Problem Solving with AI Techniques
Refresher on Probability Theory

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UM-SJTU Joint Institute

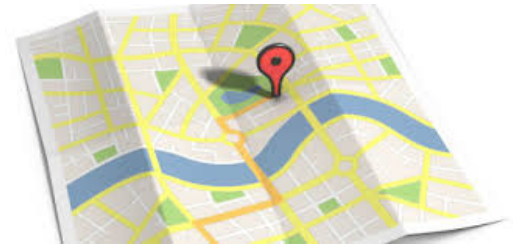
VE593, Fall 2018

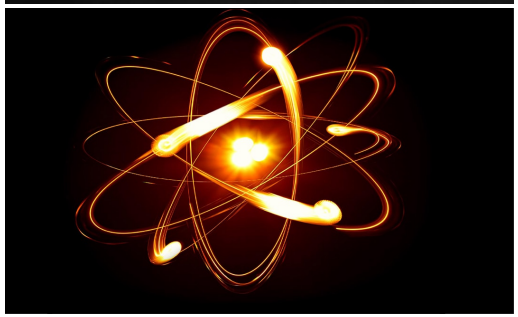
For more, see VE401 and VE501

1. Introduction of Probability
2. Formal Definitions
3. Family of Probability of Distributions
4. Some Notions From Information Theory

Why is there Uncertainty?

- Knowledge is generally uncertain because
  - Full vs partial/non observable world
  - Observation comes from imperfect sensors
  - Data often imprecise, missing or contradictory
  - World is stochastic?
- Uncertainty can be handled with probability distribution
- Other uncertainty models, e.g., possibility theory, belief functions





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Proposition	Belief	Bet	Stakes
a	.4	a	4 to 6
b	.3	b	3 to 7
$a \vee b$	.8	$\neg(a \vee b)$	2 to 8

- Empirical Justification
  - Many success stories in gambling, finance, engineering, machine learning...

Interpretation of Probability

- Objective Probability
  - probability = objective property of objects like mass
- Frequentist Interpretation
  - probability = limit of observed frequences
- Subjective Interpretation
  - probability = beliefs of an agent

Probability

- Sample space/domain Ω
  Sample point/possible world/atomic event $\omega \in \Omega$
  Event $A \subseteq \Omega$
  e.g., Roll of 6-face die: $\Omega = \{1, 2, 3, 4, 5, 6\}$

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- Probability $\mathbb{P} : A \subseteq \Omega \mapsto [0, 1]$
  e.g., $\mathbb{P}(\{1\}) = \frac{1}{6}$, $\mathbb{P}(\{2, 4\}) = \frac{1}{3}$, $\mathbb{P}(\{3, 5, 6\}) = \frac{1}{2}$

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- Kolmogorov Axioms: $\forall A, B \subseteq \Omega$
  - Nonnegativity $\mathbb{P}(A) \geq 0$
  - Normalization $\mathbb{P}(\Omega) = 1$
  - Additivity $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B)$ if $A \cap B = \{\}$

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- Implications
  - $0 \leq \mathbb{P}(A) \leq 1$
  - $\mathbb{P}(\{\}) = 0$
  - $A \subseteq B \Rightarrow \mathbb{P}(A) \leq \mathbb{P}(B)$
  - $\mathbb{P}(\Omega \backslash A) = 1 - \mathbb{P}(A)$
  - $\mathbb{P}(\cap_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mathbb{P}(A_i)$ if A_i's are disjoint

Random variable

- Random variable X: *measurable* function from Ω to some *measurable* space, e.g., $\mathbb{R}$ or $\{true, false\}$

  e.g., $X : \begin{cases} \{1,2,3,4,5,6\} \to \{1,2,3,4,5,6\} \\ x \mapsto x \end{cases}$, $Odd(\omega)$

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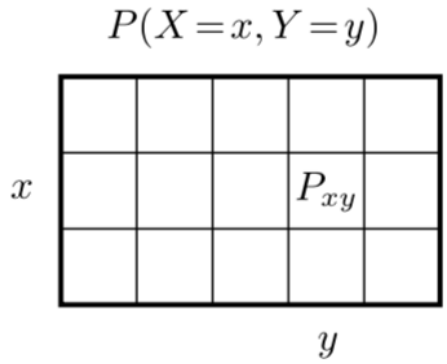
- For finite space, think of $\mathbb{P}(X)$ as a table
- Notation $\sum_X \mathbb{P}(X) = \sum_x \mathbb{P}(X = x)$

Joint Distribution

Assume we have two random variables X and Y

- Joint $\mathbb{P}(X, Y)$



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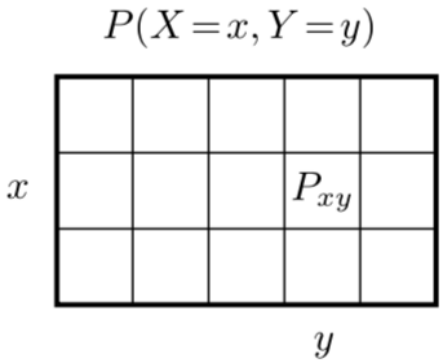

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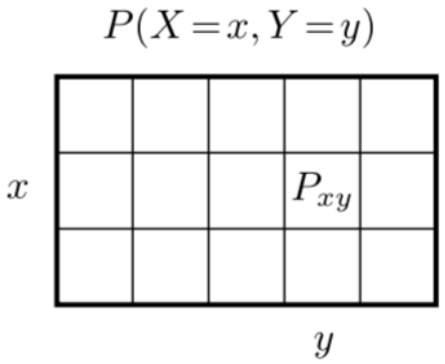
  What is the dimension of $\mathbb{P}(X \mid Y)$?
  What is the value of $\sum_X \mathbb{P}(X \mid Y)$?



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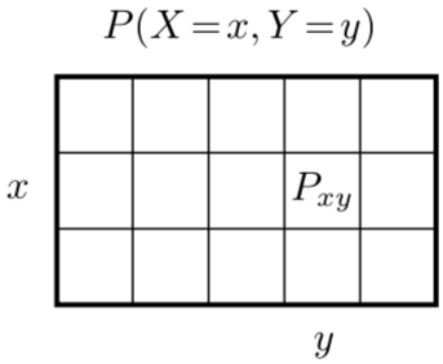
  What is the dimension of $\mathbb{P}(X \mid Y)$?
  What is the value of $\sum_X \mathbb{P}(X \mid Y)$?
- X is independent of Y iff $\mathbb{P}(X \mid Y) = \mathbb{P}(X)$

  What does it mean for $\mathbb{P}(X \mid Y)$ and $\mathbb{P}(Y \mid X)$?



Important Rules

- Product rule

$$\mathbb{P}(X, Y) = \mathbb{P}(X \mid Y)\mathbb{P}(Y) = \mathbb{P}(Y \mid X)\mathbb{P}(X)$$

- Bayes' rule

$$\mathbb{P}(X \mid Y) = \frac{\mathbb{P}(Y \mid X)\mathbb{P}(X)}{\mathbb{P}(Y)}$$

$\mathbb{P}(X \mid Y)$ posterior $\mathbb{P}(X)$ prior

$\mathbb{P}(Y \mid X)$ likelihood $\mathbb{P}(Y)$ normalization

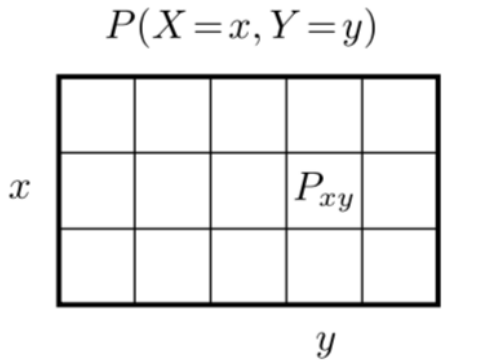
- Extended form

$$\mathbb{P}(X \mid Y) = \frac{\mathbb{P}(Y \mid X)\mathbb{P}(X)}{\sum_X \mathbb{P}(Y \mid X)\mathbb{P}(X)}$$

Joint Distribution: General Case

Assume we have n random variables $X_{1:n}$

- Joint $\mathbb{P}(X_{1:n})$
- Marginal $\mathbb{P}(X_1) = \sum_{X_{2:n}} \mathbb{P}(X_{1:n})$
- Conditional $\mathbb{P}(X_1 \mid X_{2:n}) = \frac{\mathbb{P}(X_{1:n})}{\mathbb{P}(X_{2:n})}$
- X is independent of Y given Z iff $\mathbb{P}(X \mid Y, Z) = \mathbb{P}(X \mid Z)$



$\Downarrow$

n-th order tensor

Expectation

- Discrete case: $\mathbb{E}[X] = \sum_x x\mathbb{P}(X = x)$
- Continuous case: $\mathbb{E}[X] = \int_x xp(x)dx$

- More generally, $\mathbb{E}[f(X)]$ for some function f
- $\mathbb{E}[\boldsymbol{X}]$ if $\boldsymbol{X} \in \mathbb{R}^n$

- Operator $\mathbb{E}$ is linear:
  - $\mathbb{E}(\lambda X) = \lambda \mathbb{E}(X)$
  - $\mathbb{E}(X + Y) = \mathbb{E}(X) + \mathbb{E}(Y)$

Variance

- $\mathbb{V}[X] = \mathbb{E}[(X - \mathbb{E}[X])^2]$
- Covariance $cov(X, Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$
- If X, Y are independent, then $cov(X, Y) = 0$. But opposite generally not true!
- Useful identities:
  - $\mathbb{V}(\lambda X) = \lambda^2 V(X)$
  - $\mathbb{V}(X + Y) = \mathbb{V}(X) + 2cov(X, Y) + \mathbb{V}(Y)$
- If $\boldsymbol{X} \in \mathbb{R}^n$, $\mathbb{V}[\boldsymbol{X}] = \mathbb{E}[(\boldsymbol{X} - E[\boldsymbol{X}])(\boldsymbol{X} - E[\boldsymbol{X}])^\intercal]$

Bernoulli & Binomial Distribution

- Random variable $X \in \{0, 1\}$ follows a Bernoulli distribution $Bern(p)$

$$\mathbb{P}(X = 1 \mid p) = p, \mathbb{P}(X = 0 \mid p) = 1 - p, \quad \mathbb{P}(X = x \mid p) = p^x (1-p)^{1-x}$$

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- Dataset of i.i.d. random variables $D = (X_1, X_2, \ldots, X_n)$ where $X_i \sim Bern(p)$

$$\mathbb{P}(D = (x_1, \ldots, x_n) \mid p) = \prod_{i=1}^{n} p^{x_i}(1-p)^{1-x_i}$$

$$\arg\max_p \mathbb{P}(D \mid p) = \arg\max_p \sum_{i=1}^{n} x_i \ln p + (1 - x_i) \ln(1 - p) = \frac{1}{n} \sum_{i=1}^{n} x_i$$

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- Random variable $M = \sum_{i=1}^{n} X_i \sim$ Binomial distribution $Bin(n, p)$

$$\mathbb{P}(M = m \mid n, p) = \binom{n}{m} p^m (1-p)^{n-m}, \quad \binom{n}{m} = \frac{n!}{(n-m)!m!}$$

How to Express Uncertainty over a Bernoulli Parameter p?

- Beta distribution $Beta(\alpha, \beta)$ with $\alpha, \beta > 0 =$ distribution over $[0, 1]$

$$Beta(p \mid \alpha, \beta) = \frac{1}{B(\alpha, \beta)} p^{\alpha-1}(1-p)^{\beta-1} \quad \text{with mean } \frac{\alpha}{\alpha+\beta}$$

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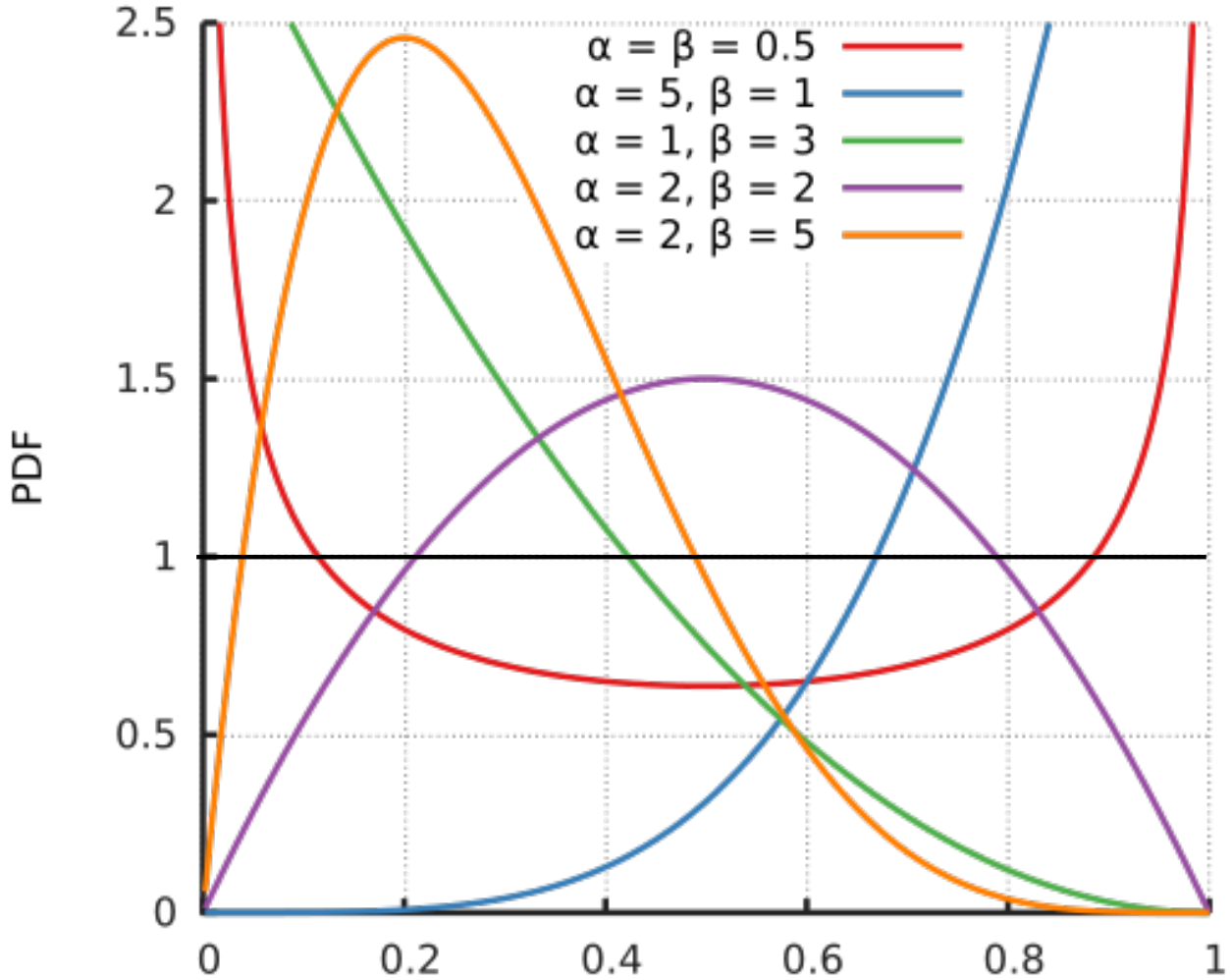
- After observing $D = (x_1, \ldots, x_n)$, with counts $a = \sum_i x_i$ and $b = \sum_i (1 - x_i)$, the belief about p can be updated:

$$\begin{aligned}
\mathbb{P}(p \mid D) &= \frac{\mathbb{P}(D \mid p)\mathbb{P}(p)}{\mathbb{P}(D)} \propto Bin(D \mid p) Beta(p \mid \alpha, \beta) \\
&\propto p^a (1-p)^b p^{\alpha-1}(1-p)^{\beta-1} = p^{\alpha-1+a}(1-p)^{\beta-1+b} \\
&= Beta(\alpha + a, \beta + b)
\end{aligned}$$

Examples of Beta Distribution



Categorical & Multinomial Distribution

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$$\mathbb{P}(D = (x_1, \dots, x_n) \,|\, \boldsymbol{p}) = \prod_{i=1}^{n} p_{x_i} = \prod_{i=1}^{n} \prod_{k=1}^{K} p_k^{[x_i = k]} = \prod_{k=1}^{K} p_k^{m_k}$$

where $m_k = \sum_{i=1}^{n} [x_i = k]$.

$$\arg\max_{\boldsymbol{p}} \mathbb{P}(D \,|\, \boldsymbol{p}) = \frac{1}{n}(m_1, m_2, \dots, m_K)$$

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- Random variable $\boldsymbol{M} = (\sum_{i=1}^{n} [X_i = k])_{k=1,\dots K}$ follows a multinomial distribution $Mult(n, \boldsymbol{p})$

$$\mathbb{P}(\boldsymbol{M} = (m_1, \dots, m_K) \,|\, n, \boldsymbol{p}) \propto \prod_{k=1}^{K} p_k^{m_k}$$

How to Express Uncertainty over Multinomial Parameter $\boldsymbol{p}$?

- Dirichlet distribution $Dir(\boldsymbol{\alpha})$ with $\alpha_k > 0$ = distribution over $(K-1)$-simplex: $\{\boldsymbol{p} \mid \boldsymbol{p} \geq 0, \sum_k p_k = 1\}$

$$Dir(\boldsymbol{p} \mid \boldsymbol{\alpha}) \propto \prod_{k=1}^{K} p_k^{\alpha_k - 1} \quad \text{with mean } \left(\frac{\alpha_k}{\sum_j \alpha_j}\right)_{k=1,\ldots,K}$$

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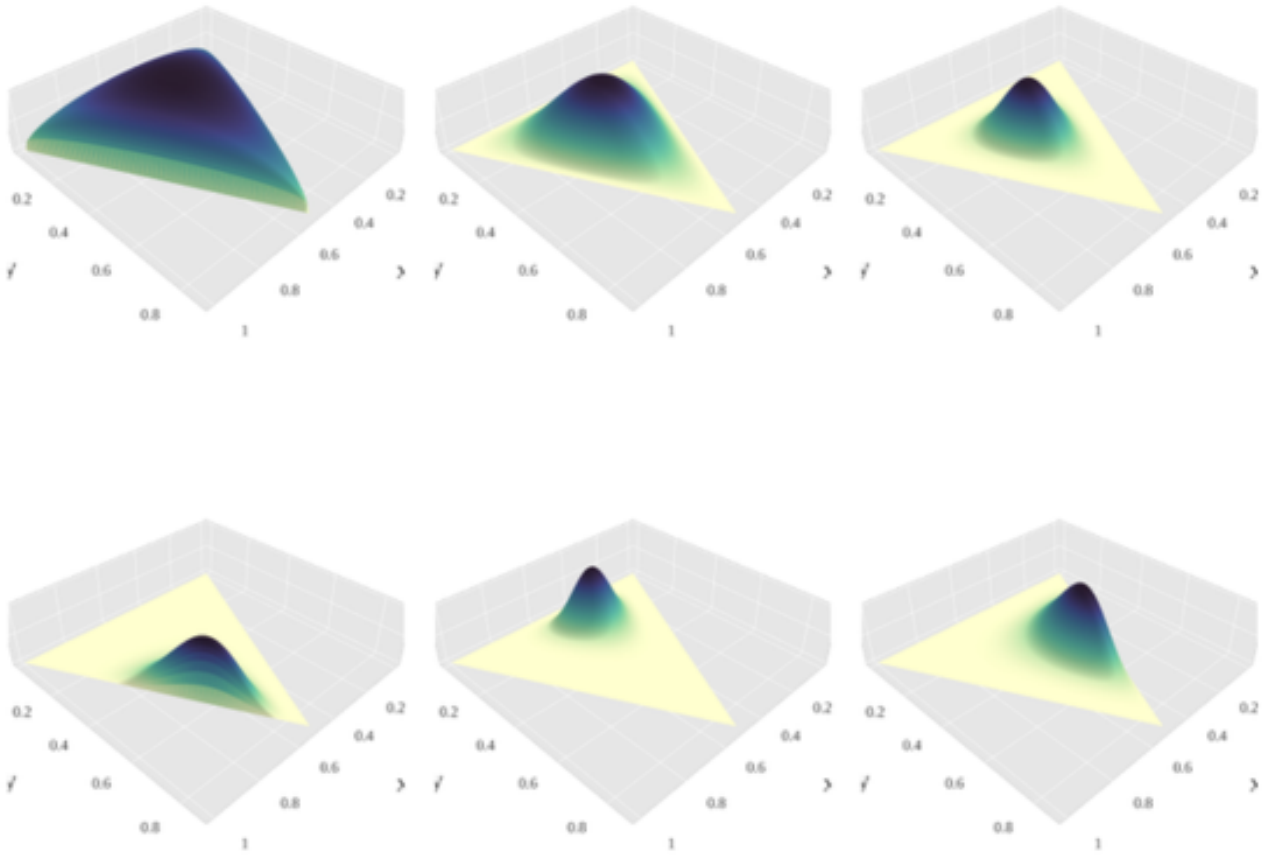
$$\begin{aligned}
\mathbb{P}(\boldsymbol{p} \mid D) &= \frac{\mathbb{P}(D \mid \boldsymbol{p})\mathbb{P}(\boldsymbol{p})}{\mathbb{P}(D)} \propto Mult(D \mid \boldsymbol{p}) Dir(\boldsymbol{p} \mid \boldsymbol{\alpha}) \\
&\propto \prod_{k=1}^{K} p_k^{a_k} \prod_{k=1}^{K} p_k^{\alpha_k - 1} = \prod_{k=1}^{K} p_k^{\alpha_k - 1 + a_k} \\
&= Dir(\boldsymbol{\alpha} + \boldsymbol{a})
\end{aligned}$$

Examples of Dirichlet Distribution



Other Discrete Distributions

- Uniform discrete distribution
- Geometric distribution: How many Bernoulli trials before a success?
- Negative binomial distribution: How many successes in a sequence of Bernoulli trials with a fixed number of failures?
- Poisson distribution: How many successes in a duration of time if they occur at known constant rate?

Distributions over Continuous Domains

- Probability for $X \in \mathbb{R}$ determined by probability density function $p(x) \in [0, \infty)$:

$$\mathbb{P}(X \in [a, b]) = \int_a^b p(x)dx \in [0, 1]$$

- Cumulative probability distribution:

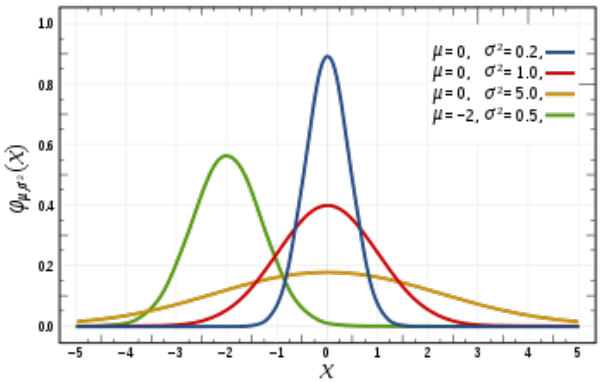
$$F(y) = \mathbb{P}(X \leq y) = \int_{-\infty}^{y} p(x)dx$$

- Note: for continuous probability distribution, $\mathbb{P}(X = x) = 0$

Gaussian (or Normal) Distribution

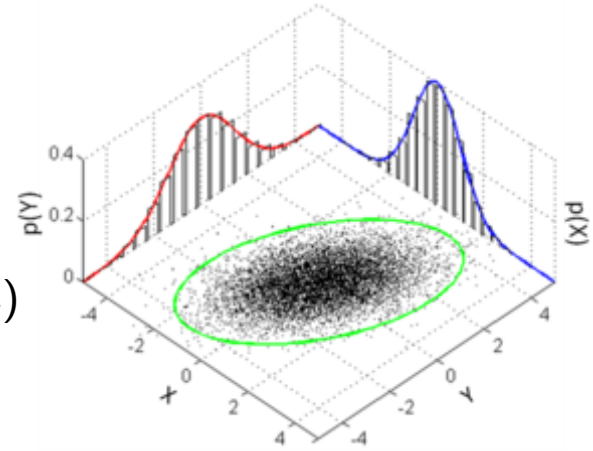
- on $\mathbb{R}$:

$$\mathcal{N}(x \mid \mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$



- on $\mathbb{R}^n$:

$$\mathcal{N}(\boldsymbol{x} \mid \boldsymbol{\mu}, \boldsymbol{\Sigma}) = \frac{1}{|2\pi\boldsymbol{\Sigma}|^{1/2}} e^{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu})^\intercal \boldsymbol{\Sigma}^{-1}(\boldsymbol{x}-\boldsymbol{\mu})}$$



Why are Gaussian Distributions Important?

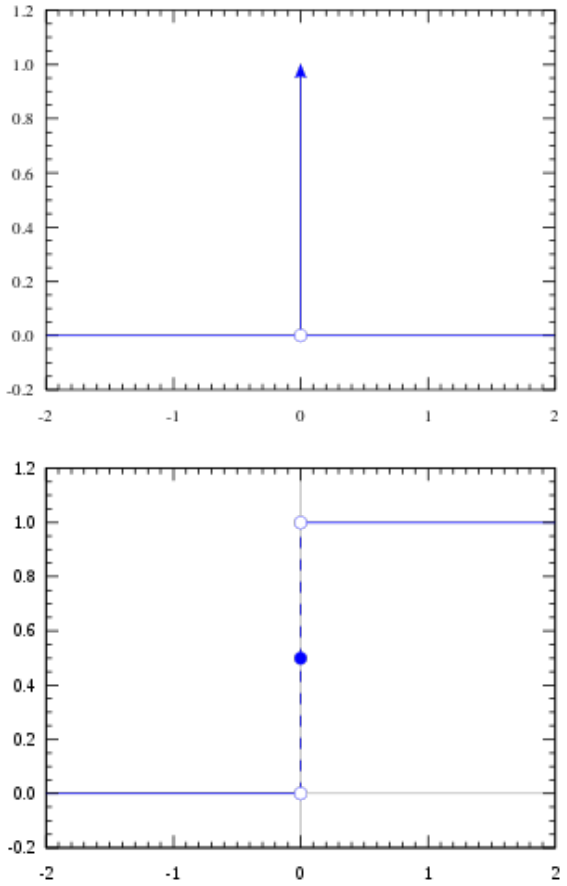
- Central limit theorem: Averages of n i.i.d random variables (with mean μ and variance σ^2) $\sim \mathcal{N}(\mu, \frac{\sigma^2}{n})$
- If only mean and variance are known, it is the distribution that maximizes *entropy*
- It makes math simpler, e.g., weighted sum of independent Gaussian r.v.s is also Gaussian

Dirac Distribution

- Dirac distribution $\delta(x) = 0$ except at $x = 0$ such that

$$\int \delta(x)dx = 1$$

- $\delta(x) = \frac{\delta}{\delta x} H(x)$ where $H(x) = [x \geq 0]$, Heaviside step function
- Limit of $\mathcal{N}(0, \frac{\sigma^2}{n})$ as $n \to \infty$
- Can represent certainty



Other Continuous Distributions

- Beta and Dirichlet distributions
- Continuous uniform distribution over a compact set (e.g., interval)
- Exponential distribution: How long before an event happens if it occurs at some known rate?
- Logistic distribution: distribution whose CDF is the logistic function $\frac{1}{1+e^{-\frac{x-\mu}{s}}}$
- χ^2 distribution: distribution of sum of n squared Gaussian

For more, see VE550

Entropy

- Neg-log of a distribution $(-\log p(x))$ reflects something like "error":
  - neg-log of Gaussian $\leftrightarrow$ squared error
  - neg-log of likelihood $\leftrightarrow$ prediction error
- Term $-\log p(x)$ is "optimal" coding length you should assign to symbol x. This will minimize the expected length of an encoding:

$$H(p) = \int_x p(x)\big(-\log p(x)\big)dx \geq 0$$

- Entropy $H(X) = \mathbb{E}[-\log p(X)]$ = measure of uncertainty, or lack of information, we have about X
- Note: Uniform distribution has highest entropy and *Dirac* distribution has lowest entropy

Relative Entropy or Kullback-Leibler Divergence

- Assume distribution $q(x)$ used to decide on coding length of symbols drawn from $p(x)$. Expected length of encoding is given by *cross-entropy*:

$$\int_x p(x)\big(-\log q(x)\big)dx \geq H(p)$$

- Difference

$$D(p||q) = \int_x p(x)\big(\log \frac{p(x)}{q(x)}\big)dx \geq 0$$

  is called relative entropy or Kullback-Leibler divergence

- Note: Although not a distance, it can be used to measure how different two distributions are