Biostat 250A HW2

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Biostat 250A Homework 2 due 11/19/20

Read Appendix A.1-A.7 in the text.

1 Let A be a given matrix and let B be any matrix B such that ABA = A. Prove or disprove that

a rank(AB) = rank(A)

b rank(BA) = rank(A)

c rank(B) = rank(A).

d rank(A | C) is not less than rank(A) for any conformable matrix C, i.e. when A has more columns, the resulting matrix cannot have a smaller rank.

2 Answer TRUE or FALSE for each of the following questions, with justification.
Suppose x is any n x 1 vector and A is a nxn matrix.

a $x'Ax = 0$ for all x and $A' = A$ imply $A = 0$.

b If $PA'A = QA'A$, then $PA' = QA'$ for any conformable matrices P and Q.

3 Determine if these three vectors are linearly independent: $s_1' = (0, 0, 1)$, $s_2' = (-1, 1, 1)$ and $s_3' = (1, 0, 0)$. If they are, use Gram-Schmidt orthogonalization process to transform them into a set of orthonormal vectors.

4 Let A be a 2x2 matrix; the first row is (4,2) and the second row is (-6,-3), and B = I - A.

a Show that $A = A^2$ and $B = B^2$ and their column spaces are not orthogonal.

b Define $[x,y] = x'Cy$, where the first row of C is (72,42) and the second row is (42,25). Show that the 3 properties of inner products are satisfied by [.,.]. Are the column spaces of A and B orthogonal now under this new inner product?

5a Let $A^2 = A$ and let A be symmetric. Find the determinant, inverse (if it exists) and all its eigenvalues. Describe the eigenvectors of such a matrix.

5b Can you deduce the rank of A from its eigenvalues? If so, how?

5c If $B = CAC^{-1}$, describe the relationship between eigenvalues of B and eigenvalues of A. What about the relationship of eigenvectors, if any, between B and A?

6 Let 1_n be the nx1 vector with all entries equal to unity and let B be a nxn matrix given by
$B = 2b/(2b-1)\, I_n - 1_n 1_n'/(2b-1)$ and $b > 1/2$.

a Is B always positive definite? What is the trace of B?

b Show that the maximum eigenvalue of B is 2b/(2b-1).

7a Let v be a nonzero vector. The Householder transformation matrix is defined by
$H_v = I - 2vv'/v'v$. Find the determinant and all eigenvalues of such a matrix.

7b Show that given any nonzero x, there is a Householder matrix such that $Hx = ||x||e_1$ where $e_1 = (1,0,0,\ldots,0)'$.

7c Show that $||x|| = ||y||$ if and only if there exists an orthogonal matrix T such that $Tx = y$.

8a Find the maximum of $x'Ax/x'x$ if A is a kxk symmetric matrix and x is orthogonal to all eigenvectors associated with the s largest eigenvalues of A and $s \leq k$.

8b If $L = L'$ and $x'Lx>0$ for all x, show that for any fixed vector b, $\sup_{h:h\neq 0} \frac{(h'b)^2}{h'Lh} = b'L^{-1}b$.

8c Suppose Y is a univariate random variable with variance a^2, X is a px1 vector of random variables with covariance matrix V, and cov(Y, X) = W. If b is any non-zero vector, what is the maximum correlation of $b'X$ with Y? What choice of b will ensure that the maximum correlation is attained? Express your answer in terms of W, V and a.

9 Find the spectral decomposition of AA' where A is a matrix whose first row is (1, 1, -1) and its second row is (0, -1, 0).

10 Find the singular value decomposition of the matrix X whose first row is (1, 0, 0, 0) and the second row is (-1, 0, 0, 0).

Problem 1: (For details, see next page)

Read Appendix A.1-A.7 in the text.

1 Let A be a given matrix and let B be any matrix B such that ABA = A. Prove or disprove that
a rank(AB) = rank(A)
b rank(BA) = rank(A)
c rank(B) = rank(A).
d rank(A | C) is not less than rank(A) for any conformable matrix C, i.e. when A has more columns, the resulting matrix cannot have a smaller rank.

Solution:

(a): $\text{rank}(ABA) \le \text{rank}(AB) \le \text{rank}(A)$

$\Rightarrow \text{rank}(AB) = \text{rank}(A)$ Prove.

(b): $\text{rank}(BA) = \text{rank}(AB)$ Prove.

(c): Should be $\text{rank}(B) \ge \text{rank}(A)$ Disprove

(d): Prove.

Problem 2: (For details, see next page)

2 Answer TRUE or FALSE for each of the following questions, with justification.
Suppose x is any n x 1 vector and A is a nxn matrix.
a x'Ax = 0 for all x and A' = A imply A = 0.
b If PA'A = QA'A, then PA' = QA' for any conformable matrices P and Q.

Sol.

(a): $A^T = A \Rightarrow A = P\Lambda P^T$ (SD)

$x^T A x = x^T P \Lambda P^T x = \tilde{x}^T \Lambda \tilde{x} = 0$

$\forall x \Rightarrow \Lambda = 0 \Rightarrow A = 0$ TRUE.

(b): $PA^TA = QA^TA \Rightarrow$

$(PA^T - QA^T)A = 0 \Rightarrow$

$(AP^T - AQ^T) \in \mathcal{N}(A^T)$

But $\mathcal{N}(A^T) = C(A)^\perp \Rightarrow$

$(AP^T - AQ^T) \in C(A)^\perp$ &

$A(P^T - Q^T) \in C(A)$ & $C(A) \cap C(A)^\perp = \{0\}$

$\Rightarrow AP^T = AQ^T$ or $PA^T = QA^T$ TRUE

Problem 3:

3 Determine if these three vectors are linearly independent: $s_1' = (0, 0, 1)$, $s_2' = (-1, 1, 1)$ and $s_3' = (1, 0, 0)$. If they are, use Gram-Schmidt orthogonalization process to transform them into a set of orthonormal vectors.

Sol. $a_1 s_1 + a_2 s_2 + a_3 s_3 = \vec{0} \Rightarrow \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$.

Let $v_1 = s_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$,

$v_2 = s_2 - \frac{\langle v_1, s_2 \rangle}{\langle v_1, v_1 \rangle} v_1 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix} - \frac{1}{1}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$v_3 = s_3 - \frac{\langle v_1, s_3 \rangle}{\|v_1\|_2^2} v_1 - \frac{\langle v_2, s_3 \rangle}{\|v_2\|_2^2} v_2$

$= \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} - 0 - \frac{-1}{2}\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$

$= \begin{pmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 0 \end{pmatrix}$

$\Rightarrow \tilde{v}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ $\tilde{v}_2 = \begin{pmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix}$ $\tilde{v}_3 = \begin{pmatrix} \frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} \\ 0 \end{pmatrix}$

Problem 4:

4 Let A be a 2x2 matrix; the first row is (4,2) and the second row is (-6,-3), and B = I - A.
a Show that A = A^2 and B = B^2 and their column spaces are not orthogonal.
b Define [x,y] = x'Cy, where the first row of C is (72,42) and the second row is (42,25). Show that the 3 properties of inner products are satisfied by [.,.]. Are the column spaces of A and B orthogonal now under this new inner product?

Sol. (a): $A = \begin{bmatrix} 4 & 2 \\ -6 & -3 \end{bmatrix}$ $B = \begin{bmatrix} -3 & -2 \\ 6 & 4 \end{bmatrix}$

$A^2 = \begin{bmatrix} 4 & 2 \\ -6 & -3 \end{bmatrix}\begin{bmatrix} 4 & 2 \\ -6 & -3 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ -6 & -3 \end{bmatrix} = A$

$B^2 = (I - A)^2 = I - A = B$.

& $\langle \underset{a_1}{\begin{pmatrix} 4 \\ -6 \end{pmatrix}}, \underset{b_1}{\begin{pmatrix} -3 \\ 6 \end{pmatrix}} \rangle > 0$ & $\langle \begin{pmatrix} 2 \\ -3 \end{pmatrix}, \begin{pmatrix} -2 \\ 4 \end{pmatrix} \rangle > 0$

(b): $C = \begin{bmatrix} 72, 42 \\ 42, 25 \end{bmatrix}$ is p.s.d. $\Rightarrow$

$\forall$ p.s.d. C, $\langle x, y \rangle_C = x^T C y$ is an inner product.

Pf: $C = P \Lambda P^T$ & Diagonal elements of Λ are all positive $\Rightarrow x^T P \Lambda P^T y = \tilde{x}^T \tilde{y}$.

where $\tilde{x} = \Lambda^{\frac{1}{2}} P^T x$ & $\tilde{y} = \Lambda^{\frac{1}{2}} P^T y$. So it is indeed an inner product.

• ① Let $x = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$, $y = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$, $\langle x, y \rangle_C = 0$

② $x = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, $y = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$, $\langle x, y \rangle_C = 0$

③ $x = \begin{pmatrix} 2 \\ -3 \end{pmatrix}$, $y = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$, $\langle x, y \rangle_C = 0$

④ $x = \begin{pmatrix} 4 \\ -6 \end{pmatrix}$, $y = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$,

$\langle x, y \rangle_C = 0$.

Problem 1:

Read Appendix A.1-A.7 in the text.

1 Let A be a given matrix and let B be any matrix B such that ABA = A. Prove or disprove that

nxn

nxm mxn

a rank(AB) = rank(A)

b rank(BA) = rank(A)

c rank(B) = rank(A).

d rank(A | C) is not less than rank(A) for any conformable matrix C, i.e. when A has more columns, the resulting matrix cannot have a smaller rank.

assumption

Sol. (a): $\text{rank}(A) = \text{rank}(ABA)$

lemma in class

$\leq \text{rank}(AB)$

$\leq \text{rank}(A)$

$\Rightarrow \text{rank}(AB) = \text{rank}(A)$ Prove.

(b): $\text{rank}(A) = \text{rank}(ABA)$

$\leq \text{rank}(BA)$

$\leq \text{rank}(A)$

$\Rightarrow \text{rank}(BA) = \text{rank}(A)$ Prove.

(c): Let $A = O_{p\times p}$, $B = I_{p\times p}$ then

$ABA = A^2 = O = A$ But

$\text{rank}(A) = 0 < \text{rank}(B) = p$. Disapprove

(d): $\text{rank}(A|C) = \#$ independent columns of $[A|C]$.

But if C contains at least a column that cannot be represented by linear combinations of columns in A, then $\text{rank}(A|C) > \text{rank}(A)$.

Otherwise, $\text{rank}(A|C) = \text{rank}(A)$.

Prove.

Problem 2:

2 Answer TRUE or FALSE for each of the following questions, with justification.
Suppose x is any n x 1 vector and A is a nxn matrix.

a x'Ax = 0 for all x and A' = A imply A = 0.

b If PA'A = QA'A, then PA' = QA' for any conformable matrices P and Q.

Sol. (a) Let the spectral Decomp. of A be

$A = T\Lambda T^T$ where

$TT^T = T^T T = I$

Then $x^T A x = (T^T x)^T \Lambda (T^T x)$

$= \sum_{i=1}^{p} \lambda_i y_i^2$ (*)

where λ_i is the i^{th} diagonal element of Λ and y_i^2 is the square of ith element in the vector $T^T x$.

Since (*) = 0 $\forall x$ thus (*) = 0 $\forall y$ [T is full rank]. This means $\lambda_i = 0$ for $i = 1, \dots, p$. Thus.

$A = TOT^T = O$. Prove.

(b): $PA^TA = QA^TA \Rightarrow (PA^T - QA^T)A = 0$

Or $A^T(AP^T - AQ^T) = 0$ equivalently.

This means $\forall x$, we have

$(AP^T - AQ^T)x \in \mathcal{N}(A^T)$, null sp. of A^T

But in class we know that

$\mathcal{N}(A^T) = C(A)^\perp$, the orthogonal complement of A. Thus,

$(AP^T - AQ^T)x \in C(A)^\perp \quad \forall x$.

But also $(AP^T - AQ^T)x = A(P^T - Q^T)x$

$\in C(A)$ Since $(P^T - Q^T)x$ is a vector.

and note that $C(A) \cap C(A)^\perp = \{0\}$.

we have $(AP^T - AQ^T)x = 0 \ \forall x$, which means $AP^T - AQ^T = 0$ or $PA^T = QA^T$. Prove.

Problem 5:

5a	Let $A^2 = A$ and let A be symmetric. Find the determinant, inverse (if it exists) and all its eigenvalues. Describe the eigenvectors of such a matrix.
5b	Can you deduce the rank of A from its eigenvalues? If so, how?
5c	If $B = CAC^{-1}$, describe the relationship between eigenvalues of B and eigenvalues of A. What about the relationship of eigenvectors, if any, between B and A?

Sol. (a): $A^T=A \Rightarrow A=P\Lambda P^T \Rightarrow A^2=P\Lambda^2P^T=A$

$\Rightarrow \Lambda^2=\Lambda \Rightarrow$

① $\lambda_i = 1$ or 0. ② $\det|A| = \det|\Lambda| = 0$ or 1

③ If $\exists$, $A^{-1}=P\Lambda^{-1}P^T = PP^T = I$.

(b): Yes. $\text{rank}(A) = Tr(A) = \sum_i^n \lambda_i$.

(c): $B=CAC^{-1} \Rightarrow \lambda_B x = Bx = CAC^{-1}x$

$\Rightarrow \lambda_B(C^{-1}x) = A(C^{-1}x)$, on the other hand

$Ax=\lambda_A x \Rightarrow \lambda_A(Cx) = CAx = B(Cx)$

So A & B have the same eigen-vals.

As for eigen-vecs, we have

$x_B = Cx_A$ or $x_A = C^{-1}x_B$.

Problem 6:

6	Let 1_n be the nx1 vector with all entries equal to unity and let B be a nxn matrix given by $B = 2b/(2b-1)\, I_n - 1_n 1_n'/(2b-1)$ and $b > 1/2$.
a	Is B always positive definite? What is the trace of B?
b	Show that the maximum eigenvalue of B is 2b/(2b-1).

Sol. $B = \frac{2b}{2b-1}I - \frac{11^T}{2b-1}$

(a): $\forall x$, $x^TBx = \frac{2b}{2b-1}\|x\|_2^2 - \frac{\langle x,1\rangle^2}{2b-1} \geq$ C-S ineq.

$\frac{2b}{2b-1}\|x\|_2^2 - \frac{\|x\|_2^2\|1\|_2^2}{2b-1} = \frac{2b-n}{2b-1}\|x\|_2^2$

with "=" iff $x = c1$ for some $c \in \mathbb{R}$

Thus, $\forall n$ s.t. $n<2b$, B is positive definite.

Otherwise, it is not.

• $Tr(B) = \frac{2bn}{2b-1} - \frac{n}{2b-1} = n$

(b): Let λ, x be s.t.

$Bx = \lambda x$, then

$\lambda\|x\|_2^2 = x^TBx \leq \frac{2b}{2b-1}\|x\|_2^2$

with "=" iff $\langle x,1\rangle = 0$

$\Rightarrow \lambda \leq \frac{2b}{2b-1}$

maximum is attained when $\langle x,1\rangle = 0$

Problem 7:

7a	Let v be a nonzero vector. The Householder transformation matrix is defined by $H_v = I-2vv'/v'v$. Find the determinant and all eigenvalues of such a matrix.
7b	Show that given any nonzero x, there is a Householder matrix such that $Hx = \|x\|e_1$ where $e_1 = (1,0,0,\ldots,0)'$.
7c	Show that $\|x\| = \|y\|$ if and only if there exists an orthogonal matrix T such that $Tx = y$.

Sol. (a) Note that $H_v^2 = H_v$, $H_v^T = H_v$

$\Rightarrow \det|H_v| = \pm 1$ Since $\det|H_v^T H_v| = 1$.

• Next, $H_v^2 = H_v \Rightarrow \lambda(H_v) = 0$ or 1.

• claim: there are only two 0 & $(n-2)$ 1s.

Pf: $\sum_i \lambda_i = Tr(H_v) = n - 2Tr\left(\frac{vv^T}{\|v\|_2^2}\right) = n-2$

$\Rightarrow$ there are $(n-2)$ 1s & two 0s.

(b): Set $v = x - \|x\|e_1$,

$\left.\begin{array}{l} \|v\|_2^2 = 2\|x\|_2^2 - 2\|x\|x^Te_1 \\ v^Tx = \|x\|_2^2 - \|x\|x^Te_1 \end{array}\right\} \Rightarrow$

$\left(I - \frac{2vv^T}{\|v\|_2^2}\right)x = x - v = \|x\|e_1 = RHS$.

(c): $(\Leftarrow)$: If $Tx=y$ & $T^TT=I$, then

$\|y\|_2^2 = \|Tx\|_2^2 = x^TT^TTx = x^Tx = \|x\|_2^2$.

$(\Rightarrow)$: If $x^Tx = y^Ty$, then WLOG,

assume $x \neq 0$ & $x \in \mathbb{R}^m$, $y \in \mathbb{R}^n$, $m \leq n$. $\Rightarrow$

$\exists A \in \mathbb{R}^{n\times m}$ s.t. $Ax=y \Rightarrow y^Ty = x^TA^TAx = x^Tx$

Thus $A^TA = P\Lambda P^T$ & $(x^TP)(\Lambda - I)(P^Tx) = 0$

$\Rightarrow$ Let $\tilde{A}_{m\times n} = \begin{bmatrix} P^T_{m\times m} \\ 0 \end{bmatrix}$, this is our orthogonal mat T.

Problem 8:

8a Find the maximum of x'Ax/x'x if A is a kxk symmetric matrix and x is orthogonal to all eigenvectors associated with the s largest eigenvalues of A and s ≤ k.

8b If L = L' and x'Lx>0 for all x, show that for any fixed vector b, supremum$_{h:h\neq 0}$ $\frac{(h'b)^2}{h'Lh}$ = b'L^{-1}b.

8c Suppose Y is a univariate random variable with variance a^2, X is a px1 vector of random variables with covariance matrix V, and cov(Y, X) = W. If b is any non-zero vector, what is the maximum correlation of b'X with Y? What choice of b will ensure that the maximum correlation is attained? Express your answer in terms of W, V and a.

Let $A = P\Lambda P^T$ be the SD.

Sol. (a): $\frac{x^T P\Lambda P^T x}{x^T P P^T x} = \frac{[\tilde{x}]^T \Lambda [\tilde{x}]}{\tilde{x}^T\tilde{x}}$ (with s zeros on top, $k-s$ entries $\tilde{x}$ below)

$= \frac{\sum_{i=1}^{k-s} \tilde{x}_i^2 \lambda_{i+s}}{\tilde{x}^T\tilde{x}} \leq \lambda_{s+1}$ ($(s+1)$ largest eigen-val)

with "=" iff $\tilde{x} = [1\ 0 \cdots 0]$ iff $x = [0 \cdots 0\ 1\ 0 \cdots 0]$ ($(s+1)$ position)

(b): Let $\langle x, y\rangle = x^T y$, by generalized C-S ineq.

then $\frac{(h^T b)^2}{h^T L h} = \frac{\langle L^{\frac{1}{2}}h, L^{-\frac{1}{2}}b\rangle^2}{\langle L^{\frac{1}{2}}h, L^{\frac{1}{2}}h\rangle} \leq \frac{\|L^{\frac{1}{2}}h\|_2^2\ \|L^{-\frac{1}{2}}b\|_2^2}{\|L^{\frac{1}{2}}h\|_2^2} = b^T L^{-1} b$

with "=" iff $L^{\frac{1}{2}}h = cL^{-\frac{1}{2}}b$ or $Lh = cb$ for some $c \in \mathbb{R}$.

(c): $\mathrm{Corr}^2(b^T X, Y) = \frac{\mathrm{Cov}(b^T X, Y)^2}{\mathrm{Var}(b^T X)\mathrm{Var}(Y)} = \frac{(b^T w)^2}{a^2 b^T V b}$

by (b) $\leq \frac{1}{a^2} w^T V^{-1} w$

with "=" iff $b \propto L^{-1} w$.

Problem 9:

9 Find the spectral decomposition of AA' where A is a matrix whose first row is (1, 1, -1) and its second row is (0, -1, 0).

Sol. $AA^T = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & -1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ -1 & 1 \end{bmatrix} \triangleq B$

let $\det|B - \lambda I| = 0 \Rightarrow \begin{cases} \lambda_1 = 2+\sqrt{2} \\ \lambda_2 = 2-\sqrt{2} \end{cases}$, the corresponding eigen-vecs are (say)

$v_1 = \begin{pmatrix} 1 \\ 1-\sqrt{2} \end{pmatrix}$, $v_2 = \begin{pmatrix} 1 \\ 1+\sqrt{2} \end{pmatrix}$, by Gram Schmidt,

we have $\tilde{V} = \begin{pmatrix} \frac{1}{\|v_1\|} & \frac{1}{\|v_2\|} \\ \frac{1-\sqrt{2}}{\|v_1\|} & \frac{1+\sqrt{2}}{\|v_2\|} \end{pmatrix}$

where $\|v_1\| = \sqrt{1+(1-\sqrt{2})^2} = \sqrt{4-2\sqrt{2}}$ &

$\|v_2\| = \sqrt{4+2\sqrt{2}}$

$\Rightarrow B = \tilde{V}\begin{pmatrix} 2+\sqrt{2} & 0 \\ 0 & 2-\sqrt{2} \end{pmatrix}\tilde{V}^T$.

Problem 10:

10 Find the singular value decomposition of the matrix X whose first row is (1, 0, 0, 0) and the second row is (-1, 0, 0, 0).

Sol. $X = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \Rightarrow$

$XX^T = \begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$

eigen-vals are $\lambda_1 = 0$ & $\lambda_2 = 2$

eigen-vecs are $v_1 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$ & $v_2 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$

let $u_2 = \lambda_2^{-\frac{1}{2}} X^T v_2 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow$

$X = \underset{2\times 1}{V}\ \underset{1\times 1}{\Lambda}\ \underset{1\times 4}{U^T}$

$\Rightarrow V = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$, $\Lambda = (\sqrt{2})$

$U^T = (1\ 0\ 0\ 0)$

• $X^T X = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,

$\lambda_1 = \lambda_2 = \lambda_3 = 0$, $\lambda_4 = 2$

$v_4^T = (1, 0, 0, 0) \Rightarrow$

let $u_4 = \lambda_4^{-\frac{1}{2}} X v_4 = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix} \Rightarrow$

$X = UDV^T$ where

$U = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \end{pmatrix}$, $\Lambda = (\sqrt{2})$,

$V^T = (1, 0, 0, 0)$

□.