Problem Solving with AI Techniques
Informed Search

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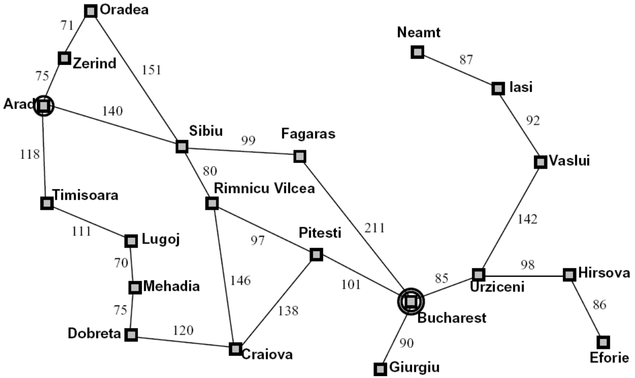
UM-SJTU Joint Institute

VE593, Fall 2018

Examples


Oradea
Zerind
Arad
Sibiu
Fagaras
Timisoara
Rimnicu Vilcea
Lugoj
Pitesti
Mehadia
Dobreta
Craiova
Neamt
Iasi
Vaslui
Urziceni
Hirsova
Bucharest
Giurgiu
Eforie
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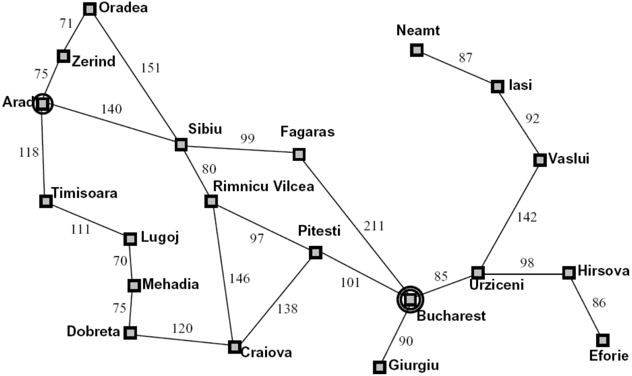
Examples



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+ & & M & O & R & E \\
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$$

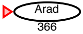
Heuristic Search

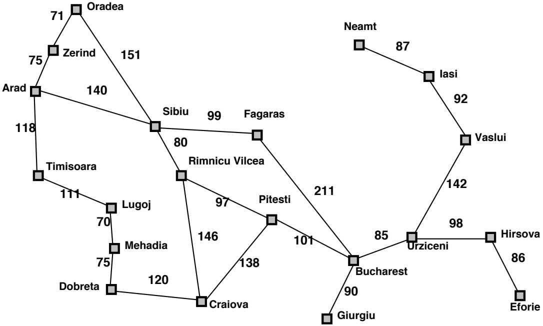
- Heuristic method: technique that can provide a feasible solution, but doesn't generally have any optimality guarantee
- Takes advantage of extra knowledge about goal/problem
- Extra knowledge in the form of a heuristic function h
  - Common sense rules intended to increase probability of solving
  - "Rules of thumb"
  - h estimates cost to reach goal
    e.g., h_{SLD} = straight-line distance from n to Bucharest
- Guides search by heuristic function h

Principle of Greedy Best-First Search

- Strategy: expand node that appear the closest to goal (as measured by h)
- Implementation: same as uniform-cost search using h instead of path costs (i.e., $g(n)$ = cost-so-far to reach a node n)

Example of Greedy Best-First Search





Straight–line distance to Bucharest

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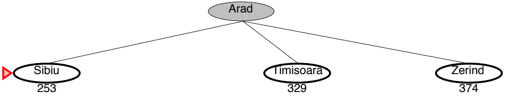

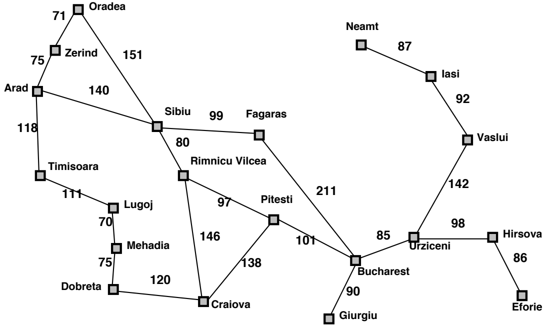

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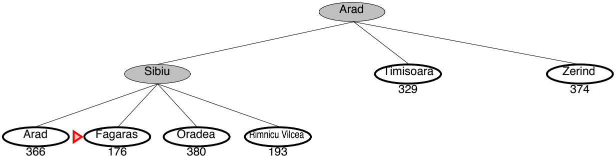

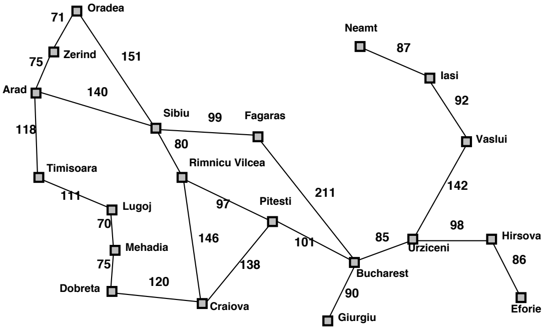

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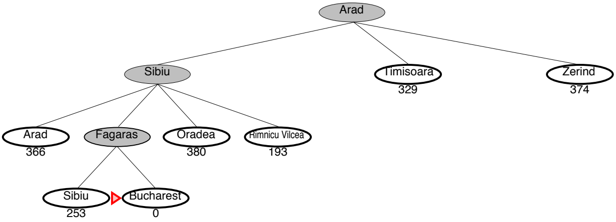

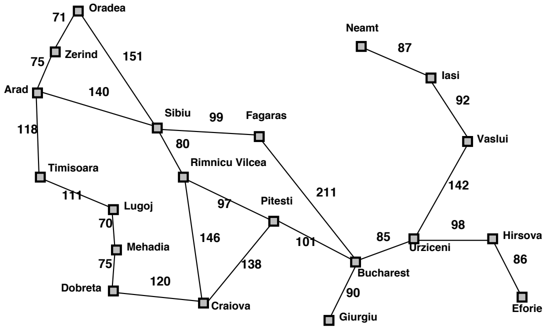

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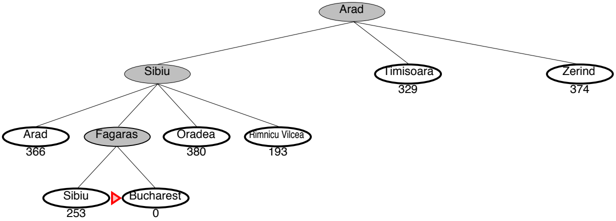

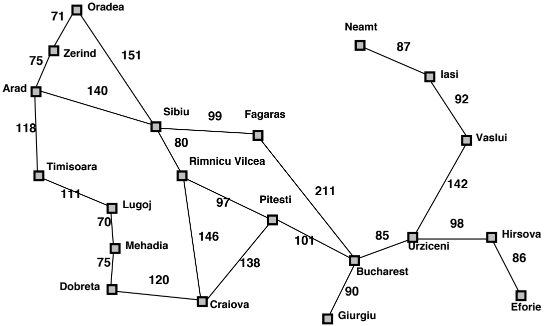

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But, going via Rimnicu Vilcea is **shorter**!!

Properties of Greedy Best-First Search

- Complete? No, can get stuck in loops, e.g., with Oradea as goal
  Iasi $\rightarrow$ Neamt $\rightarrow$ Iasi $\rightarrow$ Neamt $\rightarrow$
- Time? $\mathcal{O}(b^m)$, but good heuristic can give dramatic improvement
- Space? $\mathcal{O}(b^m)$, keeps all nodes in memory
- Optimal? No

- Greedy best-first search doesn't care about the "past" (the cost-so-far)

Principle of A* Search

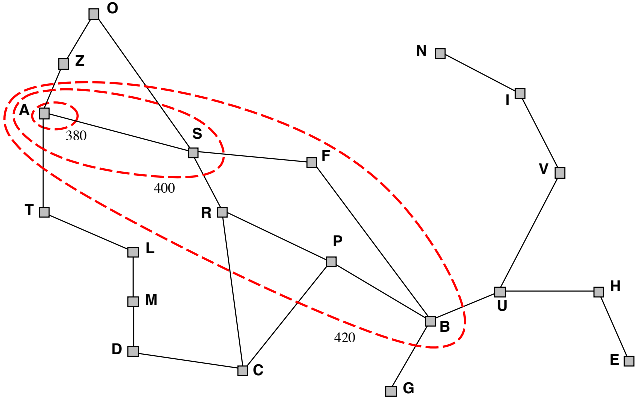
- Greedy best-first search doesn't exploit the information of $g(n)$
- Idea: avoid expanding paths that are already expensive
- Strategy: expand node that appears to be promising wrt

$$f(n) = g(n) + h(n) \quad \text{where}$$

  - $g(n) =$ cost-so-far to reach n
  - $h(n) =$ estimated cost-to-go from n
  - $f(n) =$ estimated total cost of path through n to goal
- Implementation: same as uniform-cost search using $f(n)$

Intuition of A*

- A* expands nodes in order of increasing f value
- Gradually adds "f-contours" of node
- Contour i has all nodes with $f = f_i$ where $f_i < f_{i+1}$



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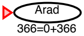

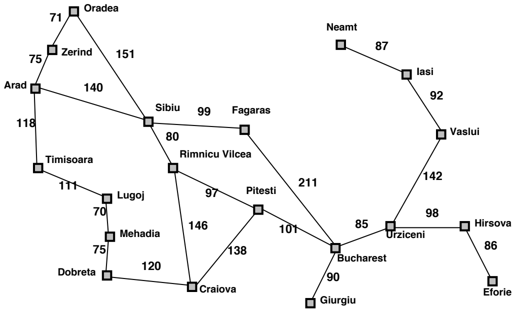

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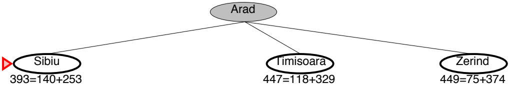

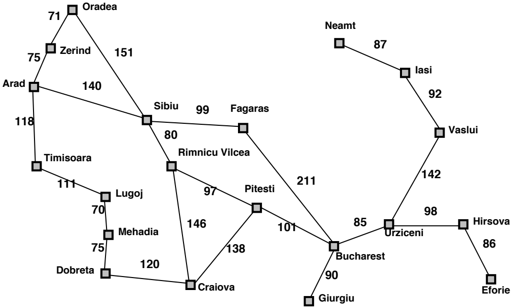

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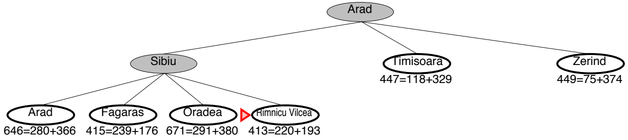

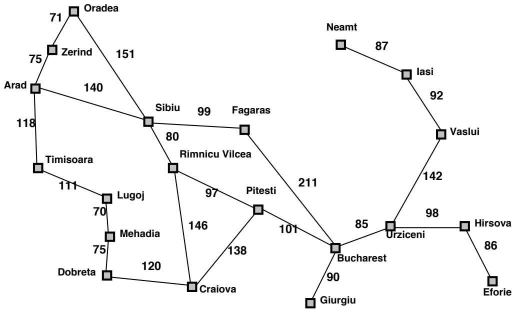

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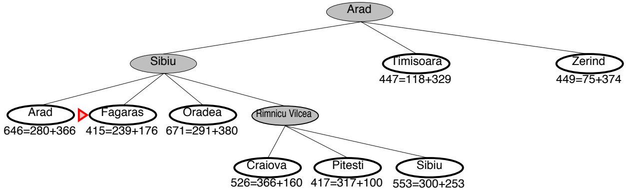

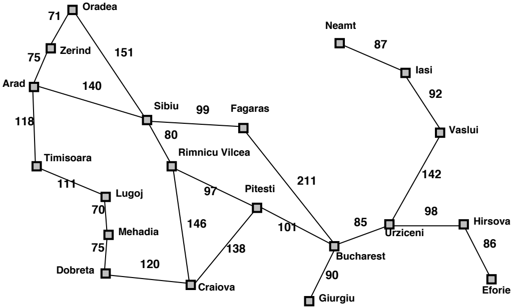

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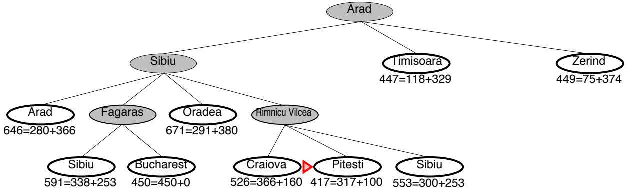

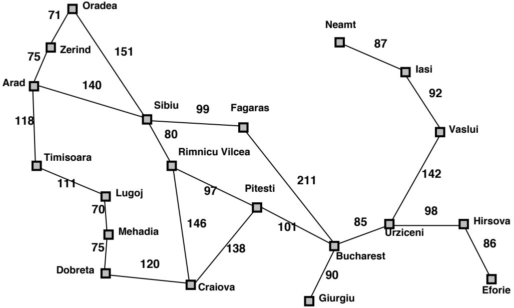

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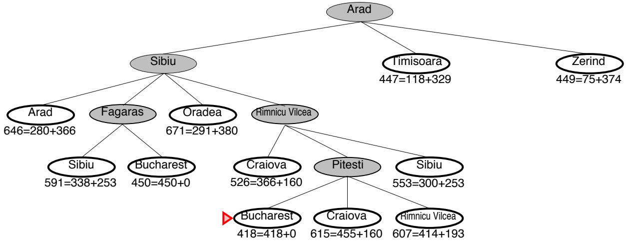

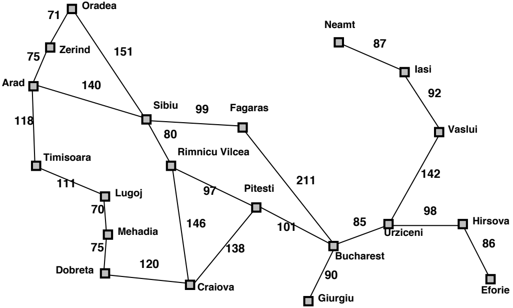

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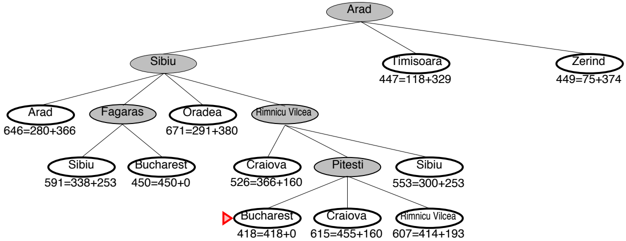

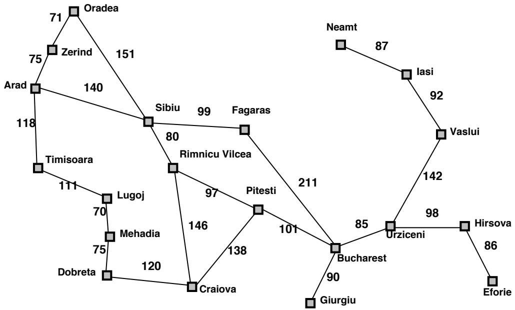

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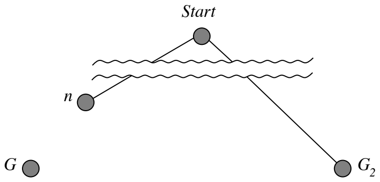
But, is A* always optimal?

Admissible Heuristics

- Heuristic $h(n) \geq 0$ is admissible if for every node n, $h(n) \leq h^*(n)$ where $h^*(n)$ is true cost to reach goal from n
- Admissible heuristic never overestimates cost to reach goal, i.e., it is optimistic
- Example: $h_{SLD}(n)$ never overestimates actual road distance. Why?
- Theorem: If $h(n)$ is admissible, A* using `TREE-SEARCH` is optimal

Proof of Optimality of A*

Suppose some suboptimal goal G_2 has been generated and is in the fringe. Let n be an unexpanded node in the fringe such that n is on the shortest path to an optimal goal G.



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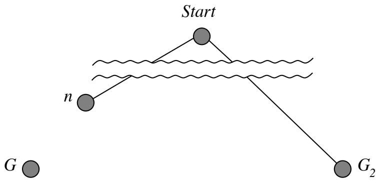

- $f(G_2) = g(G_2)$

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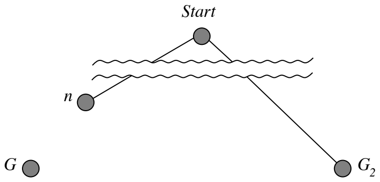

- $f(G_2) = g(G_2)$ since $h(G_2) = 0$

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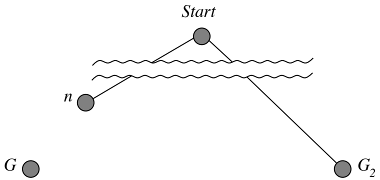

- $f(G_2) = g(G_2)$ since $h(G_2) = 0$
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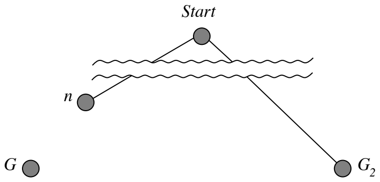

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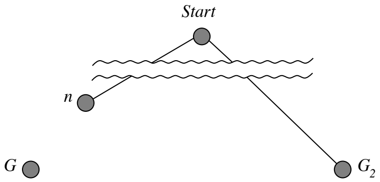

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- $f(G) = g(G)$

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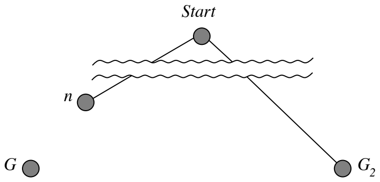

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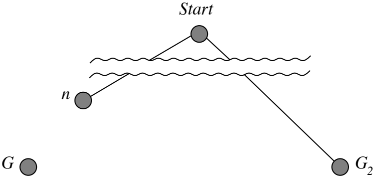

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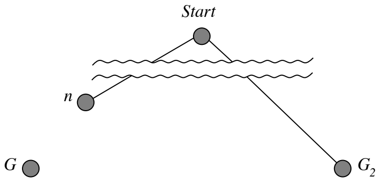

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- $f(G) = g(G)$ since $h(G) = 0$
- $f(G_2) > f(G)$ from above

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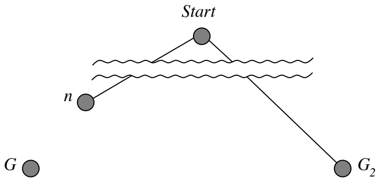

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- $f(G_2) > f(G)$ from above
- $h(n) \leq h^*(n)$

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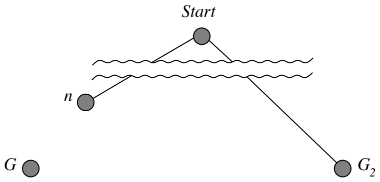

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- $f(G) = g(G)$ since $h(G) = 0$
- $f(G_2) > f(G)$ from above
- $h(n) \leq h^*(n)$ since h is admissible

Proof of Optimality of A*

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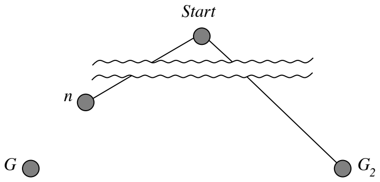

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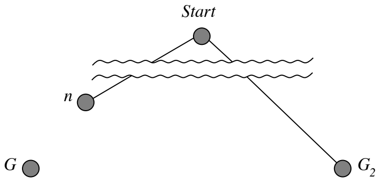

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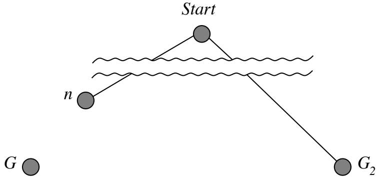

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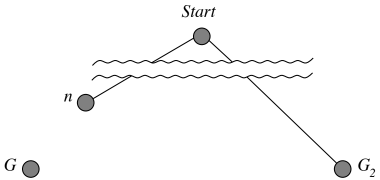

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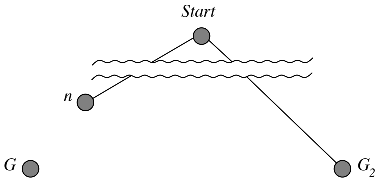

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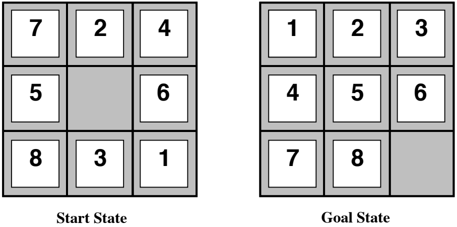
Hence $f(G_2) > f(n)$ and A* would never select G_2 for expansion.

Properties of A* Search

- Complete? Yes, unless there are infinitely many nodes n with $f(n) \leq f(G) = c^*$
- Time? Exponential in [relative error in $h \times$ length of soln]
- Space? Exponential, keeps all nodes in memory
- Optimal? Yes

- A* expands all nodes with $f(n) \leq c*$
- A* expands some nodes with $f(n) = c*$
- A* expands no nodes with $f(n) > c*$

Example for the 8-puzzle



- $h_1(n) = \#$ of misplaced tiles

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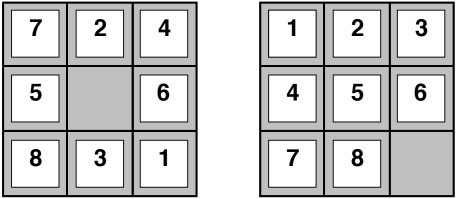

Start State **Goal State**

- $h_1(n) = \#$ of misplaced tiles
- $h_2(n) =$ total Manhattan distance, i.e., $\#$ of squares from desired location of each tile

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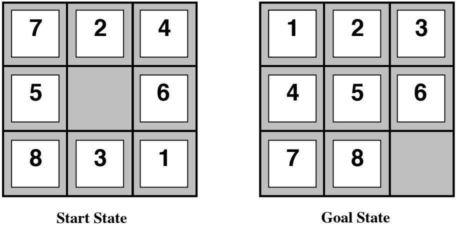

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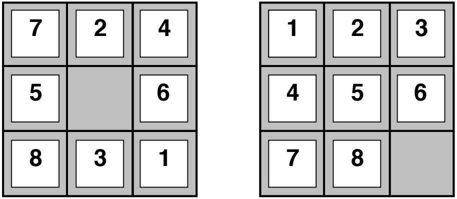

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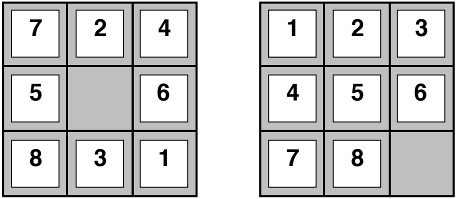

Start State Goal State

- $h_1(n) = \#$ of misplaced tiles
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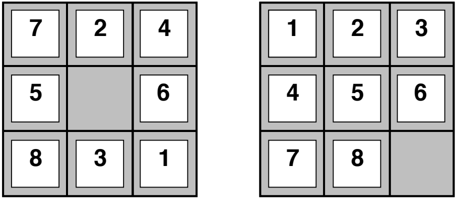
- $h_2(\text{start}) =$

Example for the 8-puzzle



Start State **Goal State**

- $h_1(n) = \#$ of misplaced tiles
- $h_2(n) =$ total Manhattan distance, i.e., # of squares from desired location of each tile

- $h_1(\text{start}) = 6$
- $h_2(\text{start}) = 4+0+3+3+1+0+2+1$
  $=14$

Dominance Relation Between Heuristics

- h_2 dominates h_1 if $\forall n, h_2(n) \geq h_1(n)$ (both admissible)
- h_2 better for search
  e.g., in previous example, h_2 dominates h_1
- Typical search costs (average # of nodes expanded)

d=12	IDS = 3,644,035 nodes
	A*(h_1) = 227 nodes
	A*(h_2) = 73 nodes
d=24	IDS = too many nodes
	A*(h_1) = 39,135 nodes
	A*(h_2) = 1,641 nodes

- Given any two admissible heuristics h_a and h_b,

$$h(n) = \max(h_a(n), h_b(n))$$

  is also admissible and dominates h_a, h_b

Relaxed Problems

- Problem with fewer restrictions on actions is called relaxed problem
- Cost of an optimal solution to relaxed problem = admissible heuristic for original problem
- h_1 optimal cost if rules of 8-puzzle relaxed so that a tile can move anywhere
- h_2 optimal cost if rules of 8-puzzle relaxed so that a tile can move to any adjacent square
- Key point: the optimal solution cost of a relaxed problem is no greater than the optimal solution cost of the real problem

1. A Priori Knowledge

2. Heuristic Search
   - Motivation
   - Greedy Best-First Search
   - A* Search
   - How to Define Good Heuristics?
   - Variants of A*

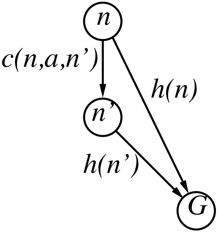
Consistent Heuristics

- Heuristic h is consistent if for every node n, every successor n' of n generated by any action a,

$$h(n) \leq cost(n, a, n') + h(n')$$

- if h is consistent, we have

$$\begin{aligned} f(n') &= g(n') + h(n') \\ &= g(n) + c(n, a, n') + h(n') \\ &\geq g(n) + h(n) = f(n) \end{aligned}$$



- i.e., $f(n)$ is non-decreasing along any path (hence, monotone)
- Theorem: If $h(n)$ is consistent, A* using `GRAPH-SEARCH` is optimal
Keeps all checked nodes in memory to avoid repeated states.

Principle of Simple Memory-Bounded A* (SMA*)

- As with BFS, A* has exponential space complexity
- Strategy:
  - Expand as usual until a memory bound is reach
  - Then, whenever adding a node, remove the *worst* node n' from tree
  - Worst means: n' with highest $f(n')$
  - To not lose information, *backup* the measured step-cost $cost(\tilde{n}, a, n')$ to improve the heuristic $h(\tilde{n})$ of its parent
- Properties: complete and optimal if the depth of the optimal path is within the memory bound