Double Pendulum Inverted on a Cart: System Modeling

1 System

The Double Pendulum Inverted on a Cart (DPIC) system is a pendulum system with two joints, which typically start suspended vertically in the air, the pendulumup position. The pendulums are connected to a cart which can be powered externally to move horizontally and manipulate the system. This is our controller. See Figure 1.5 for a diagram. This problem has been used extensively to test linear and nonlinear control laws [6], see some of this research in [6], [7] and [8].

1.1 Modeling

The system is modeled so that the center of the cart begins at $(0, 0)$, in Cartesian coordinates. Following the notation in [9], m_0 is the mass of the cart, m_1, m_2 are the masses of the first and second pendulum link. x is the horizontal position of the cart, θ_1, θ_2 are the angles between each pendulum link and the vertical. l_1, l_2 are the distances from the base of each pendulum link to the center of mass for that pendulum link, L_1, L_2 are the lengths of the each pendulum link. I_1, I_2 are the moments of inertia of each pendulum link with respect to its center of mass. g is the gravitational constant and the force $u(t)$ is the control. Each pendulum link can move through all 360°. When released from suspension the DPIC will collapse, oscillating indefinitely. If friction is present, the DPIC will decelerate over time and converge to the only stable fixed point, the pendulum-down position, $\theta_1 = \theta_2 = \pi$. This occurs when released from any combination of initial angles unless control is used. However, the DPIC system provides a difficult problem with regard to control theory, there is only a single controlling force $u(t)$ and three degrees of freedom x, θ_1, θ_2. The control force $u(t)$ acts only horizontally on the cart through a motor.

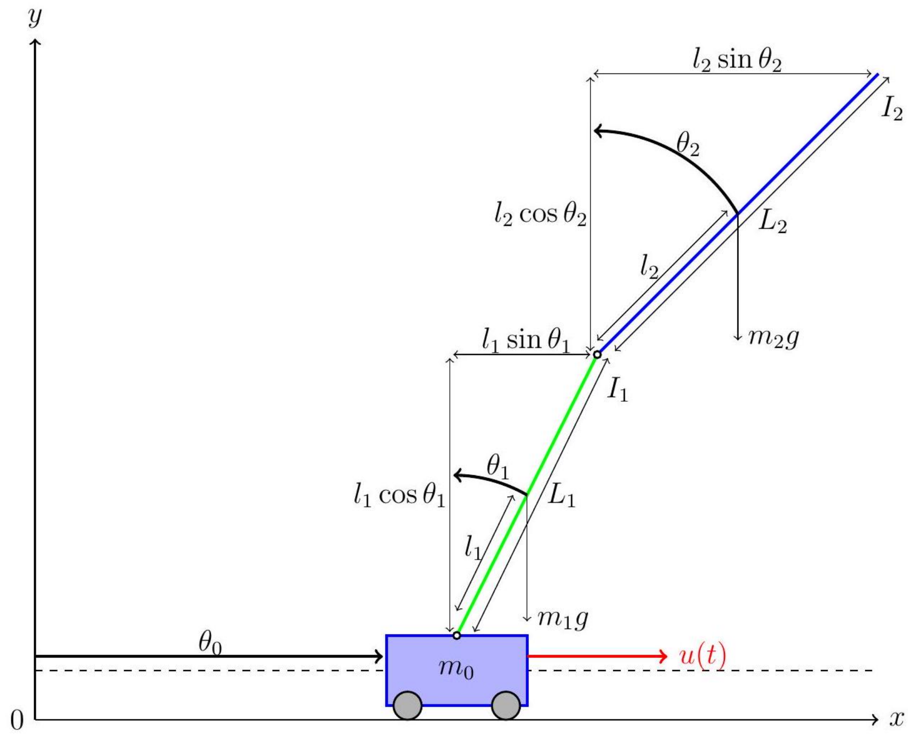

Figure 1.5: Double Pendulum Inverted on a Cart (DPIC)

1.2 Derivation via Lagrange

Next, it is necessary to calculate the equations of motion for the DPIC system. There are multiple methods to derive the system of equations modeling the DPIC system. For example, a Newtonian approach regarding forces would eventually calculate the system equations. However, it is most elegant and efficient to use the Lagrange equations for a particular Lagrangian [10].

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\boldsymbol{\theta}}}\right) - \frac{\partial L}{\partial \boldsymbol{\theta}} = \mathbf{q}$$

L is the Lagrangian, $\mathbf{q}$ is a vector of generalized forces which act in the direction of each component in θ, these forces are not included in the kinetic and potential energies of the cart and each pendulum link. The control force $u(t)$ is one of these forces. This method of derivation is similar to that of [9] but explains it in full. Let

$$\boldsymbol{\theta} = \begin{pmatrix} x \\ \theta_1 \\ \theta_2 \end{pmatrix}$$

Recall that x is the horizontal position of the cart and θ_1 and θ_2 are the angles of the first and second pendulum links to the vertical. Because of these components, the control force $u(t)$, which is the force acting on the cart, only acts horizontally and does not affect the angles directly. Negating other external forces, $\mathbf{q}$ is simply

$$\mathbf{q} = \begin{pmatrix} u(t) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u(t) = \mathbf{H}u(t)$$

Thus, looking at the Lagrange equation (1.1) for each component of θ the following system of equations is obtained.

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= u(t) \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} &= 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} &= 0 \end{aligned}$$

The Lagrangian, L is the difference between the total kinetic energy, E_{kin}, and the total gravitational potential energy, E_{pot} of the system.

$$L = E_{kin} - E_{pot}$$

These energies can be broken down into energies of the specific components. The cart, the first pendulum link, and the second pendulum link.

$$\begin{aligned} E_{kin} &= E_{kin}^{(0)} + E_{kin}^{(1)} + E_{kin}^{(2)} \\ E_{pot} &= E_{pot}^{(0)} + E_{pot}^{(1)} + E_{pot}^{(2)} \end{aligned}$$

Here the subscript notation denotes either kinetic or potential energy and the superscript indicates which component of the system is being referred to. $E_{kin}^{(0)}$ is the kinetic energy of the cart and $E_{pot}^{(2)}$ is the gravitational potential energy of the second, or top, pendulum link. Using Figure 1.5, the specific coordinates of each component in the DPIC system can be calculated, this will help calculate the Lagrangian (1.2). For the position of the cart one has

$$\begin{aligned} x_0 &= x \\ y_0 &= 0 \end{aligned}$$

The position of the midpoint of the first pendulum link

$$\begin{aligned} x_1 &= x + l_1 \sin\theta_1 \\ y_1 &= l_1 \cos\theta_1 \end{aligned}$$

The position of the midpoint of the second pendulum link

$$x_2 = x + L_1 \sin\theta_1 + l_2 \sin\theta_2$$
$$y_2 = L_1 \cos\theta_1 + l_2 \cos\theta_2$$

Using these coordinates the energy components can be calculated. First, using standard results from Newtonian mechanics [11], calculate the kinetic and potential energy, $E_{kin}^{(0)}$ and $E_{pot}^{(0)}$, for the cart

$$E_{kin}^{(0)} = \frac{1}{2} m_0 \dot{x}^2$$
$$E_{\text{pot}}^{(0)} = 0$$

Next look at the bottom pendulum link. Due to modeling each pendulum link with the center of mass at the midpoint of that link, the kinetic energy has two components. These are translational kinetic energy and rotational kinetic energy [11]

$$
\begin{aligned}
E_{kin}^{(1)} &= E_{kin}^{(1)}(\text{ trans }) + E_{kin}^{(1)}(\text{ rot })\\
E_{kin}^{(1)} &= \frac{1}{2} m_1 \left(\dot{x}_1^2 + \dot{y}_1^2\right) + \frac{1}{2} I_1 \dot{\theta}_1^2\\
E_{kin}^{(1)} &= \frac{1}{2} m_1 \left\{ \left(\frac{d}{dt}\left[x + l_1 \sin\theta_1\right] \right)^2 + \left(\frac{d}{dt}\left[l_1 \cos\theta_1\right] \right)^2 \right\} + \frac{1}{2} I_1 \dot{\theta}_1^2\\
E_{kin}^{(1)} &= \frac{1}{2} m_1 \left[\left(\dot{x} + l_1 \dot{\theta}_1 \cos\theta_1\right)^2 + \left(-l_1 \dot{\theta}_1 \sin\theta_1\right)^2 \right] + \frac{1}{2} I_1 \dot{\theta}_1^2\\
E_{kin}^{(1)} &= \frac{1}{2} m_1 \left[\dot{x}^2 + 2 l_1 \dot{x} \dot{\theta}_1 \cos\theta_1 + l_1^2 \dot{\theta}_1^2 \cos^2\theta_1 + l_1^2 \dot{\theta}_1^2 \sin^2\theta_1 \right] + \frac{1}{2} I_1 \dot{\theta}_1^2\\
E_{kin}^{(1)} &= \frac{1}{2} m_1 \left[\dot{x}^2 + 2 l_1 \dot{x} \dot{\theta}_1 \cos\theta_1 + l_1^2 \dot{\theta}_1^2 \right] + \frac{1}{2} I_1 \dot{\theta}_1^2\\
E_{kin}^{(1)} &= \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2} \left(m_1 l_1^2 + I_1 \right) \dot{\theta}_1^2 + m_1 l_1 \dot{x} \dot{\theta}_1 \cos\theta_1
\end{aligned}
$$

Furthermore

$$E_{pot}^{(1)} = m_1 g y_1$$
$$E_{pot}^{(1)} = m_1 g l_1 \cos\theta_1$$

Finally, calculate the energies for the top pendulum link. As before, for the same reasons, the kinetic energy has two components, translational and rotational.

$$
\begin{aligned}
E_{\text{kin}}^{(2)} &= \frac{1}{2}m_2\left(\dot{x}_2^2+\dot{y}_2^2\right)+\frac{1}{2}I_2\dot{\theta}_2^2\\
&= \frac{1}{2}m_2\left\{\left(\frac{d}{dt}\left[x+L_1\sin\theta_1+l_2\sin\theta_2\right]\right)^2+\left(\frac{d}{dt}\left[L_1\cos\theta_1+l_2\cos\theta_2\right]\right)^2\right\}\\
&+\frac{1}{2}I_2\dot{\theta}_2^2\\
&= \frac{1}{2}m_2\left[\left(\dot{x}+L_1\dot{\theta}_1\cos\theta_1+l_2\dot{\theta}_2\cos\theta_2\right)^2+\left(-L_1\dot{\theta}_1\sin\theta_1-l_2\dot{\theta}_2\sin\theta_2\right)^2\right]\\
&+\frac{1}{2}I_2\dot{\theta}_2^2\\
&= \frac{1}{2}m_2\left[\dot{x}^2+2L_1\dot{x}\dot{\theta}_1\cos\theta_1+2l_2\dot{x}\dot{\theta}_2\cos\theta_2+L_1^2\dot{\theta}_1^2\cos^2\theta_1\right.\\
&+2L_1l_2\dot{\theta}_1\dot{\theta}_2\cos\theta_1\cos\theta_2+l_2^2\dot{\theta}_2^2\cos^2\theta_2+L_1^2\dot{\theta}_1^2\sin^2\theta_1\\
&\left.+2L_1l_2\dot{\theta}_1\dot{\theta}_2\sin\theta_1\sin\theta_2+l_2^2\dot{\theta}_2^2\cos\theta_2\right]+\frac{1}{2}I_2\dot{\theta}_2^2\\
&= \frac{1}{2}m_2\left[\dot{x}^2+2L_1\dot{x}\dot{\theta}_1\cos\theta_1+2l_2\dot{x}\dot{\theta}_2\cos\theta_2+L_1^2\dot{\theta}_1^2\left(\cos^2\theta_1+\sin^2\theta_1\right)\right.\\
&\left.+2L_1l_2\dot{\theta}_1\dot{\theta}_2\left(\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2\right)+l_2^2\dot{\theta}_2^2\left(\cos^2\theta_2+\sin^2\theta_2\right)\right]\\
&+\frac{1}{2}I_2\dot{\theta}_2^2\\
&= \frac{1}{2}m_2\left[\dot{x}^2+2L_1\dot{x}\dot{\theta}_1\cos\theta_1+2l_2\dot{x}\dot{\theta}_2\cos\theta_2+L_1^2\dot{\theta}_1^2\right.\\
&\left.+2L_1l_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1-\theta_2\right)+l_2^2\dot{\theta}_2^2\right]+\frac{1}{2}I_2\dot{\theta}_2^2\\
&= \frac{1}{2}m_2\left[\dot{x}^2+2L_1\dot{x}\dot{\theta}_1\cos\theta_1+2l_2\dot{x}\dot{\theta}_2\cos\theta_2+L_1^2\dot{\theta}_1^2\right.\\
&\left.+2L_1l_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1-\theta_2\right)+l_2^2\dot{\theta}_2^2\right]+\frac{1}{2}I_2\dot{\theta}_2^2\\
&= \frac{1}{2}m_2\dot{x}^2+\frac{1}{2}m_2L_1^2\dot{\theta}_1^2+\frac{1}{2}\left(m_2l_2^2+I_2\right)\dot{\theta}_2^2+m_2L_1\dot{x}\dot{\theta}_1\cos_1\theta_1\\
&+m_2l_2\dot{x}\dot{\theta}_2\cos\theta_2+m_2L_1l_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1-\theta_2\right)
\end{aligned}
$$

2

Also

$$
\begin{aligned}
E_{\text{pot}}^{(2)} &= m_1gy_2\\
&= m_1g\left(L_1\cos\theta_1+l_2\cos\theta_2\right)
\end{aligned}
$$

Adding all of these kinetic and potential energies together gives the overall energies for the system.

$$
\begin{aligned}
E_{kin} =& E_{\text{kin}}^{(0)} + E_{\text{kin}}^{(1)} + E_{\text{kin}}^{(2)} \\
=& \frac{1}{2} m_0 \dot{x}^2 + \frac{1}{2} m_1 \dot{x}^2 + \frac{1}{2}\left(m_1 l_1^2 + I_1\right)\dot{\theta}_1^2 + m_1 l_1 \dot{x}\dot{\theta}_1 \cos\theta_1 \\
& + \frac{1}{2} m_2 \dot{x}^2 + \frac{1}{2} m_2 L_1^2 \dot{\theta}_1^2 + \frac{1}{2}\left(m_2 l_2^2 + I_2\right)\dot{\theta}_2^2 + m_2 L_1 \dot{x}\dot{\theta}_1 \cos\theta_1 \\
& + m_2 l_2 \dot{x}\dot{\theta}_2 \cos\theta_2 + m_2 L_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\left(\theta_1 - \theta_2\right) \\
=& \frac{1}{2}\left(m_0 + m_1 + m_2\right)\dot{x}^2 + \frac{1}{2}\left(m_1 l_1^2 + m_2 L_1^2 + I_1\right)\dot{\theta}_1^2 + \frac{1}{2}\left(m_2 l_2^2 + I_2\right)\dot{\theta}_1^2 \\
& + \left(m_1 l_1 + m_2 L_1\right)\dot{x}\dot{\theta}_1 \cos\theta_1 + m_2 l_2 \dot{x}\dot{\theta}_2 \cos\theta_2 + m_2 L_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos\left(\theta_1 - \theta_2\right)
\end{aligned}
$$

Also

$$
\begin{aligned}
E_{\text{pot}} &= E_{\text{pot}}^{(0)} + E_{\text{pot}}^{(1)} + E_{\text{pot}}^{(2)} \\
&= 0 + m_1 g l_1 \cos\theta_1 + m_1 g\left(L_1 \cos\theta_1 + l_2 \cos\theta_2\right) \\
&= g\left(m_1 l_1 + m_2 L_1\right)\cos\theta_1 + m_2 g l_2 \cos\theta_2
\end{aligned}
$$

The Lagrangian (1.2) is

$$
\begin{aligned}
L =& E_{\text{kin}} - E_{\text{pot}} \\
=& \frac{1}{2}\left(m_0 + m_1 + m_2\right)\dot{x}^2 + \frac{1}{2}\left(m_1 l_1^2 + m_2 L_1^2 + I_1\right)\dot{\theta}_1^2 + \frac{1}{2}\left(m_2 l_2^2 + I_2\right)\dot{\theta}_2^2 \\
& + \left(m_1 l_1 + m_2 L_1\right)\cos\theta_1 \dot{x}\dot{\theta}_1 + m_2 l_2 \cos\theta_2 \dot{x}\dot{\theta}_2 + m_2 L_1 l_2 \cos\left(\theta_1 - \theta_2\right)\dot{\theta}_1 \dot{\theta}_2 \\
& - g\left(m_1 l_1 + m_2 L_1\right)\cos\theta_1 - m_2 g l_2 \cos\theta_2
\end{aligned}
$$

Now that the Lagrangian is known, explicitly calculate the partial, and full derivatives for the system of equations (1.1).

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= u(t) \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} &= 0 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} &= 0
\end{aligned}
$$

Now, calculating the derivatives

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{x}} &= (m_0+m_1+m_2)\,\dot{x} + (m_1l_1+m_2L_1)\cos\theta_1\dot{\theta}_1 + m_2l_2\cos\theta_2\dot{\theta}_2 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) &= (m_0+m_1+m_2)\,\ddot{\theta}_0 + (m_1l_1+m_2L_1)\cos\theta_1\ddot{\theta}_1 + m_2l_2\cos\theta_2\ddot{\theta}_2 \\
&\quad - (m_1l_1+m_2L_1)\sin\theta_1\dot{\theta}_1^2 - m_2l_2\sin\theta_2\dot{\theta}_2^2 \\
\frac{\partial L}{\partial x} &= 0 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= (m_0+m_1+m_2)\,\ddot{\theta}_0 + (m_1l_1+m_2L_1)\cos\theta_1\ddot{\theta}_1 + m_2l_2\cos\theta_2\ddot{\theta}_2 \\
&\quad - (m_1l_1+m_2L_1)\sin\theta_1\dot{\theta}_1^2 - m_2l_2\sin\theta_2\dot{\theta}_2^2 \\
&= u(t)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{\partial L}{\partial \dot{\theta}_1} = \left(m_1l_1^2+m_2L_1^2+I_1\right)\dot{\theta}_1 + (m_1l_1+m_2L_1)\cos\theta_1\dot{x} \\
&+ m_2L_1l_2\cos(\theta_1-\theta_2)\,\dot{\theta}_2 \\
&\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) = \left(m_1l_1^2+m_2L_1^2+I_1\right)\ddot{\theta}_1 + (m_1l_1+m_2L_1)\cos\theta_1\ddot{\theta}_0 \\
&+ m_2L_1l_2\cos(\theta_1-\theta_2)\,\ddot{\theta}_2 - (m_1l_1+m_2L_1)\sin\theta_1\dot{x}\dot{\theta}_1 \\
&- m_2L_1l_2\sin(\theta_1-\theta_2)\left(\dot{\theta}_1-\dot{\theta}_2\right)\dot{\theta}_2 \\
&\frac{\partial L}{\partial \theta_1} = -(m_1l_1+m_2L_1)\sin\theta_1\dot{x}\dot{\theta}_1 - m_2L_1l_2\sin(\theta_1-\theta_2)\,\dot{\theta}_1\dot{\theta}_2 \\
&+ g\,(m_1l_1+m_2L_1)\sin\theta_1 \\
&\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = \left(m_1l_1^2+m_2L_1^2+I_1\right)\ddot{\theta}_1 + (m_1l_1+m_2L_1)\cos\theta_1\ddot{\theta}_0 \\
&+ m_2L_1l_2\cos(\theta_1-\theta_2)\,\ddot{\theta}_2 - (m_1l_1+m_2L_1)\sin\theta_1\dot{x}\dot{\theta}_1 \\
&- m_2L_1l_2\sin(\theta_1-\theta_2)\left(\dot{\theta}_1-\dot{\theta}_2\right)\dot{\theta}_2 \\
&+ (m_1l_1+m_2L_1)\sin\theta_1\dot{x}\dot{\theta}_1 + m_2L_1l_2\sin(\theta_1-\theta_2)\,\dot{\theta}_1\dot{\theta}_2 \\
&- g\,(m_1l_1+m_2L_1)\sin\theta_1 \\
&\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} = \left(m_1l_1^2+m_2L_1^2+I_1\right)\ddot{\theta}_1 + (m_1l_1+m_2L_1)\cos\theta_1\ddot{\theta}_0 \\
&+ m_2L_1l_2\cos(\theta_1-\theta_2)\,\ddot{\theta}_2 + m_2L_1l_2\sin(\theta_1-\theta_2)\,\dot{\theta}_2^2 \\
&- g\,(m_1l_1+m_2L_1)\sin\theta_1 \\
&= 0
\end{aligned}
$$

finally,

$$
\begin{aligned}
\frac{\partial L}{\partial \dot{\theta}_2} =& m_2 l_2 \cos\theta_2 \dot{x} + m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\dot{\theta}_1 + \left(m_2 l_2^2 + I_2\right)\dot{\theta}_2 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) =& m_2 l_2 \cos\theta_2 \ddot{\theta}_0 + m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_1 + \left(m_2 l_2^2 + I_2\right)\ddot{\theta}_2 \\
& - m_2 l_2 \sin\theta_2 \dot{x}\dot{\theta}_2 - m_2 L_1 l_2 \sin(\theta_1 - \theta_2)\left(\dot{\theta}_1 - \dot{\theta}_2\right)\dot{\theta}_1 \\
\frac{\partial L}{\partial \theta_2} =& - m_2 l_2 \sin\theta_2 \dot{x}\dot{\theta}_2 + m_2 L_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 \\
& + m_2 g l_2 \sin\theta_2 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} =& m_2 l_2 \cos\theta_2 \ddot{\theta}_0 + m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_1 + \left(m_2 l_2^2 + I_2\right)\ddot{\theta}_2 \\
& - m_2 l_2 \sin\theta_2 \dot{x}\dot{\theta}_2 - m_2 L_1 l_2 \sin(\theta_1 - \theta_2)\left(\dot{\theta}_1 - \dot{\theta}_2\right)\dot{\theta}_1 \\
& + m_2 l_2 \sin\theta_2 \dot{x}\dot{\theta}_2 - m_2 L_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1\dot{\theta}_2 \\
& - m_2 g l_2 \sin\theta_2 \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} =& m_2 l_2 \cos\theta_2 \ddot{\theta}_0 + m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_1 + \left(m_2 l_2^2 + I_2\right)\ddot{\theta}_2 \\
& - m_2 L_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1^2 - m_2 g l_2 \sin\theta_2 \\
=& 0
\end{aligned}
$$

This gives the calculated system, in full

$$
\begin{aligned}
&(m_0 + m_1 + m_2)\ddot{\theta}_0 + (m_1 l_1 + m_2 L_1)\cos\theta_1 \ddot{\theta}_1 + m_2 l_2 \cos\theta_2 \ddot{\theta}_2 \\
&- (m_1 l_1 + m_2 L_1)\sin\theta_1 \dot{\theta}_1^2 - m_2 l_2 \sin\theta_2 \dot{\theta}_2^2 = u(t) \\
&\left(m_1 l_1^2 + m_2 L_1^2 + I_1\right)\ddot{\theta}_1 + (m_1 l_1 + m_2 L_1)\cos\theta_1 \ddot{\theta}_0 \\
&+ m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_2 + m_2 L_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_2^2 \\
&- g(m_1 l_1 + m_2 L_1)\sin\theta_1 = 0 \\
&m_2 l_2 \cos\theta_2 \ddot{\theta}_0 + m_2 L_1 l_2 \cos(\theta_1 - \theta_2)\ddot{\theta}_1 + \left(m_2 l_2^2 + I_2\right)\ddot{\theta}_2 \\
&- m_2 L_1 l_2 \sin(\theta_1 - \theta_2)\dot{\theta}_1^2 - m_2 g l_2 \sin\theta_2 = 0
\end{aligned}
$$

1.3 2nd-Order System

The system (1.3-1.5) is a nonlinear second-order system of the form

$$\mathbf{D}(\boldsymbol{\theta})\ddot{\boldsymbol{\theta}} + \mathbf{C}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})\dot{\boldsymbol{\theta}} + \mathbf{G}(\boldsymbol{\theta}) = \mathbf{H}u$$

where

$$
\mathbf{D}(\boldsymbol{\theta}) = \begin{bmatrix} m_0+m_1+m_2 & (m_1 l_1 + m_2 L_1)\cos\theta_1 & m_2 l_2 \cos\theta_2 \\ (m_1 l_1 + m_2 L_1)\cos\theta_1 & m_1 l_1^2 + m_2 L_1^2 + I_1 & m_2 L_1 l_2 \cos(\theta_1-\theta_2) \\ m_2 l_2\cos\theta_2 & m_2 L_1 l_2 \cos(\theta_1-\theta_2) & m_2 l_2^2 + I_2 \end{bmatrix}
$$

$$
\mathbf{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \begin{bmatrix} 0 & -(m_1 l_1 + m_2 L_1)\sin\theta_1\dot\theta_1 & -m_2 l_2 \sin\theta_2\dot\theta_2 \\ 0 & 0 & m_2 L_1 l_2 \sin(\theta_1-\theta_2)\dot\theta_2 \\ 0 & -m_2 L_1 l_2 \sin(\theta_1-\theta_2)\dot\theta_1 & \end{bmatrix}
$$

$$
\mathbf{G}(\boldsymbol{\theta}) = \begin{bmatrix} -(m_1 l_1 + m_2 L_1) g\sin\theta_1 \\ -m_2 g l_2 \sin\theta_2 \end{bmatrix}
$$

$$
\mathbf{H} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
$$

To simplify the system, we adopt the choices from [9]. Here

$$
\begin{aligned} \Longrightarrow l_1 &= \frac{1}{2}L_1 \qquad & l_2 &= \frac{1}{2}L_2 \\ I_1 &= \frac{1}{12}m_1 L_1^2 & I_2 &= \frac{1}{12}m_2 L_2^2 \end{aligned}
$$

this updates the system with $\mathbf{D}(\boldsymbol{\theta})$

$$
= \begin{bmatrix} m_0+m_1+m_2 & \left(\frac{1}{2}m_1+m_2\right)L_1\cos\theta_1 & \frac{1}{2}m_2 L_2\cos\theta_2 \\ \left(\frac{1}{2}m_1+m_2\right)L_1\cos\theta_1 & \left(\frac{1}{3}m_1+m_2\right)L_1^2 & \frac{1}{2}m_2 L_1 L_2 \cos(\theta_1-\theta_2) \\ \frac{1}{2}m_2 L_2\cos\theta_2 & \frac{1}{2}m_2 L_1 L_2\cos(\theta_1-\theta_2) & \frac{1}{3}m_2 L_2^2 \end{bmatrix}
$$

and

$$
\mathbf{C}(\boldsymbol{\theta},\dot{\boldsymbol{\theta}}) = \begin{bmatrix} 0 & -\left(\frac{1}{2}m_1+m_2\right)L_1\sin\theta_1\dot\theta_1 & -\frac{1}{2}m_2 L_2\sin\theta_2\dot\theta_2 \\ 0 & 0 & \frac{1}{2}m_2 L_1 L_2\sin(\theta_1-\theta_2)\dot\theta_2 \\ 0 & -\frac{1}{2}m_2 L_1 L_2 \sin(\theta_1-\theta_2)\dot\theta_1 & 0 \end{bmatrix}
$$

$$
\mathbf{G}(\boldsymbol{\theta}) = \begin{bmatrix} 0 \\ -\frac{1}{2}(m_1+m_2)L_1 g\sin\theta_1 \\ -\frac{1}{2}m_2 g L_2\sin\theta_2 \end{bmatrix}
$$

$$
\mathbf{H} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}
$$

Note. $\mathbf{D}(\boldsymbol{\theta})$ is symmetric and nonsingular, $\Longrightarrow$ $\mathbf{D}^{-1}(\boldsymbol{\theta})$ exists and is also symmetric [9].