Analysis of Polar Codes: Channel Polarization and Code Rate

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1 Introduction

Polar Codes, introduced by Erdal Arikan in 2009, are a class of block codes that can achieve the capacity of binary-input symmetric memoryless channels (BI-DMC) using a low-complexity encoding and decoding algorithm. Polar Codes utilize a method of channel polarization which enables them to achieve the capacity of a wide variety of binary-input symmetric memoryless channels. In this document, we analyze how the presence of a fraction of channels that remain unpolarized affects the rate of convergence of the code rate to the channel capacity and how polar codes are first ever provably capacity achieving codes.

2 Background and Basic Concepts

2.1 Shannon's Channel Capacity Theorem

Shannon's Channel Capacity Theorem states that for any Discrete Memoryless channel W, there exists an associated non-negative constant $I(W)$, known as channel capacity such that

- For any $\epsilon > 0$, one can communicate over a channel W at asymptotic rate $R = I(W)$ - ϵ with vanishing probability of miscommunication.
- For any rate $R > I(W)$, it is impossible to communicate at rate R without non-negligible probability of miscommunication *i.e.* , the probability of error at decoding approaches 1 as the number of channel uses increases.
- For rate $R = I(W) - \epsilon$ where ϵ is reffered to gap to capacity , we need to minimise it as much as we can for required error - correcting codes.

2.2 Performance of Linear Block Codes

Shannon's Thoerem shows that random linear block codes perform well such that in order to get rate $R = I(W) - \epsilon$, we can take block length $N = \frac{1}{\epsilon^2}$ with probability of decoding error $e^{-\epsilon^2 N}$.

But for random-codes there are disadvantages like :

- Non-Constructive Algorithms
- Exponential Time Decoding

So there is a need of explicit error - correcting codes with efficient encoding and decoding algorithms to communicate even at rates approaching capacity.

3 Channel Capacity and Binary Polar Codes

For a binary symmetric channel (BSC) with crossover probability p, the channel capacity C is given by the formula:

$$C = 1 - H(p)$$

where $H(p)$ is the binary entropy function defined as:

$$H(p) = -p\log_2(p) - (1-p)\log_2(1-p)$$

Polar codes employ a transformation that polarizes the synthesized bit channels over iterations, causing them to become either very good (reliable) or very bad (unreliable) at transmitting information.

Also , Polar codes have property of achieveing capacity at $N \to \infty$, with :

- poly(N) construction of message
- $O(N \log N)$ Encoding and Decoding Time Complexity

4 Enhanced Polar Transformation and Channel Behavior

4.1 Transformation and Channel Definition

Given the polar transformation $U \cdot G_N = X$, where U are the original bits and X are the transformed bits. The bits U_i are transformed into X_i such that U_i depends on $U_1, \ldots, U_{i-1}$ in specific ways due to the recursive structure of G_N.

4.2 Channel Passing and Receiving

After modulation (e.g., BPSK) and transmission through an AWGN channel, we receive a vector Y^N. Each bit U_i influences the formation of the entire received vector Y^N, but each virtual channel W_i associated with U_i is defined as:

$$W_i : U_i \to (Y^N, U^{i-1})$$

where U^{i-1} represents the sequence $U_1, \ldots, U_{i-1}$.

5 Entropy Analysis

5.1 Entropy of U_i Given Past U Values and Y^N

$$H(U_i|U_{i-1}, Y^N)$$

This is not generally equal for each U_i due to the different ways each bit is polarized. Polarization aims to make some channels (bits) very reliable (low entropy) and others very unreliable (high entropy).

5.2 Polarization Effect

As a result of the polarization transform, for most indices i, as N (the total number of channels) becomes large, the entropy $H(U_i|U_{i-1}, Y^N)$ tends towards 0 or 1. This is due to the nature of the transformation, which increasingly separates the transformed bits into those that are nearly deterministic and those that remain highly random.

6 Relating Entropy to Virtual Channels W_i

The entropy of each virtual channel W_i, which carries U_i over the noise and the past bits, is then defined as:

$$H(W_i) = H(U_i|U^{i-1}, Y^N)$$

This definition aligns with the behavior of $H(U_i|U_{i-1}, Y^N)$, which suggests that the polarization effect causes $H(W_i)$ to polarize similarly:

- For most i, $H(W_i)$ approaches 0 or 1.
- But in practical condition this might not be the case always , so lets consider that δ is the fraction of unpolarized channels *i.e.* entropy close to not either 0 or 1.

Let W be a binary-input symmetric channel. Then the polar transformation generates two channels:

- W^+ - This is typically the more reliable channel.
- W^- - This is typically the less reliable channel.

The transformation seeks to satisfy the following inequality in terms of entropy:

$$H(W^+) < H(W) < H(W^-)$$

This inequality highlights the effectiveness of polarization where W^+ becomes more reliable, and W^- less reliable because higher the entropy higher the randomness and higher the error in polarizing , thus allowing for the differentiation of channel capacities and their better utilization in coding.

7 Main Analysis

Claim 1. *If a delta fraction δ of channels remains unpolarized, the resulting polar code has a rate R that approaches the channel capacity C as the block length N tends to infinity and δ tends to zero.*

Proof. By the theory of channel polarization, as N (the block length of the code) increases, the fraction of channels that polarize towards having a capacity close to either 0 or 1 increases. The rate R of the Polar Code can be defined as $R = \frac{k}{N}$, where k is the number of reliable channels used for transmitting information. For large N, we have $k \approx N(1 - H(p))$, thus $R \approx 1 - H(p)$, which aligns with the capacity C.

The presence of δ fraction of unpolarized channels, having capacities not close to 0 or 1, implies that:

$$R = \frac{N(1-\delta)(1-H(p))}{N} = (1-\delta)(1-H(p))$$

As $N \to \infty$ and $\delta \to 0$, R approaches $1 - H(p)$ matching the capacity C. The fraction $1 - \delta$ of channels that are highly reliable converges to 1, allowing the code rate to approach the channel capacity. $\square$

8 Conclusion

The analysis confirms that Polar Codes are capable of achieving channel capacity under the assumption of large block lengths and negligible fractions of unpolarized channels. This property is fundamental for the utilization of Polar Codes in practical communication systems where maximizing data transmission rate is crucial.

References

1. E. Arıkan, “Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels,” *IEEE Transactions on Information Theory*, vol. 55, no. 7, pp. 3051–3073, Jul. 2009.

2. E. Arıkan, “Polar coding for lossless compression,” *IEEE Transactions on Information Theory*, vol. 56, no. 10, pp. 5448–5464, Oct. 2010.

3. E. Arıkan, “Polar coding theorem,” *IEEE Transactions on Information Theory*, vol. 57, no. 12, pp. 8313–8326, Dec. 2011.