
$[(X_n-\mu)^2 + (X_1-\mu)^2 - 2(X_n-\mu)(X_1-\mu)]$

$= 2\sum_i^n (X_i-\mu)^2 - 2\sum_{i=1}^n (X_i-\mu)(X_{i+1}-\mu)$

$(X_{n+1} \triangleq X_1)$

$3Q_1 - Q_2 = \sum_i^n (X_i-\mu)^2 - 3n(\bar{X}-\mu)^2$
$+ 2\sum_{i=1}^n (X_i-\mu)(X_{i+1}-\mu)$

ETS

$\mathbb{E}\left(n^2(\bar{X}-\mu)^2 - 3n(\bar{X}-\mu)^2\right) =$

$\mathbb{E}\Big(\sum_i^n (X_i-\mu)^2 - 3n(\bar{X}-\mu)^2$
$+ 2\sum_{i=1}^n (X_i-\mu)(X_{i+1}-\mu)\Big)$

But

$\mathbb{E}\left(n^2(\bar{X}-\mu)^2\right) =$

$\mathbb{E}\left(\left(\sum_i^n (X_i-\mu)\right)^2\right)$

$= \mathbb{E}\Big[\sum_i^n (X_i-\mu)^2 + 2\sum_{i=1}^n (X_i-\mu)(X_{i+1}-\mu)$
$+ \underbrace{\sum_{|i-j|>1} (X_i-\mu)(X_{i+1}-\mu)}_{(*)}\Big]$

By assumption

$\mathbb{E}(*) = 0.$

Thus, we have

$\mathbb{E}(3Q_1 - Q_2) = n(n-3)\operatorname{Var}(\bar{X})$

$\square$.