VE281

Data Structures and Algorithms

Red-black Trees

Learning Objectives:

- Know what a red-black tree is and its properties
- Know how to do insertion for a red-black tree

Outline

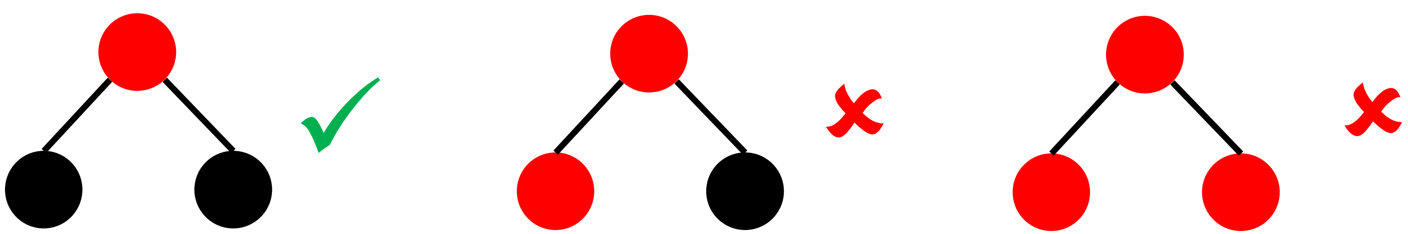
- Red-black Trees: Basics

- Red-black Trees: Insertion

Red-Black Tree

- A binary search tree. The data structure requires an extra one-bit color field in each node.
- Property

1. Every node is either red or black.
2. **Root rule**: The root is black.
3. **Red rule**: Red node can **only have** black children.
    - Can't have two consecutive red nodes on a path.

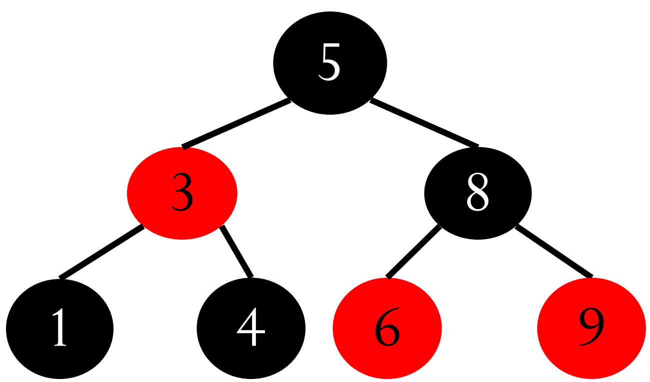


4. **Path rule**: **Every** path from a node x to NULL must have the **same number** of black nodes (including x itself).

Red-Black Tree Example

- Property

1. A binary search tree
2. Every node is either red or black.
3. **Root rule**: The root is black.
4. **Red rule**: Red node can **only have** black children.
5. **Path rule**: **Every** path from a node x to NULL must have the **same number** of black nodes (including x itself).

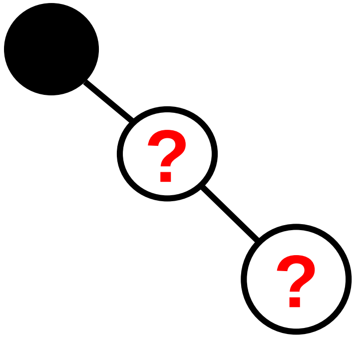


Counter Example

- Property

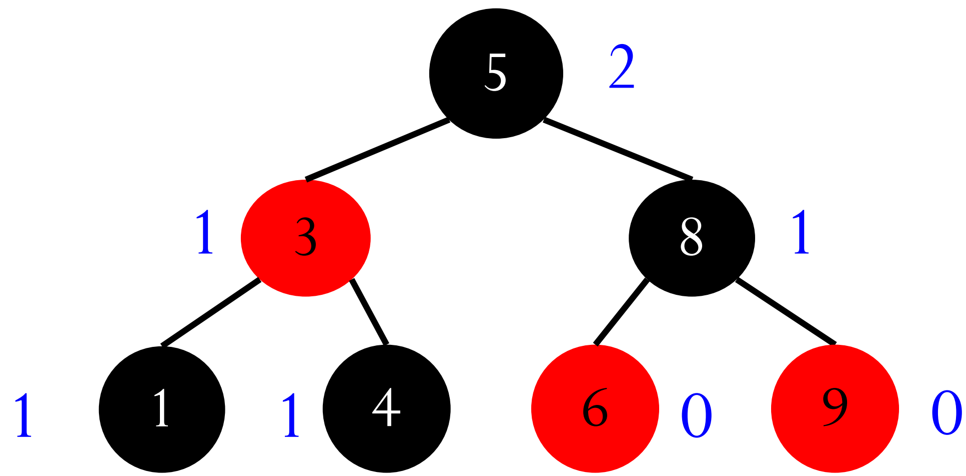
1. A binary search tree
2. Every node is either red or black.
3. **Root rule**: The root is black.
4. **Red rule**: Red node can **only have** black children.
5. **Path rule**: **Every** path from a node x to NULL must have the **same number** of black nodes (including x itself).

- **Claim**: a chain of length 3 cannot be a red-black tree



Black Height

- **Black height** of a node x is the number of black nodes on the path from x to NULL, **including** x itself.



Which Statements Are Correct?

A. It is possible for a **red** node to have a single child.

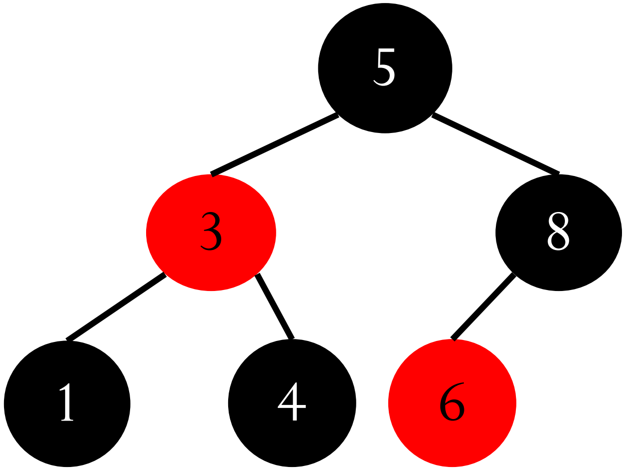
B. It is possible for a **black** node to have a single child.

C. It is possible for a node to have two children of different colors.

D. It is possible for a node to have two children and the node and its children are all of the same color.

Implication of the Rules

- If a **red** node has **at least one** child, it **<u>must have</u>** **two children** and they must be **black**.
  - Why?
    - A red node's child can only be black.
    - If has only one black child, then violate the **path rule**.
- If a black node has **only one** child, that child **<u>must be</u>** a **red leaf**.
  - Why?
    - Can't be black.
    - Must be a leaf.



Height Guarantee

- **<u>Claim</u>**: every red-black tree with n nodes has height $\leq 2\log_2(n+1)$.
- Proof:
  - In a binary tree with n nodes, there is a root-NULL path with **<u>at most</u>** $\log_2(n+1)$ nodes. (why?)
    - **<u>Thus</u>**: # black nodes on that path $\leq \log_2(n+1)$.
  - By **path rule**: every root-NULL path has $\leq \log_2(n+1)$ **black nodes**.
  - By **red rule**: every root-NULL path has $\leq 2\log_2(n+1)$ **total nodes**.

Q.E.D.

Operations on Red-Black Trees

- All **query operations** (e.g., search, min, max, succ, pred) work just like those on general BST.
  - They run in $O(\log n)$ time on a red-black trees with n nodes in the **worst case**.

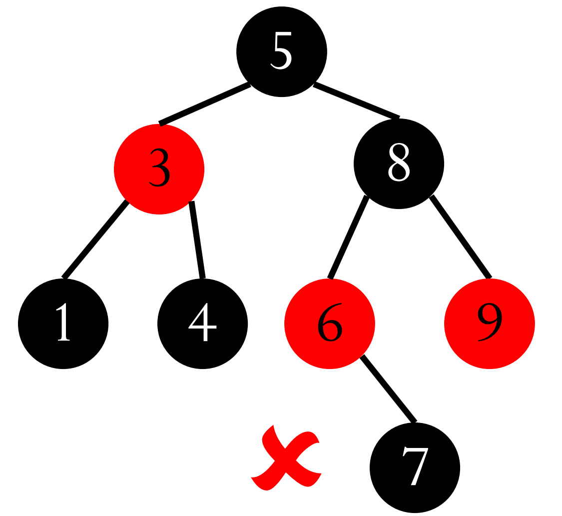
- The **modifying** operations "insertion" and "removal" must maintain the red-black tree properties.
  - They are complex.

Outline

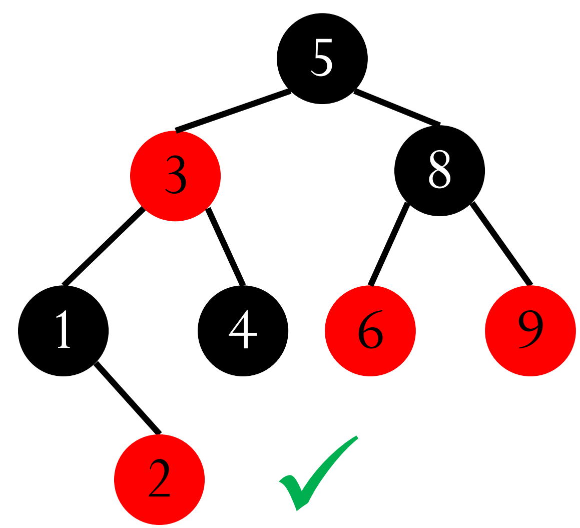
- Red-black Trees: Basics
- Red-black Trees: Insertion

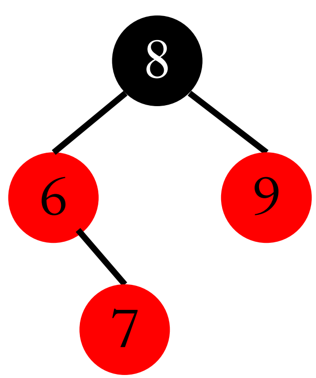
Insertion

- New node is always a **leaf**.
  - However, it can't be **black**!
    - Otherwise, violate path rule.
  - Therefore the new leaf must be **red**.



- If parent is black, done (trivial case).
- If parent is red, violate the **red rule**!

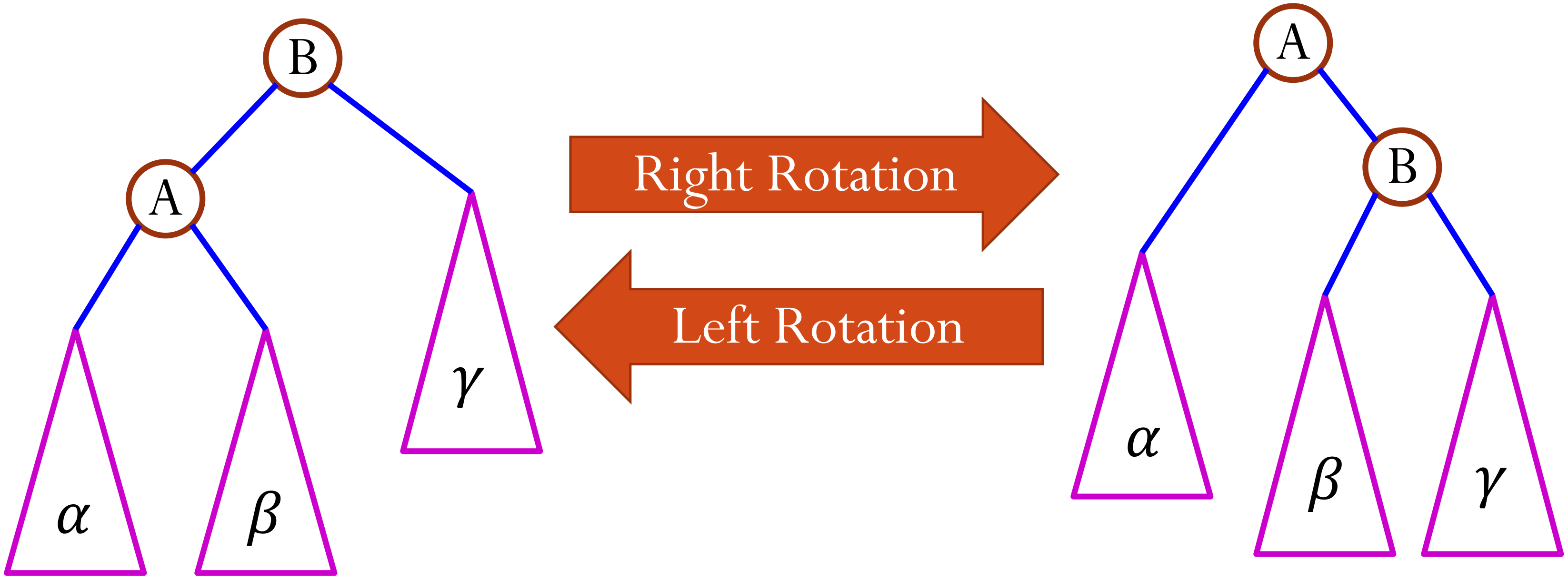




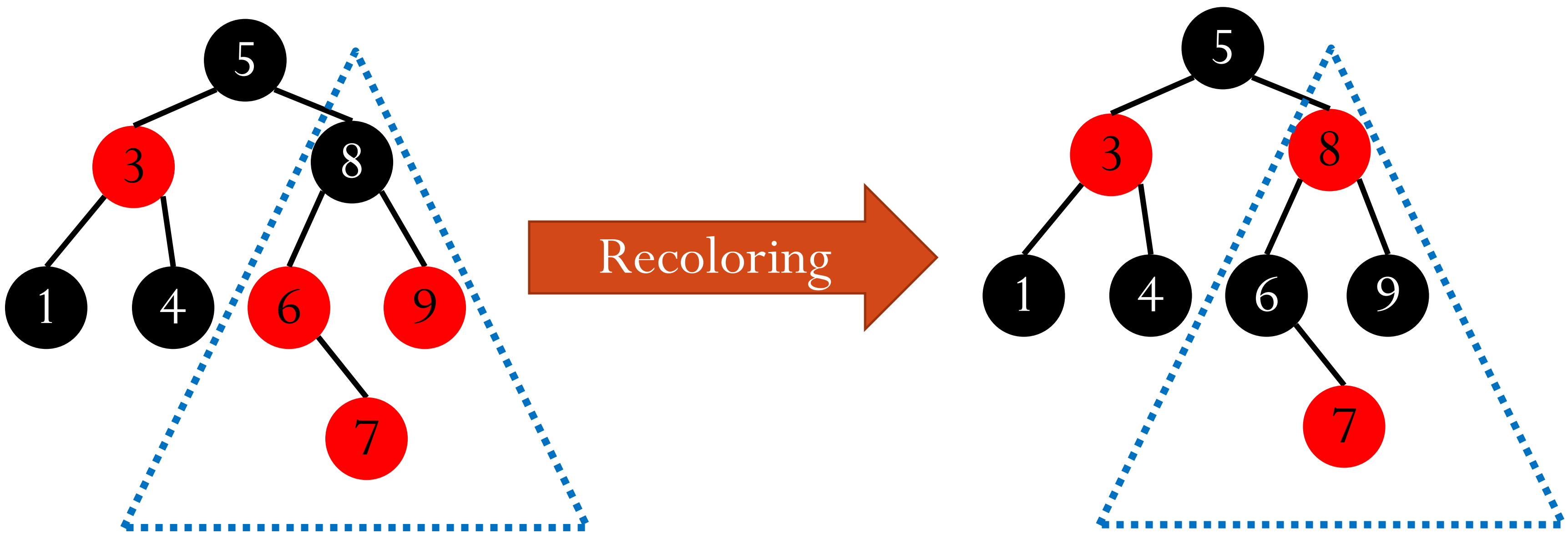
We have to do some work…

Modification: Rotation

- Maintain the binary search tree property.
- Can be done in $O(1)$ time.



Modification: Recoloring



Insertion: Sketch

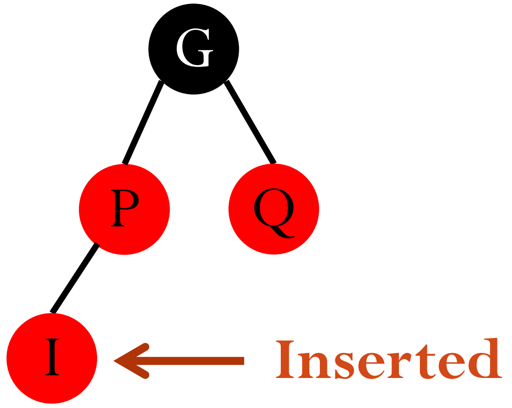
- Insert x as a **leaf**.

- Color x **red**.
  - Only **red rule** may be violated.

- Move the violation **up the tree** by recoloring/rotation.
  - At some point, the violation will be fixed.

Violation at Leaf

- **Note**: only **red rule** may be violated by inserting a (red) node as a leaf.
- When violating, its **parent** is **red** and its **grandparent** is **black**.
- **Denote**: the inserted node as "I", its parent as "P", its grandparent as "G".



Which Statements Are Correct?

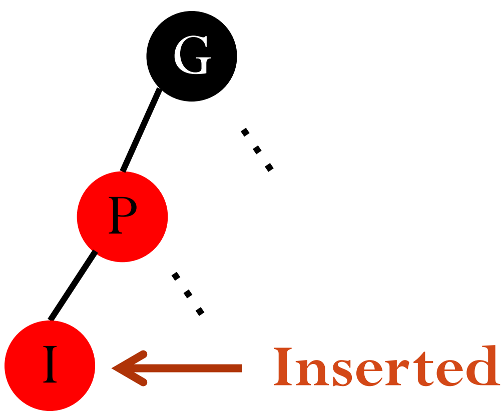
- Suppose there is a violation at the leaf. Suppose the parent of the inserted node is "P". Select all the correct statements.

A. P could be a non-leaf in the original tree.

B. P could have a sibling.

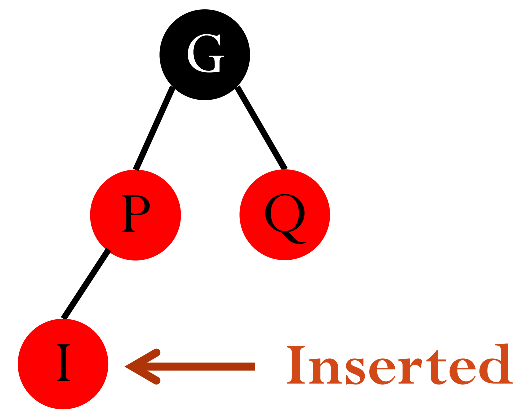
C. P cannot have a sibling.

D. P could have a sibling and that sibling must be a leaf node.



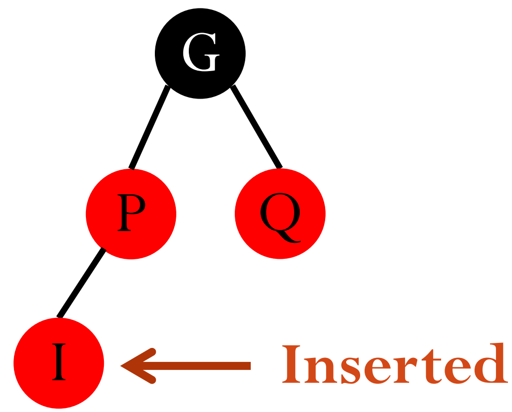
Violation at Leaf

- **Note**: only **red rule** may be violated by inserting a (red) node as a leaf.
- When violating, its **parent** is **red** and its **grandparent** is **black**.
- **Denote**: the inserted node as "I", its parent as "P", its grandparent as "G".
- **Claim**: in the old tree, "P" is a leaf, i.e., has no children.



Violation at Leaf

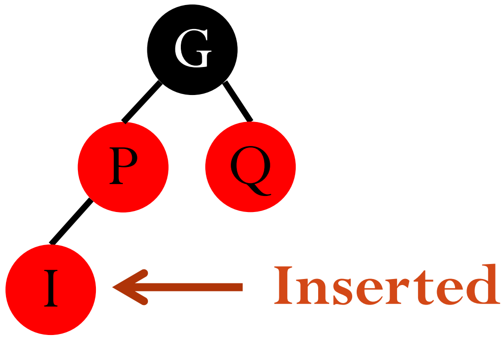
- **<u>Assume</u>**: the parent "P" is the **left child** of the grandparent "G".
  - The "right child" case is **symmetric**.
- **<u>Denote</u>**: the right child of the grandparent to be Q.
- **<u>Claim</u>**: Q is either a red leaf or a NULL.
  - Why?



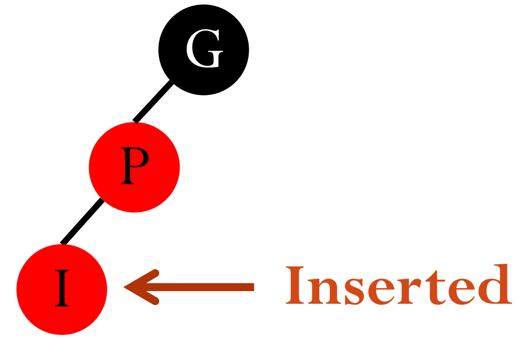
Violation at Leaf

- Three cases:

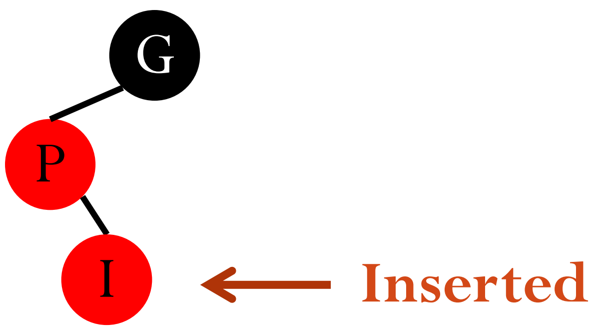
1. Q is a **red leaf**.



2. Q is empty; I is P's **left** child.



3. Q is empty; I is P's **right** child.



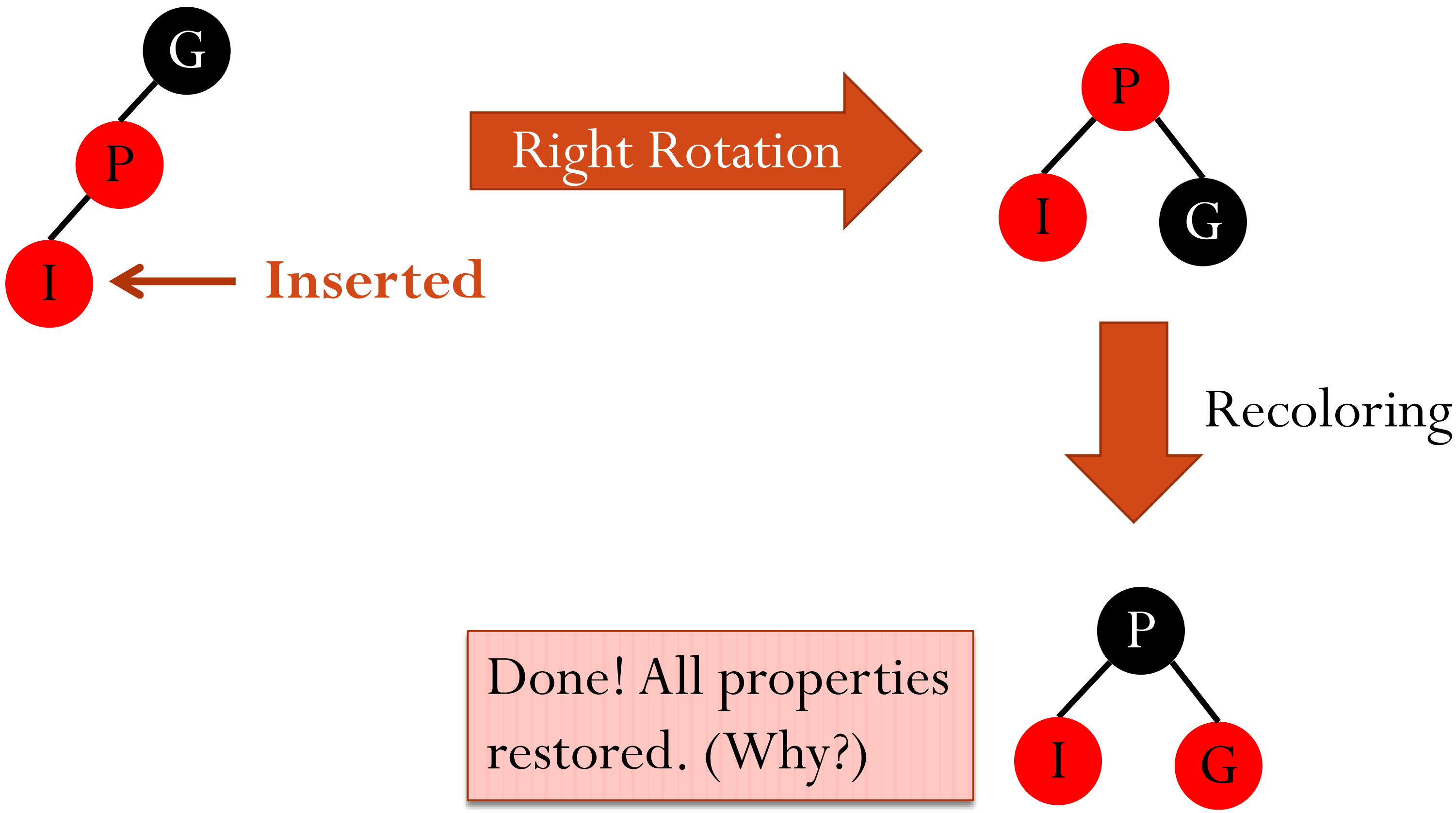
Violation at Leaf

- Case 1: Q is a **red leaf**.

May **recurse**, since G's parent may be red.

Violation at Leaf

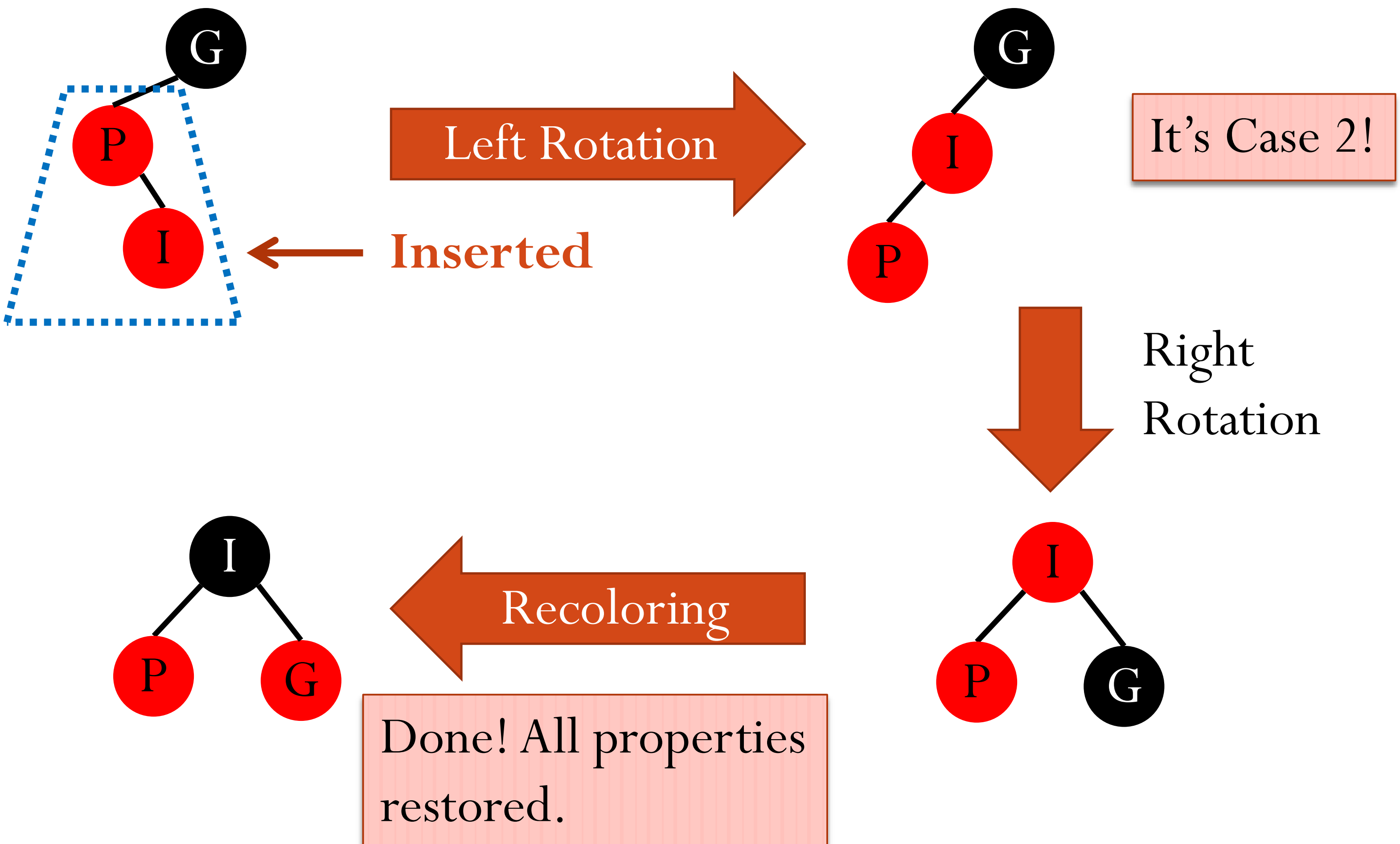
- Case 2: Q is empty; I is P's **left** child.



Done! All properties restored. (Why?)

Violation at Leaf

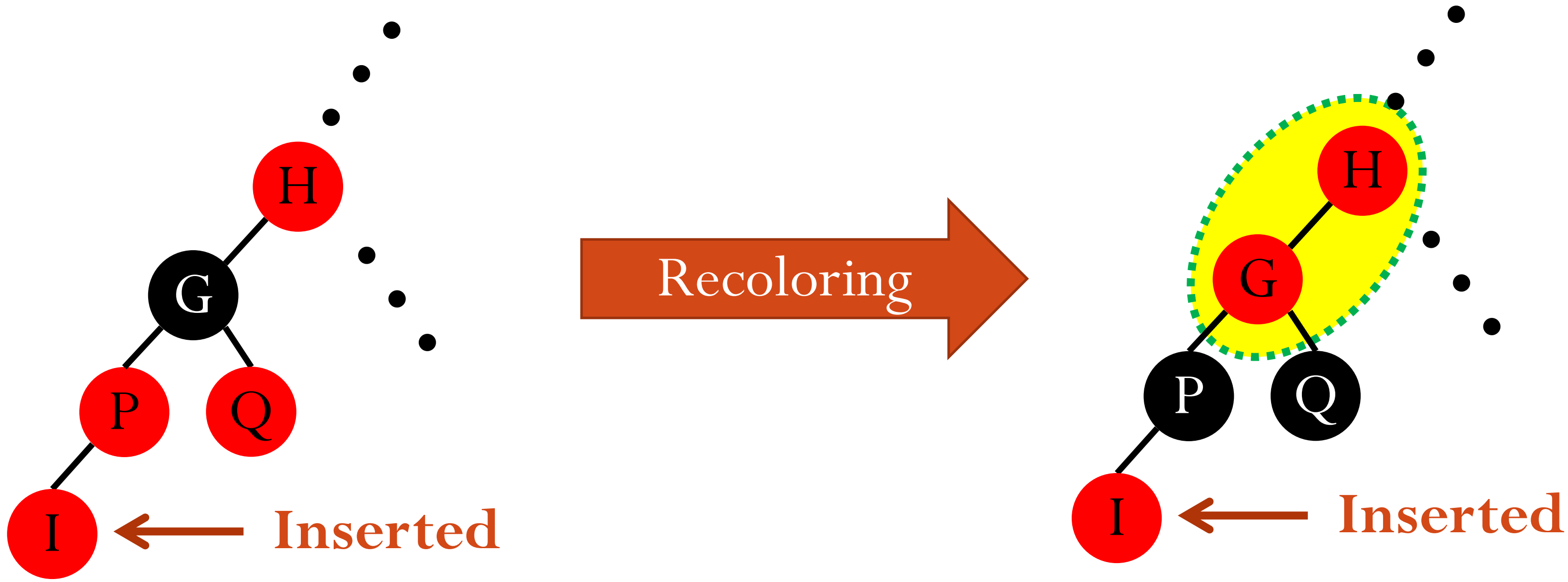
- Case 3: Q is empty; I is P's **right** child.



Violation at Leaf: Summary

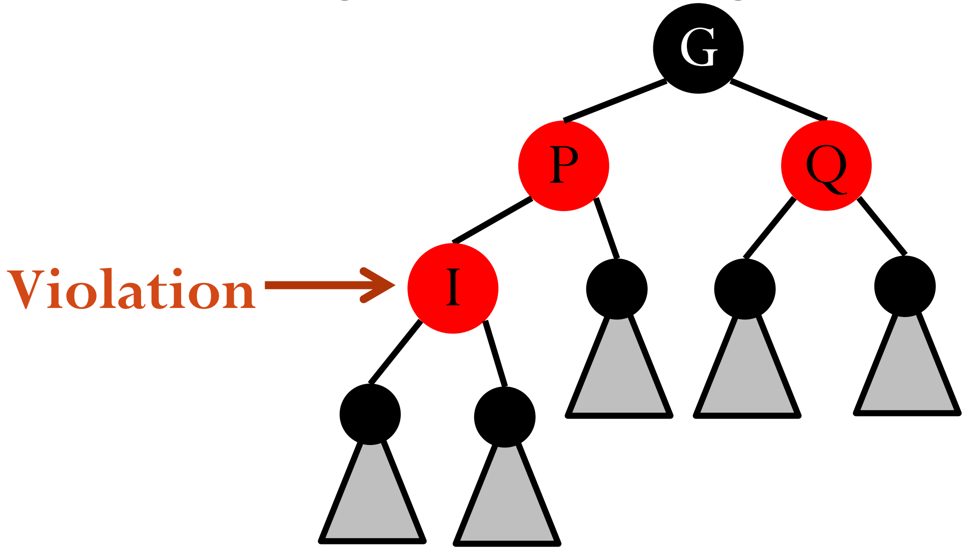
- For Case 2 (Q is empty; I is P's **left** child) and Case 3 (Q is empty; I is P's **right** child), **we're done**.

- For Case 1 (Q is a **red leaf**), we may recurse.
  - Violation of **red rule**.



Violation at Internal Nodes

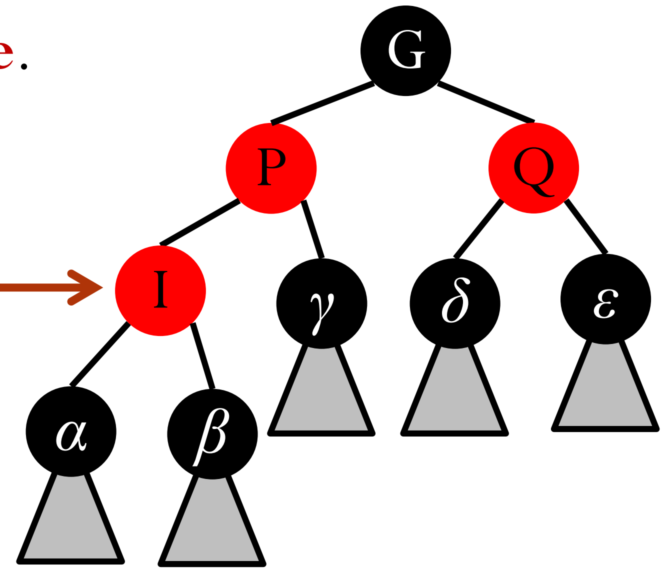
- Caused by **moving the violation up** the tree.
- When violating, its **parent** is **red** and its **grandparent** is **black**.
- **Assume**: the parent "P" is the **left child** of the grandparent "G". (The "right child" case is **symmetric**.)
- **Denote**: the right child of the grandparent to be Q.



Violation at Internal Nodes

- Three Cases:
  1. Q is a **red node**.

Violation



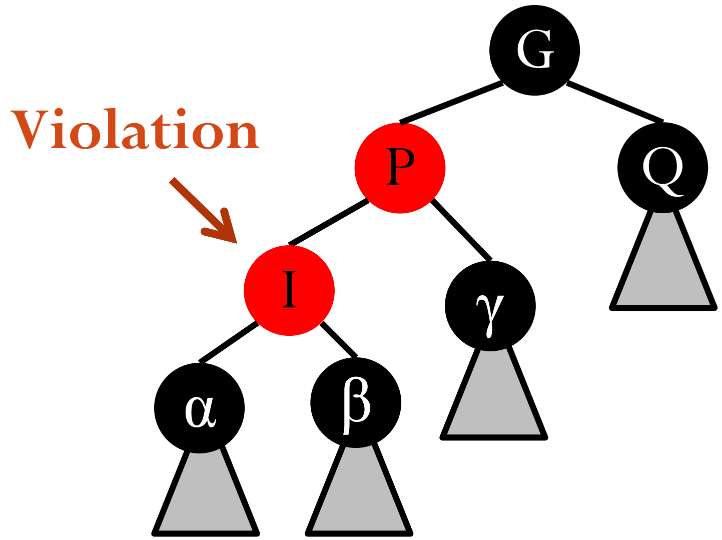
- **Claim**:
  - α, β, γ, δ, ε are trees with **black root**.
  - α, β, γ, δ, ε have the **same** **black height**.

Violation at Internal Nodes

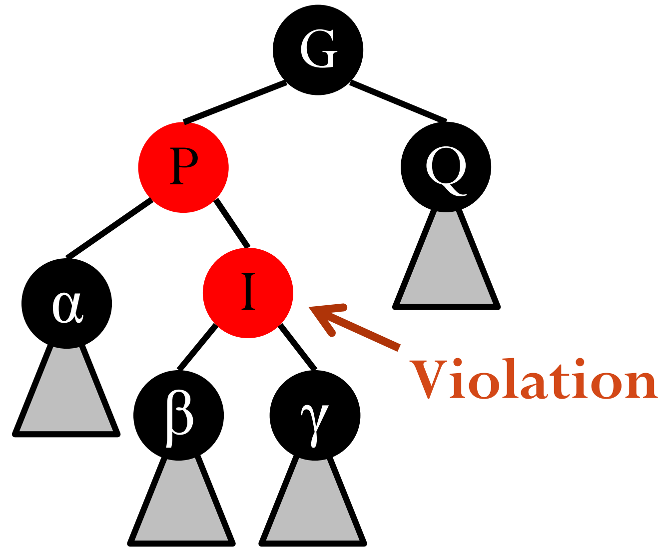
- Three Cases:

  2. Q is a **black node**; I is P's **left** child.

  3. Q is a **black node**; I is P's **right** child.

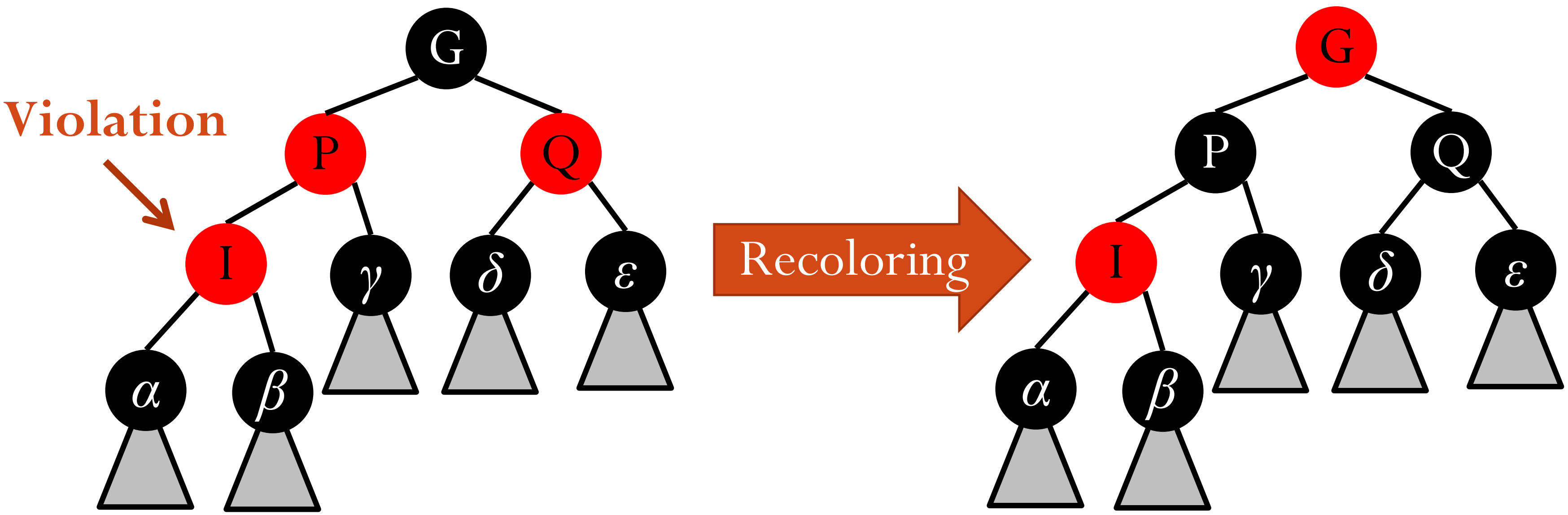


- **Claim** for Case 2 and 3:
  - α, β, γ, Q are trees with **black root**.
  - α, β, γ, Q have the **same** **black height**.



Violation at Internal Nodes

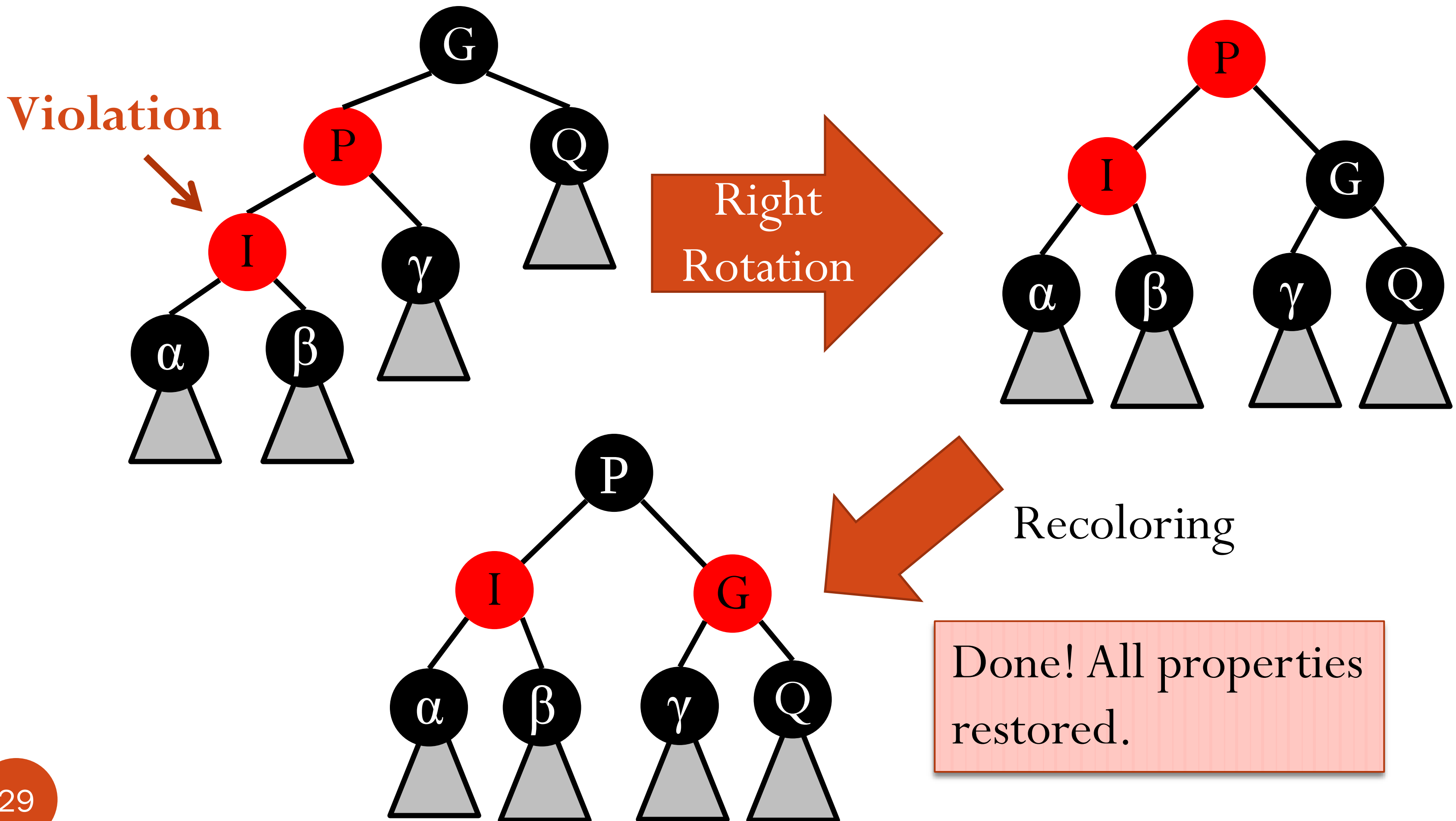
- Case 1: Q is a **red node**.



May **recurse**, since G's parent may be red.

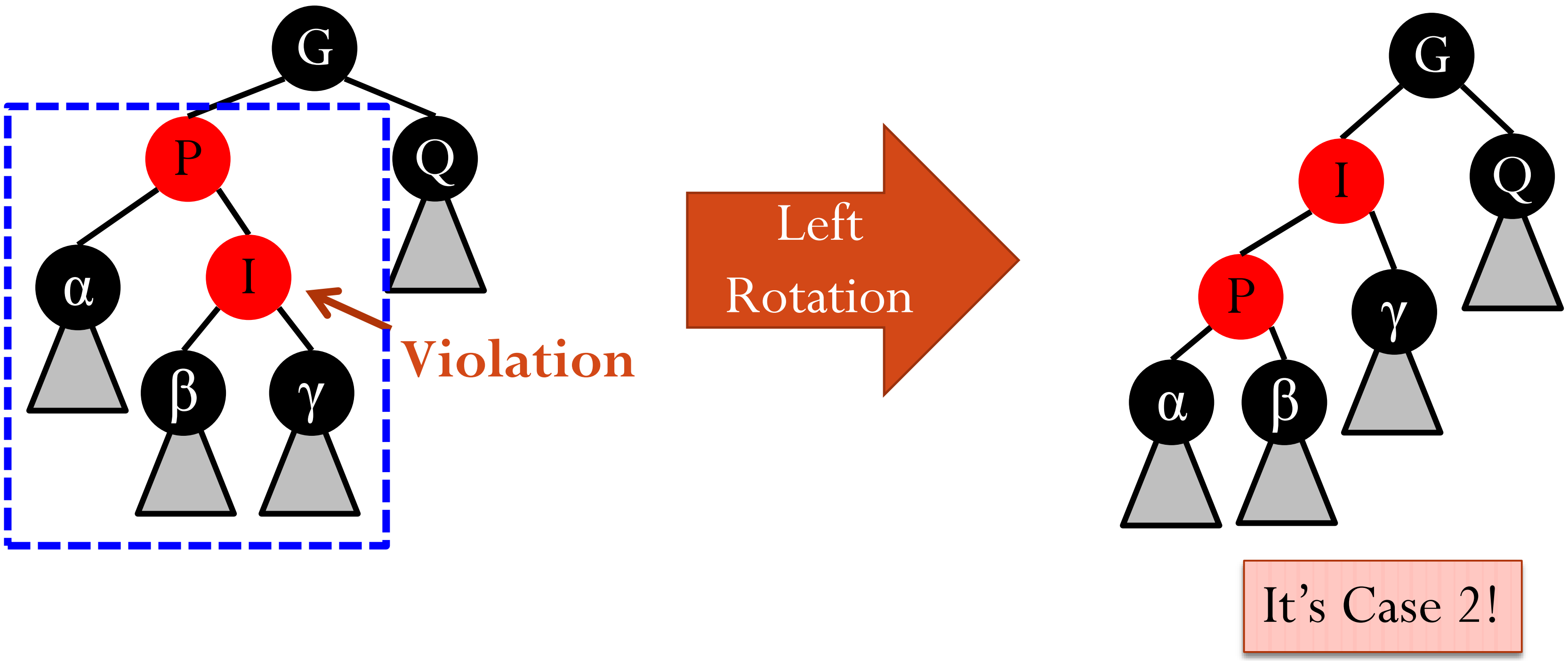
Violation at Internal Nodes

- Case 2: Q is a **black node**; I is P's **left** child.

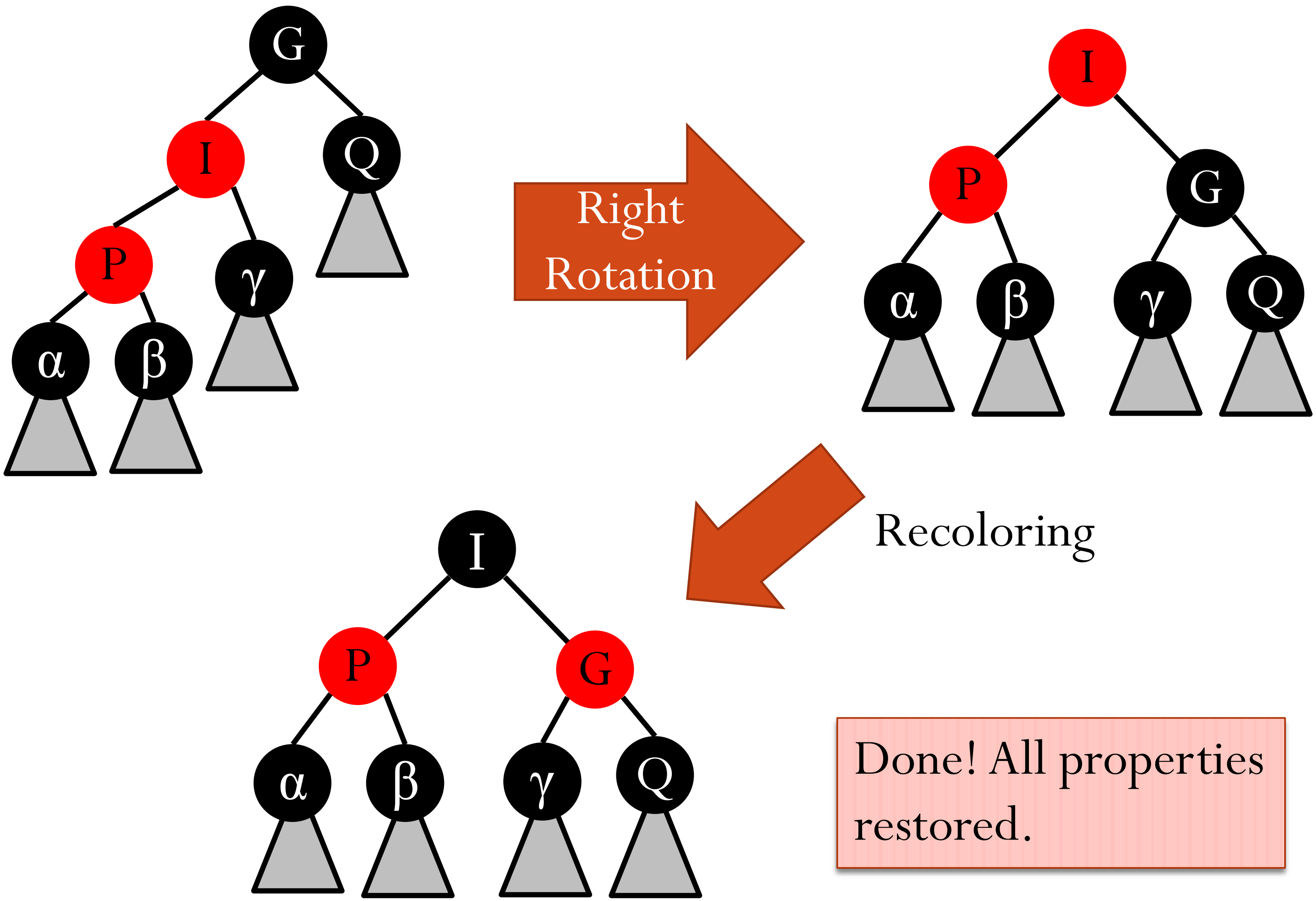


Violation at Internal Nodes

- Case 3: Q is a **black node**; I is P's **right** child.



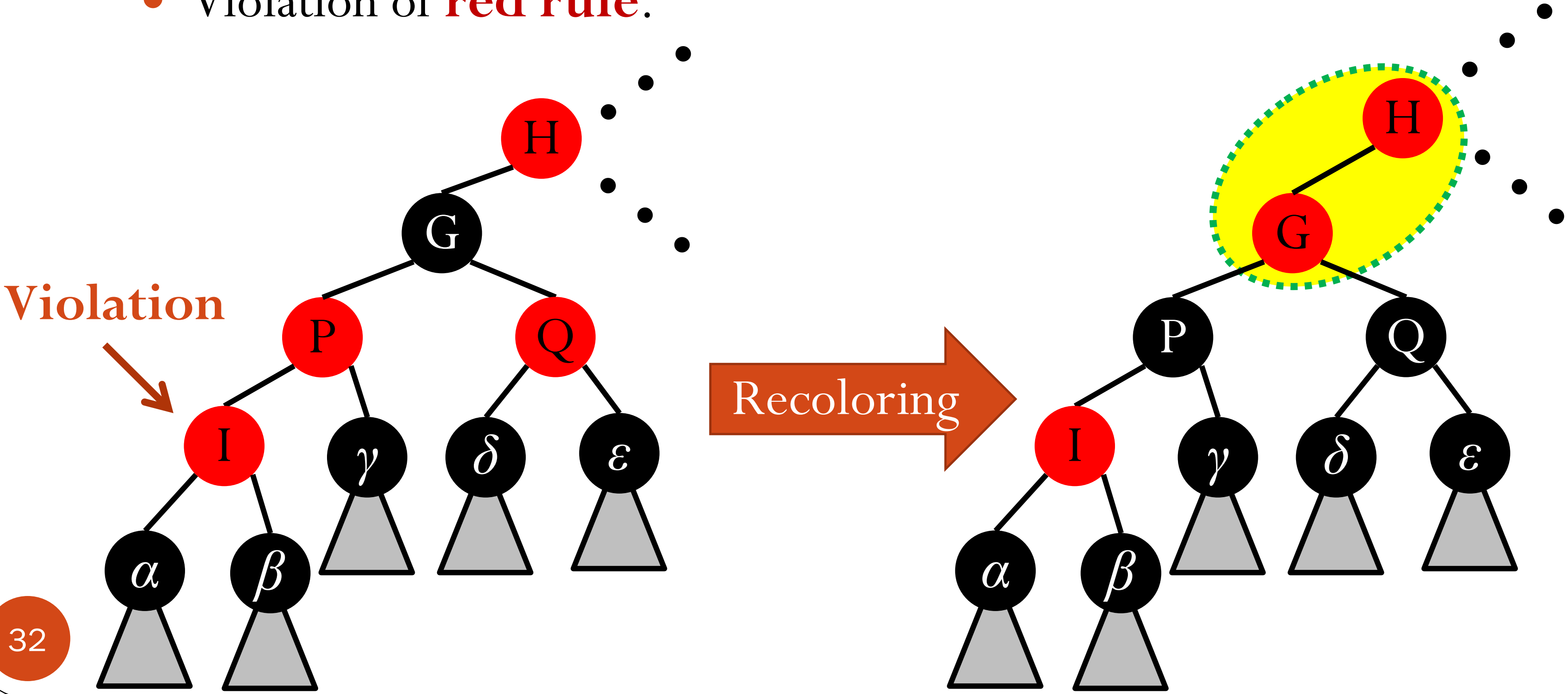
Violation at Internal Nodes: Case 3 (cont.)



Done! All properties restored.

Violation at Internal Nodes: Summary

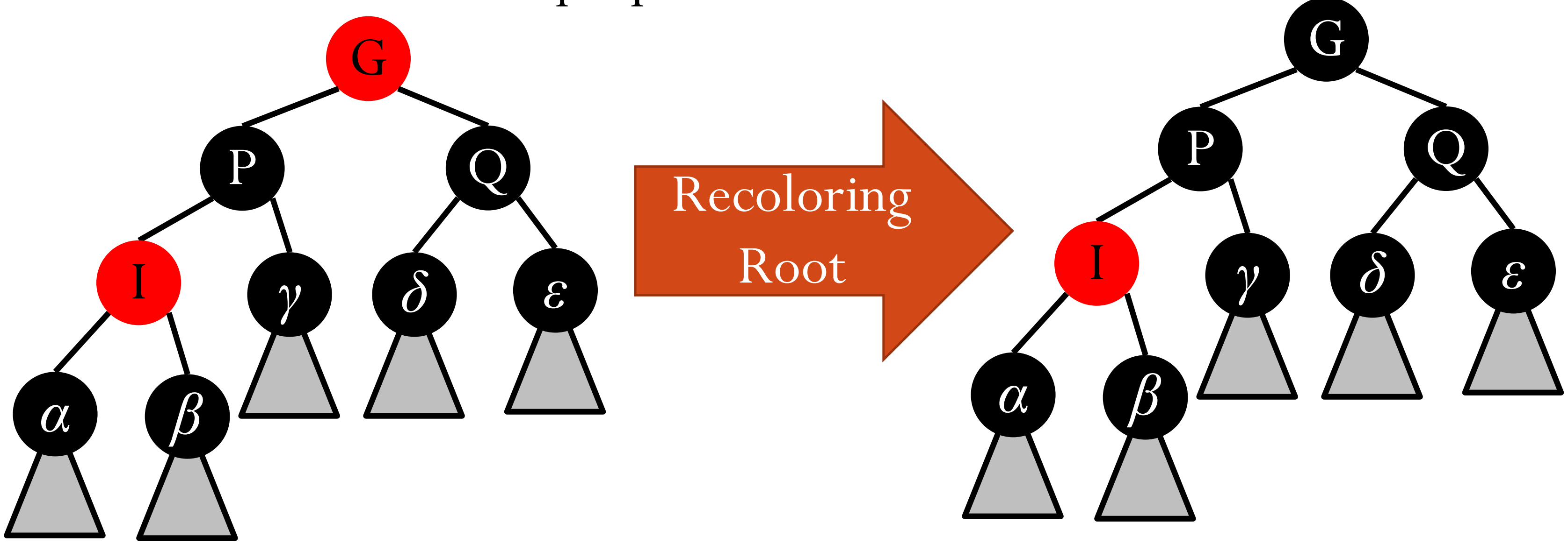
- For Case 2 (Q is a **black node**; I is P's **left** child) and Case 3 (Q is a **black node**; I is P's **right** child), **we're done**.
- For Case 1 (Q is a **red node**), we may recurse.
  - Violation of **red rule**.



Final Step: Violation Fix at the Root

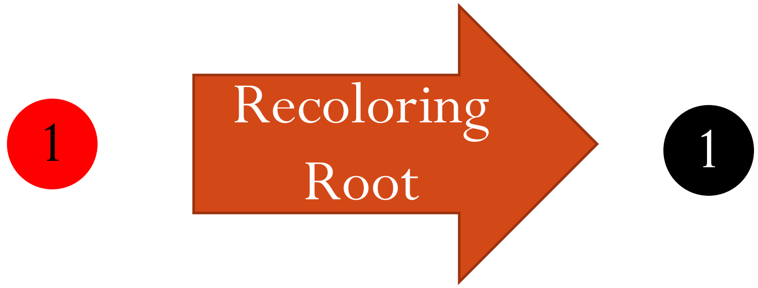
- By **moving the violation up** the tree …
  - … the root may become **red**.

- Final step: set root to be **black**.
  - All red-black tree properties are now **restored**.

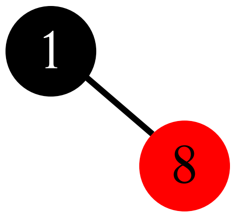


Example

- Insert 1

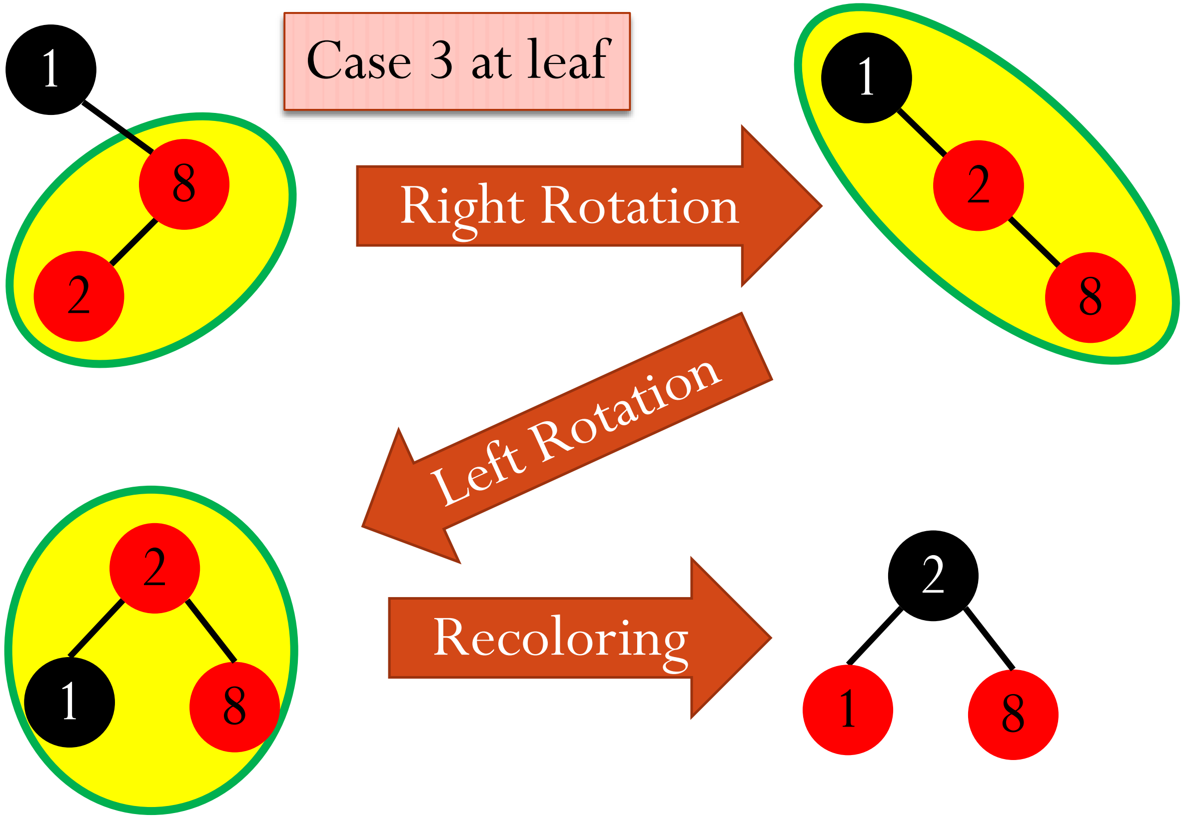


- Insert 8



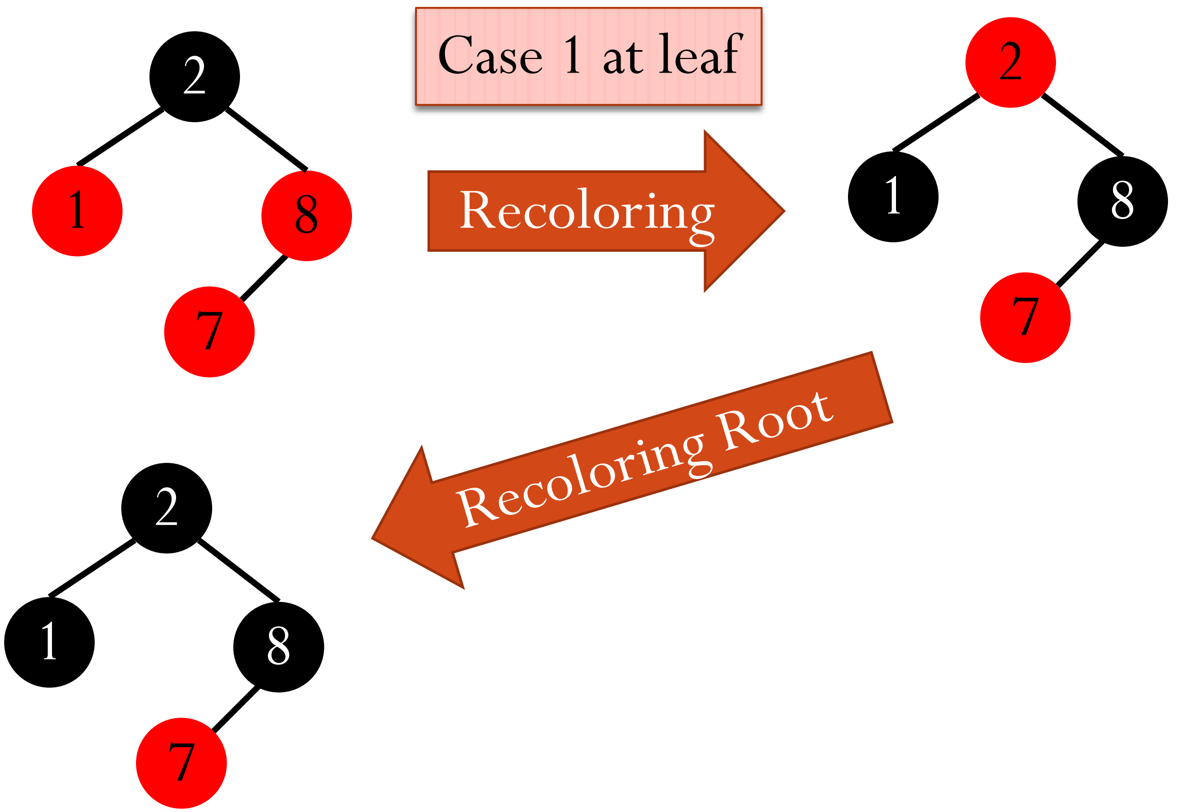
Example (cont.)

- Insert 2



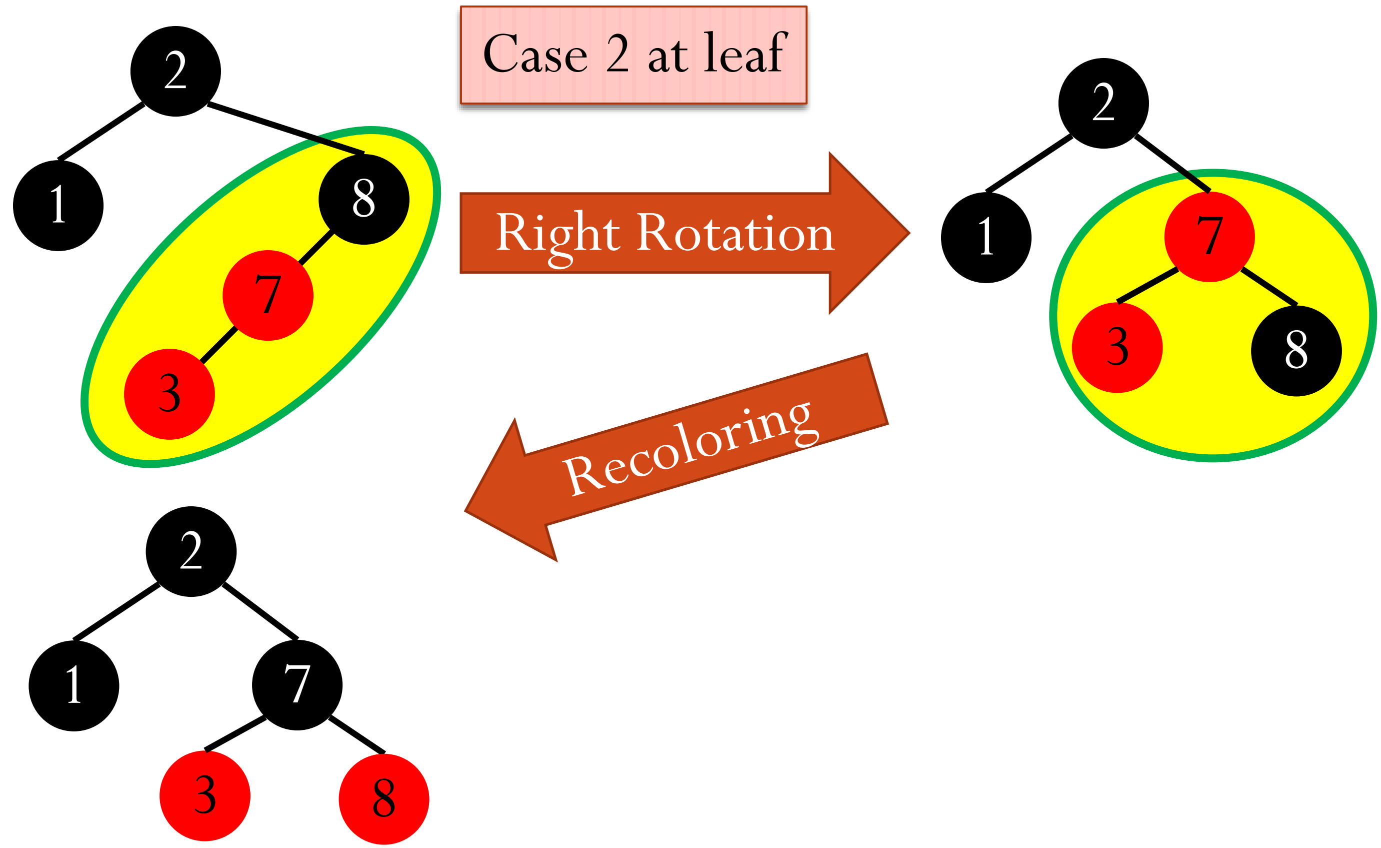
Example (cont.)

- Insert 7



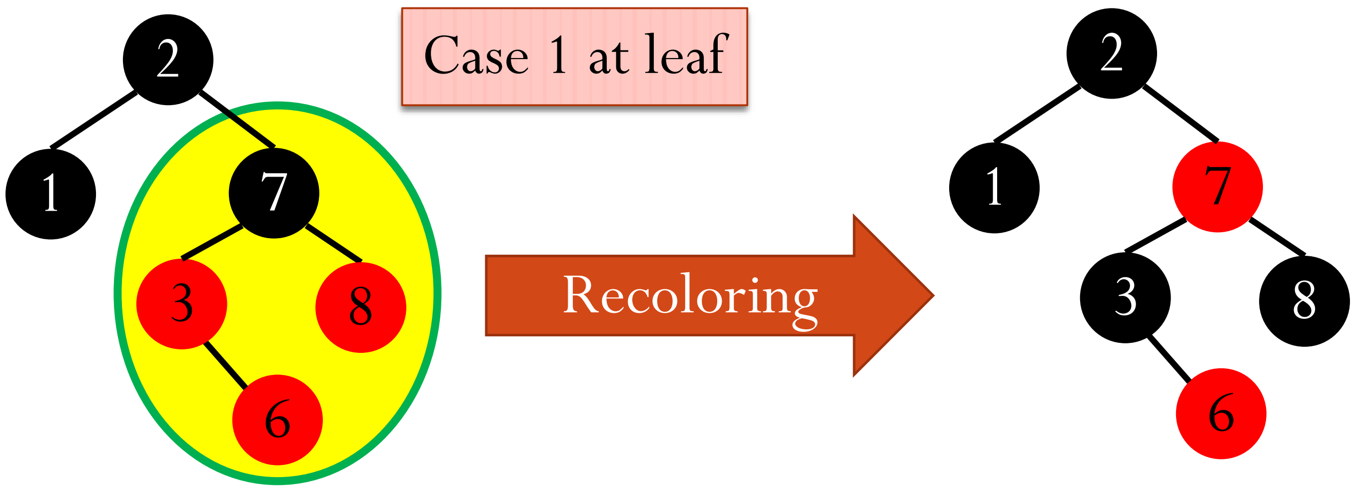
Example (cont.)

- Insert 3



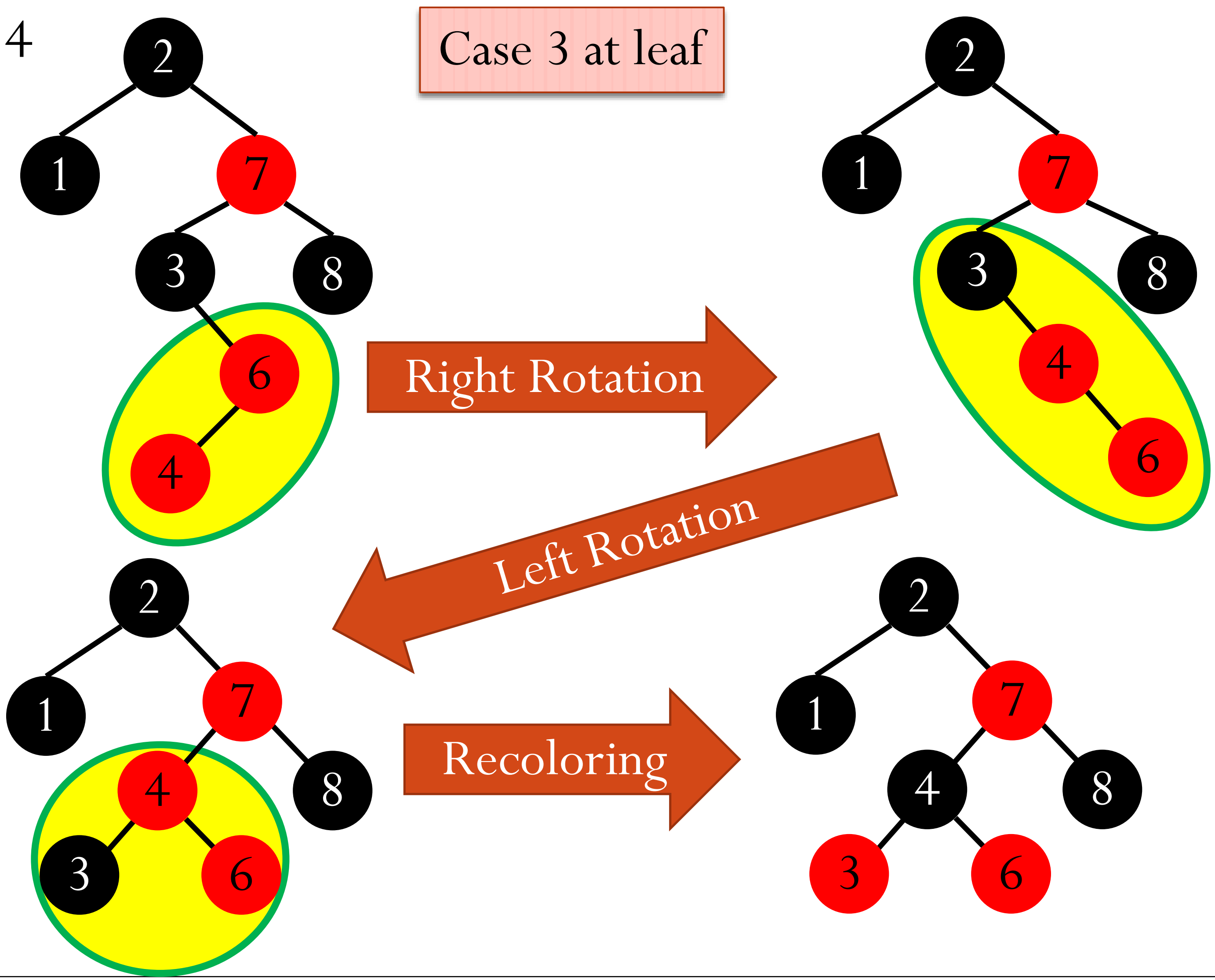
Example (cont.)

- Insert 6



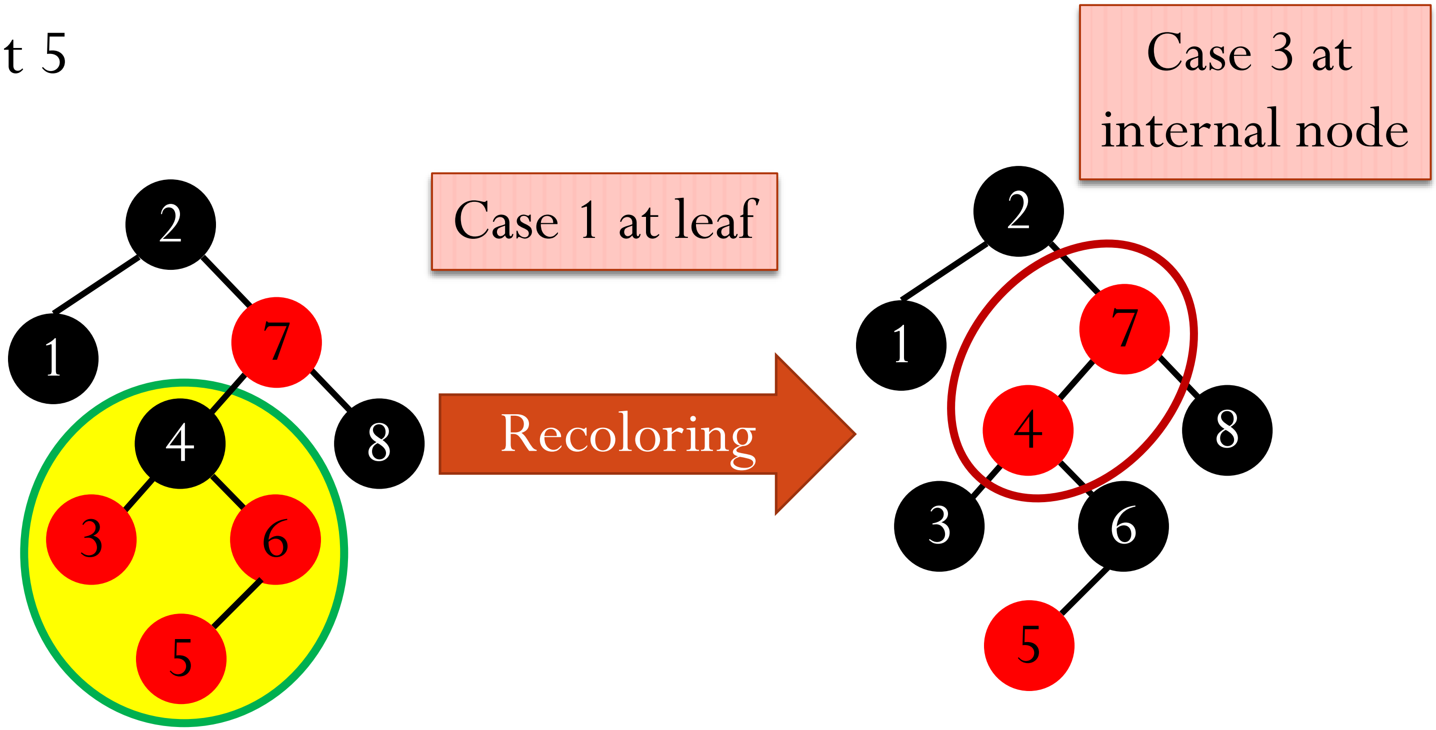
Example (cont.)

- Insert 4



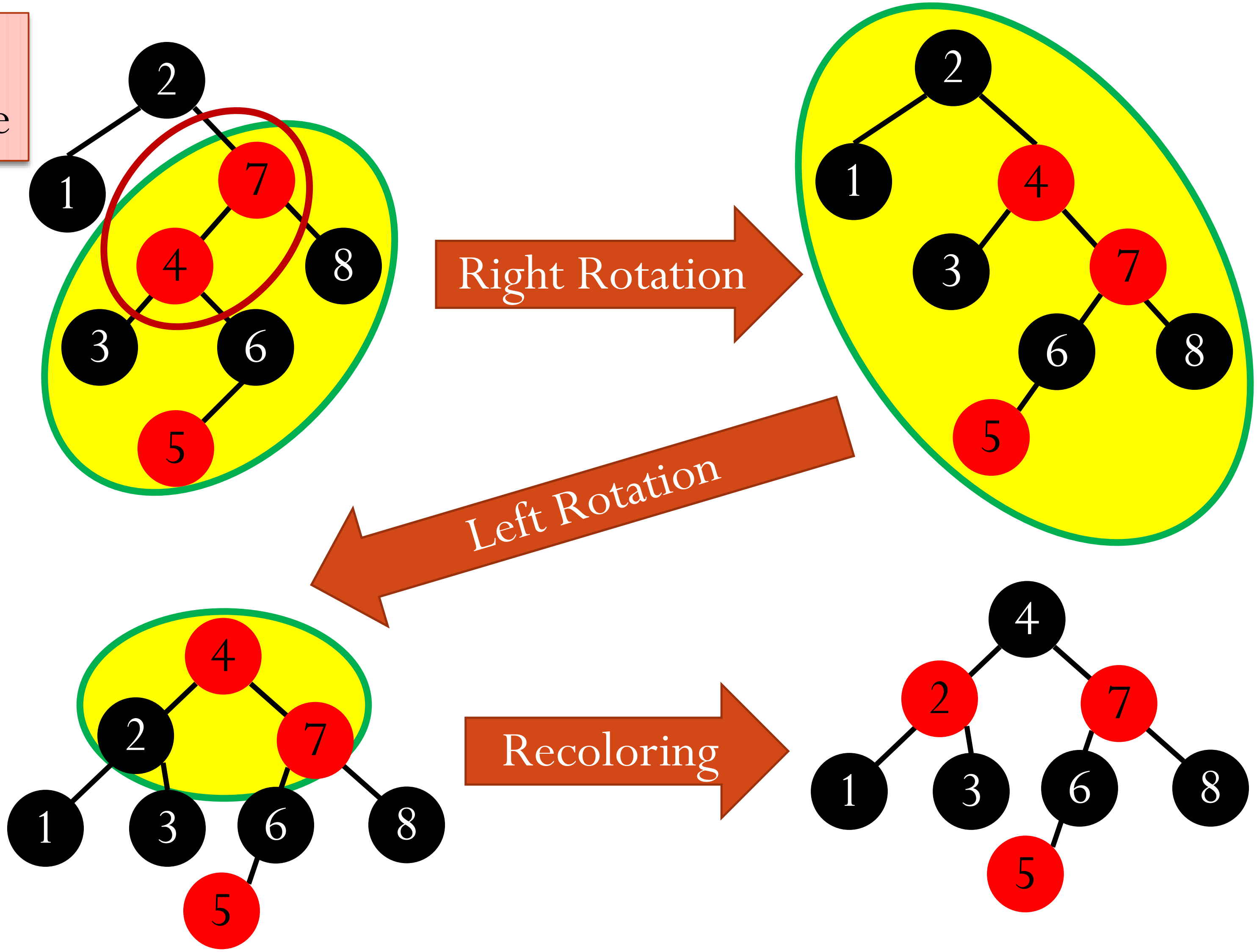
Example (cont.)

- Insert 5



Example (cont.)

Case 3 at internal node



Runtime Complexity

- Number of rotations required
  - For case 1, only need to recolor, **no** rotation.
  - For case 2 or 3, perform 1 or 2 rotations and terminate.
  - **Thus**: # rotations $= O(1)$.

- Number of recoloring required
  - Worst case: $O(\log n)$

- Runtime complexity is $O(\log n)$.