VE281

Data Structures and Algorithms

AVL Trees

Learning Objectives:

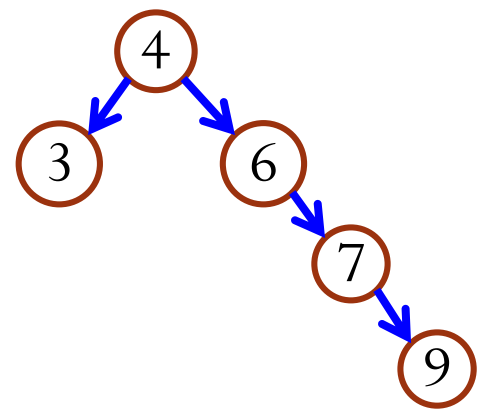
- Know the general balanced condition for a balanced search tree
- Know the balance condition of an AVL tree and balance factor
- Know the four types of rotation operations for an AVL tree and how to apply them during insertion

Outline

- Balanced Search Trees
  - AVL Trees
- AVL Tree Insertion
- Supporting Data Members and Functions of AVL Tree

Motivation

- Given n nodes, the **average case** time complexities for search, insertion, and removal on BST are all $O(\log n)$.
- However, the **worst case** time complexities are still $O(n)$.
  - The reason is that a tree could become "**unbalanced**" after a number of insertions and removals.



- We want to maintain the tree as a "**balanced**" tree.

Balanced Search Trees

- What are the requirements to call a tree a balanced tree?

- Would you require a tree to be perfect/complete to call it balanced?
  - No! They are too restrictive.

Balanced Search Trees

- We need another definition of "balanced condition."
- We want the definition to satisfy the following two criteria:
  1. Height of a tree of n nodes $= O(\log n)$.
  2. Balance condition can be maintained **efficiently**: $O(\log n)$ time to **rebalance** a tree.
- Several balanced search trees, each with its own balance condition
  - AVL trees
  - 2-3 trees
  - red-black trees

AVL Trees

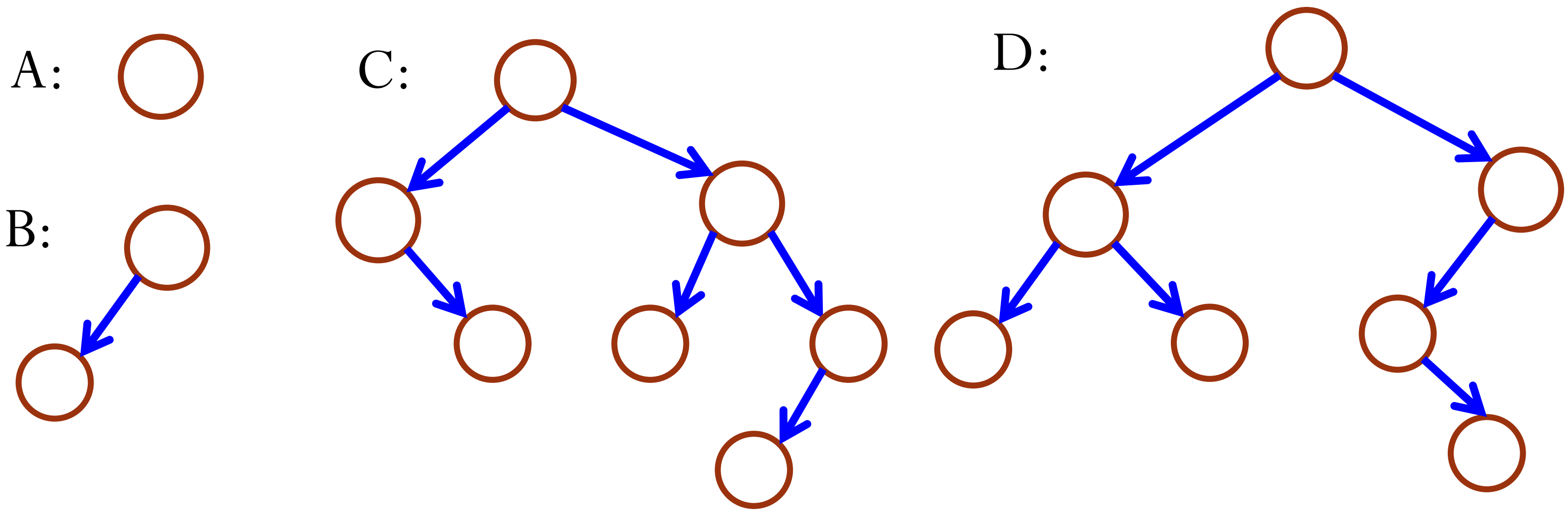
- Adelson-Velsky and Landis' trees
  - AVL tree is a **binary search tree**.

- AVL trees' balance condition:
  - An empty tree is **AVL balanced**.
  - A non-empty binary tree is **AVL balanced** if

1. Both its left and right subtrees are AVL balanced, and
2. The height of left and right subtrees differ by **at most 1**.

Which of the Following Trees Are AVL Balanced?

- Select all the AVL balanced trees.



AVL trees' balance condition:

- An empty tree is **AVL balanced**.
- A non-empty binary tree is **AVL balanced** if
  1. Both its left and right subtrees are AVL balanced, and
  2. The height of left and right subtrees differ by **at most 1**.

Properties of AVL Trees

- The height h of an AVL balanced tree with n internal nodes satisfies
$$\log_2(n+1) - 1 \leq h \leq 1.44 \log_2(n+2)$$

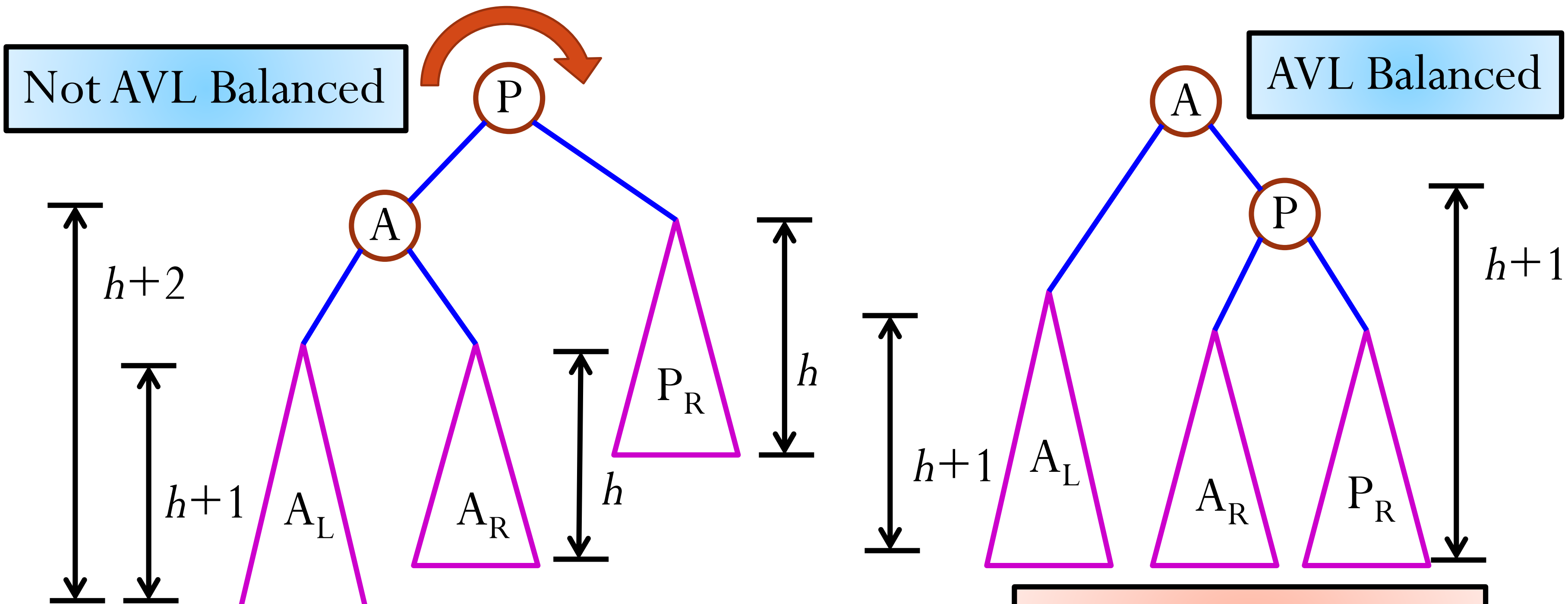
- AVL trees satisfies the general "balanced condition" 1:
  - The height of a tree of n nodes is $O(\log n)$.
  - Search is guaranteed to always be $O(\log n)$ time!
- We will also show that AVL trees satisfy the general "balance condition" 2:
  - Balance condition can be maintained **efficiently**.

AVL Trees Operations

- Search, insertion, and removal all work exactly the same as with BST.

- However, after each insertion or removal, we must check whether the tree is still **AVL balanced**.
  - If not, we need to "**re-balance**" the tree.

Re-Balance the Tree via Rotation

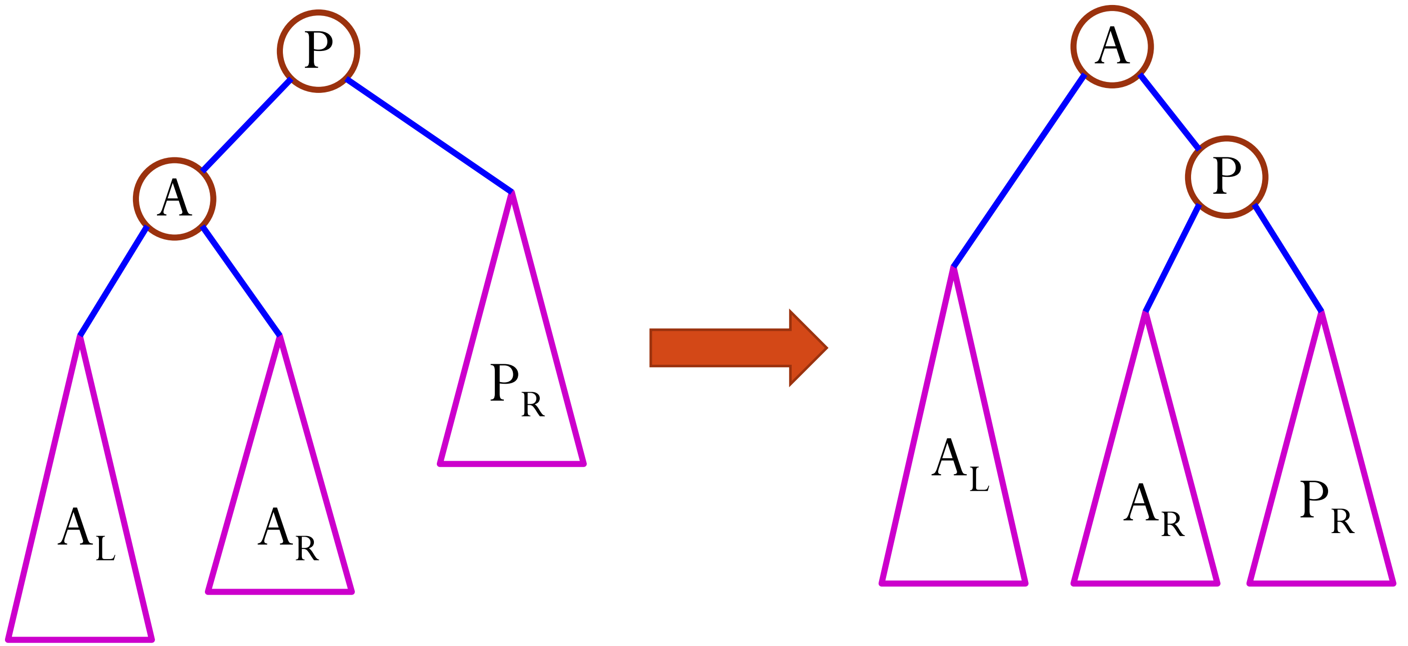


The BST ordering on keys are preserved.

- The rotation operation:
  - Interchange the role of **a parent** and **one of its children**, while still preserving the BST ordering on the keys.

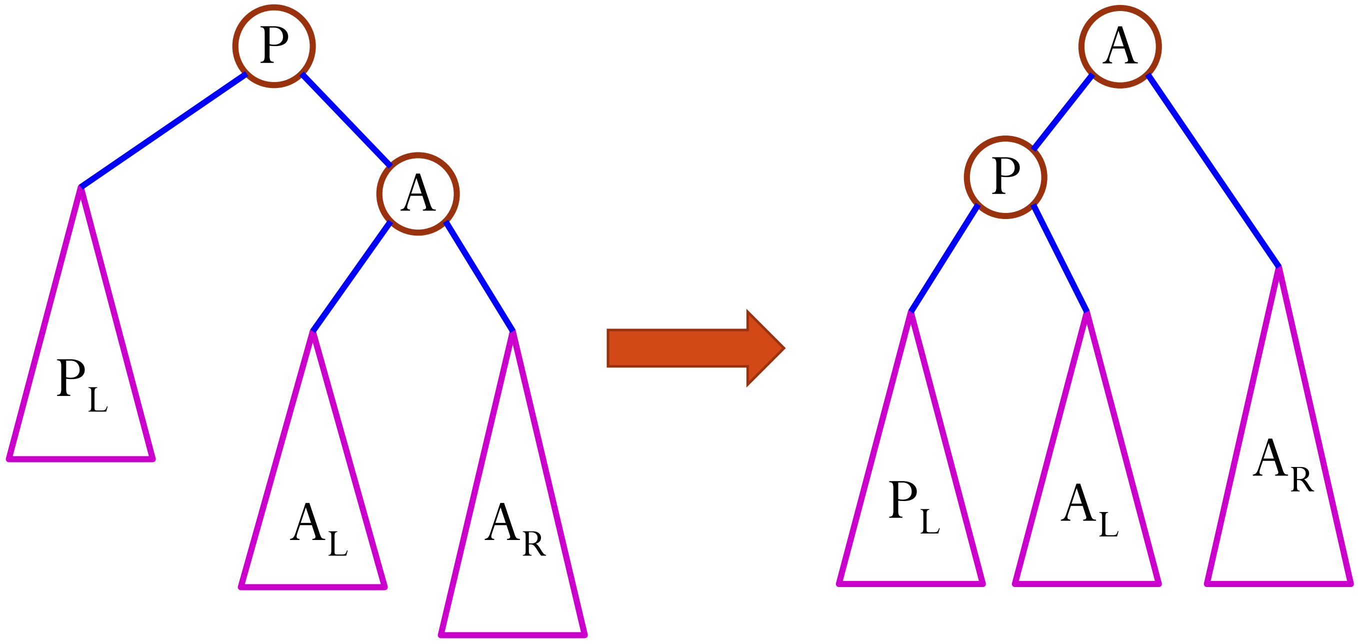
Right Rotation

1. The right link of the **left child** becomes the left link of the **parent**.
2. **Parent** becomes right child of the **old left child**.



Left Rotation

- The left link of the **right child** becomes the right link of the **parent**.
- **Parent** becomes left child of the **old right child**.

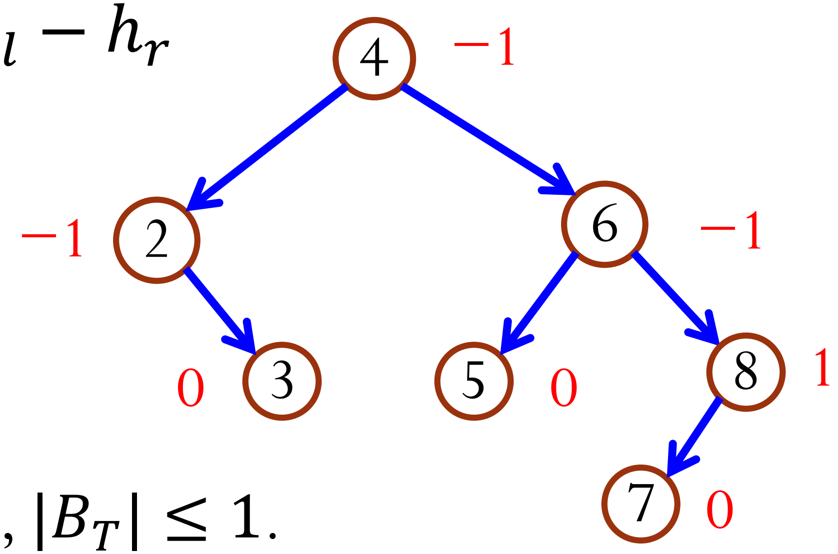


Balance Factor

- Let T_l and T_r be the left and right subtrees of a tree rooted at node T.
- Let h_l be the height of T_l and h_r be the height of T_r.
- Define the **balance factor** (B_T) of node T as

$$B_T = h_l - h_r$$

Balance Factor Example



- AVL tree's balance condition:
  - For **every node** T in the tree, $|B_T| \leq 1$.

Outline

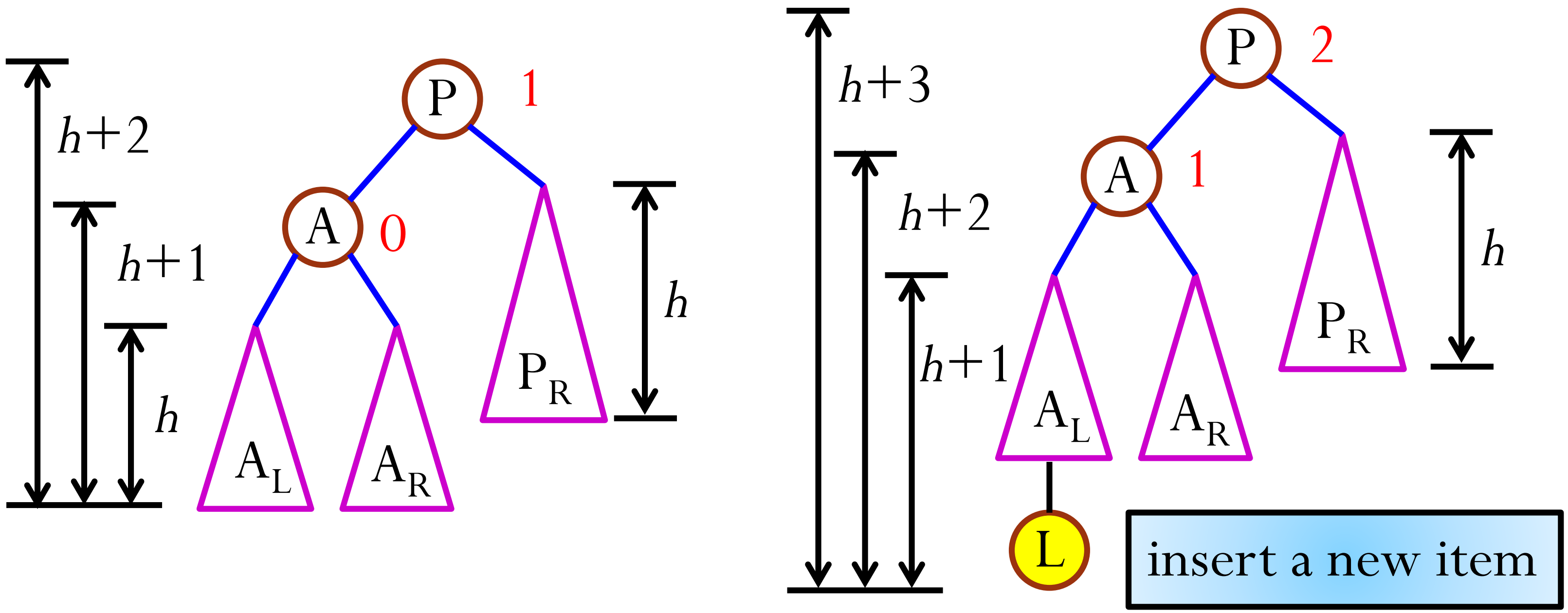
- Balanced Search Trees
  - AVL Trees
- AVL Tree Insertion
- Supporting Data Members and Functions of AVL Tree

Insertion

- Inserting an item in a tree affects potentially the heights of all of the nodes along the **access path**, i.e., the path from the root to that leaf.
- When an item is inserted in a tree, the height of any node on the access path may increase by one.
- To ensure the resulting tree is still AVL balanced, the heights of all the nodes along the access path must be **recomputed** and the AVL balance condition must be **checked**.
  - Sometimes, increasing the height by one does not violate the AVL balance condition.
  - In other cases, the AVL balance condition is violated.
  - We will fix **the first unbalanced node** in the access path **from the leaf**.

Breaking AVL Balance Condition

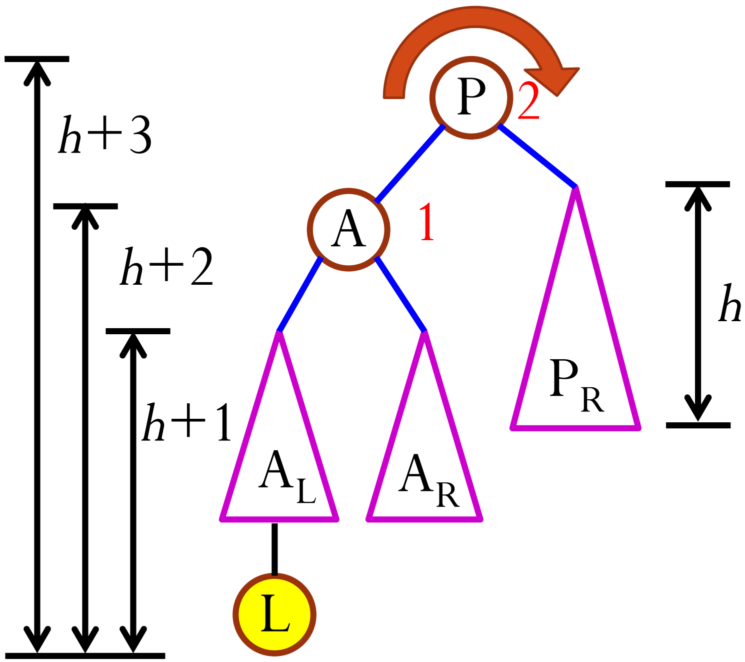
Left-Left Insertion



Left-left insertion: the first two edges in the insertion path from node P both go to the left.

Restoring AVL Balance Condition

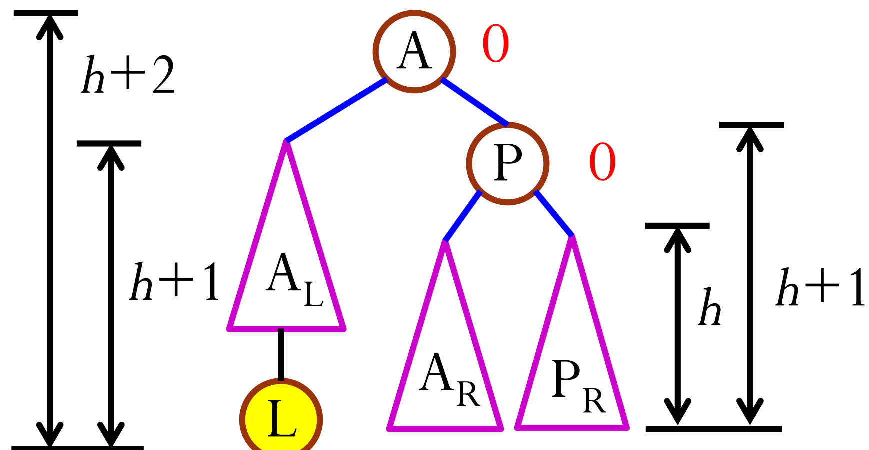
Left-Left Rotation



How to restore AVL balance?

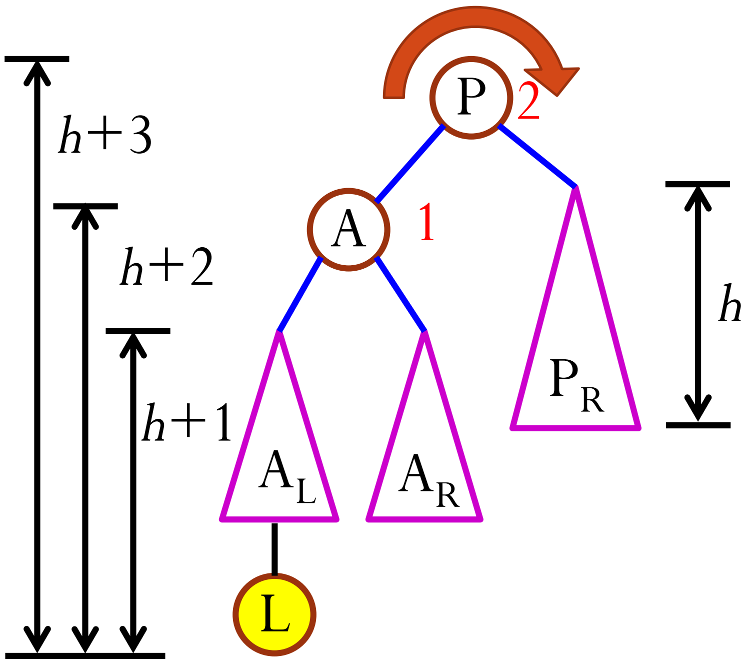
Do a right rotation at node P.

The rotation is also called **left-left (LL) rotation**.



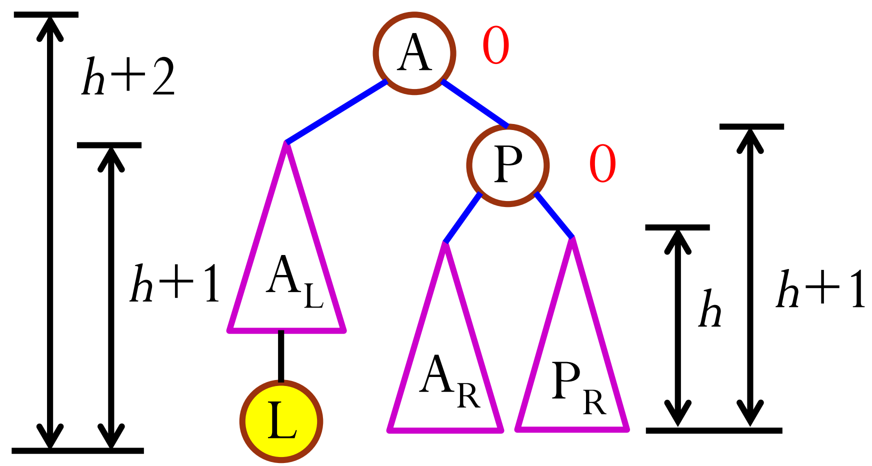
Restoring AVL Balance Condition

Left-Left Rotation



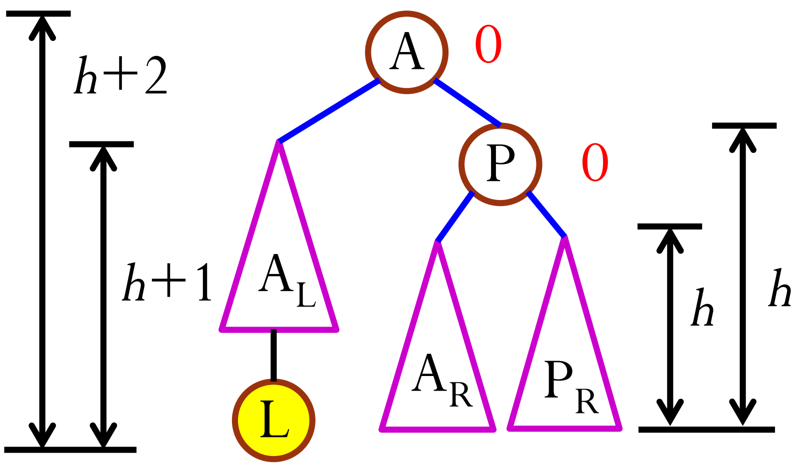
An **LL rotation** is called for when the node becomes unbalanced with a **positive** balance factor and the left subtree of the node also has a **positive** balance factor.

The rotation is also called **left-left (LL) rotation**.



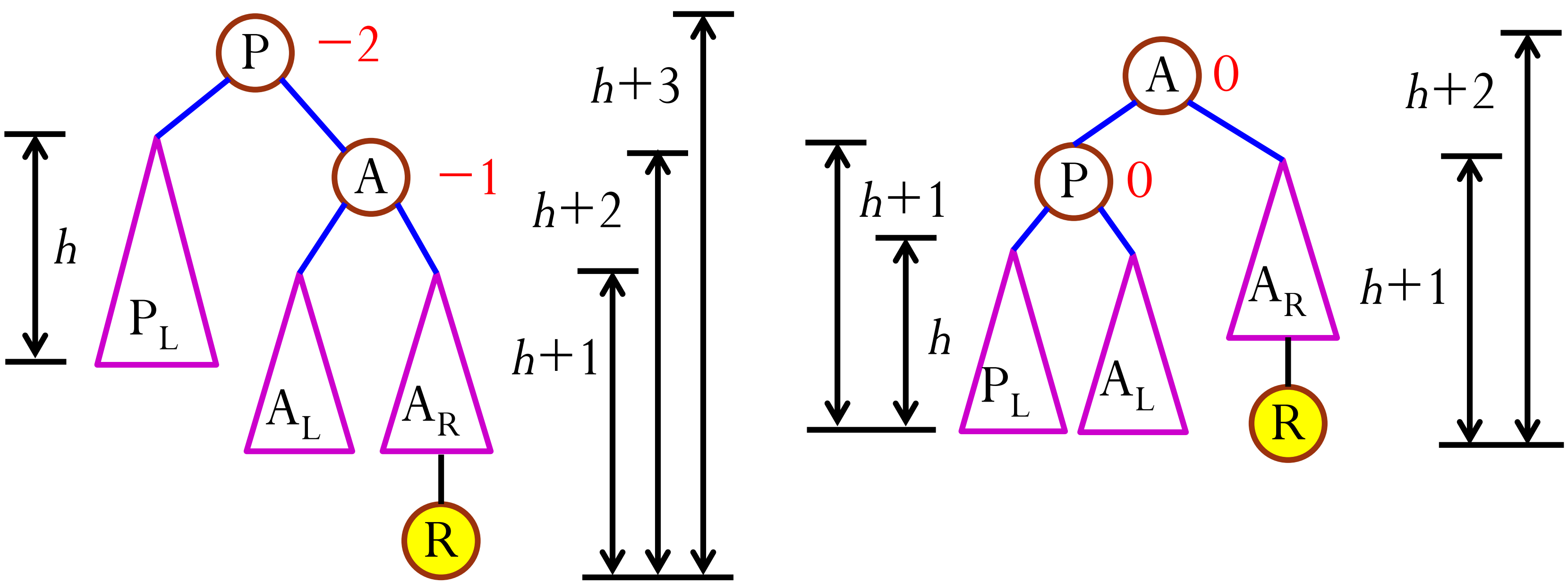
Properties of Left-Left Rotation

- The ordering property of BST is kept.
- Both nodes A and P have balance factor of 0.
- The height of the tree **after the rotation** is the same as the height of the tree before insertion.



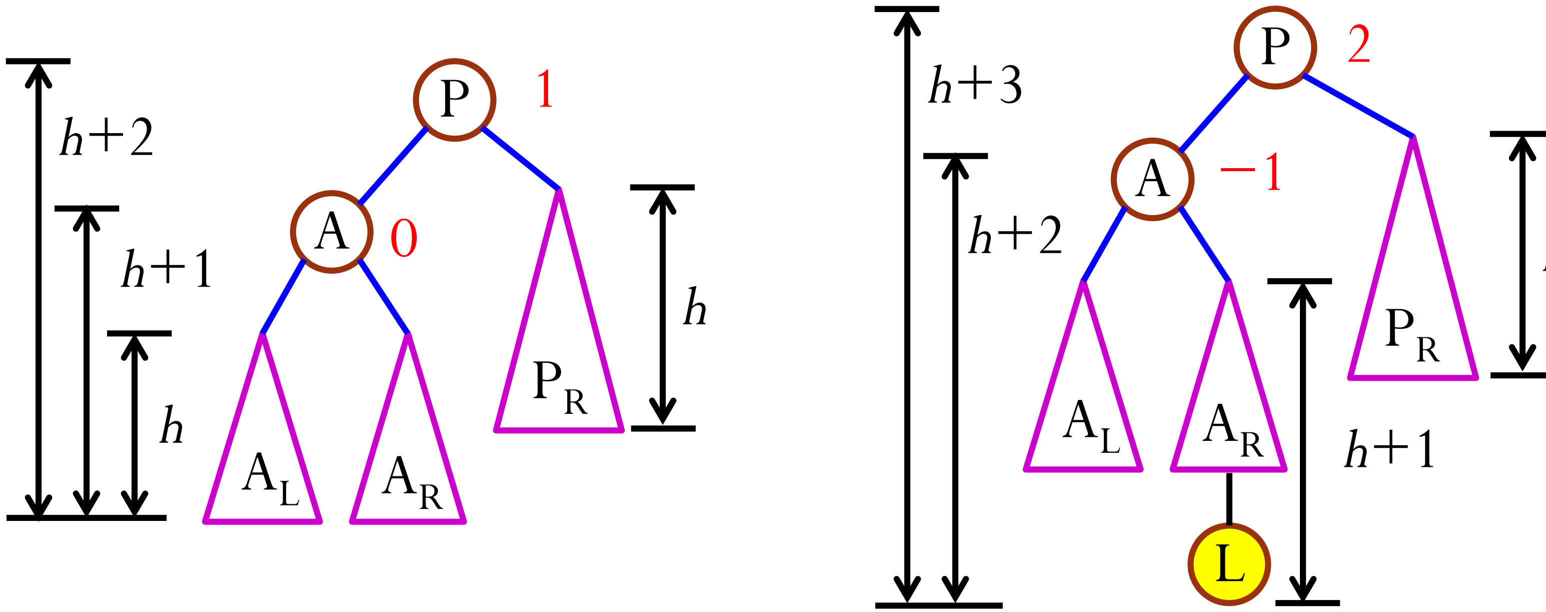
Right-Right (RR) Rotation

- Symmetric to left-left rotation.
- An RR rotation is called for when the node becomes unbalanced with a **negative** balance factor and the right subtree of the node also has a **negative** balance factor.



Breaking AVL Balance Condition

Left-Right Insertion

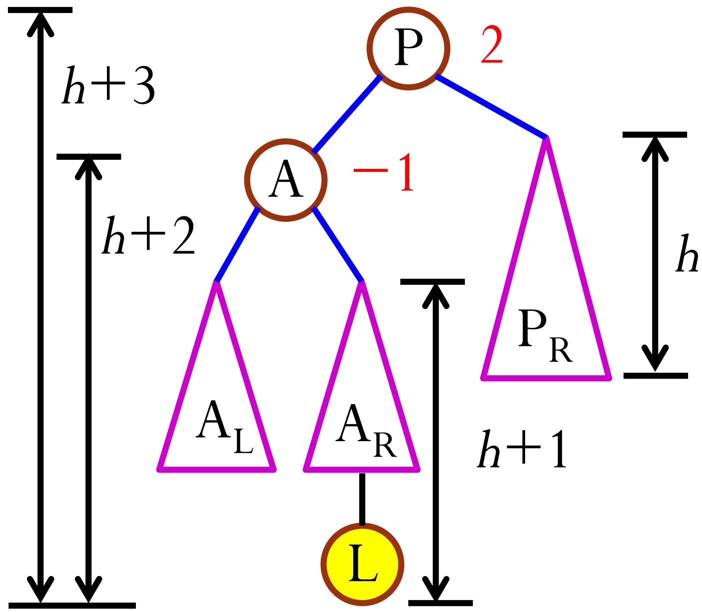


insert a new item

Left-right insertion: the first edge in the insertion path goes to the left and the second edge goes to the right.

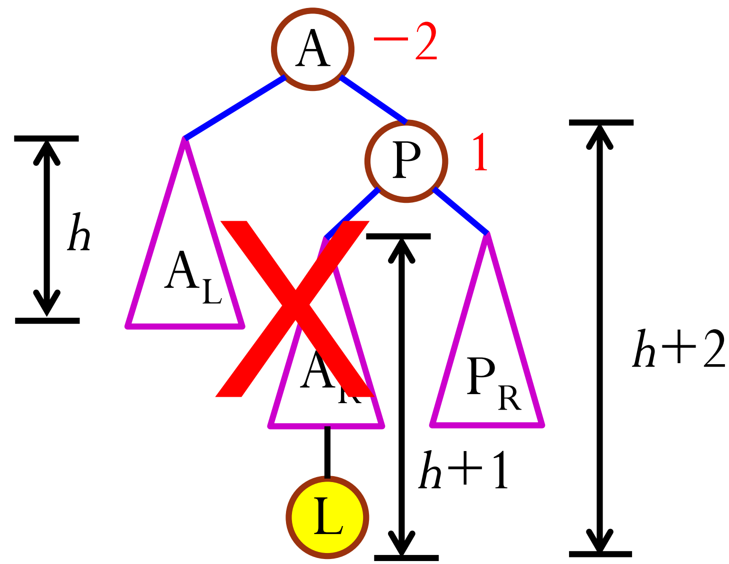
Restoring AVL Balance Condition

Left-Right Insertion

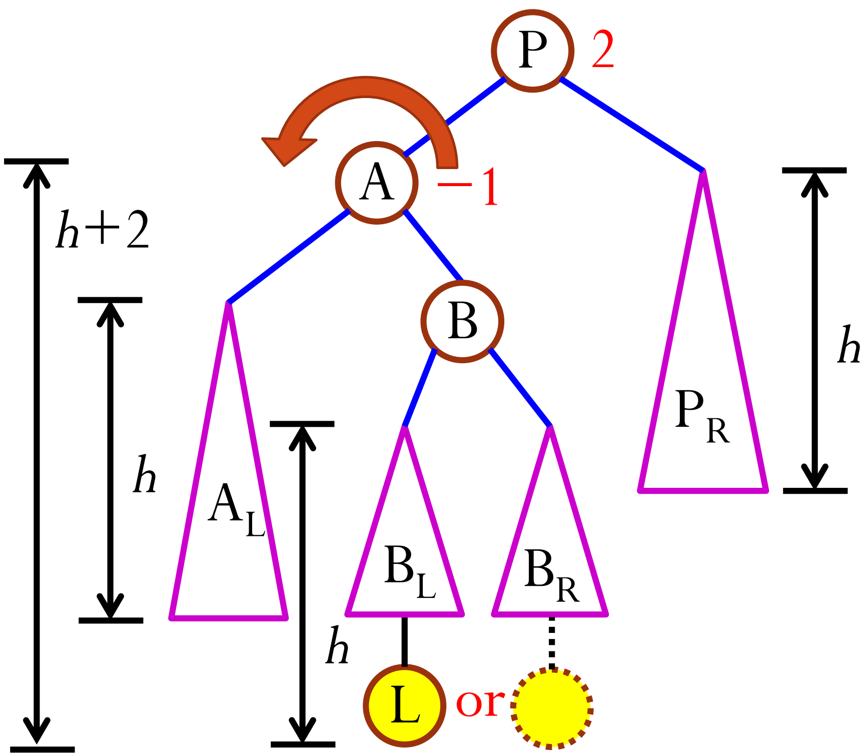


How to restore AVL balance?

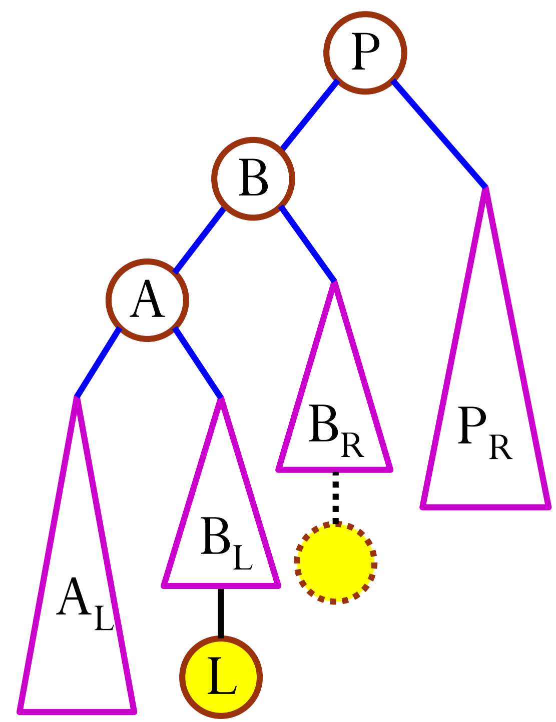
A right rotation at node P does not work!



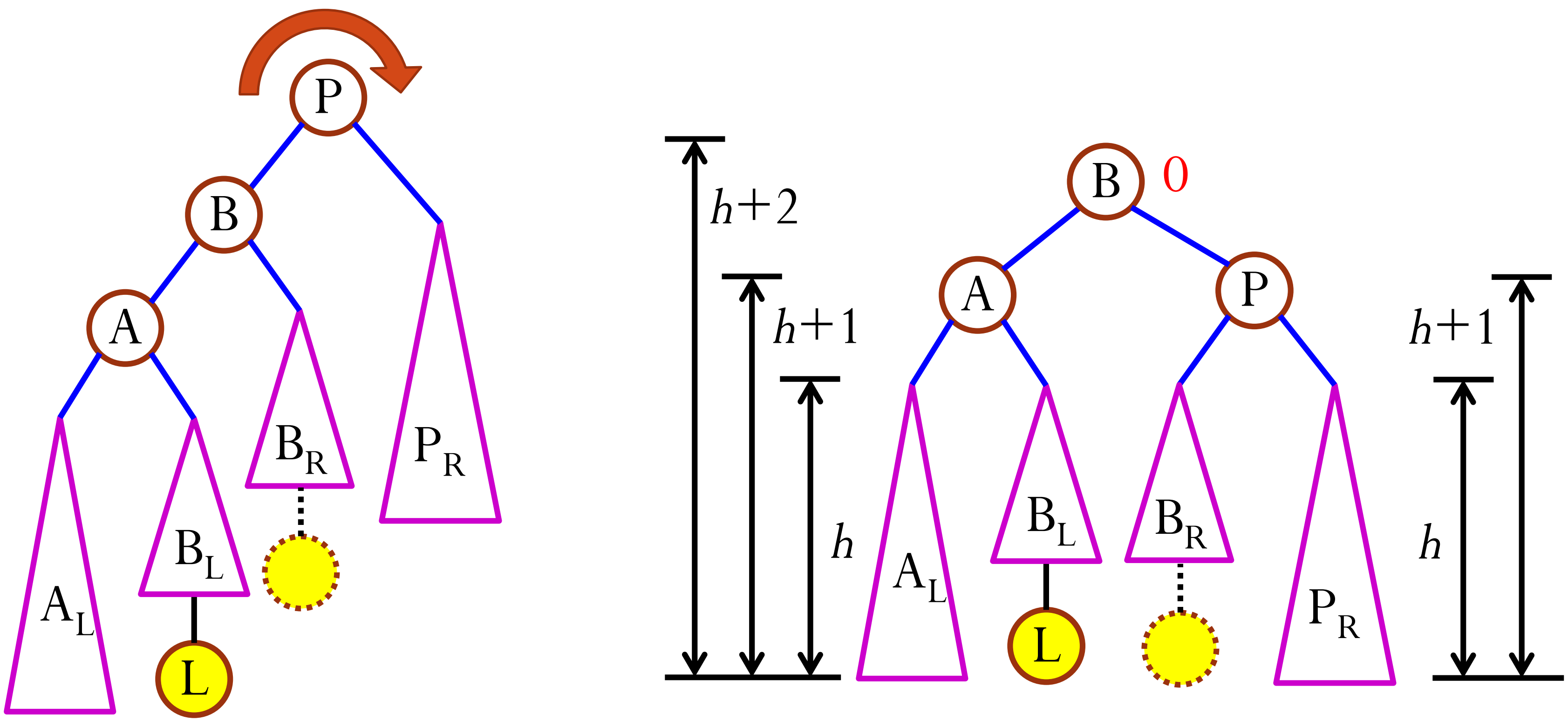
Left-Right (LR) Rotation



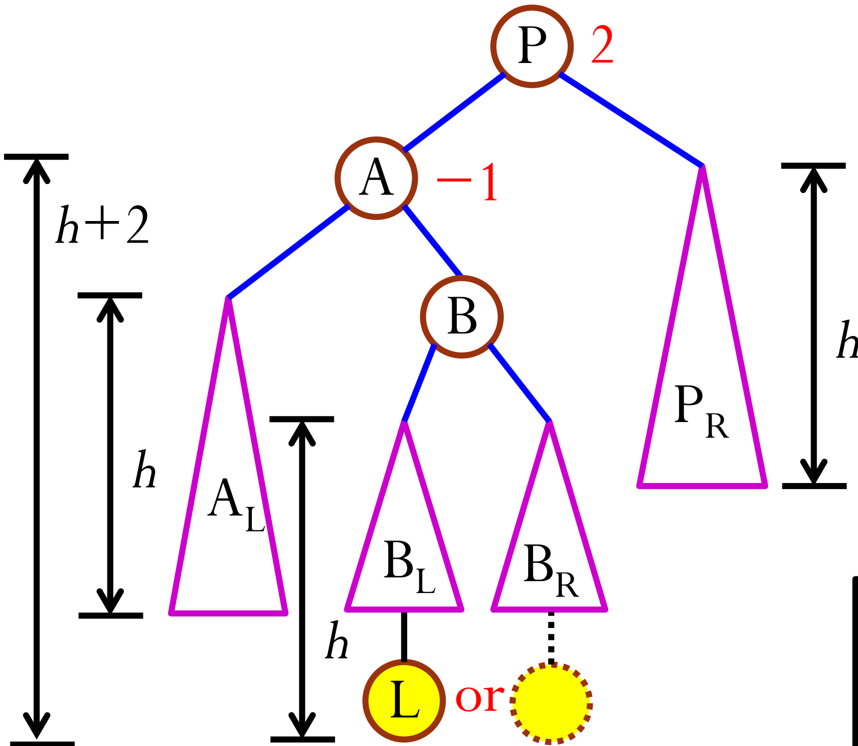
A **double rotation** to re-balance:
Do a **left** rotation on node A;
then a **right** rotation on node P
(next slide).



Left-Right (LR) Rotation



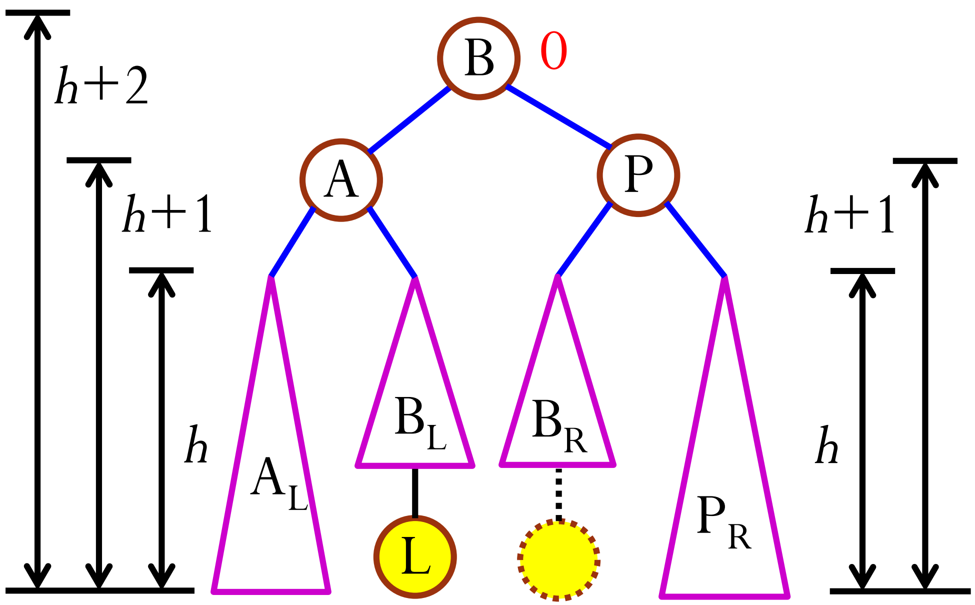
Left-Right (LR) Rotation



An **LR rotation** is called for when the node becomes unbalanced with a **positive** balance factor but the left subtree of the node has a **negative** balance factor.

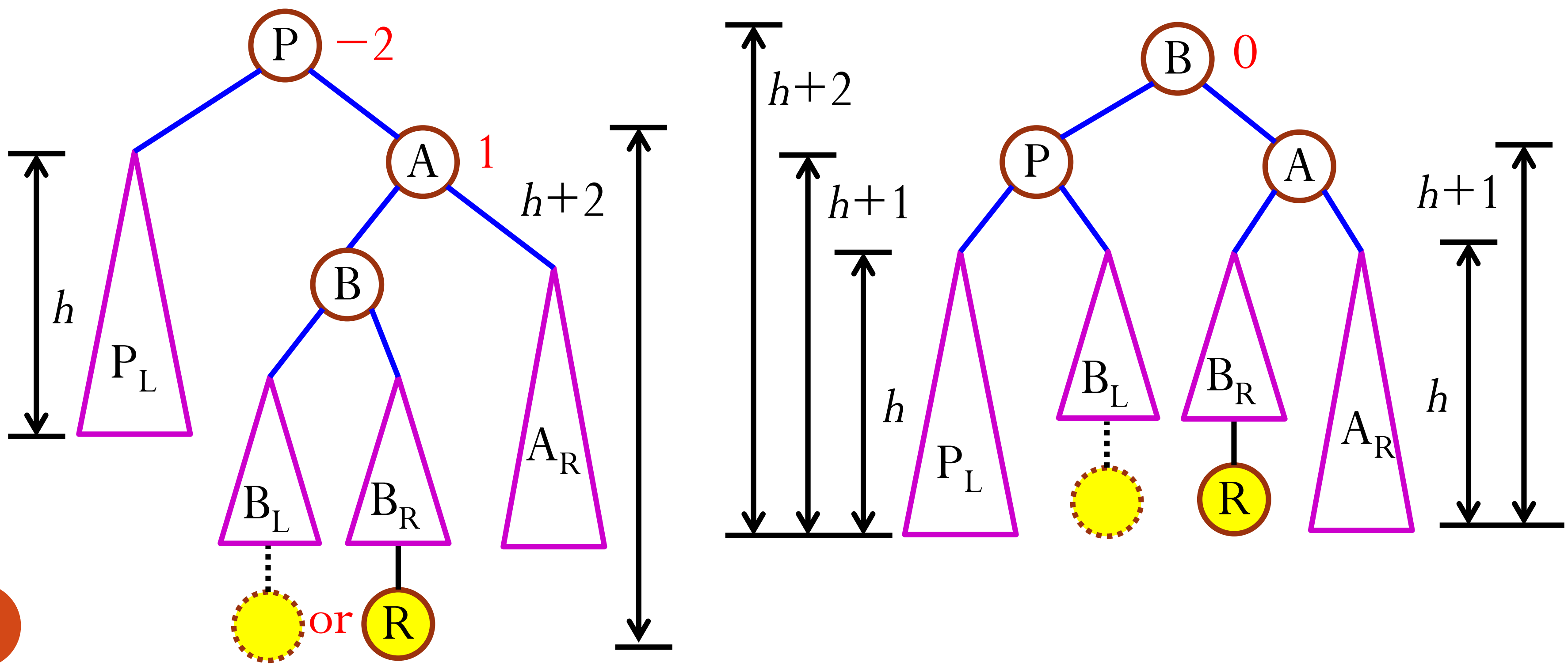
Properties of Left-Right Rotation

- The ordering property of BST is kept.
- Node B has a balance factor of 0.
- The height of the tree **after the rotation** is the same as the height of the tree before insertion.



Right-Left (RL) Rotation

- Symmetric to left-right rotation; also a double rotation.
- An **RL rotation** is called for when the node becomes unbalanced with a **negative** balance factor but the right subtree of the node has a **positive** balance factor.



Rotation Summary

- When an AVL tree becomes unbalanced, there are four cases to consider depending on the **direction** of the first two edges on the insertion path from the **unbalanced node**:
  - Left-left LL Rotation
  - Right-right RR Rotation
  - Left-right LR Rotation
  - Right-left RL Rotation

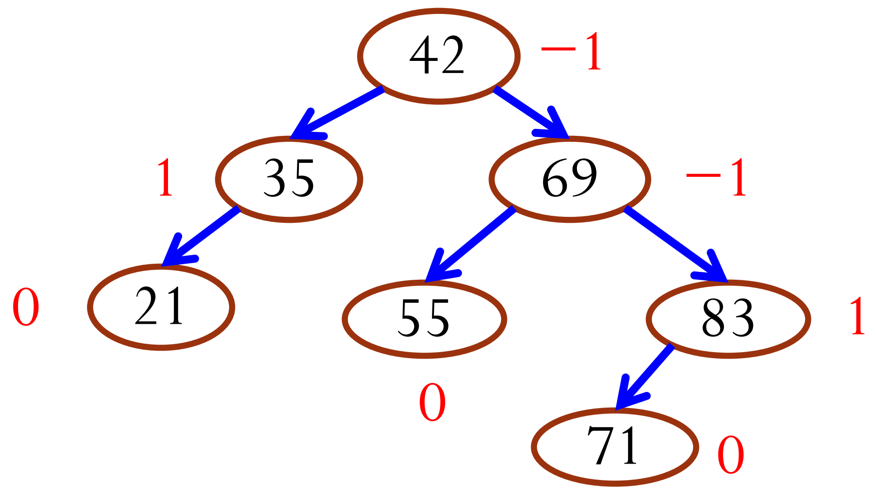
LL Rotation, RR Rotation: single rotation

LR Rotation, RL Rotation: double rotation

Note: We fix **the first unbalanced node** in the access path **from the leaf**.

Exercises

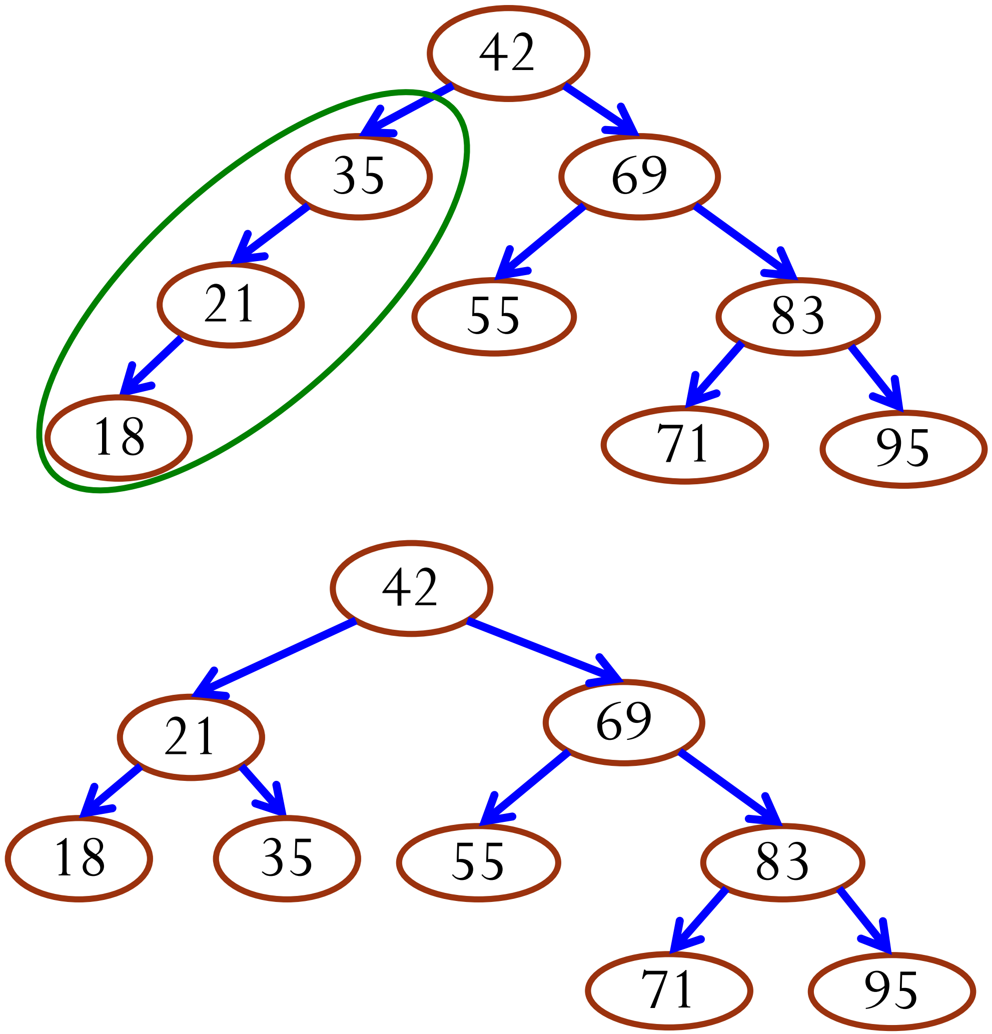
- Insert into an empty BST: 42, 35, 69, 21, 55, 83, 71.
  - Compute the balance factors.
  - Is the tree AVL balanced?



- Insert 95, 18, 75?

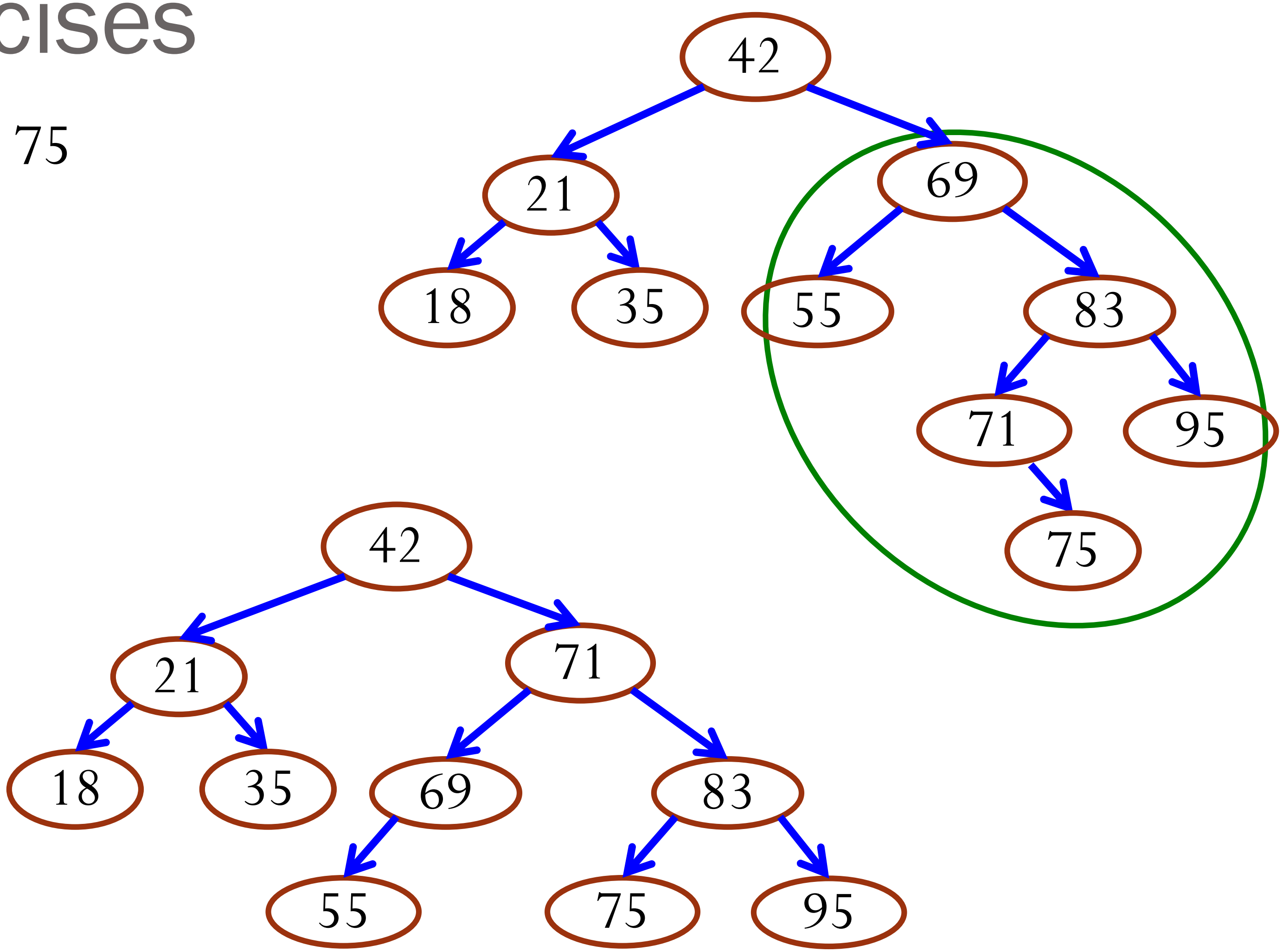
Exercises

- Insert 95, 18



Exercises

- Insert 75



The Number of Rotations Required

- When an AVL tree **becomes unbalanced after an insertion**, **exactly one** single or double rotation is required to balance the tree.
  - Before the insertion, the tree is balanced.
  - Only nodes on the access path of the insertion can be unbalanced. All other nodes are balanced.
  - We rotate at the first unbalanced node **from the leaf**.
  - By the properties of rotation, the height of the node after rotation is the same as that before insertion.
  - All ancestors of that node on the access path should now be balanced.

Outline

- Balanced Search Trees
  - AVL Trees
- AVL Tree Insertion
- Supporting Data Members and Functions of AVL Tree

AVL Trees

Supporting Data Members and Functions

```
struct node {
   Item item;
   int height;
   node *left;
   node *right;
};
```

```
int Height(node *n) {
   if(!n) return -1;
   return n->height;
}

void AdjustHeight(node *n) {
   if(!n) return;
   n->height = max( Height(n->left),
      Height(n->right) ) + 1;
}

int BalFactor(node *n) {
   if(!n) return 0;
   return (Height(n->left) -
      Height(n->right));
}
```

AVL Trees

Supporting Functions

```
void LLRotation(node *&n);
void RRRotation(node *&n);
void LRRotation(node *&n);
void RLRotation(node *&n);


void Balance(node *&n) {
   if(BalFactor(n) > 1) {
      if(BalFactor(n->left) > 0) LLRotation(n);
      else LRRotation(n);
   }
   else if(BalFactor(n) < -1) {
      if(BalFactor(n->right) < 0) RRRotation(n);
      else RLRotation(n);
   }
}
```

AVL Trees

Changes to Insertion

```
void insert(node *&root, Item item)
{
  if(root == NULL) {
    root = new node(item);
    return;
  }
  if(item.key < root->item.key)
    insert(root->left, item);
  else if(item.key > root->item.key)
    insert(root->right, item);

  Balance(root);
  AdjustHeight(root);
}
```

Removal

- First remove node as with BST

- Then update the balance factors of those ancestors in the access path and rebalance as needed.